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# Hadamard's variation and Poincaré's lemma on a certain non-convex domain

By

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**Abstract**

Let $(X,\mu_X)$ and $(B,\mu_B)$ be Wiener spaces. Let $\Omega$ be a subset in a product Wiener space $W = X \times B$ with the product measure $\mu_X \times \mu_B$. Let $\Omega_x = \{z \in B \mid (x,z) \in \Omega\}$ for $x \in X$ and $U = \{x \in X \mid \Omega_x \neq \emptyset\}$. We assume that $U$ and $\Omega_x$ $(x \in U)$ are convex sets. Let $\alpha$ be a closed 1-form on $\Omega$. We give a representation formula of $f$ to the equation $df = \alpha$ in terms of $\alpha$ and an estimate for the $L^2$-norm of $f$ using Green operators which are inverse operators of the Hodge-Kodaira operators on $\Omega_x, U$ and Hadamard's variation of them.

## § 1. Introduction

Let $\Omega$ be a bounded domain with smooth boundary of a Euclidean space. Suppose that there exists $w_0 \in \Omega$ such that the segment between $w_0$ and any point $w \in \Omega$ belongs to $\Omega$. Let $\alpha$ be a smooth closed 1-form on $\Omega$. Setting $f(w) = \int_0^1 \Big(\alpha(w_0 + t(w-w_0)), w - w_0\Big) dt$, we obtain that $df(w) = \alpha(w)$. However, this representation of $f$ cannot be extended to infinite dimensional cases. Let $(W,H,\mu)$ be an abstract Wiener space. Consider an $H$-open subset $\Omega \subset W$. We call a map from $\Omega$ to $\wedge^p H^*$ a $p$-form. If the domain $\Omega$ is convex and satisfies some good properties, the Hodge-Kodaira type operator $\Box$ with absolute boundary condition can be defined on $L^2(\Omega \to \wedge^p H^*, d\mu)$. We have $-dd^*\Box^{-1}\alpha = \alpha$ and $\|d^*\Box^{-1}\alpha\|_{L^2(\Omega,d\mu)} \le p^{-1/2}\|\alpha\|_{L^2(\Omega,d\mu)}$ for a closed $p$-form $\alpha$ on $\Omega$, where $d^*$ is the adjoint operator of the exterior differential operator $d$ in $L^2(\mu)$. In this estimate, the key is that the spectral bottom of $-\Box$ is strictly positive which follows from the convexity. We give a representation formula of $f$ in terms of the closed 1-form $\alpha$ on a certain non-convex subset $\Omega$ of a product Wiener space which may give an estimate for the Poincaré constant. See (3.17) and (3.18). In particular,

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we are interested in dimension independent estimate. Note that even if the domain $\Omega$ is contractible open set, it is not trivial that the closed form on $\Omega$ is exact in the Sobolev space category.

Now, we explain what kind of sets we are interested in. Let $X$ and $B$ be Wiener spaces with Wiener measures $\mu_X, \mu_B$. Let $W = X \times B$ be the product space with the product measure $\mu_W = \mu_X \times \mu_B$. We denote the elements in $X$ and $B$ by $x$ and $z$ respectively and $w = (x, z) \in W = X \times B$. Consider a subset $\Omega \subset W$ which satisfies the following.

**Assumption 1.1.** (1) *$\Omega$ is an $H$-open set.*
(2) *For $x \in X$, set $\Omega_x = \{z \in B \mid (x, z) \in \Omega\}$ and $U = \{x \in X \mid \Omega_x \neq \emptyset\}$. Then $\Omega_x$ is a $H$-convex set for any $x \in U$. $U$ is also a $H$-convex set with $\mu_X(U) > 0$ and it holds that $\varepsilon_U = \mathrm{essinf}\,\{\mu_B(\Omega_x) \mid x \in U\} > 0$.*

When $U \subset X$ and $V \subset B$ are $H$-convex sets, the product space $\Omega = U \times V$ satisfies Assumption 1.1 and $\Omega$ itself is an $H$-convex set. Suppose that $\Omega$ is an open set and $\Omega_x$ are usual open convex sets. For example, if $W$ is finite dimension, this holds. In this case, $\Omega$ is $C^\infty$-homotopy equivalent to $U$. So the de Rham cohomologies for all dimensions are trivial. So it might be natural to conjecture that for any closed form $\alpha$, there exists $\beta$ such that $d\beta = \alpha$ on $\Omega$ which satisfies Assumption 1.1.

Kusuoka [7, 8] gave sufficient conditions of $\Omega$ on which Poincaré's type vanishing lemma holds in local Sobolev space category. He gave a representation formula of $\beta$ in terms of $\alpha$. See [9] also. Our strategy is different from [7]. For a closed 1-form $\alpha$ on $\Omega$, we give an explicit expression of $f$ to the equation $df = \alpha$ in terms of $\alpha$ by using the Green operators on $\Omega_x$ and their Hadamard's variation in Section 3. Note that our representation is limited to 1-form at the moment. The Hodge-Kodaira operator on convex domain in Wiener space is studied by Shigekawa [14, 13]. In Section 2, we recall necessary properties of Hodge-Kodaira operator on a convex domain based on his papers. In Section 4, we give Hadamard's variational formula in Wiener spaces and explain a proof of a key estimate (3.14).

Here, we show an example of $\Omega$.

