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Kyoto University
Characterization of certain spaces of $C^\infty$-vectors of irreducible representations of solvable Lie groups

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Abstract

Let $\pi$ be an irreducible unitary representation of an exponential solvable Lie group $G$. Realizing $\pi$ on $L^2(G/H, \chi_l)$ as an induced representation from a unitary character $\chi_l$ of a subgroup $H$ of $G$, we are concerned with certain subspaces of $C^\infty$-vectors. We describe the subspace $S\mathcal{E}$ of vectors with a certain property of rapidly decreasing at infinity as the space of $C^\infty$-vectors of an irreducible unitary representation of an exponential solvable Lie group $F$ containing $G$. Furthermore, the space $AS\mathcal{E}$ introduced by Ludwig in [8] is expressed by our space $S\mathcal{E}$. Here we shall announce some results in [5], and we shall give brief discussions on fundamental examples.

1 Introduction

Let $G$ be an exponential solvable Lie group with Lie algebra $\mathfrak{g}$, and $\pi$ be an irreducible unitary representation of $G$. By the orbit method, which associates $\pi$ with a coadjoint orbit, we realize $\pi$ as an induced representation from a unitary character of a subgroup as follows: There exists a linear form $l \in \mathfrak{g}^*$ and a real polarization $\mathfrak{h}$ at $l$ such that $\pi \simeq \text{ind}^G_H \chi_l$, where $H = \exp \mathfrak{h}$ is the connected and simply connected subgroup corresponding to $\mathfrak{h}$, and $\chi_l$ is the unitary character of $H$ defined by $\chi_l(X) = e^{il(X)} (X \in \mathfrak{h})$.

We give the standard construction of the induced representation $\pi = \pi_{l,H} = \text{ind}^G_H \chi_l$: Let $\mathcal{K}(G/H)$ be the space of continuous functions $f$ on $G$ with compact support modulo $H$ such that $f(gh) = \Delta_{H,G}(h)f(g)$ for all $g \in G, h \in H$, where $\Delta_G$ and $\Delta_H$ are the

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modular functions of $G$ and $H$, respectively, and $\Delta_{H,G}(h) = \frac{\Delta_{H}(h)}{\Delta_{G}(h)}$. Then there exists a positive left invariant linear functional

\begin{equation}
(1.1) \quad f \mapsto \mu(f) = \oint_{G/H} f(g) d\mu_{G/H}(g)
\end{equation}

uniquely up to a constant factor (see [3]). Let $C(G/H, \chi_l)$ be the space of continuous functions $\phi$ on $G$ with compact support modulo $H$ such that

$$
\phi(gh) = \chi_l(h)^{-1} \Delta_{H,G}(h)^{1/2} \phi(g), \quad \forall g \in G, h \in H,
$$

and let $\mathcal{H}_{\pi} = L^2(G/H, \chi_l)$ be the completion of the space $C(G/H, \chi_l)$ with respect to the norm

$$
\|\phi\|_{\pi} := \left( \oint_{G/H} |\phi(g)|^2 d\mu_{G/H}(g) \right)^{1/2}.
$$

Then we define the action of $g \in G$ in $\mathcal{H}_{\pi}$ by

$$
\pi_{l,H}(g) \phi(x) = \phi(g^{-1} x), \quad \phi \in L^2(G/H, \chi_l), \quad g, x \in G.
$$

Let us briefly recall some well-known facts of the case of nilpotent Lie groups. Suppose $G$ is nilpotent, and taking a supplementary Malcev basis for $\mathfrak{h}$ in $\mathfrak{g}$, identify $G/H$ with $\mathbb{R}^k$, where $k = \dim(G/H)$, and realize $\pi$ on $L^2(\mathbb{R}^k)$. Then by results of Kirillov [6] and Corwin-Greenleaf-Penney [4], the actions of the enveloping algebra $\mathfrak{u}(\mathfrak{g})$ form the algebra of differential operators on $\mathbb{R}^k$ with polynomial coefficients. Thus the space of $C^\infty$-vectors $\mathcal{H}_{\pi}^\infty$ coincides with the space of Schwartz functions $S(\mathbb{R}^k)$ on $\mathbb{R}^k$ as a Fréchet space.

