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Drinfeld second realization of the quantum affine superalgebras of $D^{(1)}(2,1;x)$ via the Weyl groupoid

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Abstract

We obtain Drinfeld second realization of the quantum affine superalgebras associated with the affine Lie superalgebra $D^{(1)}(2,1;x)$. Our results are analogous to those obtained by Beck for the quantum affine algebras. Beck's analysis uses heavily the (extended) affine Weyl groups of the affine Lie algebras. In our approach the structures are based on a Weyl groupoid.

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1 Introduction

In this paper we study the quantum deformation of the affine Lie superalgebra $D^{(1)}(2,1;x)$, where $x \in \mathbb{C} \setminus \{0,-1\}$. In Definition 4.2, for any $q \in \mathbb{C}$ such that $q(q^n-1)(q^{nx}-1)(q^{n(x+1)}-1) \neq 0$ for all $n \in \mathbb{N}$, we define the quantized enveloping algebras of $D^{(1)}(2,1;x)$ by the defining relations (cf. [Y] [1]) in terms of the Chevalley-Serre generators. In Theorem 4.7 we attach to any simple (even *and* odd) reflection (cf. [DP,S]) a Lusztig type isomorphism between two such algebras. These isomorphisms satisfy Coxeter type relations, see Theorem 4.8. In Definition 6.1 we give Drinfeld second realization of quantum $D^{(1)}(2,1;x)$. Our main result is Theorem 6.6, see also Theorems 6.8 and 6.10, where we show that the two realizations are isomorphic as algebras. See [D] for the original Drinfeld second realization of the quantum affine algebras. The argument in this paper was inspired by Beck's work [Bec] and we utilize the Weyl groupoid instead of the Weyl group. Khoroshkin and Tolstoy [KT] obtained results concerning quantum affine superalgebras relevant to this paper.

Our work was motivated by recent results in Hopf algebra theory and in theoretical physics, in particular the AdS/CFT correspondence. We sketch those aspects of these developments which are relevant for our work.

[1] They were originally given in [Y, Remark 7.1.1] (or Prop.6.3.1(vii),(viii) in q-alg/9603015).

The Lie superalgebra $D(2,1;x)$ has a very interesting relation to $A(1,1) = \mathfrak{psl}(2|2)$, which is the only classical basic Lie superalgebra allowing for a non-trivial universal central extension [IK] with three central elements (see Section 2 for $D(2,1;x)$ and $A(1,1)$). One can obtain this centrally extended algebra $\mathfrak{psl}(2|2)\oplus\mathbb{C}^3$ by a contraction from $D(2,1;x)$ in the limit $x \to -1$. The Lie superalgebra $\mathfrak{psl}(2|2)$ and its central extensions have recently become important in the context of the AdS/CFT correspondence [M,W,GKP] (for comprehensive reviews, the reader is referred to [AGMOO,DF]). This conjecture relates the maximal supersymmetric Yang-Mills theory in four dimensions to string theory formulated on $\mathrm{AdS}_5 \times S_5$. On the gauge theory side of this correspondence one can think of a certain class of operators as integrable spin chains, and apply the Bethe ansatz technique to calculate their energy spectrum [MZ, BS]. The symmetry algebra of the gauge theory, which is the superconformal algebra $\mathfrak{psu}(2,2|4)$, is reduced to $\mathfrak{u}(1)\oplus(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2))\oplus\mathfrak{u}(1)$ upon choosing an appropriate vacuum for the spin chain. Excitations transform under $\mathfrak{u}(1)\oplus(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2))\oplus\mathfrak{u}(1)$, and the S-matrix which intertwines two modules is physically interpreted as the scattering matrix of those excitations (detailed descriptions are contained in [Bei1]). Interestingly, the S-matrix is already fixed, up to a scalar prefactor, by vanishing of its commutators with the generators of the centrally extended $(\mathfrak{psu}(2|2) \times \mathfrak{psu}(2|2))\oplus\mathbb{C}^3$ algebra [Bei2, Bei3], when one twists the universal enveloping algebra with an additional braiding element [J,GH,PST]. The complete symmetry algebra has been recently related to a twisted Yangian [Bei4]. The spectral parameter of the Yangian, the eigenvalues of the central charges and the braiding are all linked on the fundamental evaluation representation.

Due to its close relation to $\mathfrak{psl}(2|2)$ it is very promising to study the affine Lie superalgebra $D^{(1)}(2,1;x)$ (see Section 2 for $D^{(1)}(2,1;x)$). Since one can obtain Yangians from quantum affine algebras one can consider physical models with quantum $D^{(1)}(2,1;x)$ symmetry as deformations of models with Yangian $\mathfrak{psl}(2|2)$ symmetry. In this paper we do the first steps by deriving Drinfeld's second realization of quantum $D^{(1)}(2,1;x)$, which we need for further investigations of finite dimensional representations and studies of the universal R-matrix. Our key tool is the Weyl groupoid of (quantum) $D^{(1)}(2,1;x)$. The notion of the Weyl groupoids was initiated and has intensively been studied by the first author [H] in order to classify Nichols algebras of diagonal type with a finite set of Poincaré-Birkhoff-Witt generators. The interest in Nichols algebras arose with a fundamental paper of Andruskiewitsch and Schneider [AS1] where they developed a method to classify pointed Hopf algebras. The results of many papers culminated in a fairly general classification result [AS2] on finite dimensional pointed Hopf algebras with abelian coradical over the complex numbers. In the heart of the theory the Weyl groupoid seems to play one of the fundamental roles. Guided by this observation the first and fourth authors started to investigate the Weyl groupoids in more detail, and obtained a Matsumoto-type theorem [HY] for them. The fourth author [Y] essentially used the Weyl groupoids to get Serre-type defining relations of the quantum affine superalgebras and the Drinfeld second realization of the quantum $A^{(1)}(m,n)$ (see Remark 2.1 for the notation $A^{(1)}(m,n)$). The fourth author [Y] utilized the quantum deformation of the universal central extension of $[A^{(1)}(1,1), A^{(1)}(1,1)]$ to get a new R-matrix.

In this paper we use the following notation. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively, and let $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The symbol δ_{ij}, or $\delta_{i,j}$, denotes Kronecker's δ, that is, $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

2 The simple Lie superalgebra $D(2,1;x)$ and the affine Lie superalgebra $D^{(1)}(2,1;x)$ $(x \neq 0,-1)$

As for the terminology concerning affine Lie superalgebras, we refer to [K], or to [IK], [vdL].

Let $\mathfrak{v} = \mathfrak{v}(0) \oplus \mathfrak{v}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space. If $i \in \{0,1\}$ and $j \in \mathbb{Z}$ such that $j - i \in 2\mathbb{Z}$ then let $\mathfrak{v}(j) = \mathfrak{v}(i)$. If $X \in \mathfrak{v}(0)$ (resp. $X \in \mathfrak{v}(1)$) then we write

$$\deg(X) = 0 \text{ (resp. } \deg(X) = 1) \tag{2.1}$$

and we say that X is an *even* (resp. *odd*) element. If $X \in \mathfrak{v}(0) \cup \mathfrak{v}(1)$, then we say that X is a *homogeneous* element and that $\deg(X)$ is the *parity* (or *degree*) of X. If $\mathfrak{w} \subset \mathfrak{v}$ is a subspace and $\mathfrak{w} = (\mathfrak{w} \cap \mathfrak{v}(0)) \oplus (\mathfrak{w} \cap \mathfrak{v}(1))$ (resp. $\mathfrak{w} \subset \mathfrak{v}(0)$, resp. $\mathfrak{w} \subset \mathfrak{v}(1)$), then we say that $\mathfrak{w}$ is a *graded* (resp. *even*, resp. *odd*) subspace.

Let $\mathfrak{a} = \mathfrak{a}(0) \oplus \mathfrak{a}(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space equipped with a bilinear map $[\,,\,] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ such that $[\mathfrak{a}(i), \mathfrak{a}(j)] \subset \mathfrak{a}(i+j)$ $(i, j \in \mathbb{Z})$; we recall from the above paragraph that

$$\mathfrak{a}(i) = \{X \in \mathfrak{a} \mid \deg(X) = i\}. \tag{2.2}$$

We say that $\mathfrak{a} = (\mathfrak{a}, [\,,\,])$ is a ($\mathbb{C}$-)*Lie superalgebra* if for all homogeneous elements X, Y, Z of $\mathfrak{a}$ the following equations hold.

$$\begin{aligned} [Y,X] &= -(-1)^{\deg(X)\deg(Y)}[X,Y], && \text{(skew-symmetry)} \\ [X,[Y,Z]] &= [[X,Y],Z] + (-1)^{\deg(X)\deg(Y)}[Y,[X,Z]]. && \text{(Jacobi identity)} \end{aligned}$$

Let $\mathfrak{a}$ be a Lie superalgebra. We say that a bilinear form $(\cdot,\cdot) : \mathfrak{a} \times \mathfrak{a} \to \mathbb{C}$ is a *supersymmetric invariant form* on $\mathfrak{a}$ if for all homogeneous elements X, Y, Z of $\mathfrak{a}$ one has

$$(Y,X) = (-1)^{\deg(X)\deg(Y)}(X,Y) \text{ and } (X,[Y,Z]) = ([X,Y],Z).$$

A graded subspace $\mathfrak{i}$ of $\mathfrak{a}$ is called an *ideal* if one has $[X,Y] \in \mathfrak{i}$ for all homogeneous elements X of $\mathfrak{a}$ and all homogeneous elements Y of $\mathfrak{i}$.

Let $\mathbb{I}$ be a finite set. Let $\mathbb{A} = (\mathbb{A}_{ij})_{i,j\in\mathbb{I}}$ be an $|\mathbb{I}| \times |\mathbb{I}|$ matrix with coefficients in $\mathbb{C}$. Suppose that we are given an $|\mathbb{I}| \times |\mathbb{I}|$ diagonal matrix $\mathbb{D} = (\delta_{ij}\mathbb{D}_i)_{i,j\in\mathbb{I}}$ with $\mathbb{D}_i \in \mathbb{C} \setminus \{0\}$ satisfying the condition that $\mathbb{D}^{-1}\mathbb{A}$ is a symmetric matrix, that is, ${}^t(\mathbb{D}^{-1}\mathbb{A}) = \mathbb{D}^{-1}\mathbb{A}$. Let $\mathbb{I}^{\mathrm{odd}}$ be a subset of $\mathbb{I}$. Let $\widetilde{\mathfrak{g}}' = \widetilde{\mathfrak{g}}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ be the $\mathbb{C}$-Lie superalgebra generated by the (homogeneous) elements $\mathbb{H}_i$, $\mathbb{E}_i$, $\mathbb{F}_i$ $(i \in \mathbb{I})$ with

$$\begin{aligned}
\deg(\mathbb{H}_i) &= 0 \qquad (i \in \mathbb{I}),\\
\deg(\mathbb{E}_j) &= \deg(\mathbb{F}_j) = 0 \qquad (j \in \mathbb{I} \setminus \mathbb{I}^{\mathrm{odd}}),\\
\deg(\mathbb{E}_j) &= \deg(\mathbb{F}_j) = 1 \qquad (j \in \mathbb{I}^{\mathrm{odd}}).
\end{aligned}$$

and defined by the relations

$$[\mathbb{H}_i, \mathbb{H}_j] = 0, \quad [\mathbb{H}_i, \mathbb{E}_j] = \mathbb{A}_{ij}\mathbb{E}_j, \quad [\mathbb{H}_i, \mathbb{F}_j] = -\mathbb{A}_{ij}\mathbb{F}_j, \quad [\mathbb{E}_i, \mathbb{F}_j] = \delta_{ij}\mathbb{H}_i \quad (i, j \in \mathbb{I}).$$

Let $\widetilde{\mathfrak{h}}' = \widetilde{\mathfrak{h}}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$, $\widetilde{\mathfrak{n}}_+$ and $\widetilde{\mathfrak{n}}_-$ be the Lie subsuperalgebras generated by the sets $\{\mathbb{H}_i | i \in \mathbb{I}\}$, $\{\mathbb{E}_i | i \in \mathbb{I}\}$ and $\{\mathbb{F}_i | i \in \mathbb{I}\}$, respectively. Then $\{\mathbb{H}_i | i \in \mathbb{I}\}$ is a $\mathbb{C}$-basis of $\widetilde{\mathfrak{h}}'$ and hence one has $\dim \widetilde{\mathfrak{h}}' = |\mathbb{I}|$. Further, one obtains the decomposition $\widetilde{\mathfrak{g}}' = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}' \oplus \widetilde{\mathfrak{n}}_-$ as a $\mathbb{C}$-vector space. The Lie superalgebras $\widetilde{\mathfrak{n}}_+$ and $\widetilde{\mathfrak{n}}_-$ are free Lie superalgebras generated by the sets $\{\mathbb{E}_i | i \in \mathbb{I}\}$ and $\{\mathbb{F}_i | i \in \mathbb{I}\}$, respectively. Let $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) be the largest ideal of $\widetilde{\mathfrak{g}}'$ which is contained in $\widetilde{\mathfrak{n}}_+$ (resp. $\widetilde{\mathfrak{n}}_-$). Let $\mathfrak{g}' = \mathfrak{g}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ be the quotient Lie superalgebra $\widetilde{\mathfrak{g}}'/(\mathfrak{r}_+ \oplus \mathfrak{r}_-)$. Let $\mathbb{H}_i$, $\mathbb{E}_i$, $\mathbb{F}_i$, $\mathfrak{h}' = \mathfrak{h}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$, $\mathfrak{n}_+$, and $\mathfrak{n}_-$ be the images of $\mathbb{H}_i$, $\mathbb{E}_i$, $\mathbb{F}_i$, $\widetilde{\mathfrak{h}}' = \widetilde{\mathfrak{h}}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$, $\widetilde{\mathfrak{n}}_+$, and $\widetilde{\mathfrak{n}}_-$, respectively, under the canonical projection $\widetilde{\mathfrak{g}}' \to \mathfrak{g}'$. Then $\mathfrak{g}' = \mathfrak{n}_+ \oplus \mathfrak{h}' \oplus \mathfrak{n}_-$. Further, there exists a (unique) Lie superalgebra $\mathfrak{g} = \mathfrak{g}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}}) = \mathfrak{g}(\mathbb{A}, \mathbb{D}, \mathbb{I}, \mathbb{I}^{\mathrm{odd}})$ with the following properties.

(i) $\mathfrak{g}$ includes $\mathfrak{g}'$ as a Lie subsuperalgebra.

(ii) There exists an even subspace $\mathfrak{h}'' = \mathfrak{h}''(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h}'' \oplus \mathfrak{g}'$, $\dim \mathfrak{h}'' = |\mathbb{I}| - \mathrm{rank}\mathbb{A}$, and $[\mathfrak{h}'', \mathfrak{h}''] = [\mathfrak{h}', \mathfrak{h}''] = \{0\}$.

(iii) Let $\mathfrak{h} = \mathfrak{h}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}}) := \mathfrak{h}' \oplus \mathfrak{h}''$, so $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ as a $\mathbb{C}$-vector space. Then for each $i \in \mathbb{I}$ there exists $\alpha_i \in \mathfrak{h}^*$ such that $[\mathbb{H}, \mathbb{E}_i] = \alpha_i(\mathbb{H})\mathbb{E}_i$ and $[\mathbb{H}, \mathbb{F}_i] = -\alpha_i(\mathbb{H})\mathbb{F}_i$ for all $\mathbb{H} \in \mathfrak{h}$. Further, α_i $(i \in \mathbb{I})$ are linearly independent elements of $\mathfrak{h}^*$.

For $\beta \in \mathfrak{h}^*$, let $\mathfrak{g}_\beta = \mathfrak{g}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})_\beta := \{X \in \mathfrak{g} | [\mathbb{H}, X] = \beta(\mathbb{H})X \text{ for all } \mathbb{H} \in \mathfrak{h}\}$. Let $\Phi = \Phi(\mathbb{A}, \mathbb{I}^{\mathrm{odd}}) := \{\beta \in \mathfrak{h}^* \setminus \{0\} | \dim \mathfrak{g}_\beta \neq 0\}$. The set Φ is called the *root system* of $\mathfrak{g}$ and the elements of Φ are called *roots*. For $\alpha \in \Phi$ the space $\mathfrak{g}_\alpha$ is called the *root space* of α. Note that one obtains the decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha)$ as a $\mathbb{C}$-vector space.

Note that $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

It is well-known that there exists a (unique) nondegenerate supersymmetric invariant form $(\cdot|\cdot)$ on $\mathfrak{g}$ such that

$$\begin{aligned}(\mathbb{H}_i|\mathbb{H}) = \mathbb{D}_i\alpha_i(\mathbb{H}) &\quad \text{for all } i \in \mathbb{I},\ \mathbb{H} \in \mathfrak{h},\\ (\mathbb{H}_1''|\mathbb{H}_2'') = 0 &\quad \text{for all } \mathbb{H}_1'', \mathbb{H}_2'' \in \mathfrak{h}'',\\ (\mathbb{E}_i|\mathbb{F}_j) = \delta_{ij}\mathbb{D}_i &\quad \text{for all } i,j \in \mathbb{I}.\end{aligned}$$

Assume that $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra. It is well-known that $\mathfrak{g} = \mathfrak{g}'$ (i.e., $\mathfrak{h} = \mathfrak{h}'$), and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. The (non-twisted) *affine Lie superalgebra* $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ is the $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}H_{\hat{\delta}} \oplus \mathbb{C}H_{\Lambda_0}$$

such that $\deg(X \otimes t^m) = \deg(X)$ for all homogeneous $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$ and $\deg(H_{\hat{\delta}}) = \deg(H_{\Lambda_0}) = 0$ (that is, $\hat{\mathfrak{g}}(0) = \mathfrak{g}(0) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}H_{\hat{\delta}} \oplus \mathbb{C}H_{\Lambda_0}$ and $\hat{\mathfrak{g}}(1) = \mathfrak{g}(1) \otimes \mathbb{C}[t,t^{-1}]$), together with the super-bracket

$$\begin{aligned}&[X \otimes t^m + a_1 H_{\hat{\delta}} + b_1 H_{\Lambda_0}, Y \otimes t^n + a_2 H_{\hat{\delta}} + b_2 H_{\Lambda_0}]\\ &\quad = [X,Y] \otimes t^{m+n} + m\delta_{m+n,0}(X|Y)H_{\hat{\delta}} + b_1 n Y \otimes t^n - b_2 m X \otimes t^m\end{aligned}$$

for all $m, n \in \mathbb{Z}$, $a_1, a_2, b_1, b_2 \in \mathbb{C}$ and homogeneous elements X, Y of $\mathfrak{g}$. Note that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}H_{\hat{\delta}}$.

The affine Lie superalgebra $\hat{\mathfrak{g}}$ is identified with an infinite dimensional contragredient Lie superalgebra $\mathfrak{g}(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}}) = \mathfrak{g}(\hat{\mathbb{A}}, \hat{\mathbb{D}}, \hat{\mathbb{I}}, \hat{\mathbb{I}}^{\mathrm{odd}})$ in the following way. Let θ be the (unique) highest element of $\Phi(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$, that is, $\theta \in \Phi(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ and $\theta + \alpha_i \notin \Phi(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$ for all $i \in \mathbb{I}$. Let $\mathbb{E}_\theta \in \mathfrak{g}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})_\theta \setminus \{0\}$ and $\mathbb{E}_{-\theta} \in \mathfrak{g}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})_{-\theta} \setminus \{0\}$. Then $\mathbb{E}_{\pm\theta}$ are homogeneous elements of $\mathfrak{g}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$. Further, one has $\deg(\mathbb{E}_\theta) = \deg(\mathbb{E}_{-\theta})$, $[\mathbb{E}_{-\theta}, \mathbb{E}_\theta] \in \mathfrak{h}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$, and $(\mathbb{E}_{-\theta}|\mathbb{E}_\theta) \neq 0$. Let $\mathbb{H}_o' := [\mathbb{E}_{-\theta}, \mathbb{E}_\theta]$. Then $\mathfrak{g}(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}}) = \mathfrak{g}(\hat{\mathbb{A}}, \hat{\mathbb{D}}, \hat{\mathbb{I}}, \hat{\mathbb{I}}^{\mathrm{odd}})$ is the contragredient Lie superalgebra defined with $\hat{\mathbb{A}}$, $\hat{\mathbb{D}}$, $\hat{\mathbb{I}}$, and $\hat{\mathbb{I}}^{\mathrm{odd}}$ below.

(i) $\hat{\mathbb{I}}$ is a set given by adding an element o to $\mathbb{I}$, that is, $\hat{\mathbb{I}} = \mathbb{I} \cup \{o\}$ and $|\hat{\mathbb{I}}| = |\mathbb{I}| + 1$.

(ii) $\hat{\mathbb{I}}^{\mathrm{odd}}$ is the subset of $\hat{\mathbb{I}}$ defined as follows. If $\deg(\mathbb{E}_\theta) = 0$, then let $\hat{\mathbb{I}}^{\mathrm{odd}} := \mathbb{I}^{\mathrm{odd}}$. If $\deg(\mathbb{E}_\theta) = 1$, then let $\hat{\mathbb{I}}^{\mathrm{odd}} := \mathbb{I}^{\mathrm{odd}} \cup \{o\}$.