**Example 1.1.** Let $w = (w_1, w_2)$ be the two dimensional Brownian motion. Let $x = w_1, z = w_2$. Let $0 < \theta < \theta' < 1$, $m \in \mathbb{N}$, $a > 0$ and set

$$F(x,z) = \|\bar{z}\|_{2m,\theta/2}^{2m} + \|C_{x,z}\|_{m,\theta}^{m} - a, \tag{1.1}$$

where $\bar{z}(s,t) = z(t) - z(s)$ $C_{x,z}(s,t) = \int_s^t (x(u) - x(s))dz(u)$ for $0 \le s \le t \le 1$. We define $\Omega = \{w \mid F(x,z) < 0, \|\bar{x}\|_{2m,\theta'/2}^{2m} < a\}$. In this case,

$$U = \left\{x \mid \|\bar{x}\|_{2m,\theta'/2}^{2m} < a\right\}, \tag{1.2}$$

$$\Omega_x = \{z \mid F(x,z) < 0\}. \tag{1.3}$$

We explain the norm $\| \|_{m,\theta}$. We denote
$\Delta = \{(s,t) \in \mathbb{R}^2 \mid 0 \le s \le t \le 1\}$. Let $E$ be a normed linear space. For a map $\phi : \Delta \to E$, we define

$$\|\phi\|_{m,\theta} = \left[\int_0^1 \left\{\int_0^t \frac{|\phi(s,t)|^m}{(t-s)^{2+m\theta}} ds\right\} dt\right]^{1/m}. \tag{1.4}$$

The iterated integral $C_{x,z}$ is important in rough path analysis. See [2] in which we prove weak Poincaré inequalities on a variant of the above $\Omega$. Actually, it is not difficult to prove the Poincaré inequality on the above $\Omega$ by Remark 3.2 in [2]. However, the existence of the spectral gap does not imply the solubility of $df = \alpha$ for a closed 1-form.

The subset $\Omega$ in Example 1.1 is a subset of 2-dimensional Wiener space. To study the de Rham cohomology in Sobolev space's category of loop spaces over compact Riemannian manifolds, we need to consider higher dimensional version of the above example.

In this note, we give just ideas and sketches of the proofs and some necessary conditions are not clearly stated and some statements are not proved yet at the moment. We will publish complete statements and proofs and further studies in the near future.

## § 2. Hodge-Kodaira operator on a convex domain

In this section, we consider a Hodge-Kodaira operator on a convex domain in a Wiener space $B$. Let $D$ denote the $H$-derivative and let $d$ be the exterior differential operator based on $H$-derivative. Let $F \in \mathbb{D}^{\infty}_{\infty-}(B,\mathbb{R})$ be an $H$-$C^{\infty}$- function on $B$ and assume that $D^2F(z)$ is non-negative definite for almost all $z$ and $|DF(z)|^{-1} \in L^p(B, d\mu_B)$ for sufficiently large $p$. We consider a positive measure domain $\Omega = \{z \in B \mid F(z) < 0\}$ which is thought as a convex domain. We refer to [14, 12, 5] for basic results of Hodge-Kodaira operator and the Poincaré inequality on this set. Let $d^*$ be the adjoint of $d$ in $L^2(\Omega, \mu_B)$ and we define $L = -d^*d(= -D^*D)$ which acts on functions and $\Box_\Omega = -(dd^* + d^*d)$, where we impose the Neumann boundary condition and the absolute boundary condition. That is, their cores are given by

$$\mathrm{D}_L = \left\{f \in \mathbb{D}^{\infty}_{\infty-}(B,\mathbb{R}) \mid (Df(z), n(z)) = 0 \ \nu\text{-a.s. z}\right\}$$
$$\mathrm{D}_{\Box_\Omega} = \{\alpha \in \mathbb{D}^{\infty}_{\infty-}(B, \wedge^p H^*) \mid \iota(n)\alpha|_S(z) = 0,\ \iota(n)d\alpha|_S(z) = 0\ \nu\text{-a.s. } z\},$$

where $n(z)$ is the unit outer normal vector field on the boundary of $\Omega$, $S = \{z \in B \mid F(z) = 0\}$ and $\iota(n)$ denotes the interior product.
$d\nu(z) = |DF(z)|\delta(F(z))d\mu_B(z)$ is the induced Gaussian surface measure and $\delta$ denotes the $\delta$-function which has the mass 1 at 0. See [3, 4, 10, 14]. $n(z)$ is explicitly written as

$$n(z) = \frac{DF(z)}{|DF(z)|}, \tag{2.1}$$

where we use the natural identification by the Riesz theorem. If $B$ is finite dimension and $\Omega$ is a bounded domain with smooth boundary, the essential self-adjointness of the above operator is known. However, it seems that essential self-adjointness of them in infinite dimensional cases are not well studied. We assume that $L$ and $\Box_\Omega$ are essentially self-adjoint on the above cores. The result (2) is proved in [14]. We refer to [14, 10] for (3).

**Theorem 2.1.** (1) *$0$ is an eigenvalue of $-L$ with multiplicity $1$ and there exists a spectral gap at $0$.*
(2) $\inf \sigma(-\Box_\Omega) \geq p$ *when $\Box_\Omega$ acts on $p$-form.*
(3) *Let $\theta$ and $\eta$ be a smooth $(p-1)$-form and a $p$-form on $\Omega$ respectively. Then the following integration by parts formula holds:*

$$\int_\Omega (d\theta(z), \eta(z))\, d\mu_B(z) = \int_\Omega (\theta(z), d^*\eta(z))\, d\mu_B(z) + \int_S (\theta(z), \iota(n)\eta(z))\, d\nu(z). \tag{2.2}$$