We next observe an example of exponential groups which are not nilpotent, where the specific descriptions of $C^\infty$-vectors are different from those of nilpotent groups.

**Example 1.1.** (ax + b group) Let $\mathfrak{g}$ be a two-dimensional Lie algebra with basis $\{X, Y\}$ whose bracket relation is $[X,Y] = Y$, and let $G = \exp \mathfrak{g}$. Then with the dual basis $\{X^*, Y^*\}$ in $\mathfrak{g}^*$, the coadjoint orbits of $G$ are described as follows:

$$
\mathcal{O}_+ := \{l \in \mathfrak{g}^*; l(Y) > 0\}, \quad \mathcal{O}_- := \{l \in \mathfrak{g}^*; l(Y) < 0\},
$$

$$
\{\xi X^*\}, \quad \xi \in \mathbb{R}.
$$

Let $l_\varepsilon := \varepsilon Y^* \ (\varepsilon = \pm 1)$ and $\mathfrak{h} := \mathbb{R}Y$, $H := \exp \mathfrak{h}$. Then $\mathfrak{h}$ is a polarization at $l_\varepsilon$ and $\pi_{l_\varepsilon} := \ind_H^G \chi_{l_\varepsilon}$ is an irreducible representation of $G$. We realize $\pi_{l_\varepsilon}$ on $L^2(\mathbb{R})$ identifying $\mathbb{R}$ with $G/H$ by $\mathbb{R} \ni x \mapsto \exp(xX)H$, as follows:

$$
\pi(\exp aX) \phi(x) = \phi(x - a)
$$

$$
\pi(\exp bY) \phi(x) = e^{i\varepsilon b x} \phi(x), \quad \phi \in L^2(\mathbb{R}), \quad a, b \in \mathbb{R}.
$$

Then the actions of $\mathfrak{g}$ are described by

$$
d\pi(X) \phi(x) = -\frac{d\phi}{dx}
$$

$$
d\pi(Y) \phi(x) = i\varepsilon e^{-x} \phi(x).
$$
It shows that if \( \phi \) is a \( C^\infty \) vector, then \( \phi \) decreases rapidly at \( x \to -\infty \), but it does not necessarily decrease so rapidly at \( x \to +\infty \) as at \( x \to -\infty \).

Here we shall announce some results in [5], giving brief discussions on fundamental examples. For an exponential solvable group \( G \) and an irreducible unitary representation \( \pi \) of \( G \), we construct \( \pi \) in \( L^2(G/H, \chi_l) \) by taking \( l \in \mathfrak{g}^* \) and a suitable polarization \( \mathfrak{h} \). Then we shall define a subspace \( \mathcal{S}E(G, n, l, \mathfrak{h}) \) of vectors with some property of rapidly decreasing at infinity and show that it can be identified with the space of \( C^\infty \) vectors of an irreducible representation of an exponential solvable group \( F \) containing \( G \). Next, using our \( \mathcal{S}E \) space, we shall describe the space \( \mathcal{A}S\mathcal{E} \) introduced by Ludwig in [8].

2 The space \( \mathcal{S}E(G, n, l, \mathfrak{h}) \)

In the sequel, let \( G \) be an exponential solvable Lie group with Lie algebra \( \mathfrak{g} \). Let \( n \) be a nilpotent ideal of \( \mathfrak{g} \) such that \([\mathfrak{g}, \mathfrak{g}] \subset n\). For example, we can take the nilradical of \( \mathfrak{g} \) as \( n \), or we can also take \( n = [\mathfrak{g}, \mathfrak{g}] \). Let \( l \in \mathfrak{g}^* \) as above, and let

\[
n' := \{X \in \mathfrak{g}; \ l([X, n]) = \{0\}\}.
\]

**Definition 2.1.** (See [9].) We say that a polarization \( \mathfrak{h} \) at \( l \) is adapted to \( n \) if it satisfies (1) and (2).

1. The subalgebra \( \mathfrak{h} \cap n \) is a polarization at \( l|_\mathfrak{n} \) in \( n \).
2. \([n', \mathfrak{h} \cap n] \subset \mathfrak{h} \cap n\).  