(iii) $\hat{\mathbb{D}} = (\delta_{ij}\hat{\mathbb{D}}_i)_{i,j\in\hat{\mathbb{I}}}$ is the $|\hat{\mathbb{I}}| \times |\hat{\mathbb{I}}|$ diagonal matrix defined by $\hat{\mathbb{D}}_i = \mathbb{D}_i$ ($i \in \mathbb{I}$) and $\hat{\mathbb{D}}_o = (\mathbb{E}_{-\theta}|\mathbb{E}_\theta)$.

(iv) $\hat{\mathbb{A}} = (\hat{\mathbb{A}}_{ij})_{i,j\in\hat{\mathbb{I}}}$ is the $|\hat{\mathbb{I}}| \times |\hat{\mathbb{I}}|$ matrix defined by $\hat{\mathbb{A}}_{ij} = \mathbb{A}_{ij}$, $\hat{\mathbb{A}}_{oj} = \hat{\mathbb{D}}_j^{-1}(\mathbb{H}_o'|\mathbb{H}_j)$, $\hat{\mathbb{A}}_{io} = \hat{\mathbb{D}}_o^{-1}(\mathbb{H}_i|\mathbb{H}_o')$, and $\hat{\mathbb{A}}_{oo} = \hat{\mathbb{D}}_o^{-1}(\mathbb{H}_o'|\mathbb{H}_o')$ for $i,j \in \mathbb{I}$. (Note that ${}^t(\hat{\mathbb{D}}^{-1}\hat{\mathbb{A}}) = \hat{\mathbb{D}}^{-1}\hat{\mathbb{A}}$.)

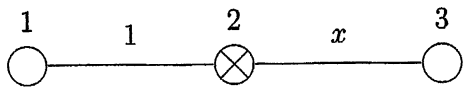

Figure 1: (Standard) Dynkin diagram of $D(2,1;x)$ $(x \neq 0,-1)$

More precisely, there exists an isomorphism $\varphi : \mathfrak{g}(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}}) \to \hat{\mathfrak{g}}$ such that $\varphi(\mathbb{H}_i) = \mathbb{H}_i \otimes 1$, $\varphi(\mathbb{E}_i) = \mathbb{E}_i \otimes 1$, $\varphi(\mathbb{F}_i) = \mathbb{F}_i \otimes 1$ $(i \in \mathbb{I})$, $\varphi(\mathbb{H}_o) = \mathbb{H}'_o \otimes 1 + \hat{\mathbb{D}}_o H_{\hat{\delta}}$, $\varphi(\mathbb{E}_o) = \mathbb{E}_{-\theta} \otimes t$, $\varphi(\mathbb{F}_o) = \mathbb{E}_\theta \otimes t^{-1}$, and $\varphi(\mathfrak{h}''(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}})) = \mathbb{C}(H_{\Lambda_0} - \frac{1}{2}(X|X)H_{\hat{\delta}} + X)$ for some $X \in \mathfrak{h}'(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$.

Let $\dot{\theta}$, $\hat{\delta}$, and Λ_0 be the elements of $\mathfrak{h}(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}})^*$ defined by $\dot{\theta}(\varphi^{-1}(H_{\hat{\delta}})) = \dot{\theta}(\varphi^{-1}(H_{\Lambda_0})) = 0$, $\hat{\delta}(\varphi^{-1}(H_{\hat{\delta}})) = \Lambda_0(\varphi^{-1}(H_{\Lambda_0})) = 0$, $\hat{\delta}(\varphi^{-1}(H_{\Lambda_0})) = \Lambda_0(\varphi^{-1}(H_{\hat{\delta}})) = 1$, $\dot{\theta}(\varphi^{-1}(\mathbb{H} \otimes 1)) = \theta(\mathbb{H})$, and $\hat{\delta}(\varphi^{-1}(\mathbb{H} \otimes 1)) = \Lambda_0(\varphi^{-1}(\mathbb{H} \otimes 1)) = 0$ for all $\mathbb{H} \in \mathfrak{h}(\mathbb{A}, \mathbb{I}^{\mathrm{odd}})$. Then one has $\alpha_o = \hat{\delta} - \dot{\theta}$ and, moreover, $(\varphi^{-1}(H_{\hat{\delta}})|\mathbb{H}) = \hat{\delta}(\mathbb{H})$ and $(\varphi^{-1}(H_{\Lambda_0})|\mathbb{H}) = \Lambda_0(\mathbb{H})$ for all $\mathbb{H} \in \mathfrak{h}(\hat{\mathbb{A}}, \hat{\mathbb{I}}^{\mathrm{odd}})$.

Now we define the Lie superalgebra $D(2,1;x)$ and the affine Lie superalgebra $D^{(1)}(2,1;x)$. Let $\mathbb{I} = \{1,2,3\} \subset \mathbb{N}$ and $\mathbb{I}^{\mathrm{odd}} = \{2\}$. Let $x \in \mathbb{C} \setminus \{0\}$ and

$$(2.3) \qquad \mathbb{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & x \\ 0 & -1 & 2 \end{pmatrix}, \quad \mathbb{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -x^{-1} \end{pmatrix}.$$

First consider the case $x \neq -1$. Then $\mathfrak{g} = \mathfrak{g}'$ and $\dim \mathfrak{g} = 17$. and, moreover, $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1+\alpha_2), \pm(\alpha_2+\alpha_3), \pm(\alpha_1+\alpha_2+\alpha_3), \pm(\alpha_1+2\alpha_2+\alpha_3)\}$. Further, $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra and is called $D(2,1;x)$. The affine Lie superalgebra $\hat{\mathfrak{g}}$ is denoted by $D^{(1)}(2,1;x)$. As mentioned above, $D^{(1)}(2,1;x)$ is identified with a contragredient Lie superalgebra. Its Dynkin diagram is given in Figure 2 (see the one labeled $d=2$ in Figure 2 especially).

Now assume that $x = -1$. Then $\dim \mathfrak{g} = 16$, $\dim \mathfrak{g}' = 15$ and $\mathfrak{g}, \mathfrak{g}'$ are called $\mathfrak{gl}(2|2)$ and $\mathfrak{sl}(2|2)$, respectively. Further, $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_1+\alpha_2), \pm(\alpha_2+\alpha_3), \pm(\alpha_1+\alpha_2+\alpha_3)\}$. However $\mathfrak{sl}(2|2)$ is not simple, and $\mathfrak{psl}(2|2) := A(1,1) := \mathfrak{sl}(2|2)/\mathbb{C}(\mathbb{H}_1 + 2\mathbb{H}_2 + \mathbb{H}_3)$ is a 14-dimensional simple Lie superalgebra. We obtain a 17-dimensional Lie superalgebra from $D(2,1;x)$ by performing a specialization of x at -1. It is a universal central extension of $\mathfrak{psl}(2|2)$ and $\mathfrak{sl}(2|2)$. Similarly, we obtain a universal central extension of $\mathfrak{psl}(2|2) \otimes \mathbb{C}[t,t^{-1}]$ and $\mathfrak{sl}(2|2) \otimes \mathbb{C}[t,t^{-1}]$ from $[D^{(1)}(2,1;x), D^{(1)}(2,1;x)]$ by performing a specialization of x at -1, see [IK].

Remark 2.1. *The Lie superalgebras* $\mathfrak{gl}(m+1|n+1)$, $\mathfrak{sl}(m+1|n+1)$, $\mathfrak{psl}(n+1|n+1)$, $A(m,n)$ *and* $A^{(1)}(m,n)$. Let m and n be non-negative integers such that $m+n \geq 1$. For $i,j \in \{1,\ldots,m+n+2\}$, let $\mathbf{E}_{i,j}$ denote the $(m+n+2) \times (m+n+2)$ matrix having 1 in (i,j) position and 0 otherwise, that is, the (i,j)-matrix unit. Let $\mathbf{E}_{m+n+2}$ denote the $(m+n+2) \times (m+n+2)$ unit matrix, that is, $\sum_{i=1}^{m+n+2} \mathbf{E}_{i,i}$.

Denote by $\mathrm{M}_{m+n+2}(\mathbb{C})$ the $\mathbb{C}$-linear space of the $(m+n+2)\times(m+n+2)$-matrices, that is, $\oplus_{i,j=1}^{m+n+2}\mathbb{C}\mathbf{E}_{i,j}$. The Lie superalgebra $\mathfrak{gl}(m+1|n+1)$ is defined by $\mathfrak{gl}(m+1|n+1)=\mathrm{M}_{m+n+2}(\mathbb{C})$ (as a $\mathbb{C}$-linear space), $\mathfrak{gl}(m+1|n+1)(0)=(\oplus_{i,j=1}^{m+1}\mathbb{C}\mathbf{E}_{i,j})\oplus(\oplus_{i,j=m+2}^{m+n+2}\mathbb{C}\mathbf{E}_{i,j})$, $\mathfrak{gl}(m+1|n+1)(1)=(\oplus_{i=1}^{m+1}\oplus_{j=m+2}^{m+n+2}\mathbb{C}\mathbf{E}_{i,j})\oplus(\oplus_{j=1}^{m+1}\oplus_{i=m+2}^{m+n+2}\mathbb{C}\mathbf{E}_{i,j})$ and $[X,Y]=XY-(-1)^{\deg(X)\deg(Y)}YX$ for all X, $Y\in\mathfrak{gl}(m+1|n+1)(0)\cup\mathfrak{gl}(m+1|n+1)(1)$, where XY and YX mean the matrix product, that is, $\mathbf{E}_{i,j}\mathbf{E}_{k,l}=\delta_{j,k}\mathbf{E}_{i,l}$. Define the $\mathbb{C}$-linear map str : $\mathfrak{gl}(m+1|n+1)\to\mathbb{C}$ by $\mathrm{str}(\mathbf{E}_{i,j})=\delta_{i,j}(\sum_{k=1}^{m+1}\delta_{i,k}-\sum_{l=m+2}^{m+n+2}\delta_{i,l})$. The Lie subsuperalgebra $\{X\in\mathfrak{gl}(m+1|n+1)\,|\,\mathrm{str}(X)=0\}$ of $\mathfrak{gl}(m+1|n+1)$ is denoted as $\mathfrak{sl}(m+1|n+1)$. The finite dimensional simple Lie superalgebra $A(m,n)$ (cf. [K]) is defined as follows. Let $\mathfrak{z}$ be the one dimensional ideal $\mathbb{C}\mathbf{E}_{m+n+2}$ of $\mathfrak{gl}(m+1|n+1)$. If $m\neq n$, then $A(m,n)$ means $\mathfrak{sl}(m+1|n+1)$. On the other hand, $A(n,n)$ means $\mathfrak{sl}(n+1|n+1)/\mathfrak{z}$, and is also denoted as $\mathfrak{psl}(n+1|n+1)$.

Recall the Lie superalgebras $\mathfrak{g}=\mathfrak{g}(\mathbb{A},\mathbb{D},\mathbb{I},\mathbb{I}^{\mathrm{odd}})$ and $\hat{\mathfrak{g}}=\hat{\mathfrak{g}}(\mathbb{A},\mathbb{I}^{\mathrm{odd}})=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}H_{\delta}\oplus\mathbb{C}H_{\Lambda_0}$, introduced above.

Define the supersymmetric invariant form $(\,,\,)$ on $\mathfrak{gl}(m+1|n+1)$ by $(X,Y)=\mathrm{str}(XY)$. Then the infinite dimensional Lie superalgebra $\mathfrak{gl}(m+1|n+1)^{(1)}$ is defined in the same way as that for $\hat{\mathfrak{g}}$ with $\mathfrak{gl}(m+1|n+1)$ and $(\,,\,)$ in place of $\mathfrak{g}$ and $(\,|\,)$ respectively. Further, $\mathfrak{sl}(m+1|n+1)^{(1)}$ means the Lie subsuperalgebra $\mathfrak{sl}(m+1|n+1)\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}H_{\delta}\oplus\mathbb{C}H_{\Lambda_0}$ of $\mathfrak{gl}(m+1|n+1)^{(1)}$. If $m\neq n$, then $A^{(1)}(m,n)$ means $\mathfrak{sl}(m+1|n+1)^{(1)}$. On the other hand, $A^{(1)}(n,n)$ means $\mathfrak{sl}(n+1|n+1)^{(1)}/(\mathfrak{z}\otimes\mathbb{C}[t,t^{-1}])$. (See also [K], or [IK], for these notation.)

Assume $\mathbb{A}$, $\mathbb{D}$, $\mathbb{I}$, and $\mathbb{I}^{\mathrm{odd}}$ to be the $(m+n+1)\times(m+n+1)$ matrix $(-\delta_{i-j,1}+2(1-\delta_{i,m+1})\delta_{i,j}-(-1)^{\delta_{i,m+1}}\delta_{i-j,-1})_{1\leq i,j\leq m+n+1}$, the diagonal $(m+n+1)\times(m+n+1)$ matrix $(\delta_{i,j}(\sum_{k=1}^{m+1}\delta_{i,k}-\sum_{l=m+2}^{m+n+1}\delta_{i,l}))_{1\leq i,j\leq m+n+1}$, $\{1,\ldots,m+n+1\}$, and $\{m+1\}$ respectively.

Assume that $m\neq n$. Then we identify $A(m,n)$ with $\mathfrak{g}$, since there exists a unique isomorphism $\varpi:\mathfrak{g}\to A(m,n)$ such that $\varpi(\mathbb{E}_i)=\mathbf{E}_{i,i+1}$ and $\varpi(\mathbb{F}_i)=\mathbf{E}_{i+1,i}$. Further, we identify $A^{(1)}(m,n)$ with the affine Lie superalgebra $\hat{\mathfrak{g}}$, since $(\varpi(X),\varpi(Y))=(X|Y)$ for all X, $Y\in\mathfrak{g}$.

Assume that $m=n$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(n+1|n+1)$, and we identify them. Note that $\mathfrak{g}$ is not simple since $[\mathfrak{g},\mathfrak{g}]\neq\mathfrak{g}$. Nevertheless, we define $\hat{\mathbb{A}}$, $\hat{\mathbb{D}}$, $\hat{\mathbb{I}}$, and $\hat{\mathbb{I}}^{\mathrm{odd}}$ in the same way as above, and we let $\bar{\mathfrak{g}}:=\mathfrak{g}(\hat{\mathbb{A}},\hat{\mathbb{D}},\hat{\mathbb{I}},\hat{\mathbb{I}}^{\mathrm{odd}})$. Let $\overline{\mathfrak{sl}}(n+1|n+1)$ be the Lie subsuperalgebra $\mathfrak{sl}(n+1|n+1)^{(1)}\oplus\mathbb{C}\mathbf{E}_{1,1}$ of $\mathfrak{gl}(n+1|n+1)^{(1)}$. Then $\bar{\mathfrak{g}}$ is isomorphic to $\overline{\mathfrak{sl}}(n+1|n+1)/(\oplus_{r\in\mathbb{Z}\setminus\{0\}}\mathfrak{z}\otimes t^r)$ (cf. [Y, Section 1.5]).

3 Semigroups and braid semigroups

3.1 Semigroups

In this section we fix notations and terminology concerning semigroups. This will be helpful for the definition of semigroups by generators and relations.

Let K be a non-empty set. We call K a *semigroup* if it is equipped with a product $K \times K \to K$, $(x,y) \mapsto xy$, satisfying the associativity law, that is, $(xy)z = x(yz)$ for x, y, $z \in K$. If K is a semigroup, we call it a *monoid* if there exists a unit $1 \in K$, that is, $1x = x1 = x$ for $x \in K$. If K is a semigroup and does not have a unit, let $\widehat{K}$ denote the monoid obtained from K by adding a unit.

Let H be a non-empty set and $L(H)$ a set of all the finite sequences of elements of H, so, $L(H) = \{(h_1,\ldots,h_n)|n \in \mathbb{N}, h_i \in H\}$. We regard $L(H)$ as a semigroup whose product is defined by $(h_1,\ldots,h_m)(h_{m+1},\ldots,h_{m+n}) = (h_1,\ldots,h_m,h_{m+1},\ldots,h_{m+n})$. Then $L(H)$ is called a *free semigroup* on H and $\widehat{L(H)}$ a *free monoid* on H. Let $\{(x_j,y_j)|j \in J\}$ be a subset of $L(H) \times L(H)$, where J is an index set. As usual, for at most two elements g, g' of $L(H)$, we let the notation $\{g,g'\}$ mean the subset of $L(H)$ consisting of g and g'. (Hence the cardinality of $\{g,g'\}$ is $2-\delta_{g,g'}$. As for $\delta_{g,g'}$, see the last paragraph in Introduction.) For g_1, $g_2 \in L(H)$, we write $g_1 \sim_1 g_2$ if the equation $\{g_1,g_2\} = \{z_1x_jz_2, z_1y_jz_2\}$ (equality of subsets of $L(H)$) holds for some $j \in J$ and some z_1, $z_2 \in \widehat{L(H)}$. For g, $g' \in L(H)$, we write $g \sim g'$ if $g = g'$ or there exist finitely many elements $g_1,\ldots,g_r$ of $L(H)$ such that $g_1 = g$, $g_r = g'$ and $g_i \sim_1 g_{i+1}$. Then $\sim$ is an equivalence relation in $L(H)$. Let $L(H)/\sim$ be the set of the equivalence classes in $L(H)$ with respect to $\sim$. For $g \in L(H)$, let $[g]$ be the element of $L(H)/\sim$ containing g. We regard $L(H)/\sim$ as a semigroup so that the map $L(H) \to L(H)/\sim$ defined by $g \mapsto [g]$ is a homomorphism. We call $L(H)/\sim$ *the semigroup generated by H and defined by the relations $x_j = y_j$ $(j \in J)$*. When there is not fear of misunderstanding, we also denote $[g]$ simply by g.

3.2 The Weyl groupoid of $D^{(1)}(2,1;x)$

For the presentation of contragredient Lie superalgebras $\mathfrak{g}$ one can use different Dynkin diagrams. This fact leads to the definition of the Weyl groupoid as a symmetry object of $\mathfrak{g}$. General properties of such groupoids were investigated in [HY]. In this section we introduce the Weyl groupoid and the extended Weyl groupoid of the affine Lie superalgebra $D^{(1)}(2,1;x)$.

Let $\mathcal{D} := \{0,1,2,3,4\}$. Let $\triangleright$: $S_5 \times \mathcal{D} \to \mathcal{D}$ denote the usual (left) action of the symmetric group S_5 on $\mathcal{D}$ by permutations. The elements of the set $\mathcal{D}$ will be used to label different Dynkin diagrams for the affine Lie superalgebra $D^{(1)}(2,1;x)$.

Let $I := \{0,1,2,3\}$. Note that $|I| = 4$ is the rank of $D^{(1)}(2,1;x)$. This set will be used to label the vertices in a given Dynkin diagram. In order to define the Weyl groupoid we will need the following structure constants. For $d \in \mathcal{D} \setminus \{4\}$ and $i,j \in I$ with $i \neq j$ let

$$(3.1) \qquad m_{i,j;d} := \begin{cases} 2 & \text{if } i \neq d \neq j, \\ 3 & \text{otherwise,} \end{cases} \qquad\qquad m_{i,j;4} := 3.$$

The index $d=4$ is distinguished, see Figure 2. Note that

$$m_{i,j;d}=m_{j,i;d}=m_{i,j;n_i\triangleright d}=m_{i,j;n_j\triangleright d}$$

for all $d\in\mathcal{D}$ and $i,j\in I$, where $n_i=(i\,4)$ as an element in S_5. Further one has

$$n_i\triangleright d=d\quad\text{if } m_{i,j;d}=2,\qquad\qquad n_in_jn_i=n_jn_in_j=(i\,j).$$

The extended Weyl groupoid defined below contains even and odd reflections and elements corresponding to permutations of vertices of Dynkin diagrams. Note that any permutation of vertices of a given diagram can be identified with a permutation f of I. In our setting only the Klein four-group

$$\mathcal{K}_4=\{f_0:=\mathrm{id},f_1:=(01)(23),f_2:=(02)(13),f_3:=(03)(12)\}\tag{3.2}$$

will be needed. Further, any $f\in\mathcal{K}_4$ induces a permutation $\gamma(f)$ of Dynkin diagrams $d\in\mathcal{D}$. Thus one obtains a group homomorphism

$$\gamma:\mathcal{K}_4\to S_5=\mathrm{Perm}\,\mathcal{D},$$

defined by the following formula, see Figure 2.

$$\gamma(f)\triangleright d=\begin{cases}f(d) & \text{if } d\in\{0,1,2,3\},\\ 4 & \text{if } d=4\end{cases}$$

for all $f\in\mathcal{K}_4$. This formula is accidentally true in our setting, and can not be generalized to arbitrary contragredient Lie superalgebras. By abuse of notation we will also write $f\triangleright d$ instead of $\gamma(f)\triangleright d$.