Let $\beta$ be a smooth $p$-form on $\Omega$. Note that $d\Box_\Omega^{-1}\beta = \Box_\Omega^{-1}d\beta$. This is proved as follows. By the essential self-adjointness of $\Box_\Omega$, $\{\Box_\Omega\gamma \mid \gamma \in \mathrm{D}_{\Box_\Omega}\}$ is dense in $L^2$. Moreover, $\mathrm{D}(\Box_\Omega) \subset \mathrm{D}(\overline{dd^*}) \cap \mathrm{D}(\overline{d^*d}) \cap \mathrm{D}(\bar{d}) \cap \mathrm{D}(d^*)$. Let $\gamma$ be a smooth $(p+1)$- form which belongs to $\mathrm{D}_{\Box_\Omega}$. Using (2.2) and Theorem 3.1 in [14],

$$\begin{aligned}
\left(d\Box_\Omega^{-1}\beta, \Box_\Omega\gamma\right) &= \left(dd\Box_\Omega^{-1}\beta, d\gamma\right) + \left(d^*d\Box_\Omega^{-1}\beta, d^*\gamma\right) \\
&= \left(d^*d\Box_\Omega^{-1}\beta, d^*\gamma\right) \\
&= \left(d\Box_\Omega^{-1}\beta, dd^*\gamma\right) + \left(d^*\Box_\Omega^{-1}\beta, d^*d^*\gamma\right) \\
&= (\beta, d^*\gamma) \\
&= (d\beta, \gamma) \\
&= (\Box_\Omega^{-1}d\beta, \Box_\Omega\gamma).
\end{aligned} \tag{2.3}$$

This shows $d\Box_\Omega^{-1}\beta = \Box_\Omega^{-1}d\beta$. Now suppose that $\beta$ is a closed 1-form. Then using $d\Box_\Omega^{-1}\beta = \Box_\Omega^{-1}d\beta = 0$ and $-(dd^* + d^*d)\Box_\Omega^{-1}\beta = \beta$, we have

$$-dd^*\Box_\Omega^{-1}\beta = \beta. \tag{2.4}$$

By the integration by parts formula (2.2),

$$\int_\Omega d^*\Box_\Omega^{-1}\beta d\mu_B(z) = \int_S \left(\Box_\Omega^{-1}\beta(z), n(z)\right) d\nu(z) = 0. \tag{2.5}$$

Next, let $h$ be a smooth function. By applying the formula (2.4) to $\beta = dh$, we have

$$d\left(d^*\Box_\Omega^{-1}dh + h\right) = 0. \tag{2.6}$$

Since $\Omega$ is $H$-convex, $d^*\Box_\Omega^{-1}dh - h$ is a constant. By (2.5), we get

$$-d^*\Box_\Omega^{-1}dh = h - \frac{1}{\mu_B(\Omega)}\int_\Omega h(z)d\mu_B(z). \tag{2.7}$$

Also we note that

$$\begin{aligned}\int_\Omega |d^*\Box_\Omega^{-1}\beta(z)|^2 d\mu_B(z) + \int_\Omega |d\Box_\Omega^{-1}\beta(z)|^2 d\mu_B(z) &= -\int_\Omega (\Box_\Omega^{-1}\beta(z), \beta(z))d\mu_B(z)\\ &\le \|\sqrt{-\Box_\Omega}^{-1}\|_{op}^2 \|\beta\|_{L^2(\Omega,\mu_B)}^2\\ &\le \|\beta\|_{L^2(\Omega,\mu_B)}^2.\end{aligned} \tag{2.8}$$

This estimate and (2.7) give an estimate for the Poincaré constant. That is,

$$\left\| h - \frac{1}{\mu_B(\Omega)}\int_\Omega h d\mu_B \right\|_{L^2(\Omega, d\mu_B)}^2 \le \|dh\|_{L^2(\Omega,d\mu_B)}^2. \tag{2.9}$$

## § 3. A solution of $df = \alpha$

We denote the $H$-derivative, exterior derivative on $B$ by $D_z, d_z$ and them on $X$ by $D_x, d_x$. We also denote the Hodge-Kodaira operator on $\Omega_x$ by $\Box_{\Omega_x}$. In this section and the next section, we consider a domain $\Omega$ of $W = X \times B$ which satisfies Assumption 1.1 and the following.

**Assumption 3.1.** (1) *$U$ is a convex domain with positive measure in the sense of Section* 2.
(2) *There exists an $H$-$C^\infty$ function $F(w) = F(x,z) \in \mathbb{D}^\infty_{\infty-}(W,\mathbb{R})$ such that $\Omega = \{w = (x,z) \mid F(w) < 0, x \in U\}$.*
(3) *For sufficiently large $p > 1$,*

$$\int_W |D_z F(x,z)|^{-p} d\mu_W(w) < +\infty. \tag{3.1}$$

(4) *For all $x \in U$, $D_z^2 F(x,z)$ is non-negative for almost all $z$.*

Let $\alpha$ be a closed 1-form on $\Omega$. Suppose that there exists $f \in \mathbb{D}^1_2(\Omega,\mathbb{R})$ such that $df = \alpha$ on $\Omega$. We aim to get an explicit expression and estimate of $f$ in terms of $\alpha$. We have

$$\begin{aligned}\alpha &= \sum_{i=1}^\infty \alpha_{1,i}(x,z)dz_i + \sum_{j=1}^\infty \alpha_{2,j}(x,z)dx_j\\ &=: \alpha^x + \alpha^z \qquad (x \in U, z \in V),\end{aligned} \tag{3.2}$$

where $dz_i \in B^*, dx_j \in X^*$. It is easy to check that for each $x \in U$, $\alpha^x$ is a closed 1-form on $\Omega_x$. Also it holds that on $\Omega$ for $v$ which is an element of the Cameron-Martin subspace of $X$,

$$(D_x)_v\alpha^x = d_z(\alpha^z, v). \tag{3.3}$$

Let

$$g(x,z) = -d_z^*\Box_{\Omega_x}^{-1}\alpha^x. \tag{3.4}$$

Then it holds that

$$\int_\Omega g(w)^2 d\mu(w) \le \int_U \left(\int_{\Omega_x} |\alpha^x(z)|^2 d\mu_B(z)\right) d\mu_X(x) \le \int_\Omega |\alpha(w)|^2 d\mu_W(w). \tag{3.5}$$