**Remark 2.2.** (1) Suppose that a polarization \( \mathfrak{h} \) at \( l \) is adapted to \( n \). Then there exists a polarization \( \mathfrak{h}_0 \subset n' \) at \( l|_{\mathfrak{n}'} \) such that \( \mathfrak{h} = \mathfrak{h}_0 + (\mathfrak{h} \cap n) \) and \( \mathfrak{h}_0 = \mathfrak{h} \cap n' \).

Furthermore, it satisfies the Pukanszky condition

\[
\text{Ad}^*(H)l = \mathfrak{h}^\perp + l,
\]

where \( \mathfrak{h}^\perp := \{f \in \mathfrak{g}^*; f(\mathfrak{h}) = \{0\}\} \), and thus we obtain a realization of the irreducible representation corresponding to the orbit \( \text{Ad}^*(G)l \) by \( \text{ind}_{\mathfrak{h}}^\mathfrak{g} \chi_l \).

(2) For any \( l \) and \( n \), there exists a polarization \( \mathfrak{h} \) at \( l \) adapted to \( n \). For example, a Vergne polarization associated with a refinement of the sequence of ideals \( \{0\} \subset n \subset \mathfrak{g} \) satisfies the condition (1) and (2) of Definition 2.1 above.

Starting from \( n \), \( l \) and a polarization \( \mathfrak{h} \) at \( l \) adapted to \( n \), we realize the irreducible representation \( \pi = \pi_{l,H} = \text{ind}_{\mathfrak{h}}^\mathfrak{g} \chi_l \) in \( L^2(\mathbb{R}^n) \) \((n = \dim(G/H))\) as follows.

Let \( \{T_1, \cdots, T_m, R_1, \cdots, R_v\} \) be a coexponential basis for \( \mathfrak{h} \) in \( \mathfrak{g} \) such that

\[
G = \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_m \cdot NH,
\]

\[
NH = \exp \mathbb{R}R_1 \cdots \exp \mathbb{R}R_v \cdot H,
\]
and identify
\[ \mathbb{R}^m \times \mathbb{R}^v \cong (G/NH) \times (NH/H) \cong G/H \]
by
\[ \mathbb{R}^m \times \mathbb{R}^v \ni (t, r) = (t_1, \ldots, t_m, r_1, \ldots, r_v) \mapsto E(t, r) := \exp t_1 T_1 \cdots \exp t_m T_m \exp r_1 R_1 \cdots \exp r_v R_v \quad \text{(modulo } H). \]

Then the left invariant functional (1.1) is described by
\[ \mu(f) = \int_{\mathbb{R}^{m+v}} f(E(t, r))dt\,dr, \quad f \in \mathcal{K}(G/H) \]
(see [7]), and we have \( L^2(G/H, \chi_l) \cong L^2(\mathbb{R}^{m+v}) \).

Denoting by \( \mathcal{D}_{t,r} \) the algebra of differential operators on \( \mathbb{R}^{m+v} \) with polynomial coefficients, we define the space \( \mathcal{S}(G, n, l, h) \) as follows:

**Definition 2.3.** Let \( \mathcal{S}(G, n, l, h) \) be the space of vectors \( \phi \in \mathcal{H}_{\pi_l, H} = L^2(G/H, \chi_l) \) such that

1. \( \phi \) is a \( C^\infty \) function.
2. \[ \|\phi\|^2_{a,D} := \int_{\mathbb{R}^{m+v}} e^{\|t\|} |D(\phi \circ E)(t, r)|^2 dt\,dr < \infty, \quad \forall a \in \mathbb{R}^+, \forall D \in \mathcal{D}_{t,r}, \]
   where \( \|t\| \) denotes a norm on \( \mathbb{R}^m \).

Let us remark that the space \( \mathcal{S}(G, n, l, h) \) is independent of the choice of coexponential basis.