Let W^{ext} be the semigroup generated by

$$\{0\}\cup\{e_d\,|\,d\in\mathcal{D}\}\cup\{s_{i,d}\,|\,i\in I,d\in\mathcal{D}\}\cup\{\tau_{f,d}\,|\,f\in\mathcal{K}_4,d\in\mathcal{D}\}\tag{3.3}$$

and defined by the following relations (3.4)–(3.12):

$$\begin{aligned}
&(3.4) && 0=0u=u0 \text{ for all elements } u \text{ in (3.3)},\\
&(3.5) && e_de_d=e_d,\quad e_de_{d'}=0 \text{ for } d\neq d',\\
&(3.6) && e_{n_i\triangleright d}s_{i,d}=s_{i,d},\quad s_{i,d}e_d=s_{i,d},\\
&(3.7) && s_{i,n_i\triangleright d}s_{i,d}=e_d,\\
&(3.8) && s_{i,d}s_{j,d}=s_{j,d}s_{i,d} && \text{if } m_{i,j;d}=2,\\
&(3.9) && s_{i,n_jn_i\triangleright d}s_{j,n_i\triangleright d}s_{i,d}=s_{j,n_in_j\triangleright d}s_{i,n_j\triangleright d}s_{j,d} && \text{if } m_{i,j;d}=3,\\
&(3.10) && e_{f\triangleright d}\tau_{f,d}=\tau_{f,d},\quad \tau_{f,d}e_d=\tau_{f,d},\quad \tau_{f_0,d}=e_d,\\
&(3.11) && \tau_{f,f'\triangleright d}\tau_{f',d}=\tau_{ff',d}\quad\text{for } f,f'\in\mathcal{K}_4,\\
&(3.12) && \tau_{f,n_i\triangleright d}s_{i,d}=s_{f(i),f\triangleright d}\tau_{f,d}.
\end{aligned}$$

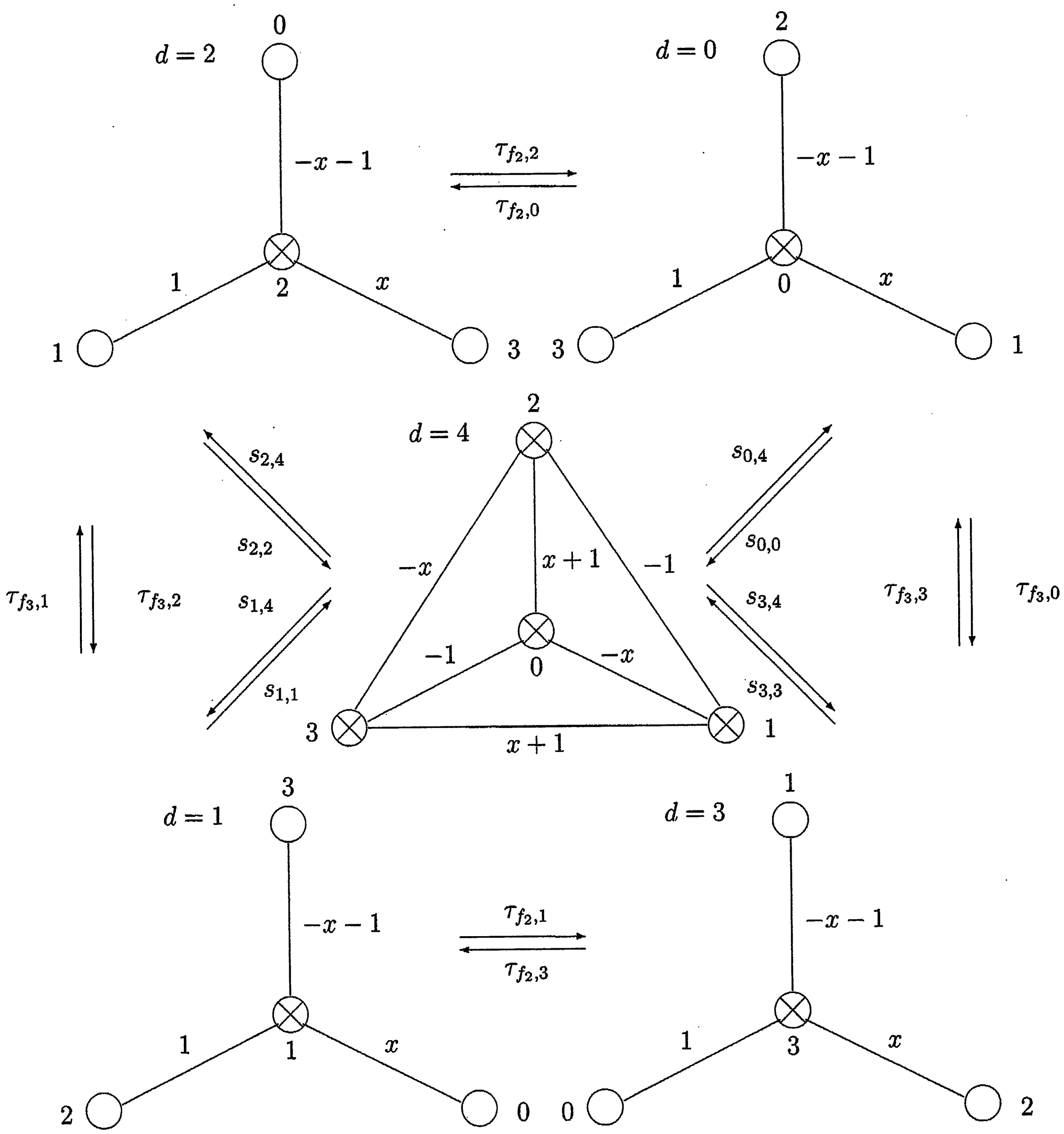

Figure 2: Dynkin diagrams of $D^{(1)}(2,1;x)$ $(x \neq 0,-1)$

Definition 3.1. The semigroup W^{ext} is called the *extended Weyl groupoid* of the affine Lie superalgebra $D^{(1)}(2,1;x)$. [2] The subgroupoid of W^{ext} generated by the set

$$\{0\} \cup \{e_d \,|\, d \in \mathcal{D}\} \cup \{s_{i,d} \,|\, i \in I, d \in \mathcal{D}\} \tag{3.13}$$

is called the *Weyl groupoid* of $D^{(1)}(2,1;x)$ and will be denoted by W.

For an element $w = s_{i_r,d_r} \cdots s_{i_2,d_2} s_{i_1,d_1}$ of W, where $d_u := n_{i_{u-1}} \cdots n_{i_1} \triangleright d_1$ for $1 \le u \le r$, we also use the abbreviations

$$w = s_{i_r} \cdots s_{i_2} s_{i_1,d_1}, \qquad \tau_{f,d_r} w = \tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d_1}, \qquad w\tau_{f,f\triangleright d} = s_{i_r} \cdots s_{i_2} s_{i_1} \tau_{f,f\triangleright d}. \tag{3.14}$$

If $r = 0$, let $s_{i_r} \cdots s_{i_2} s_{i_1,d} := e_d$, $\tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d} := \tau_{f,d}$ and $s_{i_r} \cdots s_{i_1} \tau_{f,d} := \tau_{f,d}$. Note that

$$W^{\text{ext}} = \{0\} \cup \{\tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d} \,|\, d \in \mathcal{D}, f \in \mathcal{K}_4, r \in \mathbb{N}_0, i_1, \dots, i_r \in I, \; i_u \ne i_{u+1} \text{ for } 1 \le u \le r-1\}. \tag{3.15}$$

Now we prove that the elements $\tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d}$ in Eq. (3.15) are nonzero. Let $\mathbb{R}^{\mathcal{D}}$ be an $\mathbb{R}$-vector space of dimension $|\mathcal{D}|$, and let $\{v_d \,|\, d \in \mathcal{D}\}$ be a fixed basis of $\mathbb{R}^{\mathcal{D}}$. Then there is a unique semigroup homomorphism

$$\widetilde{\operatorname{sgn}} : W^{\text{ext}} \to \operatorname{End}(\mathbb{R}^{\mathcal{D}})$$

such that

$$\begin{aligned} \widetilde{\operatorname{sgn}}(0)(v_{d'}) &= 0, & \widetilde{\operatorname{sgn}}(e_d)(v_{d'}) &= \delta_{dd'} v_{d'}, \\ \widetilde{\operatorname{sgn}}(s_{i,d})(v_{d'}) &= (-1)\delta_{dd'} v_{n_i \triangleright d}, & \widetilde{\operatorname{sgn}}(\tau_{f,d})(v_{d'}) &= \delta_{dd'} v_{f\triangleright d'} \end{aligned} \tag{3.16}$$

for all $d, d' \in \mathcal{D}$, $i \in I$, $f \in \mathcal{K}_4$. In particular $\tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d} \ne 0$ for all $d \in \mathcal{D}$, $f \in \mathcal{K}_4$, $r \in \mathbb{N}_0$, and $i_1, \dots, i_r \in I$. Let

$$e_d^{-1} := e_d, \qquad \left(\tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d}\right)^{-1} := s_{i_1} \cdots s_{i_{r-1}} s_{i_r} \tau_{f, f n_{i_r} \cdots n_{i_2} n_{i_1} \triangleright d}.$$

One says that an expression $w = \tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d}$ is *reduced* if $w = \tau_{f'} s_{j_s} \cdots s_{j_2} s_{j_1,d}$ implies that $s \ge r$. In this case one defines $\ell(w) := r$.

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[2] Here we use the less standard terminology concerning groupoids, in which the multiplication is globally defined, but may be zero (instead of nondefined). Then all groupoids are semigroups. Removing 0 from the semigroup gives a groupoid in the standard sense.

3.3 The braid semigroup

Lusztig [L1] defined automorphisms of quantized enveloping algebras of Kac-Moody Lie algebras attached to all simple reflections of the corresponding Weyl group. These automorphisms are not involutions, but nevertheless they satisfy some Coxeter relations. Analogously there exist isomorphisms between different realizations of quantized $D^{(1)}(2,1;x)$, see Section 4.2, which also satisfy Coxeter relations. At this place we introduce the abstract semigroup which forms a bridge between the aforementioned isomorphisms and the Weyl groupoid of $D^{(1)}(2,1;x)$.

Let $\widetilde{W}^{\text{ext}}$ be the semigroup generated by

$$\{0\} \cup \{\tilde{e}_d \,|\, d \in \mathcal{D}\} \cup \{\tilde{s}_{i,d} \,|\, i \in I, d \in \mathcal{D}\} \cup \{\tilde{\tau}_{f,d} \,|\, f \in \mathcal{K}_4, d \in \mathcal{D}\} \tag{3.17}$$

and defined by the relations analogous to (3.4)-(3.6), (3.8)-(3.12). Let $\widetilde{W}$ be the subsemigroup of $\widetilde{W}^{\text{ext}}$ generated by

$$\{0\} \cup \{\tilde{e}_d \,|\, d \in \mathcal{D}\} \cup \{\tilde{s}_{i,d} \,|\, i \in I, d \in \mathcal{D}\}. \tag{3.18}$$

Notation 3.2. For elements of $\widetilde{W}$ and $\widetilde{W}^{\text{ext}}$ we use a notation analogous to the one in Eq. (3.14).

Similarly to Eq. (3.15) one has

$$\widetilde{W}^{\text{ext}} = \{0\} \cup \{\tilde{\tau}_f \tilde{s}_{i_r} \cdots \tilde{s}_{i_2} \tilde{s}_{i_1,d} \,|\, d \in \mathcal{D}, f \in \mathcal{K}_4,\ r \in \mathbb{N}_0,\ i_1, \dots, i_r \in I\}. \tag{3.19}$$

Note that there exists a canonical semigroup homomorphism $\mathfrak{p} : \widetilde{W}^{\text{ext}} \to W^{\text{ext}}$ such that

$$\mathfrak{p}(0) = 0, \qquad \mathfrak{p}(\tilde{\tau}_f \tilde{s}_{i_r} \cdots \tilde{s}_{i_2} \tilde{s}_{i_1,d}) = \tau_f s_{i_r} \cdots s_{i_2} s_{i_1,d}$$

for all $d \in \mathcal{D}$, $f \in \mathcal{K}_4$, $r \in \mathbb{N}_0$, and $i_1, \dots, i_r \in I$.

3.4 Special elements of $\widetilde{W}^{\text{ext}}$

The extended Weyl group of an affine Lie algebra $\hat{\mathfrak{g}}$, which is the group generated by the Weyl group and by diagram automorphisms of $\hat{\mathfrak{g}}$, can be written as the semidirect product of the (finite) Weyl group of $\mathfrak{g}$ and a free abelian group, and the latter can be identified with the weight lattice corresponding to $\mathfrak{g}$. We expect that a similar decomposition holds for the extended Weyl groupoid of any affine Lie superalgebra. Since we are mainly interested in the (quantum) affine Lie superalgebra $D^{(1)}(2,1;x)$, we will not work out here the details of such a decomposition, but concentrate on those formulas which are necessary to obtain Theorems 4.7, 4.8, and 4.10. Nevertheless it may be helpful to think about the elements $\omega^\vee_{i,d}$ introduced below as the generators of the weight lattice in the extended Weyl groupoid W^{ext}. Further, the analog of the Weyl group of $\mathfrak{g}$ will be the subgroupoid of W^{ext} generated by the reflections $s_{i,d}$, where $i \in I \setminus \{0\}$, $d \in \mathcal{D} \setminus \{0\}$.

Define $\omega^\vee_{i,d} \in W^{\mathrm{ext}}$, where $i \in I \setminus \{0\}$ and $d \in \mathcal{D} \setminus \{0\}$, as follows. Let i, j, $k \in I \setminus \{0\}$ be such that $\{i,j,k\} = \{1,2,3\}(= I \setminus \{0\})$. Let

$$\begin{aligned} \omega^\vee_{i,4} &:= \tau_{f_i,4} s_{i,i} s_{k,i} s_{j,i} s_{i,4} = \tau_{f_i} s_i s_k s_j s_{i,4} = \tau_{f_i} s_i s_j s_k s_{i,4}, \\ \omega^\vee_{j,i} &:= \tau_{f_j,k} s_{j,k} s_{k,4} s_{i,i} s_{j,i} = \tau_{f_j} s_j s_k s_i s_{j,i}, \\ \omega^\vee_{i,i} &:= s_{0,i} s_{i,4} s_{j,j} s_{k,j} s_{j,4} s_{i,i} = s_0 s_i s_j s_k s_j s_{i,i} = s_0 s_i s_k s_j s_k s_{i,i}. \end{aligned} \tag{3.20}$$

It is easy to check [HY] that all of the above expressions are reduced. Define also $\tilde{\omega}^\vee_{i,d} \in \widetilde{W}^{\mathrm{ext}}$, where $i \in I \setminus \{0\}$ and $d \in \mathcal{D} \setminus \{0\}$ by the following formulas:

$$\begin{aligned} \tilde{\omega}^\vee_{i,4} &:= \tilde{\tau}_{f_i,4} \tilde{s}_{i,i} \tilde{s}_{k,i} \tilde{s}_{j,i} \tilde{s}_{i,4} = \tilde{\tau}_{f_i} \tilde{s}_i \tilde{s}_k \tilde{s}_j \tilde{s}_{i,4} = \tilde{\tau}_{f_i} \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_{i,4}, \\ \tilde{\omega}^\vee_{j,i} &:= \tilde{\tau}_{f_j,k} \tilde{s}_{j,k} \tilde{s}_{k,4} \tilde{s}_{i,i} \tilde{s}_{j,i} = \tilde{\tau}_{f_j} \tilde{s}_j \tilde{s}_k \tilde{s}_i \tilde{s}_{j,i}, \\ \tilde{\omega}^\vee_{i,i} &:= \tilde{s}_{0,i} \tilde{s}_{i,4} \tilde{s}_{j,j} \tilde{s}_{k,j} \tilde{s}_{j,4} \tilde{s}_{i,i} = \tilde{s}_0 \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_j \tilde{s}_{i,i} = \tilde{s}_0 \tilde{s}_i \tilde{s}_k \tilde{s}_j \tilde{s}_k \tilde{s}_{i,i}. \end{aligned} \tag{3.21}$$

Note that $\mathfrak{p}(\tilde{\omega}^\vee_{i,d}) = \omega^\vee_{i,d}$ for all $i \in I \setminus \{0\}$ and $d \in \mathcal{D} \setminus \{0\}$.

Remark 3.3. In order to understand the above definitions it is important to note that for all $i \in I$ the vertex i is playing a special role in the Dynkin diagram labeled by i.

The elements of $\widetilde{W}^{\mathrm{ext}}$ defined above satisfy the equations

$$\tilde{e}_d \tilde{\omega}^\vee_{i,d} = \tilde{\omega}^\vee_{i,d} \tilde{e}_d = \tilde{\omega}^\vee_{i,d}$$

for all $i \in I \setminus \{0\}$ and $d \in \mathcal{D} \setminus \{0\}$. Further, for i, j, k as above let

$$\tilde{\nu}_{i,i} := \tilde{\tau}_{f_i} \tilde{s}_i \tilde{s}_k \tilde{s}_{j,i}, \qquad \tilde{\nu}_{j,i} := \tilde{\tau}_{f_j} \tilde{s}_j \tilde{s}_k \tilde{s}_{i,i}, \qquad \tilde{\nu}_{i,4} := \tilde{s}_0 \tilde{s}_i \tilde{s}_j \tilde{s}_k \tilde{s}_{j,4}. \tag{3.22}$$

Lemma 3.4. *One has*

$$\tilde{\nu}_{i,n_i \triangleright d} \tilde{s}_{i,d} = \tilde{\omega}^\vee_{i,d}$$

for all $i \in I \setminus \{0\}$ *and* $d \in \mathcal{D} \setminus \{0\}$.

Proof. This follows immediately from Eq.s (3.22) and (3.21). □

3.5 Some commutation relations in $\widetilde{W}^{\mathrm{ext}}$

In [L2, Lemma 2.7] Lusztig studies the braid group of the extended Weyl group of an affine Lie algebra. In our setting the following related formulas are valid.

Theorem 3.5. *(1) For all* $i, j \in I \setminus \{0\}$ *and* $d \in \mathcal{D} \setminus \{0\}$ *one has*

$$\tilde{\omega}^\vee_{i,d} \tilde{\omega}^\vee_{j,d} = \tilde{\omega}^\vee_{j,d} \tilde{\omega}^\vee_{i,d} \tag{3.23}$$

(2) Assume that $\{i,j,k\} = \{1,2,3\}$ *and* $d \in \mathcal{D} \setminus \{0\}$. *Then one has*

$$\tilde{\nu}_{i,d} \tilde{\omega}^\vee_{i,d} = \tilde{s}_{i,d} (\tilde{\omega}^\vee_{j,d})^{m_{i,j;d}-2} (\tilde{\omega}^\vee_{k,d})^{m_{i,k;d}-2}, \tag{3.24}$$

$$\tilde{\omega}^\vee_{i,n_j \triangleright d} \tilde{s}_{j,d} = \tilde{s}_{j,d} \tilde{\omega}^\vee_{i,d}. \tag{3.25}$$

Proof. Suppose that $i \neq j$ and $d = 4$. One calculates

$$
\begin{aligned}
\tilde{\omega}^\vee_{i,d}\tilde{\omega}^\vee_{j,d} =& \tilde{\tau}_{f_i,4}\underline{\tilde{s}_{i,i}\tilde{s}_{k,i}\tilde{s}_{j,i}\tilde{s}_{i,4}\tilde{\tau}_{f_j,4}}\tilde{s}_{j,j}\tilde{s}_{k,j}\tilde{s}_{i,j}\tilde{s}_{j,4} = \underline{\tilde{\tau}_{f_i,4}\tilde{\tau}_{f_j,4}}\tilde{s}_{k,k}\underline{\tilde{s}_{i,k}\tilde{s}_{0,k}}\tilde{s}_{k,4}\tilde{s}_{j,j}\tilde{s}_{k,j}\tilde{s}_{i,j}\tilde{s}_{j,4} \\
=& \tilde{\tau}_{f_k,4}\tilde{s}_{k,k}\tilde{s}_{0,k}\tilde{s}_{i,k}\underline{\tilde{s}_{k,4}\tilde{s}_{j,j}\tilde{s}_{k,j}}\tilde{s}_{i,j}\tilde{s}_{j,4} = \tilde{\tau}_{f_k,4}\tilde{s}_{k,k}\tilde{s}_{0,k}\underline{\tilde{s}_{i,k}\tilde{s}_{j,k}}\tilde{s}_{k,4}\underline{\tilde{s}_{j,j}\tilde{s}_{i,j}\tilde{s}_{j,4}} \\
=& \tilde{\tau}_{f_k,4}\tilde{s}_{k,k}\tilde{s}_{0,k}\tilde{s}_{j,k}\tilde{s}_{i,k}\tilde{s}_{k,4}\tilde{s}_{i,i}\tilde{s}_{j,i}\tilde{s}_{i,4}, \\
\tilde{\omega}^\vee_{j,d}\tilde{\omega}^\vee_{i,d} =& \tilde{\tau}_{f_j,4}\underline{\tilde{s}_{j,j}\tilde{s}_{k,j}\tilde{s}_{i,j}\tilde{s}_{j,4}\tilde{\tau}_{f_i,4}}\tilde{s}_{i,i}\tilde{s}_{k,i}\tilde{s}_{j,i}\tilde{s}_{i,4} = \underline{\tilde{\tau}_{f_j,4}\tilde{\tau}_{f_i,4}}\tilde{s}_{k,k}\underline{\tilde{s}_{j,k}\tilde{s}_{0,k}}\tilde{s}_{k,4}\tilde{s}_{i,i}\tilde{s}_{k,i}\tilde{s}_{j,i}\tilde{s}_{i,4} \\
=& \tilde{\tau}_{f_k,4}\tilde{s}_{k,k}\tilde{s}_{0,k}\tilde{s}_{j,k}\underline{\tilde{s}_{k,4}\tilde{s}_{i,i}\tilde{s}_{k,i}}\tilde{s}_{j,i}\tilde{s}_{i,4} = \tilde{\tau}_{f_k,4}\tilde{s}_{k,k}\tilde{s}_{0,k}\tilde{s}_{j,k}\tilde{s}_{i,k}\tilde{s}_{k,4}\tilde{s}_{i,i}\tilde{s}_{j,i}\tilde{s}_{i,4}.
\end{aligned}
$$

The statement of part (1) for $d \in \{1,2,3\}$ and part (2) can be obtained analogously. □

3.6 Symmetric bilinear forms

The affine Lie superalgebra $D^{(1)}(2,1;x)$ can be described with help of different Dynkin diagrams. In this section we define symmetric bilinear forms associated to all of these diagrams.