It is plausible that if the map $x \to \Omega_x$ is smooth and $\alpha$ is also smooth, then $g(x,z)$ is a smooth function on $\Omega$ in the sense of Malliavin. Of course, this is the subject of Hadamard's variation. By the result in Section 2, $d_z g(x,z) = \alpha^x(z)$. Hence $d_z f(x,z) - d_z g(x,z) = \alpha^x(z) - \alpha^x(z) = 0$. Since $\Omega_x$ is a convex set, the difference $f(x,z) - g(x,z)$ is a constant $\mu_B - a.s.\ z$. That is, there exists a smooth function $h$ such that

$$f(x,z) - g(x,z) = h(x) \qquad \text{for almost all } (x,z) \in \Omega. \tag{3.6}$$

Actually $h(x) = \mu_B(\Omega_x)^{-1}\int_{\Omega_x} f(x,z)d\mu_B(z)$ holds. Since we have already shown that $\|g\|_{L^2(\Omega,d\mu_W)}$ is bounded by $\|\alpha\|_{L^2(\Omega,d\mu_W)}$, we need to estimate $\|h\|_{L^2(\Omega,d\mu_W)}$ for the estimate of $\|f\|_{L^2(\Omega,d\mu_W)}$. By the Fubini theorem,

$$\int_\Omega |h(x)|^2 d\mu_W(w) = \int_U |h(x)|^2 \mu_B(\Omega_x) d\mu_X(x). \tag{3.7}$$

So we estimate $\|h\|_{L^2(U,d\mu_X)}$. If $h$ is in the domain of $d$ and $dh \in L^2(U, d\mu_W)$, then we can define

$$h_U(x) := -d_x^*\Box_U^{-1} d_x h(x), \qquad x \in U \tag{3.8}$$

and

$$\|h_U\|_{L^2(U,d\mu_X)} \le \|dh\|_{L^2(U,d\mu_X)}. \tag{3.9}$$

Moreover, we have $dh_U(x) = dh(x)\ x \in U$. Because $U$ is $H$-convex, $h_U(x) - h(x)$ is almost surely constant on $U$. Hence $h \in L^2(U, d\mu_X)$ and

$$h_U(x) = h(x) - \frac{1}{\mu_X(U)}\int_U h(x)d\mu_X(x) = -d_x^*\Box_U^{-1} d_x h(x) \qquad x \in U. \tag{3.10}$$

We need to show that $dh \in L^2(U, d\mu_X)$ to use the representation (3.10). Since

$$\begin{aligned}\int_\Omega |dh(x)|^2 d\mu(w) &= \int_U |dh(x)|^2 \mu_B(\Omega_x) d\mu_X(x)\\ &\geq \varepsilon_U \int_U |dh(x)|^2 d\mu_X(x),\end{aligned} \tag{3.11}$$

we need only to estimate $\|dh\|_{L^2(\Omega, d\mu_W)}$. By (3.6),

$$\begin{aligned}(D_x)_v h(x) &= (D_x)_v \left(f(x,z) - g(x,z)\right)\\ &= (\alpha^z(x), v) + (D_x)_v d_z^* \Box_{\Omega_x}^{-1} \alpha^x\\ &= (\alpha^z, v) + d_z^* \left(D_x(\Box_{\Omega_x}^{-1}), v\right) \alpha^x + d_z^* \Box_{\Omega_x}^{-1} (D_x)_v \alpha^x\\ &= (\alpha^z, v) + d_z^* \left(D_x(\Box_{\Omega_x}^{-1}), v\right) \alpha^x + d_z^* \Box_{\Omega_x}^{-1} d_z(\alpha^z, v), \qquad (x,z) \in \Omega,\end{aligned} \tag{3.12}$$

where we have used (3.3). By (2.7),

$$(D_x)_v h(x) = d_z^* \left\{\left(D_x(\Box_{\Omega_x}^{-1}), v\right) \alpha^x\right\} + \frac{1}{\mu_B(\Omega_x)} \int_{\Omega_x} (\alpha^z(x), v) d\mu_B(z). \tag{3.13}$$

If $\Omega = U \times V$, then the term $d_z^* \left\{\left(D_x(\Box_{\Omega_x}^{-1}), v\right) \alpha^x\right\}$ vanishes. If it is not the case, $\left(D_x(\Box_{\Omega_x}^{-1}), v\right)$ can be calculated by the Hadamard's variational formula. Note that (3.13) shows $d_z^* \left\{D_x \left(\Box_{\Omega_x}^{-1}) \alpha^x\right)\right\}(w)$ is independent of $z$ actually. It is plausible that there exists a nonnegative function $\Phi_F(w) \in \cap_{p>1} L^p(\Omega, d\mu_W)$ which is independent of $\alpha$ such that

$$\int_\Omega |\left(d_z^* \left\{D_x(\Box_{\Omega_x}^{-1}) \alpha^x\right)\right\}(w)|^2 d\mu_W(w) \leq \int_\Omega |\alpha(w)|^2 \Phi_F(w) d\mu_W(w). \tag{3.14}$$

We show a rough proof of this inequality in Lemma 4.1. For the second term on the right-hand side of (3.12), we have

$$\int_\Omega \frac{1}{\mu_B(\Omega_x)^2} \left(\int_{\Omega_x} (\alpha^z(x), v) d\mu_B(z)\right)^2 d\mu_W(w) \leq |v|^2 \int_\Omega |\alpha(w)|^2 d\mu_W(w). \tag{3.15}$$

Consequently $dh \in L^2(U, d\mu_X)$. Now we estimate $h_U$. Again by the Fubini theorem,

$$\begin{aligned}\int_\Omega |h_U(x)|^2 d\mu_W(w) &\leq \int_U |h_U(x)|^2 d\mu_X(x)\\ &\leq \int_U |dh(x)|^2 d\mu_X(x)\\ &\leq \varepsilon_U^{-1} \int_\Omega |dh(x)|^2 d\mu_W(w).\end{aligned} \tag{3.16}$$