In [5], we obtained the following result. There exists an exponential solvable Lie group \( F \) containing \( G \), and an irreducible representation \( \pi_0 \) of \( F \) such that \( \pi_0|_G \cong \pi \) and the space \( \mathcal{S}(G, n, l, h) \) is naturally identified with the space of \( C^\infty \) vectors of \( \pi_0 \). More specifically, we can construct an exponential solvable Lie algebra \( \mathfrak{f} \) which has the properties (i), (ii) and (iii):

1. \( \mathfrak{f} \) is described as \( \mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a} \), where \( \mathfrak{a} \) is an abelian ideal of \( \mathfrak{f} \) satisfying
   \[ [\mathfrak{a}, n + h] = \{0\}. \]
2. \( \dim \mathfrak{a} = 2 \dim(g/(n+h)) = 2m \), and there exist a coexponential basis \( \{X_1, \ldots, X_m\} \) for \( n + h \) in \( \mathfrak{g} \) and a basis \( \{A_1, \ldots, A_m, B_1, \ldots, B_m\} \) of \( \mathfrak{a} \) such that
   \[ [X_j, A_k] = \delta_{jk} A_k, \quad [X_j, B_k] = -\delta_{jk} B_k, \quad 1 \leq j, k \leq m. \]
(iii) For all extension $l_1 \in f^*$ of $l \in g^*$, we have that
\[ \dim(f(l_1)) = \dim(g(l)) + \dim(\mathfrak{a}), \]
where $f(l_1) := \{X \in f; l_1([X, f]) = \{0\}\}$, $g(l) := \{X \in g; l([X, g]) = \{0\}\}$. Thus the subalgebra $\mathfrak{p} := \mathfrak{h} + \mathfrak{a}$ is a polarization at $l_1$ adapted to the nilpotent ideal $n + \mathfrak{a}$ of $f$.

Let $F = \exp f$, $P = \exp \mathfrak{p}$, and $\chi_{l_1}$ be the unitary character of $F$ defined by $\chi_{l_1}(\exp X) = e^{il_1(X)}$ for $X \in \mathfrak{p}$. Then by (iii) above, the induced representation $\pi_{l_1, P} := \text{ind}_P^G \chi_{l_1}$ is irreducible and $\pi_{l_1, P}|_G \simeq \pi$. In fact, the intertwining operator
\[ \mathcal{R}_{l_1} : \mathcal{H}_{\pi_{l_1, P}} \rightarrow \mathcal{H}_{\pi_{l_1}, H} = L^2(G/H, \chi_l) \]
is defined by
\[ \mathcal{R}_{l_1} \psi = \psi|_G, \quad \psi \in L^2(F/P, \chi_{l_1}), \]
and the inverse $S_{l_1} := \mathcal{R}_{l_1}^{-1}$ is
\[ S_{l_1} \phi(g \exp Y) := e^{-il_1(Y)} \phi(g), \quad \phi \in L^2(G/H, \chi_l), \quad g \in G, \ Y \in \mathfrak{a}. \]
It can be seen easily that
\[ S_{l_1}(\mathcal{E}(G, n, l, \mathfrak{h})) \subset \mathcal{H}_{\pi_{l_1}, P}^\infty. \]

Now we define another family of seminorms $\{\| \cdot \|_{l_0, U}\}$ on $\mathcal{E}(G, n, l, \mathfrak{h})$:
\[ \|\phi\|_{l_1, U} := \|d\pi_{l_1, P}(U)S_{l_1} \phi\|_{\pi_{l_1, P}}, \quad U \in \mathfrak{u}(f). \]

**Theorem 2.4.** ([5]) Let $G, n, l, \mathfrak{h}$ be as above. Then there exists an exponential solvable Lie algebra $f$ having the property (i), (ii), (iii) above and satisfying the following:

(iv) There exists an extension $l_0 \in f^*$ of $l$ such that the family of seminorms $\{\| \cdot \|_{l_0, U}, U \in \mathfrak{u}(f)\}$ is equivalent to the family of seminorms $\{\| \cdot \|_{a, D}, a \in \mathbb{R}_+, D \in \mathfrak{D}_{x}\}$; and thus we have
\[ \mathcal{E}(G, n, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l_0}, P}). \]