For any $d \in \mathcal{D}$ let V_d be a four dimensional $\mathbb{C}$-vector space, and let $\Pi_d = \{\alpha_{i,d} \,|\, i \in I\}$ be a basis of V_d. Let $x \in \mathbb{C} \setminus \{0,-1\}$. According to the Dynkin diagrams in Figure 2, for each $d \in \mathcal{D}$ define a symmetric bilinear form $(\,,\,) = (\,,\,)_d : V_d \times V_d \to \mathbb{C}$ as follows:

$$
\begin{aligned}
(\alpha_{i,4},\alpha_{i,4}) =&0 \quad \text{for } i \in I, & (\alpha_{0,4},\alpha_{3,4}) =&(\alpha_{1,4},\alpha_{2,4}) = -1, \\
(\alpha_{0,4},\alpha_{1,4}) =&(\alpha_{2,4},\alpha_{3,4}) = -x, & (\alpha_{0,4},\alpha_{2,4}) =&(\alpha_{1,4},\alpha_{3,4}) = x+1,
\end{aligned}
$$

$$
\begin{aligned}
(\alpha_{0,0},\alpha_{0,0}) =&0, & (\alpha_{i,0},\alpha_{j,0}) =&0 \quad \text{for } i,j \in \{1,2,3\},\ i \neq j, \\
(\alpha_{1,0},\alpha_{1,0}) =&-2x, & (\alpha_{1,0},\alpha_{0,0}) =&x, \\
(\alpha_{2,0},\alpha_{2,0}) =&2(x+1), & (\alpha_{2,0},\alpha_{0,0}) =&-x-1, \\
(\alpha_{3,0},\alpha_{3,0}) =&-2, & (\alpha_{3,0},\alpha_{0,0}) =&1,
\end{aligned}
$$

and

$$
(\alpha_{i,d},\alpha_{j,d}) = (\alpha_{f_d(i),0},\alpha_{f_d(j),0})
$$

for $d \in \{1,2,3\}$. For $d \in \mathcal{D}$ define a $\mathbb{Z}$-module map $p = p_d : \mathbb{Z}\Pi_d \to \mathbb{Z}$ by

$$
p(\alpha_{i,d}) := \begin{cases} 0 & \text{if } (\alpha_{i,d},\alpha_{i,d}) \neq 0, \\ 1 & \text{if } (\alpha_{i,d},\alpha_{i,d}) = 0 \end{cases}
$$

and call $p(\alpha)$ the *parity* of $\alpha \in \mathbb{Z}\Pi_d$. The next lemma follows immediately from Eq. (3.1) and the definition of $(\,,\,)$.

Lemma 3.6. *One has*

$$m_{i,j;d} = 2 \quad \text{if } (\alpha_{i,d},\alpha_{j,d}) = 0, \qquad m_{i,j;d} = 3 \quad \text{if } (\alpha_{i,d},\alpha_{j,d}) \neq 0.$$

for all $d \in \mathcal{D}$ and $i,j \in I$ with $i \neq j$.

In the following lemma we give a representation of W^{ext} which is compatible with the symmetric bilinear form defined above.

Lemma 3.7. *Let $\mathbf{V} := \oplus_{d=0}^4 V_d$. Then there exists a unique semigroup homomorphism $\mathsf{t} : W^{\mathrm{ext}} \to \mathrm{End}_{\mathbb{C}}(\mathbf{V})$ such that*

$$\mathsf{t}(0) = 0, \quad \mathsf{t}(e_d) = \mathrm{id}_{V_d}, \qquad \mathsf{t}(\tau_{f,d})(\alpha_{i,d}) = \alpha_{f(i),f\triangleright d},$$
$$\mathsf{t}(s_{i,d})(\alpha_{i,d}) = -\alpha_{i,n_i\triangleright d}, \qquad \mathsf{t}(s_{i,d})(\alpha_{j,d}) = \alpha_{j,n_i\triangleright d} + (m_{i,j;d}-2)\alpha_{i,n_i\triangleright d}$$

for all $i,j \in I$ and $d \in \mathcal{D}$ with $i \neq j$. Moreover, for $w \in W^{\mathrm{ext}}$ with $we_d = w$ and for v, $v' \in V_d$ and $\mu \in \mathbb{Z}\Pi_d$ we have

$$(\mathsf{t}(w)(v), \mathsf{t}(w)(v')) = (v,v') \quad \text{and} \quad (-1)^{p(\mathsf{t}(w)(\mu))} = (-1)^{p(\mu)}. \tag{3.26}$$

Further, if $p(\alpha_{i,d}) = 0$ then $n_i \triangleright d = d$, $(\alpha_{i,d},\alpha_{i,d}) \neq 0$, and

$$\mathsf{t}(s_{i,d})(\alpha_{j,d}) = \alpha_{j,d} - \frac{2(\alpha_{i,d},\alpha_{j,d})}{(\alpha_{i,d},\alpha_{i,d})}\alpha_{i,d}$$

for all $i,j \in I$ and $d \in \mathcal{D}$.

Proof. One has to check that the definition of t is compatible with the relations (3.4)–(3.12). This is trivial for all relations different from (3.8) and (3.9), but also easy for the latter. For example, if $i,j,k \in I$ are pairwise distinct then one gets

$$\begin{aligned} s_{i,i}s_{j,i}s_{i,4}(\alpha_{k,4}) = s_{i,i}s_{j,i}(\alpha_{k,i}+\alpha_{i,i}) &= s_{i,i}(\alpha_{k,i}+\alpha_{i,i}+\alpha_{j,i}) \\ &= \alpha_{k,4}+\alpha_{i,4}+\alpha_{j,4} = s_{j,j}s_{i,j}s_{j,4}(\alpha_{k,4}). \end{aligned} \tag{3.27}$$

Further, Eq. (3.26) has to be checked for generators w of W^{ext}. Again all calculations are easily done. □

For affine Lie (super)algebras there exists a distinguished root $\hat{\delta}$ (see Section 2), which should be considered here. For all $d \in \mathcal{D}$ set

$$\hat{\delta}_4 := \sum_{i=0}^{3} \alpha_{i,4}, \qquad \hat{\delta}_d := \alpha_{d,d} + \sum_{i=0}^{3} \alpha_{i,d} \quad \text{for } d \in \{0,1,2,3\}. \tag{3.28}$$

Then these are the elements corresponding to $\hat{\delta}$. Note that for all $d \in \mathcal{D}$ one has

$$\mathbb{C}\hat{\delta}_d = \{\lambda \in V_d \mid (\lambda,\mu) = 0 \text{ for all } \mu \in V_d\}.$$

Further, using Eq. (3.20) and Lemma 3.7 one gets by computations similar to the one in Eq. (3.27) the following formulas:

$$\mathsf{t}(\omega^\vee_{i,d})(\alpha_{i,d}) = \alpha_{i,d} - \hat{\delta}_d, \quad \mathsf{t}(\omega^\vee_{i,d})(\alpha_{j,d}) = \alpha_{j,d}, \quad \mathsf{t}(w)(\hat{\delta}_d) = \hat{\delta}_{d'} \tag{3.29}$$

for all $i,j \in I \setminus \{0\}$ with $i \neq j$, and all $w \in W^{\mathrm{ext}}$ and $d,d' \in \mathcal{D} \setminus \{0\}$ such that $e_{d'}we_d = w$.

4 Quantum affine superalgebras of $D^{(1)}(2,1;x)$

Drinfeld [D] gave a second realization of quantum affine algebras $U_q(\hat{\mathfrak{g}})$. He identified the generators of his algebra as loop-like generators in $U_q(\hat{\mathfrak{g}})$. We follow Beck's method [Bec] to define the analog of Drinfeld's second realization for $D^{(1)}(2,1;x)$. First we introduce the quantum affine superalgebra of $D^{(1)}(2,1;x)$ for any Dynkin diagram. Then in Section 4.2 we will give Lusztig type isomorphisms between these algebras, and observe that these isomorphisms satisfy the relations of the braid groupoid.

4.1 Quantum affine superalgebras U'_d

Fix $\hbar \in \mathbb{C} \setminus \mathbb{Z}\pi\sqrt{-1}$. For any $u \in \mathbb{C}$ let

$$q^u := \exp(u\hbar) = \sum_{n=0}^{\infty} \frac{(u\hbar)^n}{n!}, \qquad q := q^1, \qquad [u]_q := \frac{q^u - q^{-u}}{q - q^{-1}}.$$

In this paper we assume that

$$(4.1) \qquad q^{ku} \neq 1 \quad \text{for all } u \in \{1, x, x+1\} \text{ and } k \in \mathbb{N}.$$

Let $d \in \mathcal{D}$. Let $\mathcal{U}'_d$ be the $\mathbb{C}$-algebra (with 1) generated by the elements

$$(4.2) \qquad \sigma_d, \quad K_{i,d}^{\frac{1}{2}}, \quad K_{i,d}^{-\frac{1}{2}}, \quad E_{i,d}, \quad F_{i,d}, \quad \text{where } i \in I$$

and defined by the relations (4.3)–(4.7) below:

$$(4.3) \qquad XY = YX \text{ for } X, Y \in \{\sigma_d, K_{i,d}^{\frac{1}{2}}, K_{i,d}^{-\frac{1}{2}} \mid i \in I\},$$

$$(4.4) \qquad \sigma_d^2 = 1, \; K_{i,d}^{\frac{1}{2}} K_{i,d}^{-\frac{1}{2}} = K_{i,d}^{-\frac{1}{2}} K_{i,d}^{\frac{1}{2}} = 1,$$

$$(4.5) \qquad \sigma_d E_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} E_{i,d}, \qquad \sigma_d F_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} F_{i,d},$$

$$(4.6) \qquad K_{i,d}^{\frac{1}{2}} E_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{(\alpha_{i,d},\alpha_{j,d})/2} E_{j,d}, \quad K_{i,d}^{\frac{1}{2}} F_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{-(\alpha_{i,d},\alpha_{j,d})/2} F_{j,d},$$

$$(4.7) \qquad E_{i,d} F_{j,d} - (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})} F_{j,d} E_{i,d} = \delta_{ij} \frac{(K_{i,d}^{\frac{1}{2}})^2 - (K_{i,d}^{-\frac{1}{2}})^2}{q - q^{-1}}$$

for all $i, j \in I$. Note that for all $i \in I$ the equations $(K_{i,d}^{\frac{1}{2}})^{-1} = K_{i,d}^{-\frac{1}{2}}$ and $(K_{i,d}^{\frac{1}{2}})^0 = 1$ hold. Later on we will also use the abbreviations

$$(4.8) \qquad K_{i,d} := (K_{i,d}^{\frac{1}{2}})^2, \quad K_{i,d}^{\frac{m}{2}} := (K_{i,d}^{\frac{1}{2}})^m, \quad K_{i,d}^m := (K_{i,d}^{\frac{m}{2}})^2 \quad \text{for all } m \in \mathbb{Z},$$

$$(4.9) \qquad K_{\mu;d} := \prod_{i=0}^{3} K_{i,d}^{\frac{m_i}{2}}, \quad \text{for all } \mu = \frac{1}{2}\sum_{i=0}^{3} m_i \alpha_{i,d} \in \frac{1}{2}\mathbb{Z}\Pi_d \text{ with } m_i \in \mathbb{Z}.$$

In particular, according to the definition of $\hat{\delta}_d$ in Eq. (3.28) we have

$$(4.10)\qquad K_{\hat{\delta}_d;4}=\prod_{i=0}^{3}K_{i,4},\qquad K_{\hat{\delta}_d;d}=K_{d,d}\prod_{i=0}^{3}K_{i,d}\quad\text{for } d\in\{0,1,2,3\}.$$

The algebra $\mathcal{U}'_d$ admits a unique $\mathbb{Z}\Pi_d$-grading (see Section 3.6 for the definition of Π_d)

$$\mathcal{U}'_d=\bigoplus_{\lambda\in\mathbb{Z}\Pi_d}\mathcal{U}'_{d,\lambda},\qquad 1\in\mathcal{U}'_{d,0},\qquad \mathcal{U}'_{d,\mu}\mathcal{U}'_{d,\lambda}\subset\mathcal{U}'_{d,\mu+\lambda}\quad\text{for all }\mu,\lambda\in\mathbb{Z}\Pi_d,$$

such that $\sigma_d, K_{i,d}^{\pm\frac{1}{2}}\in\mathcal{U}'_{d,0}$, $E_{i,d}\in\mathcal{U}'_{d,\alpha_{i,d}}$, and $F_{i,d}\in\mathcal{U}'_{d,-\alpha_{i,d}}$ for all $i\in I$. Further, there exists a unique algebra automorphism Ψ_d of $\mathcal{U}'_d$ such that

$$\Psi_d(\sigma_d)=\sigma_d,\quad \Psi_d(K_{i,d}^{\pm\frac{1}{2}})=K_{i,d}^{\mp\frac{1}{2}},\quad \Psi_d(E_{i,d})=(-1)^{p(\alpha_{i,d})}F_{i,d},\quad \Psi_d(F_{i,d})=E_{i,d}.$$

Notice that $\Psi_d^2(X)=\sigma_d X\sigma_d$ for all $x\in\mathcal{U}'_d$.

Notation 4.1. (Super-bracket and q-super-bracket) For elements of $\mathcal{U}'_d$ we use the *super-bracket* $[\,,\,]$ and the *q-super-bracket* $[\![\,,\,]\!]$ defined as follows. For any $\mu,\lambda\in\mathbb{Z}\Pi_d$, $a\in\mathbb{C}$, and $X_\mu\in\mathcal{U}'_{d,\mu}$, $Y_\lambda\in\mathcal{U}'_{d,\lambda}$ we let

$$(4.11)\qquad [X_\mu,Y_\lambda]_a:=X_\mu Y_\lambda-(-1)^{p(\mu)p(\lambda)}aY_\lambda X_\mu,$$

$$(4.12)\qquad [X_\mu,Y_\lambda]:=[X_\mu,Y_\lambda]_1,\qquad [\![X_\mu,Y_\lambda]\!]:=[X_\mu,Y_\lambda]_{q^{-(\mu,\lambda)}}$$

Now we define the quantum affine superalgebras U'_d of $D^{(1)}(2,1;x)$ for the Dynkin diagrams labeled by $d\in\mathcal{D}$.

Definition 4.2. For any $d\in\mathcal{D}$ let U'_d be the quotient of the algebra $\mathcal{U}'_d$ by the ideal generated by the following elements (see also [Y, Remark 7.1.1] (or Prop.6.3.1(vii),(viii) in q-alg/9603015)):

$$(4.13)\qquad E_{i,d}^2,\quad\text{where } i\in I\text{ and } p(\alpha_{i,d})=1,$$

$$(4.14)\qquad [\![E_{i,d},E_{j,d}]\!],\quad\text{where } i,j\in I,\ i\neq j,\text{ and } (\alpha_{i,d},\alpha_{j,d})=0,$$

$$(4.15)\qquad [\![E_{i,d},[\![E_{i,d},E_{j,d}]\!]]\!],\quad\text{where } i,j\in I,\ i\neq j,\ p(\alpha_{i,d})=0,\text{ and } (\alpha_{i,d},\alpha_{j,d})\neq 0,$$

$$(4.16)\qquad [(\alpha_{i,4},\alpha_{k,4})]_q[\![[\![E_{i,4},E_{j,4}]\!],E_{k,4}]\!]-[(\alpha_{i,4},\alpha_{j,4})]_q[\![[\![E_{i,4},E_{k,4}]\!],E_{j,4}]\!],\quad\text{if } d=4,\text{ where } i,j,k\in I\text{ such that } i<j<k,$$

$$(4.17)\qquad \begin{aligned}&[(\alpha_{i,d}+\alpha_{d,d},\alpha_{k,d}+\alpha_{d,d})]_q[\![[\![[\![E_{d,d},E_{i,d}]\!],[\![E_{d,d},E_{j,d}]\!]]\!],[\![E_{d,d},E_{k,d}]\!]]\!]-\\&[(\alpha_{i,d}+\alpha_{d,d},\alpha_{j,d}+\alpha_{d,d})]_q[\![[\![[\![E_{d,d},E_{i,d}]\!],[\![E_{d,d},E_{k,d}]\!]]\!],[\![E_{d,d},E_{j,d}]\!]]\!]\\&\text{if } d\neq 4,\text{ where } \{i,j,k,d\}=I\text{ and } i<j<k,\end{aligned}$$

$$(4.18)\qquad \Psi_d(X),\quad\text{for all } X\text{ in (4.13)–(4.17).}$$

Notice that Ψ_d induces an automorphism of U'_d, which will also be denoted by Ψ_d. Further, U'_d inherits a $\mathbb{Z}\Pi_d$-grading from $\mathcal{U}'_d$, that is

$$U'_d = \bigoplus_{\lambda\in\mathbb{Z}\Pi_d} U'_{d,\lambda}, \qquad 1 \in U'_{d,0}, \qquad U'_{d,\mu}U'_{d,\lambda} \subset U'_{d,\mu+\lambda} \quad \text{for all } \mu,\lambda\in\mathbb{Z}\Pi_d,$$

and hence Notation 4.1 can be applied also to the elements of U'_d.

Lemma 4.3. *For any $\mu,\lambda,\xi\in\mathbb{Z}\Pi_d$ and $X_\mu\in U'_{d,\mu}$, $Y_\lambda\in U'_{d,\lambda}$, $Z_\xi\in U'_{d,\xi}$, (or, alternatively, $X_\mu\in\mathcal{U}'_{d,\mu}$, $Y_\lambda\in\mathcal{U}'_{d,\lambda}$, $Z_\xi\in\mathcal{U}'_{d,\xi}$,) one has the following formulas.*

$$\llbracket X_\mu, K_{\lambda;d}Y_\lambda\rrbracket = q^{-(\lambda,\mu)}K_{\lambda;d}[X_\mu,Y_\lambda], \qquad \llbracket K_{\mu;d}^{-1}X_\mu, Y_\lambda\rrbracket = K_{\mu;d}^{-1}[X_\mu,Y_\lambda],$$
$$\llbracket\llbracket X_\mu,Y_\lambda\rrbracket,Z_\xi\rrbracket = \llbracket X_\mu,\llbracket Y_\lambda,Z_\xi\rrbracket\rrbracket + (-1)^{p(\lambda)p(\xi)}q^{-(\lambda,\xi)}[\llbracket X_\mu,Z_\xi\rrbracket,Y_\lambda]_{q^{(\lambda,\xi-\mu)}},$$
$$\llbracket X_\mu,\llbracket Y_\lambda,Z_\xi\rrbracket\rrbracket = \llbracket\llbracket X_\mu,Y_\lambda\rrbracket,Z_\xi\rrbracket + (-1)^{p(\mu)p(\lambda)}q^{-(\mu,\lambda)}[Y_\lambda,\llbracket X_\mu,Z_\xi\rrbracket]_{q^{(\lambda,\mu-\xi)}}.$$

Proof. The first two equations follow from Eq.s (4.12) and (4.6). For the other relations one needs only the definition of the q-super-bracket. □

Lemma 4.4. *Let $d\in\mathcal{D}$ and $i,\ j\in I$ with $i\neq j$. Then the following equations hold in $\mathcal{U}'_d$ and in U'_d.*

$$[\llbracket E_{j,d},E_{i,d}\rrbracket,F_{i,d}] = -\,[(\alpha_{i,d},\alpha_{j,d})]_q K_{i,d}^{-1}E_{j,d}, \tag{4.19}$$
$$[\llbracket E_{j,d},E_{i,d}\rrbracket,F_{j,d}] = (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})}[(\alpha_{i,d},\alpha_{j,d})]_q E_{i,d}K_{j,d}, \tag{4.20}$$
$$[E_{i,d},\llbracket F_{j,d},F_{i,d}\rrbracket] = (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})}[(\alpha_{i,d},\alpha_{j,d})]_q K_{i,d}F_{j,d}, \tag{4.21}$$
$$[E_{j,d},\llbracket F_{j,d},F_{i,d}\rrbracket] = -\,[(\alpha_{i,d},\alpha_{j,d})]_q F_{i,d}K_{j,d}^{-1} \tag{4.22}$$
$$[\llbracket E_{j,d},E_{i,d}\rrbracket,\llbracket F_{j,d},F_{i,d}\rrbracket] = (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})}q^{-(\alpha_{i,d},\alpha_{j,d})}[(\alpha_{i,d},\alpha_{j,d})]_q\frac{K_{i,d}K_{j,d}-K_{i,d}^{-1}K_{j,d}^{-1}}{q-q^{-1}} \tag{4.23}$$

Proof. Eq.s (4.19),(4.20) can be checked directly by using Eq.s (4.6),(4.7). Then Eq.s (4.21),(4.22) can be obtained from (4.19),(4.20) by applying Ψ_d. Finally, Eq. (4.23) follows from formulas (4.19)–(4.22) by using (4.3),(4.6). □

Let $\mathcal{U}'^{>0}_d$, $\mathcal{U}'^{<0}_d$, and $\mathcal{U}'^0$ be the subalgebras of $\mathcal{U}'_d$ generated by the sets $\{E_{i,d}\,|\,i\in I\}$, $\{F_{i,d}\,|\,i\in I\}$, and $\{\sigma_d, K_{i,d}^{\pm\frac12}\,|\,i\in I\}$, respectively. Let $U'^{>0}_d$, $U'^{<0}_d$, and U'^0 denote the images of $\mathcal{U}'^{>0}_d$, $\mathcal{U}'^{<0}_d$, and $\mathcal{U}'^0$, respectively, under the canonical projection $\mathcal{U}'_d\to U'_d$. Notice that $U'^{<0}_d=\Psi_d(U'^{>0}_d)$ and $\Psi_d(U'^0_d)=U'^0_d$.