We have already given an estimate for the integral of the right-hand side on (3.16). Let

$$\begin{aligned}
&\tilde{f}(x,z)\\
&:= f(x,z) - \frac{1}{\mu_X(U)}\int_U \frac{1}{\mu_B(\Omega_x)}\left(\int_{\Omega_x} f(x,z)d\mu_B(z)\right)d\mu_X(x)\\
&= g(x,z) + h(x) - \frac{1}{\mu_X(U)}\int_U h(x)d\mu_X(x),\\
&= -d_z^*\Box_{\Omega_x}^{-1}\alpha^x - d_x^*\Box_U^{-1}\left\{d_z^*\left\{D_x(\Box_{\Omega_x}^{-1})\alpha^x\right\} + \frac{1}{\mu_B(\Omega_x)}\int_{\Omega_x}\alpha^z(x)d\mu_B(z)\right\}.
\end{aligned}\tag{3.17}$$

Then we have

$$\begin{aligned}
\int_\Omega\left(f(w) - \frac{1}{\mu(\Omega)}\int_\Omega f(w)d\mu(w)\right)^2 d\mu_W(w) &\le \int_\Omega \tilde{f}(x,z)^2 d\mu_W(w)\\
&\le C\varepsilon_U^{-1}\int_\Omega |\alpha(w)|^2\tilde{\Phi}_F(w)d\mu_W(w).
\end{aligned}\tag{3.18}$$

In the above argument, we assume the existence of $f$ such that $df = \alpha$ on $\Omega$. Of course, this holds if $\alpha$ is given by $df$ itself. In this case, the inequality (3.18) implies a (weak) Poincaré inequality when $\Phi_F$ is a good function. See [1]. If the closed 1-form $\alpha$ is given first, the existence of $f$ is not trivial. However, applying the estimate (3.14) and the representation formula (3.17) to the finite dimensional part of $\alpha$ which is also a closed 1-form on finite dimensional domain with infinite dimensional remaining orthogonal parameters, we may obtain $f$ itself by a limiting argument. We will study this in a forthcoming paper. We give some discussions for a justification of the estimate (3.14) in the next section after showing Hadamard's variational formula for $\Box_{\Omega_x}^{-1}$.

## § 4. Hadamard's variational formula in Wiener spaces

We denote $S_x = \{z \in B \mid F(x,z) = 0\}$ for $x \in U$. $S_x$ is the boundary of $\Omega_x$. We consider a variation of $\Omega_x$ by $\{\Omega_{x+\varepsilon v} \mid \varepsilon \in \mathbb{R}\}$. Then $S_{x+\varepsilon v} = \{z \in B \mid F(x+\varepsilon v, z) = 0\}$. We introduce a real valued function $\psi_\varepsilon(x,z)$ $(z \in S_x)$ such that $\psi_0(x,z) = 0$ and $S_{x+\varepsilon v} = \{z + \psi_\varepsilon(x,z)n_x(z) \mid z \in S_x\}$. Here $n_x(z)$ stands for the unit outer normal vector at $z \in S_x$. Using the equation, $F(x+\varepsilon v, z+\psi_\varepsilon(x,z)n_x(z)) = 0$, we get $\frac{\partial}{\partial\varepsilon}\psi_\varepsilon(x,z)|_{\varepsilon=0} = -\frac{(D_xF(x,z),v)}{|D_zF(x,z)|} =: \rho_{x,v}(z)$. That is, our variation can be approximated by the variation $\{z + \varepsilon\rho_{x,v}(z)n_x(z) \mid z \in S_x\}$. The function $\rho_{x,v}(z)$ corresponds to the function $\rho$ in [11].

For $f \in \mathbb{D}^\infty_{\infty-}(B,\mathbb{R})$, we denote

$$\mathcal{E}_x(f,f) = \int_{\Omega_x}|Df(z)|^2 d\mu_B(z).\tag{4.1}$$

We denote $d\nu_x(z) = |D_zF(x,z)|\delta(F(x,z))d\mu_B(z)$. Let $T_t^x$, $L^x$ be the semi-group and the non-positive generator of $\mathcal{E}_x$ with the Neumann boundary condition. We denote $S_t^{x,v} = (D_x)_v T_t^x$. The following theorem in the case where $B$ is finite dimension and the measure is the Lebesgue measure was proved in [11]. In Theorem 4.1, Theorem 4.2, we do not need to assume that $\Omega_x$ is convex. Also the convexity assumption on $\Omega_x$ is not necessary for the proof of the variation formula of the semi-group $e^{t\Box_{\Omega_x}}$. However we assume the convexity in Theorem 4.3 and Lemma 4.1. Note that $\Box_{\Omega_x}^{-1}$ is meaningless if $\Box_{\Omega_x}$ is not invertible.

**Theorem 4.1.** *For $f, g \in \mathbb{D}^\infty_{\infty-}(B,\mathbb{R})$, it holds that*

$$
\begin{aligned}
\int_{\Omega_x} S_t^{x,v} f(z)g(z)d\mu_B(z) =& -\int_0^t\int_{S_x}\left(D_zT_s^xf(z), D_zT_{t-s}^xg(z)\right)\rho_{x,v}(z)d\nu_x(z)ds \\
& -\frac{\partial}{\partial t}\int_0^t\left(\int_{S_x}T_s^xf(z)T_{t-s}^xg(z)\rho_{x,v}(z)d\nu_x(z)\right)ds \\
=& -\int_0^t\int_{S_x}\left(D_zT_s^xf(z), D_zT_{t-s}^xg(z)\right)\rho_{x,v}(z)d\nu_x(z)ds \\
& -\int_0^t\left(\int_{S_x}T_s^xf(z)L^xT_{t-s}^xg(z)\rho_{x,v}(z)d\nu_x(z)\right)ds \\
& -\int_{S_x}T_t^xf(z)g(z)\rho_{x,v}(z)d\nu_x(z) \\
& +\int_{S_x}f(z)T_t^xg(z)\rho_{x,v}(z)d\nu_x(z).
\end{aligned}
\tag{4.2}
$$