**Example 2.5.** ($ax + b$ group) Let $g = \mathbb{R}X + \mathbb{R}Y$ and $\mathfrak{h} = \mathbb{R}Y$ be as in Example 1.1, and let $l := Y^*$ and $n = \mathbb{R}Y$, which is the nilradical of $g$. Then the polarization $\mathfrak{h}$ is obviously adapted to $n$. We construct $\pi_l$ in $L^2(\mathbb{R})$ as in Example 1.1. Then a square integrable smooth function $\phi$ belongs to $\mathcal{E}(G, n, l, \mathfrak{h})$ if and only if
\[ \int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \ \forall D \in \mathfrak{D}_x, \]
where $\mathfrak{D}_x$ is the algebra of differential operators on $\mathbb{R}$ with polynomial coefficients. Applying Theorem 2.4 above, we have
\[ f = g \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B \]
\[ [X, A] = A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = 0. \]
Let \( l_0 \in \mathfrak{f}^* \) be an extension of \( l \) such that \( l_0(B) \neq 0 \). Then we have

\[
(2.2) \quad \mathcal{SE}(G, n, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l_0}, P}).
\]

In fact, realizing \( \pi_{l_0, P} = \text{ind}_{F}^{E} \chi \iota_{0} \) in \( L^2(F/P, \chi_{l_0}) = S_{l_0}(L^2(G/P, \chi_{l})) \cong L^2(\mathbb{R}) \), we have that \( \mathfrak{f} \) acts by

\[
(2.3) \quad d\pi_{l_0, P}(X)\phi(x) = -\frac{d\phi}{dx},
(2.4) \quad d\pi_{l_0, P}(Y)\phi(x) = i e^{-x}\phi(x),
(2.5) \quad d\pi_{l_0, P}(A)\phi(x) = i l_0(A) e^{-x}\phi(x),
(2.6) \quad d\pi_{l_0, P}(B)\phi(x) = i l_0(B) e^{x}\phi(x).
\]

Thus we can directly verify the equality (2.2).

**Remark 2.6.** In Example 2.5, replacing \( F \) with a subgroup \( F' \) of \( F \), we can also identify the space \( \mathcal{SE}(G, n, l, \mathfrak{h}) \) with the space of \( C^\infty \) vectors of an extension of \( \pi_l \). Let \( \mathfrak{a'} := \mathbb{R}B, \mathfrak{f'} := \mathfrak{g} \ltimes \mathfrak{a'}, \mathfrak{p'} := \mathfrak{h} + \mathfrak{a'}, \mathfrak{F'} := \text{exp} \mathfrak{f'} \) and \( \mathfrak{P'} := \text{exp} \mathfrak{p'} \). Then \( \mathfrak{p'} \) is a polarization at any extension \( l'_1 \in \mathfrak{f'}^* \) of \( l \) and \( \pi_{l'_1, P'|_{G}} \cong \pi_l \), where \( \pi_{l'_1, P'} = \text{ind}_{F'}^{E} \chi_{l'_1} \). We denote the intertwining operator by \( \mathcal{R}_{l_0} : L^2(F'/P', \chi_{l'_1}) \to L^2(G/H, \chi_{l}) \) as above. Suppose that an extension \( l''_0 \in \mathfrak{f''}^* \) of \( l \) satisfies \( l''_0(B) \neq 0 \). Then we also obtain that

\[
(2.7) \quad \mathcal{SE}(G, n, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_{\pi_{l''_0}, P'}).
\]

In fact, letting \( l_0 \in \mathfrak{f}^* \) be any extension of \( l''_0 \), we have \( \pi_{l_0, P}|_{\mathfrak{f'}^*} \cong \pi_{l''_0, P'} \), and using the descriptions (2.3), (2.4) and (2.6), we obtain the equality (2.7).