Theorem 4.5. (1) *The $\mathbb{C}$-algebras $\mathcal{U}'_d$ and U'_d can be regarded as Hopf algebras $(\mathcal{U}'_d,\Delta,S,\varepsilon)$ and $(U'_d,\Delta,S,\varepsilon)$ such that*

$$\Delta(X)=X\otimes X,\ S(X)=X^{-1},\ \varepsilon(X)=1 \textit{ for } X\in\{\sigma_d,K_{i,d}^{\pm\frac12}\,|\,i\in I\},$$
$$\Delta(E_{i,d}) = E_{i,d}\otimes 1 + K_{i,d}\sigma_d^{p(\alpha_{i,d})}\otimes E_{i,d}, \quad S(E_{i,d}) = -\,K_{i,d}^{-1}\sigma_d^{p(\alpha_{i,d})}E_{i,d},$$
$$\Delta(F_{i,d}) = F_{i,d}\otimes K_{i,d}^{-1} + \sigma_d^{p(\alpha_{i,d})}\otimes F_{i,d}, \quad S(F_{i,d}) = -\,(-1)^{p(\alpha_{i,d})}F_{i,d}K_{i,d}\sigma_d^{p(\alpha_{i,d})},$$
$$\varepsilon(E_{i,d})=\varepsilon(F_{i,d})=0$$

for all $i \in I$.

(2) *The algebras* $\mathcal{U}'_d$ *and* U'_d *admit a triangular decomposition. More precisely, the multiplication maps*

$$\mathcal{U}'^{>0}_d \otimes \mathcal{U}'^0 \otimes \mathcal{U}'^{<0}_d \to \mathcal{U}'_d, \qquad \mathcal{U}'^{<0}_d \otimes \mathcal{U}'^0 \otimes \mathcal{U}'^{>0}_d \to \mathcal{U}'_d,$$
$$U'^{>0}_d \otimes U'^0 \otimes U'^{<0}_d \to U'_d, \qquad U'^{<0}_d \otimes U'^0 \otimes U'^{>0}_d \to U'_d,$$

where $X_1 \otimes X_2 \otimes X_3$ *is mapped to* $X_1X_2X_3$ *for all* X_1, X_2, X_3, *are isomorphisms of* $\mathbb{Z}\Pi_d$*-graded* $\mathbb{C}$*-vector spaces. Further, the algebra* $\mathcal{U}'^{>0}_d$ *is the free algebra generated by the set* $\{E_{i,d} \mid i \in I\}$, *and* $U'^{>0}_d$ *is isomorphic to its quotient by the ideal generated by the elements in* (4.13)–(4.17). *The algebras* $\mathcal{U}'^0$ *and* U'^0 *are both isomorphic to the commutative algebra generated by the set* $\{\sigma_d, K_{i,d}^{\pm\frac{1}{2}} \mid i \in I\}$ *and defined by the relations* (4.4), *so the set*

$$\Big\{\sigma_d^m \prod_{i=0}^{3} (K_{i,d}^{\frac{1}{2}})^{n_i} \mid m \in \{0,1\}, n_i \in \mathbb{Z} \text{ for all } i \in I\Big\}$$

is a $\mathbb{C}$*-basis of both* $\mathcal{U}'^0_d$ *and* U'^0_d.

Proof. Part (2) of the theorem is standard [J, Prop. 4.16, Thm. 4.21]. The compatibility of Δ and ε with the defining relations of $\mathcal{U}'_d$, and the axioms of S can be easily checked. In order to prove that Δ is well-defined on $U'^{>0}_d$ and $U'^{<0}_d$ we used the computer algebra program Mathematica. □

4.2 Lusztig type isomorphisms

The main result of this section is Theorem 4.10, which tells that the elements of the braid semigroup $\widetilde{W}^{\rm ext}$ defined in Section 3.3 can be represented by isomorphisms between the quantum affine algebras U'_d.

Notation 4.6. For any $d \in \mathcal{D}$ and $i, j \in I$ with $i \neq j$ one has

$$(-1)^{p(\alpha_{i,d})p(\alpha_{j,d})}[(\alpha_{i,d}, \alpha_{j,d})]_q \in \{0, 1, [x]_q, [-x-1]_q\}.$$

We fix a square root of all of these four complex numbers, and write

$$r_{i,j;d} := 1\Big/\sqrt{(-1)^{p(\alpha_{i,d})p(\alpha_{j,d})}[(\alpha_{i,d}, \alpha_{j,d})]_q}$$

if $(\alpha_{i,d}, \alpha_{j,d}) \neq 0$. In this case one also has $r_{i,j;n_i \triangleright d} = r_{i,j;d} = r_{j,i;d}$. Further, let $\epsilon_{i,j;d} = (-1)^{\delta_{ij} + p(\alpha_{i,d})p(\alpha_{j,d})}$. (As for the symbol δ_{ij}, see the last paragraph in Introduction.)

Theorem 4.7. *The following statements are valid.*

(1) *For all $d \in \mathcal{D}$ and $i \in I$ there exist unique $\mathbb{C}$-algebra isomorphisms*

$$T_{i,d}, T^{-}_{i,d} : U'_d \to U'_{n_i \triangleright d} \tag{4.24}$$

satisfying Eq.s (4.25)–(4.31) *below.*

$$T_{i,d}(\sigma_d) = T^{-}_{i,d}(\sigma_d) = \sigma_{n_i \triangleright d}, \tag{4.25}$$

$$\begin{aligned} T_{i,d}(K^{\frac{1}{2}}_{i,d}) &= T^{-}_{i,d}(K^{\frac{1}{2}}_{i,d}) = K^{-\frac{1}{2}}_{i,n_i \triangleright d}, \\ T_{i,d}(K^{\frac{1}{2}}_{j,d}) &= T^{-}_{i,d}(K^{\frac{1}{2}}_{j,d}) = K^{\frac{1}{2}}_{j,n_i \triangleright d}(K^{\frac{1}{2}}_{i,n_i \triangleright d})^{m_{i,j;d}-2}, \end{aligned} \tag{4.26}$$

where $j \in I$ and $j \neq i$,

$$\begin{aligned} T_{i,d}(E_{i,d}) &= (-1)^{p(\alpha_{i,n_i \triangleright d})} q^{-\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{i,n_i \triangleright d})}{2}} F_{i,n_i \triangleright d} K_{i,n_i \triangleright d}, \\ T_{i,d}(F_{i,d}) &= q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{i,n_i \triangleright d})}{2}} K^{-1}_{i,n_i \triangleright d} E_{i,n_i \triangleright d}, \end{aligned} \tag{4.27}$$

$$\begin{aligned} T^{-}_{i,d}(E_{i,d}) &= q^{-\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{i,n_i \triangleright d})}{2}} K^{-1}_{i,n_i \triangleright d} F_{i,n_i \triangleright d}, \\ T^{-}_{i,d}(F_{i,d}) &= (-1)^{p(\alpha_{i,n_i \triangleright d})} q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{i,n_i \triangleright d})}{2}} E_{i,n_i \triangleright d} K_{i,n_i \triangleright d}, \end{aligned} \tag{4.28}$$

$$\begin{aligned} T_{i,d}(E_{j,d}) &= r_{i,j;d} q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{j,n_i \triangleright d})}{2}} [\![E_{j,n_i \triangleright d}, E_{i,n_i \triangleright d}]\!], \\ T_{i,d}(F_{j,d}) &= r_{i,j;d} q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{j,n_i \triangleright d})}{2}} [\![F_{j,n_i \triangleright d}, F_{i,n_i \triangleright d}]\!], \end{aligned} \tag{4.29}$$

$$\begin{aligned} T^{-}_{i,d}(E_{j,d}) &= r_{i,j;d} q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{j,n_i \triangleright d})}{2}} [\![E_{i,n_i \triangleright d}, E_{j,n_i \triangleright d}]\!], \\ T^{-}_{i,d}(F_{j,d}) &= r_{i,j;d} q^{\frac{(\alpha_{i,n_i \triangleright d}, \alpha_{j,n_i \triangleright d})}{2}} [\![F_{i,n_i \triangleright d}, F_{j,n_i \triangleright d}]\!], \end{aligned} \tag{4.30}$$

where in Eq.s (4.29),(4.30) *one has $j \in I$ with $j \neq i$ and $(\alpha_{i,d}, \alpha_{j,d}) \neq 0$,*

$$T_{i,d}(E_{j,d}) = T^{-}_{i,d}(E_{j,d}) = E_{j,n_i \triangleright d}, \quad T_{i,d}(F_{j,d}) = T^{-}_{i,d}(F_{j,d}) = F_{j,n_i \triangleright d} \tag{4.31}$$

where $j \in I$ with $j \neq i$ and $(\alpha_{i,d}, \alpha_{j,d}) = 0$.

(2) *One has $T^{-}_{i,n_i \triangleright d} = (T_{i,d})^{-1}$ for all $d \in \mathcal{D}$ and $i \in I$.*

(3) *The isomorphisms $T_{i,d}$ satisfy the equations $\Psi_{n_i \triangleright d} T_{i,d} = T_{i,d} \Psi_d$.*

Proof. Parts (2) and (3) of the theorem are obtained easily from the definition of $T_{i,d}$, $T^{-}_{i,d}$, and Ψ_d, and from Lemma 4.4. The uniqueness of $T_{i,d}$ and $T^{-}_{i,d}$ are obvious, but to check the compatibility of $T_{i,d}$, $T^{-}_{i,d}$ with the defining relations of U'_d we need again the computer algebra program Mathematica. □

Theorem 4.8. *For any $d \in \mathcal{D}$ the following statements hold.*

(1) *If $i, j \in I$, where $i \neq j$ and $m_{i,j;d} = 2$, then*

$$T_{i,d} T_{j,d} = T_{j,d} T_{i,d}, \qquad T_{i,d}(E_{j,d}) = E_{j,d}, \qquad T_{i,d}(F_{j,d}) = F_{j,d}. \tag{4.32}$$

(2) *If $i, j \in I$, where $i \neq j$ and $m_{i,j;d} = 3$, then the following equations hold.*

$$T_{i,n_j n_i \triangleright d} T_{j,n_i \triangleright d} T_{i,d} = T_{j,n_i n_j \triangleright d} T_{i,n_j \triangleright d} T_{j,d}, \tag{4.33}$$

$$T_{i,n_j \triangleright d} T_{j,d}(E_{i,d}) = E_{j,n_i n_j \triangleright d}, \qquad T_{i,n_j \triangleright d} T_{j,d}(F_{i,d}) = F_{j,n_i n_j \triangleright d}. \tag{4.34}$$

Proof. All statements of the theorem follow from Theorem 4.7, the definition of the algebras U'_d, and from Lemma 4.4. □

For all $d \in \mathcal{D}$ and $f \in \mathcal{K}_4$, see Eq. (3.2), let $T_{f,d} : U'_d \to U'_{f \triangleright d}$ denote the $\mathbb{C}$-algebra isomorphism satisfying the following equations.

$$\begin{aligned} T_{f,d}(\sigma_d) &= \sigma_{f \triangleright d}, & T_{f,d}(K_{i,d}^{\pm\frac{1}{2}}) &= K_{f(i),f \triangleright d}^{\pm\frac{1}{2}}, \\ T_{f,d}(E_{i,d}) &= E_{f(i),f \triangleright d}, & T_{f,d}(F_{i,d}) &= F_{f(i),f \triangleright d}, \end{aligned}$$

where $i \in I$.

Definition 4.9. Let $\mathcal{M}_{\mathcal{D}} = \cup_{d,d' \in \mathcal{D}} \mathrm{Hom}\,(U'_d, U'_{d'}) \cup \{0\}$ be a disjoint union of sets, where $\mathrm{Hom}\,(U'_d, U'_{d'})$ denotes the set of unital algebra maps from U'_d to $U'_{d'}$. The set $\mathcal{M}_{\mathcal{D}}$ admits a unique semigroup structure with the following properties.

$$0\phi = \phi 0 = 0, \quad \phi_1 \phi_2 = \begin{cases} \phi_1 \circ \phi_2 & \text{if } d_2 = d_3, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\phi \in \mathcal{M}_{\mathcal{D}}$ and all $\phi_1 \in \mathrm{Hom}\,(U'_{d_1}, U'_{d_2})$ and $\phi_2 \in \mathrm{Hom}\,(U'_{d_3}, U'_{d_4})$, where $d_1, d_2, d_3, d_4 \in \mathcal{D}$.

Note that the set $\mathcal{M}_{\mathcal{D}} \setminus \{0\}$ in the above definition can also be considered as the morphisms of a category with objects U'_d, where $d \in \mathcal{D}$.

Recall the definition of $\widetilde{W}^{\mathrm{ext}}$ from Section 3.3.

Theorem 4.10. *There exists a unique semigroup homomorphism*

$$\mathcal{T} : \widetilde{W}^{\mathrm{ext}} \to \mathcal{M}_{\mathcal{D}} \tag{4.35}$$

such that for all $d \in \mathcal{D}$, $i \in I$, and $f \in \mathcal{K}_4$ one has

$$\mathcal{T}(0) = 0, \quad \mathcal{T}(\tilde{e}_d) = \mathrm{id} : U'_d \to U'_d, \quad \mathcal{T}(\tilde{s}_{i,d}) = T_{i,d}, \quad \mathcal{T}(\tilde{\tau}_{f,d}) = T_{f,d}. \tag{4.36}$$

Moreover, for all $\tilde{w} \in \widetilde{W}^{\mathrm{ext}}$ with $\tilde{w} \neq 0$ and $\tilde{e}_{d'} \tilde{w} \tilde{e}_d = \tilde{w}$ for some $d, d' \in \mathcal{D}$, we have

$$\mathcal{T}(\tilde{w})(U'_{d,\mu}) = U'_{d',\mathfrak{t}(\mathfrak{p}(\tilde{w}))(\mu)} \qquad \textit{for all } \mu \in \mathbb{Z}\Pi_d, \tag{4.37}$$

$$\mathcal{T}(\tilde{w})(\sigma_d) = \sigma_{d'}, \ \mathcal{T}(\tilde{w})(K_{\mu;d}) = K_{\mathfrak{t}(\mathfrak{p}(\tilde{w}))(\mu);d'} \qquad \textit{for all } \mu \in \frac{1}{2}\mathbb{Z}\Pi_d, \tag{4.38}$$

$$\mathcal{T}(\tilde{w})\Psi_d = \Psi_{d'} \mathcal{T}(\tilde{w}). \tag{4.39}$$

Proof. One has to show that Eq. (4.36) is compatible with the defining relations of $\widetilde{W}^{\mathrm{ext}}$. This follows from Theorem 4.8 and the definition of the maps $T_{f,d}$, where $f \in \mathcal{K}_4$ and $d \in \mathcal{D}$. □

Note that by Eq. (3.26) also the formula

$$\mathcal{T}(\tilde{w})([\![X_\mu, Y_\lambda]\!]) = [\![\mathcal{T}(\tilde{w})(X_\mu), \mathcal{T}(\tilde{w})(Y_\lambda)]\!] \tag{4.40}$$

holds for all $\tilde{w} \in \widetilde{W}^{\mathrm{ext}}$, $X_\mu \in U'_{d,\mu}$, and $Y_\lambda \in U'_{d,\lambda}$, where $d \in \mathcal{D}$, $\mu, \lambda \in \mathbb{Z}\Pi_d$, $\tilde{w} \neq 0$, and $\tilde{w}e_d = \tilde{w}$.

5 Root vectors associated to imaginary roots

The aim of this section is to construct root vectors to the imaginary roots $k\hat{\delta}_d$ for all $k \in \mathbb{N}$ and $d \in \mathcal{D} \setminus \{0\}$. First we will give some technical results on the isomorphisms $\mathcal{T}(\tilde{\omega}^\vee_{i,d})$, where $\tilde{\omega}^\vee_{i,d}$ and $\mathcal{T}$ are as in Eq. (3.21) and Theorem 4.10 respectively. Then we define root vectors $\bar{\psi}_{i,k;d}$ of weight $k\hat{\delta}_d$ for each $i \in I \setminus \{0\}$, $d \in \mathcal{D} \setminus \{0\}$, and $k \in \mathbb{N}$. Finally we prove that these root vectors commute with each other if they belong to the same algebra U'_d.

5.1 Preliminary definition of root vectors to imaginary roots

We start with the calculation of various values of the isomorphisms $\mathcal{T}(\tilde{\omega}^\vee_{i,d})$.

Lemma 5.1. *Let $d \in \mathcal{D} \setminus \{0\}$ and $i, j \in I \setminus \{0\}$ with $i \neq j$.*

(1) *The following formulas hold.*

$$\mathcal{T}(\tilde{\omega}^\vee_{i,d})(E_{j,d}) = E_{j,d}, \qquad \mathcal{T}(\tilde{\omega}^\vee_{i,d})(F_{j,d}) = F_{j,d}, \qquad \mathcal{T}(\tilde{\omega}^\vee_{i,d})(K_{j,d}^{\pm\frac{1}{2}}) = K_{j,d}^{\pm\frac{1}{2}}.$$

(2) *We have*

$$\begin{aligned}\mathcal{T}(\tilde{\omega}^\vee_{i,d})([\![E_{i,d}, E_{j,d}]\!]) &= \mathcal{T}(\tilde{\omega}^\vee_{j,d})([\![E_{j,d}, E_{i,d}]\!]),\\ \mathcal{T}(\tilde{\omega}^\vee_{i,d})([\![F_{i,d}, F_{j,d}]\!]) &= \mathcal{T}(\tilde{\omega}^\vee_{j,d})([\![F_{j,d}, F_{i,d}]\!]).\end{aligned}$$

Proof. Part (1) can be checked easily by using Eq. (4.34) and Theorem 4.10.

(2) If $(\alpha_{i,d}, \alpha_{j,d}) = 0$ then Lemma 5.1(2) holds by Formula (4.14). Assume

now that $(\alpha_{i,d},\alpha_{j,d})\neq 0$. Using Theorem 4.10 we obtain the following formulas.

$$
\begin{aligned}
&q^{\frac{(\alpha_{i,d},\alpha_{j,d})}{2}} r_{i,j;d}\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(\llbracket E_{i,d}, E_{j,d}\rrbracket) && \\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})T_{j,n_j\triangleright d}(E_{i,n_j\triangleright d}) && \text{(by Eq. (4.29))}\\
&= T_{j,n_j\triangleright d}\mathcal{T}(\tilde{\omega}^{\vee}_{i,n_j\triangleright d})(E_{i,n_j\triangleright d}) && \text{(by Eq. (3.25))}\\
&= \mathcal{T}(\tilde{\nu}_{j,n_j\triangleright d})\mathcal{T}(\tilde{\omega}^{\vee}_{j,n_j\triangleright d})(E_{i,n_j\triangleright d}) && \text{(by Eq. (3.24), Lemma 5.1(1))}\\
&= \mathcal{T}(\tilde{\nu}_{j,n_j\triangleright d})(E_{i,n_j\triangleright d}) && \text{(by Lemma 5.1(1))}\\
&= \mathcal{T}(\tilde{\nu}_{j,n_j\triangleright d})T_{j,d}T^{-}_{j,n_j\triangleright d}(E_{i,n_j\triangleright d}) && \text{(by Theorem 4.7(2))}\\
&= q^{\frac{(\alpha_{i,d},\alpha_{j,d})}{2}} r_{i,j;d}\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})(\llbracket E_{j,d}, E_{i,d}\rrbracket) && \text{(by Lemma 3.4 and Eq. (4.29)).}
\end{aligned}
$$

Hence we have the first equation. The second one can be obtained from the first one by applying Ψ_d and using Eq. (4.39). □

Recall from Notation 4.6 that for $d\in\mathcal{D}$ and $\{i,j,k,l\}=I$ with $p(\alpha_{i,d})=1$ one has the equation $(r_{i,j;d}r_{i,k;d}r_{i,l;d})^{-2}=[x]_q[-x-1]_q$.

Lemma 5.2. *Let $d\in\mathcal{D}\setminus\{0\}$ and $i,\ j,\ k\in\{1,2,3\}$ such that $\{i,j,k\}=\{1,2,3\}$.*

(1) *The following formulas hold.*

$$
\begin{aligned}
\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(K^{-1}_{i,d}F_{i,d}) &= q^{\frac{(\alpha_{i,d},\alpha_{i,d})}{2}}\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})T^{-}_{i,n_i\triangleright d}(E_{i,n_i\triangleright d}) = q^{\frac{(\alpha_{i,d},\alpha_{i,d})}{2}}\mathcal{T}(\tilde{\nu}_{i,n_i\triangleright d})(E_{i,n_i\triangleright d})\\
&= \begin{cases} -[(\alpha_{0,4},\alpha_{k,4})]_q^{-1}\llbracket E_{j,4},\llbracket E_{k,4},E_{0,4}\rrbracket\rrbracket & \text{if } d=4,\\ r_{i,0;i}r_{i,j;i}r_{i,k;i}\llbracket E_{j,i},\llbracket E_{k,i},\llbracket E_{i,i},E_{0,i}\rrbracket\rrbracket\rrbracket & \text{if } d=i,\\ r_{j,0;j}r_{j,i;j}r_{j,k;j}\llbracket E_{j,j},\llbracket E_{k,j},\llbracket E_{j,j},E_{0,j}\rrbracket\rrbracket\rrbracket & \text{if } d=j.\end{cases}
\end{aligned}
$$

(2) *For all $l,m\in\mathbb{Z}$ we have the following equations.*

$$
\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(K^{-1}_{i,d}F_{i,d}),K^{-1}_{i,d}F_{i,d}\rrbracket=0,\qquad \llbracket E_{i,d},\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d})\rrbracket=0,
$$
$$
\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(E_{i,d}),\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}(K^{-1}_{j,d}F_{j,d})\rrbracket=0.
$$

Proof. Part (1) can be proven directly by using (4.32),(4.34). The first equation of part (2) can be proven by using part (1) and 4.20. The second equation of part (2) follows from the first one by applying the algebra map $\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}\circ\Psi_d$.