For $\lambda > 0$, we denote $R_\lambda = \int_0^\infty e^{-\lambda t}T_t^x dt = (\lambda - L^x)^{-1}$. Suppose that $(D_x)_vR_\lambda = \int_0^\infty e^{-\lambda t}S_t^{x,v}dt$. Then multiplying by $e^{-\lambda t}$ $(\lambda > 0)$ the both sides of the equality in Theorem 4.1 and integrating with respect to $t$ from 0 to $+\infty$, we get

**Theorem 4.2.**

$$
\begin{aligned}
&\int_{\Omega_x}\{((D_x)_vR_\lambda)f\}(z)g(z)d\mu_B(z) \\
&= -\int_{S_x}(D_zR_\lambda f(z), D_zR_\lambda g(z))\rho_{x,v}(z)d\nu_x(z) \\
&\quad -\int_{S_x}R_\lambda f(z)L^xR_\lambda g(z)\rho_{x,v}(z)d\nu_x(z) \\
&\quad -\int_{S_x}R_\lambda f(z)g(z)\rho_{x,v}(z)d\nu_x(z) + \int_{S_x}f(z)R_\lambda g(z)\rho_{x,v}(z)d\nu_x(z).
\end{aligned}
\tag{4.3}
$$

For the variation of the Green operator $\Box_{\Omega_x}^{-1}$, we have a similar expression. We use the notation $Q_z = d_z + d_z^*$ below.

**Theorem 4.3.** *We denote* $G_{\Omega_x} = \square_{\Omega_x}^{-1}$ *and* $H^v_{\Omega_x} = (D_x)_v G_{\Omega_x}$. *Then*

$$\begin{aligned}
&\int_{\Omega_x} (H^v_{\Omega_x}\alpha(z), \beta(z)) d\mu_B(z)\\
&= \int_{S_x} \left(Q_z G_{\Omega_x}\alpha(z), Q_z G_{\Omega_x}\beta(z)\right) \rho_{x,v}(z) d\nu_x(z)\\
&+ \int_{S_x} \left(\alpha(z), G_{\Omega_x}\beta(z)\right) \rho_{x,v}(z) d\nu_x(z)\\
&+ \int_{S_x} d_z^* G_{\Omega_x}\alpha(z) \left\{\left(D_{n_x} G_{\Omega_x}\beta(z), n_x(z)\right)\rho_{x,v}(z) - \iota(D\rho_{x,v}(z)) G_{\Omega_x}\beta(z)\right\} d\nu_x(z)\\
&+ \int_{S_x} d_z^* G_{\Omega_x}\beta(z) \left\{\left(D_{n_x} G_{\Omega_x}\alpha(z), n_x(z)\right)\rho_{x,v}(z) - \iota(D\rho_{x,v}(z)) G_{\Omega_x}\alpha(z)\right\} d\nu_x(z).
\end{aligned}$$

For the estimate of $d_z^* H^v_{\Omega_x}$, we have the following by using Theorem 4.3.

**Lemma 4.1.** *Let* $\alpha$ *be a* 1*-form on* $\Omega_x$ *with* $d_z\alpha = 0$ *on* $\Omega_x$. *Then, there exists an absolute constant* $C$ *such that*

$$\begin{aligned}
&\int_{\Omega_x} |d_z^* H^v_{\Omega_x}\alpha(z)|^2 d\mu_B(z)\\
&\le \frac{C}{\mu_B(\Omega_x)} \left\{\int_{\Omega_x} |\alpha(z)|^2 M_{v,x}(z) d\mu_B(z) + N_{v,x}^2 \|\alpha\|^2_{L^2(\Omega_x, d\mu_B)}\right\} \qquad (4.4)
\end{aligned}$$

*where*

$$\begin{aligned}
M_{v,x}(z) &= |\bar{n}^\varepsilon_{x,v}(z)|^2 + |D_z \bar{n}^\varepsilon_{x,v}(z)|^2 + |D_z^* \bar{n}^\varepsilon_{x,v}(z)|^2,\\
N_{v,x} &= \|\bar{n}^\varepsilon_{x,v}\|_{L^2(\Omega_x, d\mu_B)} + \|D_z \bar{n}^\varepsilon_{x,v}\|_{L^2(\Omega_x, d\mu_B)} + \|D_z^* \bar{n}^\varepsilon_{x,v}\|_{L^2(\Omega_x, d\mu_B)}\\
&\quad + \|D_z \rho_{x,v}\|_{L^2(\Omega_x, d\mu_B)} + \|D_z^2 \rho_{x,v}\|_{L^2(\Omega_x, d\mu_B)},\\
\tilde{n}^\varepsilon_x &= \frac{D_z F(x,z)}{|D_z F(x,z)|} \phi\left(\varepsilon^{-1} F(x,z)\right),\\
\bar{n}^\varepsilon_{x,v} &= \frac{D_z F(x,z)}{|D_z F(x,z)|} \rho_{x,v}(z) \phi\left(\varepsilon^{-1} F(x,z)\right).
\end{aligned}$$

*Here* $\varepsilon > 0$ *and* $\phi$ *is a smooth function on* $\mathbb{R}$ *with* $\phi(t) = 1$ *for* $-1 \le t \le 1$ *and* $\phi(t) = 0$ *for* $|t| \ge 2$.