**Example 2.7.** (Heisenberg group) Taking \( n = [\mathfrak{g}, \mathfrak{g}] \) instead of the nilradical, we observe an example of nilpotent groups. Let \( \mathfrak{g} = \mathbb{R}\text{span}\{X, Y, Z\} \) be the 3-dimensional Lie algebra whose non-trivial bracket relation is \([X, Y] = Z\). Let \( \mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \mathbb{R}Z, \mathfrak{l} = Z^* \) and \( \mathfrak{h} = \mathbb{R}Y + \mathbb{R}Z \). Then \( \mathfrak{h} \) is a polarization adapted to \( \mathfrak{n} \). We realize the representation \( \pi_{l, H} \) in \( L^2(\mathbb{R}) \) by the coexponential basis \( \{X\} \) for \( \mathfrak{h} \) in \( \mathfrak{g} \). Then we have

\[
(2.8) \quad d\pi_{l, H}(X)\phi(x) = -\frac{d\phi}{dx}, \quad d\pi_{l, H}(Y)\phi(x) = -ix\phi(x), \quad d\pi_{l, H}(Z) = i.
\]

We have that a smooth function \( \phi(x) \in L^2(\mathbb{R}) \) belongs to \( \mathcal{SE}(G, n, l, \mathfrak{h}) \) if and only if

\[
\int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad \forall D \in \mathcal{D}_x.
\]

Applying Theorem 2.4, we have

\[
\mathfrak{f} = \mathfrak{g} \ltimes \mathfrak{a}, \quad \mathfrak{a} = \mathbb{R}A + \mathbb{R}B,
\]

\[
[X, A] = A, \quad [X, B] = -B, \quad [Y, A] = [Y, B] = [Z, A] = [Z, B] = 0.
\]
Let \( l_0 \in \mathfrak{f}^* \) be an extension of \( l \) such that \( l_0(A) \neq 0 \) and \( l_0(B) \neq 0 \). Then we have
\[
\mathcal{S}\mathcal{E}(G, n, l, \mathfrak{h}) = \mathcal{R}_{l_0}(\mathcal{H}_\mathfrak{p})^\infty.
\]
In fact, we have
\[
\begin{align*}
\pi_{l_0, P}(A) \phi(x) &= il_0(A)e^{-x} \phi(x), \\
\pi_{l_0, P}(B) \phi(x) &= il_0(B)e^x \phi(x).
\end{align*}
\]
By the actions (2.8), (2.9) and (2.10), we can obtain the conclusion.

3 The space \( \mathcal{A}\mathcal{S}\mathcal{E} \) and the space \( \mathcal{S}\mathcal{E}^\infty \)

Let \( G, n, l, \mathfrak{h} \) be as above. As we mentioned in Remark 2.2, we have that \( \mathfrak{h} = (\mathfrak{h} \cap n^l) + (\mathfrak{h} \cap n) \), so we have \( \mathfrak{h} \subset n + n^l \). We choose a coexponential basis \( \{ T_1, \ldots, T_\nu, S_1, \ldots, S_u \} \) for \( \mathfrak{h} + n \) in \( g \) along with the sequence \( g \supset n + n^l \supset n + \mathfrak{h} \), where \( \nu + u = m \), so that
\[
G = \exp \mathbb{R}T_1 \cdots \exp \mathbb{R}T_\nu \cdot N^l N \quad N^l N = \exp \mathbb{R}S_1 \cdots \exp \mathbb{R}S_u \cdot NH.
\]
(Here we write \( N^l := \exp n^l \).) In the sequel, we identify
\[
\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v \cong (G/N^l N) \times (N^l N/NH) \times (NH/H) \cong G/H
\]
by
\[
\mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v \ni (t, s, r) = (t_1, \ldots, t_\nu, s_1, \ldots, s_u, r_1, \ldots, r_v) \quad \mapsto E(t, s, r) = E(t)E(s)E(r) \text{ (modulo } H),
\]
where
\[
E(t) = \exp t_1 T_1 \cdots \exp t_\nu T_\nu, \quad E(s) = \exp s_1 S_1 \cdots \exp s_u S_u, \quad E(r) = \exp r_1 R_1 \cdots \exp r_v R_v, \quad (t, s, r) \in \mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v.
\]
For \( \phi \in \mathcal{S}\mathcal{E}(G, n, l, \mathfrak{h}) \), let \( \hat{\phi}_s(t, s, r) \) be the partial Fourier transform of \( \phi \) in \( s \):
\[
\hat{\phi}_s(t, s, r) := \int_{\mathbb{R}^u} \phi(E(t, x, r))e^{i(x,s)}dx,
\]
where \( (x, s) \) is the standard inner product of \( \mathbb{R}^u \). Denoting by \( \mathcal{D}_{t,s,r} \) the algebra of differential operators on \( \mathbb{R}^\nu \times \mathbb{R}^u \times \mathbb{R}^v \) with polynomial coefficients, we define the space \( \mathcal{A}\mathcal{S}\mathcal{E}(G, n, l, \mathfrak{h}) \) introduced in [8], where this space has been denoted by \( \mathcal{E}S \).