Let now $l,m\in\mathbb{Z}$. One gets

$$
\begin{aligned}
&\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(E_{i,d}),\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}(K^{-1}_{j,d}F_{j,d})\rrbracket && \\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}(\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-m}\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(E_{i,d}),K^{-1}_{j,d}F_{j,d}\rrbracket) && \text{(by Eq. (4.40))}\\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}(\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-m}(E_{i,d}),K^{-1}_{j,d}F_{j,d}\rrbracket) && \text{(by Theorems 3.5(1), 4.10)}\\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}(\llbracket\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(E_{i,d}),K^{-1}_{j,d}F_{j,d}\rrbracket) && \text{(by Lemma 5.1(1))}\\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(\llbracket E_{i,d},K^{-1}_{j,d}F_{j,d}\rrbracket) && \text{(as in the previous steps)}\\
&= \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{m}\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{l}(q^{(\alpha_{i,d},\alpha_{j,d})}K^{-1}_{j,d}[E_{i,d},F_{j,d}]) && \text{(by Lemma 4.3)}\\
&= 0 && \text{(by Eq. (4.7)).}
\end{aligned}
$$

□

In order to define root vectors for integer multiples of $\hat{\delta}_d$, we now define for all $i \in I \setminus \{0\}$, $k \in \mathbb{N}$, and $d \in \mathcal{D} \setminus \{0\}$ a family $\{\bar{\psi}^{(s)}_{i,k;d} \mid s \in \mathbb{Z}\}$ of elements of U'_d. In Proposition 5.6 it will be shown that the cardinality of each of these families is one, and their unique elements will be considered as the root vectors associated to the pairs $(\alpha_{i,d}, k\hat{\delta}_d)$.

Definition 5.3. Recall the definition of $K_{\hat{\delta}_d;d}$ from Eq. (4.10). For $d \in \mathcal{D} \setminus \{0\}$, $i \in I \setminus \{0\}$, $k \in \mathbb{N}$, and $s \in \mathbb{Z}$ let

$$
\begin{aligned}
\bar{\psi}^{(s)}_{i,k;d} &=(-1)^k q^{-(\alpha_{i,d},\alpha_{i,d})} \mathcal{T}(\tilde{\omega}^\vee_{i,d})^s([\![E_{i,d}, \mathcal{T}(\tilde{\omega}^\vee_{i,d})^k (K^{-1}_{i,d} F_{i,d})]\!]) \\
&=(-1)^k q^{-(\alpha_{i,d},\alpha_{i,d})} [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^s (E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{s+k} (K^{-1}_{i,d} F_{i,d})]\!] \\
&=(-1)^k K^{s+k}_{\hat{\delta}_d;d} K^{-1}_{i,d} [\mathcal{T}(\tilde{\omega}^\vee_{i,d})^s (E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{s+k} (F_{i,d})],
\end{aligned}
\tag{5.1}
$$

where the last equation follows from Theorem 4.10, Eq. (3.29), and Lemma 4.3. Note that $\bar{\psi}^{(s)}_{i,k;d} \in U'_{d,k\hat{\delta}_d}$.

We will also need the element

$$
\bar{\psi}^{(0)}_{i,0;d} := q^{-(\alpha_{i,d},\alpha_{i,d})} [\![E_{i,d}, K^{-1}_{i,d} F_{i,d}]\!] = \frac{1 - K^{-2}_{i,d}}{q - q^{-1}} \in U'_{d,0}.
\tag{5.2}
$$

Lemma 5.4. *Let $d \in \mathcal{D} \setminus \{0\}$, $i, j \in I \setminus \{0\}$, $k \in \mathbb{N}$, and $s \in \mathbb{Z}$. Then we have*

$$
\mathcal{T}(\tilde{\omega}^\vee_{i,d})(\bar{\psi}^{(s)}_{j,k;d}) = \begin{cases} \bar{\psi}^{(s+1)}_{j,k;d} & \text{if } i = j, \\ \bar{\psi}^{(s)}_{j,k;d} & \text{if } i \neq j. \end{cases}
$$

Proof. The statement in the case $i = j$ follows from the definition of $\bar{\psi}^{(s)}_{j,k;d}$. If $i \neq j$ then one obtains the claim by Eq. (3.23) and Lemma 5.1(1). □

Lemma 5.5. *Let $d \in \mathcal{D} \setminus \{0\}$ and $i, j \in I \setminus \{0\}$. Then for all $k \in \mathbb{N}$ the following equations hold.*

$$
\begin{aligned}
[\bar{\psi}^{(-1)}_{i,k;d}, E_{j,d}] &=\epsilon_{i,j;d} q^{(\alpha_{i,d},\alpha_{j,d})} [\bar{\psi}^{(0)}_{i,k-1;d}, \mathcal{T}(\tilde{\omega}^\vee_{j,d})^{-1}(E_{j,d})]_{q^{-2(\alpha_{i,d},\alpha_{j,d})}}, \\
[\bar{\psi}^{(1-k)}_{i,k;d}, K^{-1}_{j,d} F_{j,d}] &=\epsilon_{i,j;d} q^{-(\alpha_{i,d},\alpha_{j,d})} [\bar{\psi}^{(1-k)}_{i,k-1;d}, \mathcal{T}(\tilde{\omega}^\vee_{j,d})(K^{-1}_{j,d} F_{j,d})]_{q^{2(\alpha_{i,d},\alpha_{j,d})}}.
\end{aligned}
$$

In particular, one has the formulas

$$
\begin{aligned}
[\bar{\psi}^{(-1)}_{i,1;d}, E_{j,d}] &=\epsilon_{i,j;d} [(\alpha_{i,d}, \alpha_{j,d})]_q \mathcal{T}(\tilde{\omega}^\vee_{j,d})^{-1}(E_{j,d}), \\
[\bar{\psi}^{(0)}_{i,1;d}, K^{-1}_{j,d} F_{j,d}] &= -\epsilon_{i,j;d} [(\alpha_{i,d}, \alpha_{j,d})]_q \mathcal{T}(\tilde{\omega}^\vee_{j,d})(K^{-1}_{j,d} F_{j,d}).
\end{aligned}
$$

Proof. Assume first that $i = j$. By Eq. (5.1), Lemma 5.2(2), and Lemma 4.3 we have the following.

$$
\begin{aligned}
[\bar{\psi}^{(-1)}_{i,k;d}, E_{i,d}] &= -[\![E_{i,d}, \bar{\psi}^{(-1)}_{i,k;d}]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![E_{i,d}, [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!]]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} (-1)^{p(\alpha_{i,d})+1} q^{(\alpha_{i,d},\alpha_{i,d})} \\
&\qquad [[\![E_{i,d}, \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d})]_{q^{-2(\alpha_{i,d},\alpha_{i,d})}} \\
&= \epsilon_{i,i;d} q^{(\alpha_{i,d},\alpha_{i,d})} [\bar{\psi}^{(0)}_{i,k-1;d}, \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d})]_{q^{-2(\alpha_{i,d},\alpha_{i,d})}}, \\
[\bar{\psi}^{(1-k)}_{i,k;d}, K^{-1}_{i,d} F_{i,d}] &= [\![\bar{\psi}^{(1-k)}_{i,k;d}, K^{-1}_{i,d} F_{i,d}]\!] \\
&= (-1)^{k} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{1-k}(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(K^{-1}_{i,d} F_{i,d})]\!], K^{-1}_{i,d} F_{i,d}]\!] \\
&= (-1)^{k} q^{-(\alpha_{i,d},\alpha_{i,d})} (-1)^{p(\alpha_{i,d})} q^{-(\alpha_{i,d},\alpha_{i,d})} \\
&\qquad [[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{1-k}(E_{i,d}), K^{-1}_{i,d} F_{i,d}]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(K^{-1}_{i,d} F_{i,d})]_{q^{2(\alpha_{i,d},\alpha_{i,d})}} \\
&= \epsilon_{i,i;d} q^{-(\alpha_{i,d},\alpha_{i,d})} [\bar{\psi}^{(1-k)}_{i,k-1;d}, \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(K^{-1}_{i,d} F_{i,d})]_{q^{2(\alpha_{i,d},\alpha_{i,d})}}.
\end{aligned}
$$

Assume now that $i \neq j$. First notice that because of Lemmata 5.2 and 5.1 we have the following.

$$
\begin{aligned}
[\bar{\psi}^{(-1)}_{i,k;d}, E_{j,d}] &= -[\![E_{j,d}, \bar{\psi}^{(-1)}_{i,k;d}]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![E_{j,d}, [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!]]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![[\![E_{j,d}, \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}(E_{i,d})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{-1}([\![E_{j,d}, E_{i,d}]\!]), \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![\mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-1}([\![E_{i,d}, E_{j,d}]\!]), \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} [\![[\![E_{i,d}, \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-1}(E_{j,d})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!] \\
&= (-1)^{k+1} q^{-(\alpha_{i,d},\alpha_{i,d})} (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})} q^{(\alpha_{j,d},\alpha_{i,d})} \\
&\qquad [[\![E_{i,d}, \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^{k-1}(K^{-1}_{i,d} F_{i,d})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-1}(E_{j,d})]_{q^{-2(\alpha_{j,d},\alpha_{i,d})}} \\
&= \epsilon_{i,j;d} q^{(\alpha_{i,d},\alpha_{j,d})} [\bar{\psi}^{(0)}_{i,k-1;d}, \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^{-1}(E_{j,d})]_{q^{-2(\alpha_{i,d},\alpha_{j,d})}}.
\end{aligned}
$$

The remaining equation can be proved similarly. □

5.2 Definition of type one imaginary root vectors

The main result of this subsection is the following statement.

Proposition 5.6. *Let $d \in \mathcal{D} \setminus \{0\}$.*

(1) *One has $\bar{\psi}^{(s)}_{i,k;d} = \bar{\psi}^{(0)}_{i,k;d}$ for all $i \in I \setminus \{0\}$, $k \in \mathbb{N}$, and $s \in \mathbb{Z}$.*

(2) *One has $[\bar{\psi}^{(0)}_{i,k;d}, \bar{\psi}^{(0)}_{j,r;d}] = 0$ for all i, $j \in I \setminus \{0\}$ and k, $r \in \mathbb{N}$.*

Definition 5.7. Let $\bar{\psi}_{i,k;d} := \bar{\psi}^{(0)}_{i,k;d}$ for all $i \in I \setminus \{0\}$, $k \in \mathbb{N}$, and $d \in \mathcal{D} \setminus \{0\}$. For $k \in -\mathbb{N}$ and $i \in I \setminus \{0\}$, $d \in \mathcal{D} \setminus \{0\}$ set $\bar{\psi}_{i,k;d} := \Psi_d(\bar{\psi}_{i,-k;d})$. The elements $\bar{\psi}_{i,k;d} \in U'_{d,k\hat{\delta}_d}$, where $k \in \mathbb{Z} \setminus \{0\}$, are called *type one imaginary root vectors.*

In order to prove the above proposition we need a technical lemma. We use the notation $\{k; c\} := \sum_{j=0}^{k-1} c^{2j-k+1}$ for all $k \in \mathbb{N}$ and $c \in \mathbb{C}$.

Lemma 5.8. *Let $d \in \mathcal{D}$, and let $\epsilon, \eta \in \{1, -1\} \subset \mathbb{C}$, a, b, $c \in \mathbb{C} \setminus \{0\}$, and $m, n \in \mathbb{N}$ with $m \le n$. Let $(X_u)_{0 \le u \le n}$, $(Y_u)_{0 \le u \le n}$, $(Z_v^+)_{v \in \mathbb{Z}}$, and $(Z_v^-)_{v \in \mathbb{Z}}$ be families of $\mathbb{Z}\Pi_d$-homogeneous elements of U'_d, with the following properties.*

1. *The parity of the $\mathbb{Z}\Pi_d$-degree of X_u is even for all $u \in \{0, 1, \dots, n\}$.*
2. *The families $(X_u)_{0 \le u \le n}$, $(Y_u)_{0 \le u \le n}$, $(Z_v^+)_{v \in \mathbb{Z}}$, and $(Z_v^-)_{v \in \mathbb{Z}}$ satisfy the following equations.*

$$
\begin{aligned}
(5.3)\qquad & \\
[X_u, Z_v^\pm] =& \epsilon(c^{\pm 1} X_{u-1} Z_{v\mp 1}^\pm - c^{\mp 1} Z_{v \mp 1}^\pm X_{u-1}) \qquad \text{if } 1 \le u \le m,\ v \in \mathbb{Z},\\
Y_0 =& a Z_0^+ Z_0^- - b Z_0^- Z_0^+,\\
Y_u =& \eta^u (a Z_0^+ Z_u^- - b Z_i^- Z_0^+) \qquad \text{if } 1 \le u \le n-1,\\
Y_u =& \eta^u (a Z_{-1}^+ Z_{u-1}^- - b Z_{u-1}^- Z_{-1}^+) \qquad \text{if } 1 \le u \le n-1,\\
Y_n =& \eta^n (a Z_{-1}^+ Z_{n-1}^- - b Z_{n-1}^- Z_{-1}^+).
\end{aligned}
$$

3. *One has $[X_0, Y_n] = 0$.*

Then for all u with $2 \le u \le m$ we have

$$[X_u, Y_{n-u}] = (\eta\epsilon)^{u-1}\{u; c\}[X_1, Y_{n-1}]. \tag{5.4}$$

Moreover if there exists $r \in \mathbb{C}$ such that the equations

$$[X_1, Z_0^+] = r Z_{-1}^+, \qquad [X_1, Z_{n-1}^-] = - r Z_n^- \tag{5.5}$$

hold, then one also has

$$r Y_n - \eta [X_1, Y_{n-1}] = r \eta^n (a Z_0^+ Z_n^- - b Z_n^- Z_0^+). \tag{5.6}$$

Proof. The strategy of the proof of the first part of the lemma is the following. First we prove that

$$[X_u, Y_{n-u}] = \eta\epsilon(c + c^{-1})[X_{u-1}, Y_{n-u+1}] - [X_{u-2}, Y_{n-u+2}] \tag{5.7}$$

for all $u \in \{2, \dots, m\}$. Then assumption 3 implies that

$$[X_2, Y_{n-2}] = \eta\epsilon(c + c^{-1})[X_1, Y_{n-1}],$$

and induction on u using (5.7) implies Eq. (5.4).

Now we prove Eq. (5.7). Using (5.3) and Jacobi identity for [,] one obtains for all $u \in \{2,\dots,m\}$ the following.

$$
\begin{aligned}
[X_u, Y_{n-u}] &= [X_u, \eta^{n-u} a Z_0^+ Z_{n-u}^- - \eta^{n-u} b Z_{n-u}^- Z_0^+] \\
&= \eta^{n-u} a \Big\{ \epsilon(c X_{u-1} Z_{-1}^+ - c^{-1} Z_{-1}^+ X_{u-1}) Z_{n-u}^- \\
&\qquad + Z_0^+ \epsilon(c^{-1} X_{u-1} Z_{n-u+1}^- - c Z_{n-u+1}^- X_{u-1}) \Big\} \\
&\qquad - \eta^{n-u} b \Big\{ \epsilon(c^{-1} X_{u-1} Z_{n-u+1}^- - c Z_{n-u+1}^- X_{u-1}) Z_0^+ \\
&\qquad + Z_{n-u}^- \epsilon(c X_{u-1} Z_{-1}^+ - c^{-1} Z_{-1}^+ X_{u-1}) \Big\}.
\end{aligned}
$$

We calculate the first four summands of the last expression separately by using Eq.s (5.3).

$$
\begin{aligned}
&\eta^{n-u} \epsilon a c X_{u-1} Z_{-1}^+ Z_{n-u}^- = \eta^{n-u} \epsilon a c X_{u-1} \Big(\frac{Y_{n-u+1}}{\eta^{n-u+1} a} + \frac{b}{a} Z_{n-u}^- Z_{-1}^+ \Big) \\
&\quad = \eta \epsilon c X_{u-1} Y_{n-u+1} + \eta^{n-u} \epsilon b c (Z_{n-u}^- X_{u-1} + \epsilon c^{-1} X_{u-2} Z_{n-u+1}^- - \epsilon c Z_{n-u+1}^- X_{u-2}) Z_{-1}^+, \\
&- \eta^{n-u} \epsilon a c^{-1} Z_{-1}^+ X_{u-1} Z_{n-u}^- \\
&\quad = -\eta^{n-u} \epsilon a c^{-1} Z_{-1}^+ (Z_{n-u}^- X_{u-1} + \epsilon c^{-1} X_{u-2} Z_{n-u+1}^- - \epsilon c Z_{n-u+1}^- X_{u-2}) \\
&\quad = -\eta^{n-u} \epsilon a c^{-1} \Big(\Big(\frac{Y_{n-u+1}}{\eta^{n-u+1} a} + \frac{b}{a} Z_{n-u}^- Z_{-1}^+ \Big) X_{u-1} \\
&\qquad + \epsilon c^{-1} Z_{-1}^+ X_{u-2} Z_{n-u+1}^- - \epsilon c Z_{-1}^+ Z_{n-u+1}^- X_{u-2} \Big), \\
&\eta^{n-u} \epsilon a c^{-1} Z_0^+ X_{u-1} Z_{n-u+1}^- \\
&\quad = \eta^{n-u} \epsilon a c^{-1} (X_{u-1} Z_0^+ - \epsilon c X_{u-2} Z_{-1}^+ + \epsilon c^{-1} Z_{-1}^+ X_{u-2}) Z_{n-u+1}^-, \\
&\quad = \eta^{n-u} \epsilon a c^{-1} X_{u-1} Z_0^+ Z_{n-u+1}^- + \eta^{n-u} a c^{-2} Z_{-1}^+ X_{u-2} Z_{n-u+1}^- \\
&\qquad - X_{u-2} (Y_{n-u+2} + \eta^{n-u} b Z_{n-u+1}^- Z_{-1}^+), \\
&- \eta^{n-u} \epsilon a c Z_0^+ Z_{n-u+1}^- X_{u-1} = -\eta \epsilon c (Y_{n-u+1} + \eta^{n-u+1} b Z_{n-u+1}^- Z_0^+) X_{u-1} \\
&\quad = -\eta \epsilon c Y_{n-u+1} X_{u-1} + \eta^{n-u} \epsilon b c Z_{n-u+1}^- (-X_{u-1} Z_0^+ + \epsilon c X_{u-2} Z_{-1}^+ - \epsilon c^{-1} Z_{-1}^+ X_{u-2}).
\end{aligned}
$$

Comparison of the latter formulas gives that

$$
\begin{aligned}
&[X_u, Y_{n-u}] \\
&\quad = \eta \epsilon c X_{u-1} Y_{n-u+1} - \eta \epsilon c^{-1} Y_{n-u+1} X_{u-1} \\
&\qquad + \eta^{n-u} a Z_{-1}^+ Z_{n-u+1}^- X_{u-2} + \eta^{n-u} \epsilon a c^{-1} X_{u-1} Z_0^+ Z_{n-u+1}^- - X_{u-2} Y_{n-u+2} \\
&\qquad - \eta \epsilon c Y_{n-u+1} X_{u-1} - \eta^{n-u} b Z_{n-u+1}^- Z_{-1}^+ X_{u-2} - \eta^{n-u} \epsilon b c^{-1} X_{u-1} Z_{n-u+1}^- Z_0^+ \\
&\quad = \eta \epsilon c [X_{u-1}, Y_{n-u+1}] - \eta \epsilon c^{-1} Y_{n-u+1} X_{u-1} \\
&\qquad + Y_{n-u+2} X_{u-2} + \eta \epsilon c^{-1} X_{u-1} Y_{n-u+1} - X_{u-2} Y_{n-u+2} \\
&\quad = \eta \epsilon (c + c^{-1}) [X_{u-1}, Y_{n-u+1}] - [X_{u-2}, Y_{n-u+2}].
\end{aligned}
$$

This proves Eq. (5.7).