In the above lemma, we do not use the property that $d_z^* \left\{H^v_{\Omega_x}\alpha^x\right\}$ is independent of $z$. If we use this and we assume the further property of the function $F(x,z)$, we may prove that

$$\int_{\Omega_x} |d_z^* H^v_{\Omega_x}\alpha^x(z)|^2 d\mu_B(z) \le C \int_{\Omega_x} |\alpha^x(z)|^2 d\mu_B(z), \qquad (4.5)$$

where $C$ is independent of $x$. That is, we may prove the Poincaré inequality on $\Omega$

*Proof.* Note that

$$\left\{\int_{\Omega_x} |d_z^* H_{\Omega_x}^v \alpha(z)|^2 d\mu_B(z)\right\}^{1/2}$$

$$= \sup\left\{\int_{\Omega_x} d_z^* H_{\Omega_x}^v \alpha(z) g(z) d\mu_B(z) \;\middle|\; g \text{ is a } \mathbb{D}^\infty \text{ function with } g|_{\partial\Omega_x} = 0 \right.$$

$$\left. \text{and } \|g\|_{L^2(\Omega_x,\mu_B)}=1\right\}. \tag{4.6}$$

Let $g$ be a function which satisfies the assumptions on the right-hand side of (4.6). By the integration by parts formula and Theorem 4.3, we have

$$\begin{aligned}
&\int_{\Omega_x} d_z^* H_{\Omega_x}^v \alpha(z) g(z) d\mu_B(z) \\
&\quad = \int_{\Omega_x} \left(H_{\Omega_x}^v \alpha(z), d_z g(z)\right) d\mu_B(z) \\
&= \int_{S_x} \left(Q_z G_{\Omega_x} \alpha(z), Q_z G_{\Omega_x} d_z g(z)\right) \rho_{x,v}(z) d\nu_x(z) \\
&\quad + \int_{S_x} \left(\alpha(z), G_{\Omega_x} d_z g(z)\right) \rho_{x,v}(z) d\nu_x(z) \\
&\quad + \int_{S_x} d_z^* G_{\Omega_x} \alpha(z) \left\{\left(D_{n_x} G_{\Omega_x} dg(z), n_x(z)\right) \rho_{x,v}(z) - \iota(D\rho_{x,v}(z)) G_{\Omega_x} dg(z)\right\} d\nu_x(z) \\
&\quad + \int_{S_x} d_z^* G_{\Omega_x} dg(z) \left\{\left(D_{n_x} G_{\Omega_x} \alpha(z), n_x(z)\right) \rho_{x,v}(z) - \iota(D\rho_{x,v}(z)) G_{\Omega_x} \alpha(z)\right\} d\nu_x(z) \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We estimate $I_i$. For the first term, noting that $d_z G_{\Omega_x} d_z g = G_{\Omega_x} d_z^2 g = 0$ and (2.7),

$$\begin{aligned}
Q_z G_{\Omega_x} dg(z) &= (d_z^* + d_z) G_{\Omega_x} dg(z) \\
&= d_z^* G_{\Omega_x} dg(z) \\
&= g(z) - \bar{g}.
\end{aligned} \tag{4.7}$$

Here $\bar{g} = \mu_B(\Omega_x)^{-1} \int_{\Omega_x} g(z) d\mu_B(z)$. Thus using the integration by parts formula and $g|_{S_x} = 0$,

$$\begin{aligned}
I_1 &= \bar{g} \int_{\Omega_x} D_z^* \left\{d_z^* G_{\Omega_x} \alpha(z) \bar{n}_{x,v}^\varepsilon(z)\right\} d\mu_B(z) \\
&= \bar{g} \left(-\int_{\Omega_x} (\alpha(z), \bar{n}_{x,v}^\varepsilon(z)) d\mu_B(z) + \int_{\Omega_x} d_z^* G_{\Omega_x} \alpha(z) D_z^* \bar{n}_{x,v}^\varepsilon(z) d\mu_B(z)\right).
\end{aligned} \tag{4.8}$$

Here we have used that $d_z d_z^* G_{\Omega_x}\alpha = -\alpha$ which holds because $d_z\alpha = 0$ on $\Omega_x$. Consequently,

$$|I_1| \le \|g\|_{L^2(\Omega_x,d\mu_B)}\|\alpha\|_{L^2(\Omega_x,d\mu_B)}\mu_B(\Omega_x)^{-1/2} \left(\|\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)} + \|D_z^*\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}\right). \tag{4.9}$$

We estimate $I_2$. By the integration by parts formula,

$$\begin{aligned} I_2 &= -\int_{\Omega_x} D_z^*\left\{(\alpha(z), G_{\Omega_x} d_z g(z))\,\bar{n}^{\varepsilon}_{x,v}(z)\right\} d\mu_B(z) \\ &= \int_{\Omega_x}\left\langle D_z\alpha(z)\left[\bar{n}^{\varepsilon}_{x,v}(z)\right], G_{\Omega_x} d_z g(z)\right\rangle d\mu_B(z) \\ &\quad + \int_{\Omega_x}\left\langle \alpha(z), D_z\left(G_{\Omega_x} d_z g\right)(z)\left[\bar{n}^{\varepsilon}_{x,v}(z)\right]\right\rangle d\mu_B(z) \\ &\quad - \int_{\Omega_x}\left(\alpha(z), G_{\Omega_x} d_z g(z)\right) D_z^*\bar{n}^{\varepsilon}_{x,v}(z) d\mu_B(z) \\ &:= I_{2,1} + I_{2,2} + I_{2,3}. \end{aligned} \tag{4.10}$$

For $I_{2,1}$,

$$|I_{2,1}| \le \left(\int_{\Omega_x}\left|(-G_{\Omega_x})^{1/2}\left\{D_z\alpha(z)\left[\bar{n}^{\varepsilon}_{x,v}(z)\right]\right\}\right|^2 d\mu_B(z)\right)^{1/2} \left(\int_{\Omega_x}|(-G_{\Omega_x})^{1/2} d_z g(z)|^2 d\mu_B(z)\right)^{1/2}. \tag{4.11}$$