Definition 3.1. (See [8].) Let \( \mathcal{A}\mathcal{S}\mathcal{E}(G, n, l, \mathfrak{h}) \) be the space of functions \( \phi \in L^2(G/H, \chi_l) \) such that
\(1\) \(\phi \in SE(G, n, l, h)\),

\(2\)

\[ \|\hat{\phi}_s(t, s, r)\|^2_{a, b, D} := \int_{\mathbb{R}^{v+u+v}} e^{a||t||} e^{b||s||}|D\hat{\phi}_s(t, s, r)|^2 dt ds dr < \infty, \forall (a, b) \in \mathbb{R}_+^2, \forall D \in \mathcal{D}_{t,s,r}. \]

**Remark 3.2.** The space \(\mathcal{ASE}\) is independent of the choice of coexponential bases. We write the letter \(A\) in front to indicate that the functions \(\phi \in \mathcal{ASE}(G, n, l, h)\) are analytic in the direction \(s\). It has been shown in [8] and [1] that for \(\phi\) and \(\psi\) in \(\mathcal{ASE}(G, n, l, h)\) there exists a function \(f \in L^1(G)\), more precisely in the subalgebra \(ES(G)\) (see [8]) such that

\[ \pi_{l,H} f(\xi) = \langle \xi, \psi \rangle \phi, \quad \xi \in \mathcal{H}_{\pi_{l,H}}. \]

Let \(\mathcal{P}(h)\) be the set of polarizations \(h\) at \(l\) adapted to \(n\) and satisfying \(h \cap n = h \cap n\). For \(h \in \mathcal{P}(h)\), we have \(\text{ind}_H^G \chi_l \simeq \text{ind}_{\tilde{H}}^{\hat{H}} \chi_l\), where \(\tilde{H} = \exp h\). We denote the intertwining operator by \(I_{h,\overline{h}} : L^2(G/\tilde{H}, \chi_l) \rightarrow L^2(G/H, \chi_l)\) (see [2].)

**Definition 3.3.** We define

\[ SE^{\infty}(G, n, l, h) := \bigcap_{\tilde{h} \in \mathcal{P}(h)} I_{h,\overline{h}}(SE(G, n, l, \tilde{h})). \]

Then we have the following result:

**Theorem 3.4.** ([5])

\[ SE^{\infty}(G, n, l, h) = \mathcal{ASE}(G, n, l, h). \]

**Example 3.5.** Let \(g = \mathbb{R}\)-Span\(\{X, Y, Z\}\), \(n, l, h\) be as in Example 2.7. Concerning Theorem 3.4, we have \(n' = g\) and a smooth function \(\phi \in L^2(\mathbb{R})\) belongs to \(\mathcal{ASE}(G, n, l, h)\) if and only if

\(3.11\)

\[ \int_{\mathbb{R}} e^{a|x|}|D\phi(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathcal{D}_x, \]

\(3.12\)

\[ \int_{\mathbb{R}} e^{a|x|}|D\hat{\phi}(x)|^2 dx < \infty, \quad \forall a \in \mathbb{R}_+, \quad D \in \mathcal{D}_x, \]

where

\[ \hat{\phi}(x) = \int_{\mathbb{R}} e^{ias} \phi(s) ds. \]

Since \(n = \mathbb{R}Z\) is the center of \(g\), any polarization \(h\) at \(l\) belongs to \(\mathcal{P}(h)\). Thus by Theorem 3.4, we have that the intersection of \(I_{h,\overline{h}}(SE(G, n, l, \tilde{h}))\) for all polarizations \(h\) at \(l\) consists of analytic functions \(\phi\) satisfying (3.11) and (3.12).
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References