Now we prove Eq. (5.6). We have

$$\begin{aligned}
rY_n =& r\eta^n a Z^+_{-1} Z^-_{n-1} - r\eta^n b Z^-_{n-1} Z^+_{-1}\\
=&\eta^n a (X_1 Z^+_0 - Z^+_0 X_1) Z^-_{n-1} - \eta^n b Z^-_{n-1}[X_1, Z^+_0]\\
=&\eta X_1 (Y_{n-1} + \eta^{n-1} b Z^-_{n-1} Z^+_0) - \eta^n a Z^+_0 (Z^-_{n-1} X_1 - r Z^-_n) - \eta^n b Z^-_{n-1}[X_1, Z^+_0]\\
=&\eta X_1 Y_{n-1} + \eta^n b [X_1, Z^-_{n-1}] Z^+_0 + \eta^n a r Z^+_0 Z^-_n - \eta^n (a Z^+_0 Z^-_{n-1} - b Z^-_{n-1} Z^+_0) X_1\\
=&\eta X_1 Y_{n-1} - \eta^n b r Z^-_n Z^+_0 + \eta^n a r Z^+_0 Z^-_n - \eta Y_{n-1} X_1,
\end{aligned}$$

which gives (5.6). □

Proof of Proposition 5.6. We prove both parts of the proposition simultaneously by induction on $k + r$, where $r = 0$ in part (1). Let $n \in \mathbb{N}$ and assume that we have proved (1) for all $k \in \mathbb{N}$ with $k < n$ and (2) for all $k, r \in \mathbb{N}$ with $k + r < n$.

First we assume that $i \neq j$. We want to apply Lemma 5.8 with $m = n$ and the following setting.

$$\begin{aligned}
&\epsilon := \epsilon_{i,j;d}, && \eta := -1, && r := \epsilon_{i,j;d}[(\alpha_{i,d}, \alpha_{j,d})]_q,\\
&a := q^{-(\alpha_{i,d}, \alpha_{i,d})}, && b := (-1)^{p(\alpha_{j,d})}, && c := q^{(\alpha_{i,d}, \alpha_{j,d})},
\end{aligned}$$

$$X_u := \bar{\psi}^{(0)}_{i,u;d} \quad (0 \le u \le n), \quad Y_u := \bar{\psi}^{(0)}_{j,u;d} \quad (0 \le u \le n-1), \quad Y_n := \bar{\psi}^{(-1)}_{j,n;d},$$

$$Z^+_v := \mathcal{T}(\tilde{\omega}^\vee_{j,d})^v (E_{j,d}), \qquad Z^-_v := \mathcal{T}(\tilde{\omega}^\vee_{j,d})^v (K^{-1}_{j,d} F_{j,d}),$$

for all $v \in \mathbb{Z}$. Note that the first formula in Eq. (5.3) holds by Lemmata 5.1(1), 5.4 and 5.5 and by the induction hypothesis for part (1). The fourth line in (5.3) can be shown by applying $\mathcal{T}(\tilde{\omega}^\vee_{j,d})$ to the definition of $\bar{\psi}^{(0)}_{j,u;d}$ and using the induction hypothesis for part (1). All other formulas in (5.3) hold essentially by the definition of Y_u. Eq.s (5.5) follow essentially from Lemmata 5.1(1), 5.4 and 5.5. Eq. (5.2) implies assumption 3. Thus Lemma 5.8 gives that

$$[\bar{\psi}^{(0)}_{i,u;d}, \bar{\psi}^{(0)}_{j,n-u;d}] = (-\epsilon_{i,j;d})^{u-1} \{u; q^{(\alpha_{i,d}, \alpha_{j,d})}\} [\bar{\psi}^{(0)}_{i,1;d}, \bar{\psi}^{(0)}_{j,n-1;d}], \tag{5.8}$$

for $2 \le u \le n$. Let $u = n$ in the last formula. Then from the definition of $\bar{\psi}^{(0)}_{j,0;d}$ one gets

$$0 = [\bar{\psi}^{(0)}_{i,n;d}, \bar{\psi}^{(0)}_{j,0;d}] = (-\epsilon_{i,j;d})^{n-1} \{n; q^{(\alpha_{i,d}, \alpha_{j,d})}\} [\bar{\psi}^{(0)}_{i,1;d}, \bar{\psi}^{(0)}_{j,n-1;d}].$$

Therefore the assumption in (4.1) implies that

$$[\bar{\psi}^{(0)}_{i,1;d}, \bar{\psi}^{(0)}_{j,n-1;d}] = 0. \tag{5.9}$$

This with Eq. (5.8) gives part (2) of the proposition for $i \neq j$ and $k + r = n$. In order to prove part (1) of the proposition, assume that $i \neq j$ and $(\alpha_{i,d}, \alpha_{j,d}) \neq 0$. The second part of Lemma 5.8 gives that

(5.10)

$$\epsilon_{i,j;d}[(\alpha_{i,d}, \alpha_{j,d})]_q \bar{\psi}^{(-1)}_{j,n;d} = -[\bar{\psi}^{(0)}_{i,1;d}, \bar{\psi}^{(0)}_{j,n-1;d}] + \epsilon_{i,j;d}[(\alpha_{i,d}, \alpha_{j,d})]_q \mathcal{T}(\tilde{\omega}^\vee_{j,d})(\bar{\psi}^{(-1)}_{j,n;d}).$$

and hence part (1) of the proposition follows from (5.9) and Lemma 5.4 together with assumption (4.1) and relation $[(\alpha_{i,d},\alpha_{j,d})]_q \neq 0$.

Second we assume that $i = j$. We can apply the same argument as above, together with the equation

$$\bar{\psi}^{(-1)}_{i,k;d} = \mathcal{T}(\tilde{\omega}^{\vee}_{i,d})(\bar{\psi}^{(0)}_{i,k;d}) = \bar{\psi}^{(0)}_{i,k;d}$$

for $1 \leq k \leq n$, since part (1) with $k \leq n$ has already been proved (cf. Lemma 5.4). Then one has the same equations as in (5.8)–(5.9) with $j = i$. As a result one gets part (2) for $i = j$. □

5.3 Definition of type two imaginary root vectors

We start with the definition.

Definition 5.9. Let $d \in \mathcal{D}$ and z a formal parameter. The coefficients $\bar{h}_{i,k;d} \in U'_{d,k\hat{\delta}_d}$ of the formal power series

$$\sum_{k=1}^{\infty} \bar{h}_{i,k;d} z^k := \frac{1}{q-q^{-1}} \log\Big(1 + \sum_{r=1}^{\infty}(q-q^{-1})\bar{\psi}_{i,r;d} z^r\Big) \tag{5.11}$$

together with the elements $\bar{h}_{i,-k;d} := -\Psi_d(\bar{h}_{i,k;d})$, where $k \in \mathbb{N}$, are called *type two imaginary root vectors.*

Note that the above definition is equivalent to the more common formulas

$$\begin{aligned} \exp\Big((q-q^{-1})\sum_{k=1}^{\infty}\bar{h}_{i,k;d}z^k\Big) &= 1 + \sum_{r=1}^{\infty}(q-q^{-1})\bar{\psi}_{i,r;d}z^r, \\ \exp\Big(-(q-q^{-1})\sum_{k=1}^{\infty}\bar{h}_{i,-k;d}z^k\Big) &= 1 + \sum_{r=1}^{\infty}(q-q^{-1})\bar{\psi}_{i,-r;d}z^r. \end{aligned}$$

In order to determine commutation relations between root vectors to real roots and type two imaginary root vectors, we need two technical lemmata. The first of them is standard in a more general setting.

Lemma 5.10. *Let $d \in \mathcal{D}$ and $\epsilon \in \mathbb{C}$, $c \in \mathbb{C}\setminus\{0\}$. Let $(X_u)_{u\in\mathbb{N}}$ and $(Z_v)_{v\in\mathbb{Z}}$ be families of $\mathbb{Z}\Pi_d$-homogeneous elements of U'_d such that the parities of the degrees of the elements X_u are even for all $u \in \mathbb{N}$. Then the three conditions* (i), (ii) *and* (iii) *below are equivalent.*

(i) *For all $u \geq 2$ one has*

$$[X_u, Z_v] = \epsilon(cX_{u-1}Z_{v-1} - c^{-1}Z_{v-1}X_{u-1}).$$

(ii) *For all $u \geq 1$ one has*

$$[X_u, Z_v] = (\epsilon c)^{u-1}[X_1, Z_{v-u+1}] + \epsilon(c - c^{-1}) \sum_{r=1}^{u-1} (\epsilon c)^{u-1-r} Z_{v-u+r} X_r.$$

(iii) *For all $u \geq 1$ one has*

$$[X_u, Z_v] = (\epsilon c^{-1})^{u-1}[X_1, Z_{v-u+1}] + \epsilon(c - c^{-1}) \sum_{r=1}^{u-1} (\epsilon c^{-1})^{u-1-r} X_r Z_{v-u+r}.$$

Lemma 5.11. *Let $d \in \mathcal{D}$, and let $\epsilon \in \{1, -1\} \subset \mathbb{C}$, and $c \in \mathbb{C}\setminus\{0\}$. Let $(X_u)_{u\in\mathbb{N}}$, $(Z_v^+)_{v\in\mathbb{Z}}$ and $(Z_v^-)_{v\in\mathbb{Z}}$ be families of $\mathbb{Z}\Pi_d$-homogeneous elements of U_d', with the following properties.*

1. *The parity of the $\mathbb{Z}\Pi_d$-degree of X_u is even for all $u \in \mathbb{N}$.*
2. *For all $u, u' \in \mathbb{N}$ one has $[X_u, X_{u'}] = 0$.*
3. *The families $(X_u)_{u\in\mathbb{N}}$, $(Z_v^+)_{v\in\mathbb{Z}}$, and $(Z_v^-)_{v\in\mathbb{Z}}$ satisfy the following equations.*

$$\begin{aligned} [X_u, Z_v^\pm] &= \epsilon(c^{\pm 1} X_{u-1} Z_{v\mp 1}^\pm - c^{\mp 1} Z_{v\mp 1}^\pm X_{u-1}) \quad \text{for all } u \in \mathbb{N},\ v \in \mathbb{Z}, \\ [X_1, Z_v^+] &= r Z_{v-1}^+, \quad [X_1, Z_v^-] = -r Z_{v+1}^- \quad \text{for all } v \in \mathbb{Z}. \end{aligned} \tag{5.12}$$

Let $b \in \mathbb{C} \setminus \{0\}$ such that $rb = \epsilon(c - c^{-1})$. For all $u \in \mathbb{N}$ define $\mathcal{L}_u \in U_d'$ by the following generating function in z.

$$\exp\left(b \sum_{u=1}^{\infty} \mathcal{L}_u z^u\right) = 1 + b \sum_{u=1}^{\infty} X_u z^u. \tag{5.13}$$

Then for all $u \in \mathbb{N}$ and $v \in \mathbb{Z}$ we have

$$[\mathcal{L}_u, Z_v^\pm] = \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{ub} Z_{v\mp u}^\pm. \tag{5.14}$$

Proof. We proceed by induction on u. For $u = 1$ one has $\mathcal{L}_1 = X_1$, and hence the second line in (5.12) together with the definition of b implies the claim.

Let $n \in \mathbb{N}$ and assume that Eq. (5.14) holds for $u < n$. Let $\mathcal{L}(z) = \sum_{u=1}^{\infty} \mathcal{L}_u z^u$ and $\mathcal{Z}^\pm(y) = \sum_{v=-\infty}^{\infty} Z_v^\pm y^v$, where y is a formal parameter, and let

$$\begin{aligned} \mathcal{P}(y) :=& [\mathcal{L}_n, \mathcal{Z}^\pm(y)] - \epsilon^n \frac{c^{\pm n} - c^{\mp n}}{nb} y^{\pm n} \mathcal{Z}^\pm(y) \\ =& \sum_{v=-\infty}^{\infty} \left([\mathcal{L}_n, Z_v^\pm] - \epsilon^n \frac{c^{\pm n} - c^{\mp n}}{nb} Z_{v\mp u}^\pm\right) y^v. \end{aligned}$$

In the remainder of this proof we treat equations in the algebra $U'_d[\![z]\!]/(z^{n+1})$. We have the following:

$$[\mathcal{L}_u, \mathcal{Z}^{\pm}(y)] = \sum_{v=-\infty}^{\infty} \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{ub} Z^{\pm}_{v\mp u} y^v = y^{\pm u} \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{ub} \mathcal{Z}^{\pm}(y),$$

for all $u \in \mathbb{N}$ with $u < n$ by induction hypothesis, and hence

$$[b\mathcal{L}(z), \mathcal{Z}^{\pm}(y)] = b\mathcal{P}(y)z^n + \sum_{u=1}^{\infty} \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{u} y^{\pm u} z^u \mathcal{Z}^{\pm}(y).$$

This gives

$$\begin{aligned}
&(\exp b\mathcal{L}(z))\mathcal{Z}^{\pm}(y)(\exp b\mathcal{L}(z))^{-1} = \exp(\operatorname{ad} b\mathcal{L}(z))(\mathcal{Z}^{\pm}(y)) \\
&= \mathcal{Z}^{\pm}(y) + b\mathcal{P}(y)z^n + \sum_{u=1}^{\infty} \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{u} y^{\pm u} z^u \mathcal{Z}^{\pm}(y) \\
&\quad + \sum_{v=2}^{\infty} \frac{1}{v!} \Big(\sum_{u=1}^{\infty} \epsilon^u \frac{c^{\pm u} - c^{\mp u}}{u} y^{\pm u} z^u \Big)^v \mathcal{Z}^{\pm}(y) \\
&= b\mathcal{P}(y)z^n + \exp\Big(\sum_{u=1}^{\infty} \frac{1}{u} \big((\epsilon c^{\pm 1} y^{\pm 1} z)^u - (\epsilon c^{\mp 1} y^{\pm 1} z)^u \big) \Big) \mathcal{Z}^{\pm}(y) \\
&= b\mathcal{P}(y)z^n + \exp\big(-\log(1 - \epsilon c^{\pm 1} y^{\pm 1} z) + \log(1 - \epsilon c^{\mp 1} y^{\pm 1} z) \big) \mathcal{Z}^{\pm}(y) \\
&= b\mathcal{P}(y)z^n + (1 - \epsilon c^{\mp 1} y^{\pm 1} z) \sum_{v=0}^{\infty} (\epsilon c^{\pm 1} y^{\pm 1} z)^v \mathcal{Z}^{\pm}(y) \\
&= b\mathcal{P}(y)z^n + \mathcal{Z}^{\pm}(y) + (1 - c^{\mp 2}) \sum_{v=1}^{\infty} (\epsilon c^{\pm 1} y^{\pm 1})^v z^v \mathcal{Z}^{\pm}(y).
\end{aligned}$$

Next we compute $[\exp b\mathcal{L}(z), \mathcal{Z}^{\pm}(y)]$ in two different ways. By Lemma 5.10 we obtain that

$$\begin{aligned}
[X_u, \mathcal{Z}^{\pm}(y)] = \sum_{v=-\infty}^{\infty} [X_u, Z^{\pm}_v] y^v = \sum_{v=-\infty}^{\infty} \Big(&(\epsilon c^{\pm 1})^{u-1} [X_1, Z^{\pm}_{v\mp(u-1)}] \\
&+ \epsilon(c^{\pm 1} - c^{\mp 1}) \sum_{m=1}^{u-1} (\epsilon c^{\pm 1})^{u-1-m} Z^{\pm}_{v\mp(u-m)} X_m \Big) y^v,
\end{aligned}$$

and using the second line in Equation (5.12) we get

$$\begin{aligned}
[X_u, \mathcal{Z}^{\pm}(y)] = &\pm r(\epsilon c^{\pm 1})^{u-1} y^{\pm u} \mathcal{Z}^{\pm}(y) \\
&+ \epsilon(c^{\pm 1} - c^{\mp 1}) \sum_{m=1}^{u-1} (\epsilon c^{\pm 1})^{u-1-m} y^{\pm(u-m)} \mathcal{Z}^{\pm}(y) X_m.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
[\exp b\mathcal{L}(z), \mathcal{Z}^{\pm}(y)] &= b\sum_{u=1}^{\infty}[X_u, \mathcal{Z}^{\pm}(y)]z^u \\
&= \pm br\sum_{u=1}^{\infty}(\epsilon c^{\pm 1})^{u-1}y^{\pm u}z^u\mathcal{Z}^{\pm}(y) \\
&\quad + b\epsilon(c^{\pm 1} - c^{\mp 1})\sum_{u=1}^{\infty}\sum_{m=1}^{u-1}(\epsilon c^{\pm 1})^{u-1-m}y^{\pm(u-m)}z^u\mathcal{Z}^{\pm}(y)X_m,
\end{aligned}
$$

and also

$$
\begin{aligned}
[\exp b\mathcal{L}(z), \mathcal{Z}^{\pm}(y)] &= (\exp b\mathcal{L}(z))\mathcal{Z}^{\pm}(y)(\exp b\mathcal{L}(z))^{-1}\exp b\mathcal{L}(z) - \mathcal{Z}^{\pm}(y)\exp b\mathcal{L}(z) \\
&= (b\mathcal{P}(y)z^n + \mathcal{Z}^{\pm}(y) + (1-c^{\mp 2})\sum_{v=1}^{\infty}(\epsilon c^{\pm 1}y^{\pm 1})^v z^v\mathcal{Z}^{\pm}(y))\exp b\mathcal{L}(z) \\
&\quad - \mathcal{Z}^{\pm}(y)\exp b\mathcal{L}(z) \\
&= b\mathcal{P}(y)z^n + (1-c^{\mp 2})\sum_{v=1}^{\infty}(\epsilon c^{\pm 1}y^{\pm 1})^v z^v\mathcal{Z}^{\pm}(y)(1 + b\sum_{u=1}^{\infty}X_u z^u)
\end{aligned}
$$

Comparison of the two expressions for $[\exp b\mathcal{L}(z), \mathcal{Z}^{\pm}(y)]$ using the formula $\pm br = \epsilon(c^{\pm 1} - c^{\mp 1})$ gives that $b\mathcal{P}(y)z^n = 0$, and hence $\mathcal{P}(y) = 0$. This gives the statement of the lemma. □

Next we calculate commutation relations between root vectors to real roots and type two imaginary root vectors. Let

$$
\Theta(k) = \begin{cases} 0 & \text{if } k \leq 0, \\ 1 & \text{if } k > 0, \end{cases} \tag{5.15}
$$

denote the Heaviside function.

Lemma 5.12. *Let $d \in \mathcal{D} \setminus \{0\}$.*

(1) *For all $i, j \in I \setminus \{0\}$ and $k \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{Z}$ one has*

$$
\begin{aligned}
[\bar{h}_{i,k;d}, \mathcal{T}(\tilde{\omega}_{j,d}^{\vee})^m(E_{j,d})] &= \epsilon_{i,j;d}^{k}\frac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k}K_{\hat{\delta}_d;d}^{\Theta(-k)k}\mathcal{T}(\tilde{\omega}_{j,d}^{\vee})^{m-k}(E_{j,d}), \\
[\bar{h}_{i,k;d}, \mathcal{T}(\tilde{\omega}_{j,d}^{\vee})^m(F_{j,d})] &= -\epsilon_{i,j;d}^{k}\frac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k}K_{\hat{\delta}_d;d}^{\Theta(k)k}\mathcal{T}(\tilde{\omega}_{j,d}^{\vee})^{m+k}(F_{j,d}).
\end{aligned}
$$

(2) *For all $i, j \in I \setminus \{0\}$ and $k, l \in \mathbb{Z} \setminus \{0\}$ one has*

$$
[\bar{h}_{i,k;d}, \bar{h}_{j,l;d}] = \delta_{k,-l}(-1)^k\epsilon_{i,j;d}^{k}\frac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k}\frac{K_{\hat{\delta}_d;d}^{k} - K_{\hat{\delta}_d;d}^{-k}}{q - q^{-1}}
$$

Proof. Part (1) follows immediately from Lemmata 5.4, 5.5, 5.11 and Proposition 5.6.

To part (2). If $k > l > 0$ or $k < l < 0$, then the equation $[\bar{h}_{i,k;d}, \bar{h}_{j,l;d}] = 0$ holds by Preposition 5.6(2). Assume now that $k > 0 > l$ and let $m = -l$. By Definitions 5.7, 5.3 and Preposition 5.6(1), we have for all $s \in \mathbb{Z}$ the equation

$$\bar{\psi}_{j,l;d} = \Psi(\bar{\psi}^{(s)}_{j,m;d}) = (-1)^m K^{-(s+m)}_{\hat{\delta}_d;d} K_{j,d}(-1)^{p(\alpha_{j,d})}[\mathcal{T}(\tilde{\omega}^\vee_{j,d})^s(F_{j,d}), \mathcal{T}(\tilde{\omega}^\vee_{j,d})^{s+m}(E_{j,d})].$$

By part (1) of the lemma and by Proposition 5.6(1) we have

$$\begin{aligned}
&[\bar{h}_{i,k;d}, \bar{\psi}_{j,l;d}] = [\bar{h}_{i,k;d}, (-1)^l K^l_{\hat{\delta}_d;d} K_{j,d}(-1)^{p(\alpha_{j,d})}[F_{j,d}, \mathcal{T}(\tilde{\omega}^\vee_{j,d})^m(E_{j,d})]] \\
&= (-1)^{m+p(\alpha_{j,d})} K^l_{\hat{\delta}_d;d} K_{j,d} \epsilon^k_{i,j;d} \frac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k} \{ \\
&\quad [-K^k_{\hat{\delta}_d;d}\mathcal{T}(\tilde{\omega}^\vee_{j,d})^k(F_{j,d}), \mathcal{T}(\tilde{\omega}^\vee_{j,d})^m(E_{j,d})] + [F_{j,d}, \mathcal{T}(\tilde{\omega}^\vee_{j,d})^{m-k}(E_{j,d})]\} \\
&= \begin{cases} 0 & \text{if } m < k, \\ (-1)^{k+1}\epsilon^k_{i,j;d} \dfrac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k} \dfrac{K^k_{\hat{\delta}_d;d} - K^{-k}_{\hat{\delta}_d;d}}{q - q^{-1}} & \text{if } m = k, \\ (-1)^{k+1}\epsilon^k_{i,j;d} \dfrac{[k(\alpha_{i,d}, \alpha_{j,d})]_q}{k} (K^k_{\hat{\delta}_d;d} - K^{-k}_{\hat{\delta}_d;d}) \bar{\psi}_{d,k-m;d} & \text{if } m > k. \end{cases}
\end{aligned}$$

Then part (2) of the lemma can be shown along the lines of the last part of the proof of Lemma 5.11. □

The commutation relations in the following two lemmata will also be needed in the last section.