By the integration by parts formula,

$$\begin{aligned} &\int_{\Omega_x}|(-G_{\Omega_x})^{1/2} d_z g(z)|^2 d\mu_B(z) \\ &= \int_{\Omega_x}\left(-G_{\Omega_x} d_z g(z), d_z g(z)\right) d\mu_B(z) \\ &= -\int_{\Omega_x} d_z^* G_{\Omega_x} d_z g(z) g(z) d\mu_B(z) \\ &= \int_{\Omega_x} g(z)^2 d\mu_B(z) - \frac{1}{\mu_B(\Omega_x)}\left(\int_{\Omega_x} g(x) d\mu_B(z)\right)^2. \end{aligned}$$

This implies the boundedness of $(-G_x)^{1/2} d_z$ in $L^2$ also. We consider the first term on the right-hand side of (4.11). Since $d_z\alpha = 0$ on $\Omega_x$,

$$D_z\alpha(z)\left[\bar{n}^{\varepsilon}_{x,v}(z)\right] = D_z\left(\alpha(z), \bar{n}^{\varepsilon}_{x,v}(z)\right) - \left(D_z\bar{n}^{\varepsilon}_{x,v}(z), \alpha(z)\right).$$

Therefore

$$
\begin{aligned}
&\int_{\Omega_x} \left|(-G_{\Omega_x})^{1/2}\left\{D_z\alpha(z)\left[\bar{n}^{\varepsilon}_{x,v}(z)\right]\right\}\right|^2 d\mu_B(z) \\
&\quad\le 2\int_{\Omega_x}\left|(-G_{\Omega_x})^{1/2}\Big\{D_z\Big(\alpha(z),\bar{n}^{\varepsilon}_{x,v}(z)\Big)\Big\}\right|^2 d\mu_B(z) \\
&\quad+2\int_{\Omega_x}\left|(-G_{\Omega_x})^{1/2}\Big\{\big(\alpha(z),D_z\bar{n}^{\varepsilon}_{x,v}(z)\big)\Big\}\right|^2 d\mu_B(z) \\
&\quad\le 2\int_{\Omega_x}|\alpha(z)|^2\left(|\bar{n}^{\varepsilon}_{x,v}(z)|^2+D_z\bar{n}^{\varepsilon}_{x,v}(z)|^2\right)d\mu_B(z). \qquad (4.12)
\end{aligned}
$$

Here we have used the boundedness of $(-G_{\Omega_x})^{1/2}D_z$. For $I_{2,2}$ and $I_{2,3}$, using the boundedness

$$
\max\left(\|G_{\Omega_x}d_zg\|_{L^2(\Omega_x,d\mu_B)},\|D_zG_{\Omega_x}d_zg\|_{L^2(\Omega_x,d\mu_B)}\right)\le\|g\|_{L^2(\Omega_x,d\mu_B)}, \qquad (4.13)
$$

we have

$$
\begin{aligned}
&|I_{2,2}+I_{2,3}| \\
&\le\left\{\int_{\Omega_x}|\alpha(z)|^2\left(|\bar{n}^{\varepsilon}_{x,v}(z)|^2+|D_z^*\bar{n}^{\varepsilon}_{x,v}(z)|^2\right)d\mu_B(z)\right\}^{1/2}\|g\|_{L^2(\Omega_x,d\mu_B)}
\end{aligned}
$$

which proves the lemma. It remains to show that $\|D_zG_{\Omega_x}d_zg\|_{L^2(\Omega_x,d\mu_B)}\le\|g\|_{L^2(\Omega_x,d\mu_B)}$. To this end, we use the convexity of $\Omega_x$. By the convexity of $\Omega_x$, by Theorem 3.3 in [14],

$$
\begin{aligned}
\int_{\Omega_x}|D_zG_{\Omega_x}d_zg(z)|^2d\mu_B(z)&\le\int_{\Omega_x}|(d_z+d_z^*)G_{\Omega_x}d_zg(z)|^2d\mu_B(z)\\
&=\int_{\Omega_x}|d_z^*G_{\Omega_x}d_zg(z)|^2d\mu_B(z)\\
&=\int_{\Omega_x}\left(g(z)-\bar{g}\right)^2d\mu_B(z)\le\|g\|^2_{L^2(\Omega_x,d\mu_B)}.
\end{aligned}
$$

We can estimate $I_3$ and $I_4$ in the same way and obtain

$$
\begin{aligned}
|I_3|\le&\left\{\left(\int_{\Omega_x}|\alpha(z)|^2\left(|D_z\bar{n}^{\varepsilon}_{x,v}(z)|+|\bar{n}^{\varepsilon}_{x,v}(z)|+|D_z^*\bar{n}^{\varepsilon}_{x,v}(z)|\right)^2d\mu_B(z)\right)^{1/2}\right.\\
&+\|\alpha\|_{L^2(\Omega_x,d\mu_B)}\Big(\|\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}+\|D\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}\\
&\left.+\|D_z^*\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}\Big)\right\}\|g\|_{L^2(\Omega_x,d\mu_B)}\mu_B(\Omega_x)^{-1/2},\\
|I_4|\le&\|\alpha\|_{L^2(\Omega_x,d\mu_x)}\|g\|_{L^2(\Omega_x,d\mu_x)}\mu_B(\Omega_x)^{-1/2}\\
&\times\Big(\|\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}+\|D_z\bar{n}^{\varepsilon}_{x,v}\|_{L^2(\Omega_x,d\mu_B)}+\|D_z\rho_{x,v}\|_{L^2(\Omega_x,d\mu_B)}\\
&+\|D_z^2\rho_{x,v}\|_{L^2(\Omega_x,d\mu_B)}\Big)
\end{aligned}
$$

which completes the proof. □

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