Lemma 5.13. *Let $d \in \mathcal{D} \setminus \{0\}$ and $i, j \in I \setminus \{0\}$. For all k, $l \in \mathbb{Z}$, one has*

$$[\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^k(E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{j,d})^l(E_{j,d})]\!] = (-1)^{\delta_{i,j}}[\![\mathcal{T}(\tilde{\omega}^\vee_{j,d})^{l+1}(E_{j,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k-1}(E_{i,d})]\!],$$
$$[\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^k(F_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{j,d})^l(F_{j,d})]\!] = (-1)^{\delta_{i,j}}[\![\mathcal{T}(\tilde{\omega}^\vee_{j,d})^{l+1}(F_{j,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k-1}(F_{i,d})]\!].$$

Proof. First, treat the first equation. If $i \neq j$ then the statement follows from Lemma 5.1, Theorem 4.10, and Theorem 3.5(1). Assume now that $i = j$. Without loss of generality, by Lemma 5.1(1), Theorem 4.10, and Theorem 3.5(1), let $k \geq 1$ and $l = 0$. If $k = 1$ then the statement of the lemma follows from Lemma 5.2(2). Let $l \in I \setminus \{0\}$ be such that $(\alpha_{i,d}, \alpha_{l,d}) \neq 0$. We proceed by induction on k. By Lemma 5.12(1) one has

$$\begin{aligned}
0 &= \frac{\epsilon_{i,l;d}}{[(\alpha_{i,d}, \alpha_{l,d})]_q} K^{-1}_{\hat{\delta}_d;d}[\bar{h}_{l,-1;d}, [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^k(E_{i,d}), E_{i,d}]\!] \\
&\quad + [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})(E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k-1}(E_{i,d})]\!]] \\
&= [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k+1}(E_{i,d}), E_{i,d}]\!] + \mathcal{T}(\tilde{\omega}^\vee_{i,d})([\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k-1}(E_{i,d}), E_{i,d}]\!] \\
&\quad + [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})(E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k-2}(E_{i,d})]\!]) + [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})(E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^k(E_{i,d})]\!] \\
&= [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})^{k+1}(E_{i,d}), E_{i,d}]\!] + [\![\mathcal{T}(\tilde{\omega}^\vee_{i,d})(E_{i,d}), \mathcal{T}(\tilde{\omega}^\vee_{i,d})^k(E_{i,d})]\!],
\end{aligned}$$

as desired. The second equation is obtained from the first one by applying Ψ_d. □

Lemma 5.14. *Let $d \in \mathcal{D} \setminus \{0\}$ and $i, j \in I \setminus \{0\}$.*

(1) *If $d \in \{1,2,3\}$ and $i \neq d$, then for all k, r, $l \in \mathbb{Z}$, one has*

$$\begin{aligned}
&[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(E_{i,d}), [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^r(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(E_{j,d})]\!]]\!]+\\
&\quad [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^r(E_{i,d}), [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(E_{j,d})]\!]]\!] = 0,\\
&[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(F_{i,d}), [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^r(F_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(F_{j,d})]\!]]\!]+\\
&\quad [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^r(F_{i,d}), [\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(F_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(F_{j,d})]\!]]\!] = 0.
\end{aligned}$$

(2) *If $(\alpha_{i,d}, \alpha_{j,d}) = 0$, then for all $k \in \mathbb{Z}$, one has*

$$\begin{aligned}
&[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(E_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(E_{j,d})]\!] = 0,\\
&[\![\mathcal{T}(\tilde{\omega}^{\vee}_{i,d})^k(F_{i,d}), \mathcal{T}(\tilde{\omega}^{\vee}_{j,d})^l(F_{j,d})]\!] = 0.
\end{aligned}$$

(3) *If $d = 4$, then for all k, r, $l \in \mathbb{Z}$, one has*

$$\begin{aligned}
&[(\alpha_{1,4}, \alpha_{3,4})]_q [\![[\![\mathcal{T}(\tilde{\omega}^{\vee}_{1,d})^k(E_{1,4}), \mathcal{T}(\tilde{\omega}^{\vee}_{2,d})^r(E_{2,4})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{3,d})^l(E_{3,4})]\!]+\\
&\quad [(\alpha_{1,4}, \alpha_{2,4})]_q [\![[\![\mathcal{T}(\tilde{\omega}^{\vee}_{1,4})^r(E_{1,4}), \mathcal{T}(\tilde{\omega}^{\vee}_{3,4})^k(E_{3,4})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{2,4})^l(E_{2,4})]\!] = 0,\\
&[(\alpha_{1,4}, \alpha_{3,4})]_q [\![[\![\mathcal{T}(\tilde{\omega}^{\vee}_{1,4})^k(F_{1,4}), \mathcal{T}(\tilde{\omega}^{\vee}_{2,4})^r(F_{2,4})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{3,4})^l(F_{3,4})]\!]+\\
&\quad [(\alpha_{1,4}, \alpha_{2,4})]_q [\![[\![\mathcal{T}(\tilde{\omega}^{\vee}_{1,4})^r(F_{1,4}), \mathcal{T}(\tilde{\omega}^{\vee}_{3,4})^k(F_{3,4})]\!], \mathcal{T}(\tilde{\omega}^{\vee}_{2,4})^l(F_{2,4})]\!] = 0.
\end{aligned}$$

Proof. These equations are obtained from the equations $X = 0$ in U'_d, where X are the elements having the same expressions as those in (4.13)-(4.18), by applying $\mathcal{T}(\tilde{\omega}^{\vee}_{u,d})^m$ and the $\mathbb{C}$-linear map $\mathrm{ad}\,\bar{h}_{u,m;d}$ defined by $\mathrm{ad}\,\bar{h}_{u,m;d}(Y) = [\bar{h}_{u,m;d}, Y]$ and using Lemma 5.1(1), Theorem 4.10, and Theorem 3.5(1). □

6 Second realization of the quantum affine superalgebras

6.1 Main theorem for U'_d

Here for each $d \in \mathcal{D} \setminus \{0\} = \{1,2,3,4\}$, we introduce Drinfeld second realization associated with $D^{(1)}(2,1;x)$ and prove that it is isomorphic to U'_d as a $\mathbb{C}$-algebra. We first give a modified version of the Drinfeld second realization of U'_d. Then via the version we give the Drinfeld second realization of U'_d.

Definition 6.1. Let $d \in \mathcal{D} \setminus \{0\} = \{1,2,3,4\}$. Let $DU'_d = \bigoplus_{\mu \in \mathbb{Z}\Pi_d} DU'_{d,\mu}$ be the $\mathbb{Z}\Pi_d$-graded $\mathbb{C}$-algebra generated by the elements

$$\sigma_d,\ K^{\frac{1}{2}}_{i,d},\ K^{-\frac{1}{2}}_{i,d} \in DU'_{d,0},\ (i \in I) \tag{6.1}$$

$$x^{\pm}_{i,k;d} \in DU'_{d,\pm\alpha_{i,d}+k\hat{\delta}_d},\ (i \in I \setminus \{0\},\ k \in \mathbb{Z}) \tag{6.2}$$

$$\psi_{i,r;d},\ h_{i,r;d} \in DU'_{d,r\hat{\delta}_d},\ (i \in I \setminus \{0\},\ r \in \mathbb{Z} \setminus \{0\}) \tag{6.3}$$

and defined by the relations below, where the elements $K_{i;d}$ and $K_{\hat{\delta}_d;d}$ are defined as in Eq. (4.10).

$$\sigma_d^2 = 1,\ K_{i,d}^{\frac{1}{2}} K_{i,d}^{-\frac{1}{2}} = K_{i,d}^{-\frac{1}{2}} K_{i,d}^{\frac{1}{2}} = 1, \tag{6.4}$$

$$XY = YX \quad \text{for all } X,\ Y \text{ in Eq. (6.1)} \tag{6.5}$$

$$\sigma_d X \sigma_d = (-1)^{p(\mu)} X,\ K_{i,d}^{\frac{1}{2}} X K_{i,d}^{-\frac{1}{2}} = q^{\frac{(\alpha_{i,d},\mu)}{2}} X \text{ for all } X \in DU'_{d,\mu},\ \mu \in \mathbb{Z}\Pi_d, \tag{6.6}$$

$$[x^+_{i,k;d}, x^-_{j,l;d}] = \begin{cases} 0 & \text{if } i \neq j, \\ K_{\hat{\delta}_d;d}^{-l} K_{i,d} \psi_{i,k+l;d} & \text{if } i = j \text{ and } k+l > 0, \\ \dfrac{K_{\hat{\delta}_d;d}^{k} K_{i,d} - K_{\hat{\delta}_d;d}^{-k} K_{i,d}^{-1}}{q - q^{-1}} & \text{if } i = j \text{ and } k+l = 0, \\ -K_{\hat{\delta}_d;d}^{-k} K_{i,d}^{-1} \psi_{i,k+l;d} & \text{if } i = j \text{ and } k+l < 0, \end{cases} \tag{6.7}$$

$$\exp\Big(\pm (q - q^{-1}) \sum_{k>0} z^k h_{i,\pm k;d} \Big) = 1 + (q - q^{-1}) \sum_{k>0} z^k \psi_{i,\pm k;d} \tag{6.8}$$

(as equations of generating functions in z)

$$[h_{i,k;d}, h_{j,l;d}] = \delta_{k,-l} \frac{[k(\alpha_{i,d},\alpha_{j,d})]_q}{k} \frac{K_{\hat{\delta}_d;d}^{k} - K_{\hat{\delta}_d;d}^{-k}}{q - q^{-1}} \tag{6.9}$$

$$[h_{i,k;d}, x^\pm_{j,l;d}] = \pm \frac{[k(\alpha_{i,d},\alpha_{j,d})]_q}{k} K_{\hat{\delta}_d;d}^{\Theta(\mp k)k} x^\pm_{j,k+l;d} \tag{6.10}$$

$$[\![x^\pm_{i,k;d}, x^\pm_{j,l;d}]\!] = \begin{cases} 0 & \text{if } (\alpha_{i,d},\alpha_{j,d}) = 0, \\ -[\![x^\pm_{j,l\mp1;d}, x^\pm_{i,k\pm1;d}]\!] & \text{if } d \in \{1,2,3\}, \\ [\![x^\pm_{j,l\mp1;4}, x^\pm_{i,k\pm1;4}]\!] & \text{if } d = 4 \end{cases} \tag{6.11}$$

$$[\![x^\pm_{i,r;d}, [\![x^\pm_{i,k;d}, x^\pm_{j,l;d}]\!]]\!] + [\![x^\pm_{i,k;d}, [\![x^\pm_{i,r;d}, x^\pm_{j,l;d}]\!]]\!] = 0 \text{ if } d \in \{1,2,3\} \text{ and } i \neq d, \tag{6.12}$$

(6.13)
$$[(\alpha_{1,4},\alpha_{3,4})]_q [\![[\![x^\pm_{1,r;4}, x^\pm_{2,k;4}]\!], x^\pm_{3,l;4}]\!] - [(\alpha_{1,4},\alpha_{2,4})]_q [\![[\![x^\pm_{1,r;4}, x^\pm_{3,k;4}]\!], x^\pm_{2,l;4}]\!] = 0 \text{ if } d = 4.$$

Note that if $i \in I \setminus \{0\}$ then in DU'_d, one has $[\![x^\pm_{i,k;d}, x^\pm_{i,k\pm1;d}]\!] = 0$ for all $k \in \mathbb{Z}$. Note that if $i \in I \setminus \{0\}$ and $p(\alpha_{i,d}) = 1$ then in DU'_d, one has $(x^\pm_{i,k;d})^2 = \frac{1}{2}[\![x^\pm_{i,k;d}, x^\pm_{i,k;d}]\!] = 0$ for all $k \in \mathbb{Z}$.

Proposition 6.2. *Let $d \in \mathcal{D} \setminus \{0\}$. Then there exists a unique $\mathbb{C}$-algebra homomorphism*

$$\mathcal{F}_d : DU'_d \to U'_d \tag{6.14}$$

such that for all $u \in I$, $i \in I \setminus \{0\}$, $k \in \mathbb{Z}$, and $r \in \mathbb{Z} \setminus \{0\}$, one has

$$\mathcal{F}_d(\sigma_d) = \sigma_d, \quad \mathcal{F}_d(K_{u,d}^{\pm\frac{1}{2}}) = K_{u,d}^{\pm\frac{1}{2}}, \tag{6.15}$$

$$\mathcal{F}_d(x_{i,k;d}^{+}) = (-\epsilon_{i,i;d})^k \mathcal{T}(\tilde{\omega}_{i,d}^{\vee})^{-k}(E_{i,d}), \tag{6.16}$$

$$\mathcal{F}_d(x_{i,k;d}^{-}) = (-\epsilon_{i,i;d})^k \mathcal{T}(\tilde{\omega}_{i,d}^{\vee})^{k}(F_{i,d}), \tag{6.17}$$

$$\mathcal{F}_d(h_{i,r;d}) = \epsilon_{i,i;d}^r \bar{h}_{i,r;d}, \quad \mathcal{F}_d(\psi_{i,r;d}) = \epsilon_{i,i;d}^r \bar{\psi}_{i,r;d}. \tag{6.18}$$

Proof. This follows immediately from Definitions 5.3, 5.7, 5.9, Proposition 5.6, and Lemmata 5.12, 5.13, and 5.14. □

Let $d \in \mathcal{D} \setminus \{0\}$. Define $D\Psi_d$ to be the automorphism of DU'_d such that $D\Psi_d(\sigma_d) = \sigma_d$, $D\Psi_d(K_{u,d}^{\pm\frac{1}{2}}) = K_{u,d}^{\mp\frac{1}{2}}$, $D\Psi_d(x_{i,k;d}^{\pm}) = (\mp 1)^{p(\alpha_{i,d})} x_{i,-k;d}^{\mp}$, $D\Psi_d(\psi_{i,r;d}) = \psi_{i,-r;d}$, and $D\Psi_d(h_{i,r;d}) = -h_{i,-r;d}$. Note that

$$\mathcal{F}_d D\Psi_d = \Psi_d \mathcal{F}_d. \tag{6.19}$$

Define the two elements $\mathbb{X}_d^{\pm} \in DU'_{d,\pm\alpha_{0,d}}$ by

$$\mathbb{X}_d^{\pm} = \begin{cases} -r_{d,j;d} r_{d,k;d} r_{d,0;d} K_{0,d}^{\pm 1} [\![[\![[\![x_{d,\pm 1;d}^{\mp}, x_{j,0;d}^{\mp}]\!], x_{k,0;d}^{\mp}]\!], x_{d,0;d}^{\mp}]\!] \\ \quad \text{if } d \in \{1,2,3\}, \text{ where } \{j,k,d\} = \{1,2,3\} \text{ and } j < k, \\ \pm[(\alpha_{1,4}, \alpha_{2,4})]_q^{-1} K_{0,4}^{\pm 1} [\![[\![x_{1,\pm 1;4}^{\mp}, x_{2,0;4}^{\mp}]\!], x_{3,0;4}^{\mp}]\!], \quad \text{if } d = 4. \end{cases} \tag{6.20}$$

Lemma 6.3. *Let $d \in \mathcal{D} \setminus \{0\}$. One has $\mathcal{F}_d(\mathbb{X}_d^+) = E_{0,d}$ and $\mathcal{F}_d(\mathbb{X}_d^-) = F_{0,d}$.*

Proof. Assume that $d = 4$. By Lemma 4.3, for all i, $j \in I = \{0,1,2,3\}$ and all $Y_\lambda \in U'_{4,\lambda}$, one has $[\![E_{i,4}, F_{j,4}K_{j,4}^{-1}]\!] = \delta_{ij} \frac{1-K_{i,4}^{-2}}{q-q^{-1}}$, and

$$[\![[\![E_{i,4}, Y_\lambda]\!], F_{j,4}K_{j,4}^{-1}]\!] = [\![E_{i,4}, [\![Y_\lambda, F_{j,4}K_{j,4}^{-1}]\!]]\!] + \delta_{ij}(-1)^{p(\lambda)p(\alpha_{i,4})}[(\lambda, \alpha_{i,4})]_q Y_\lambda. \tag{6.21}$$

Note that for $X_\mu \in U'_{4,\mu}$ and $Y_\lambda \in U'_{4,\lambda}$, one has

$$[\![X_\mu K_{\mu;4}, Y_\lambda K_{\lambda;4}]\!] = q^{(\alpha_{\mu,4},\alpha_{\lambda,4})} [\![X_\mu, Y_\lambda]\!] K_{\mu+\lambda;4}. \tag{6.22}$$

By Eq.s (3.28), (3.29) and (4.38), one has

$$\mathcal{T}(\tilde{\omega}_{1,4}^{\vee})(K_{1,4}^{-1}) = K_{0,4}K_{2,4}K_{3,4}. \tag{6.23}$$

Then one has

$$\begin{aligned} \mathcal{F}_4(\mathbb{X}_4^+) &= \frac{K_{0,4}[\![[\![-\mathcal{T}(\tilde{\omega}_{1,d}^{\vee})(F_{1,4}), F_{2,4}]\!], F_{3,4}]\!]}{[(\alpha_{1,4}, \alpha_{2,4})]_q} \quad \text{(by Eq.s (6.17) and (6.20))} \\ &= -[(\alpha_{1,4}, \alpha_{2,4})]_q^{-1} [\![[\![\mathcal{T}(\tilde{\omega}_{1,4}^{\vee})(F_{1,4}K_{1,4}^{-1}), F_{2,4}K_{2,4}^{-1}]\!], F_{3,4}K_{3,4}^{-1}]\!] \\ &\qquad \text{(by Eq.s (6.22), (6.23), and (4.6))} \\ &= \frac{[\![[\![[\![E_{3,4}, [\![E_{2,4}, E_{0,4}]\!]]\!], F_{2,4}K_{2,4}^{-1}]\!], F_{3,4}K_{3,4}^{-1}]\!]}{[(\alpha_{1,4}, \alpha_{2,4})]_q [(\alpha_{0,4}, \alpha_{2,4})]_q} \quad \text{(by Lemma 5.2(1))} \\ &= -[(\alpha_{1,4}, \alpha_{2,4})]_q^{-1} [\![[\![E_{3,4}, E_{0,4}]\!], F_{3,4}K_{3,4}^{-1}]\!] \quad \text{(by Eq. (6.21))} \\ &= E_{0,4} \quad \text{(by Eq. (6.21))}. \end{aligned}$$

Thus one has $\mathcal{F}_4(\mathbb{X}_4^+) = E_{0,4}$. By this and Eq. (6.19), one obtains

$$\mathcal{F}_4(\mathbb{X}_4^-) = \mathcal{F}_4(-D\Psi_d(\mathbb{X}_4^+)) = -\Psi_d(\mathcal{F}_4(\mathbb{X}_4^+)) = -\Psi_d(E_{0,4}) = F_{0,4},$$

as desired.

The lemma for $d \in \{1,2,3\}$ can be proven in an entirely similar way. □

For $i \in I \setminus \{0\}$ and $d \in \mathcal{D} \setminus \{0\}$, let $z^{\pm}_{i,k;d} := x^{\pm}_{i,k;d} K^{\pm 1}_{i,d} K^{k}_{\hat{\delta}_d;d}$.

To prove the existence of the inverse of $\mathcal{F}$, we need the following lemma, which is similar to Lemma 4.3.

Lemma 6.4. *Let $\mu, \lambda, \xi \in \mathbb{Z}\Pi_d$, $i,j \in I \setminus \{0\}$, and $k, r \in \mathbb{Z}$, and let $X_\mu \in DU'_{d,\mu}$, $Y_\lambda \in DU'_{d,\lambda}$, and $Z_\xi \in DU'_{d,\xi}$.*

(1) *One has*

$$(6.24)\quad [\![X_\mu, K_{\lambda;d} Y_\lambda]\!] = q^{-(\lambda,\mu)} K_{\lambda;d}[X_\mu, Y_\lambda], \qquad [\![K^{-1}_{\mu;d} X_\mu, Y_\lambda]\!] = K^{-1}_{\mu;d}[X_\mu, Y_\lambda],$$

$$(6.25)\quad [\![[\![X_\mu, Y_\lambda]\!], Z_\xi]\!] = [\![X_\mu, [\![Y_\lambda, Z_\xi]\!]]\!] + (-1)^{p(\lambda)p(\xi)} q^{-(\lambda,\xi)} [[\![X_\mu, Z_\xi]\!], Y_\lambda]_{q^{(\lambda,\xi-\mu)}},$$

$$(6.26)\quad [\![X_\mu, [\![Y_\lambda, Z_\xi]\!]]\!] = [\![[\![X_\mu, Y_\lambda]\!], Z_\xi]\!] + (-1)^{p(\mu)p(\lambda)} q^{-(\mu,\lambda)} [Y_\lambda, [\![X_\mu, Z_\xi]\!]]_{q^{(\lambda,\mu-\xi)}}.$$

Further, one has

$$(6.27)\quad [\![x^+_{i,k;d}, [\![Y_\lambda, z^-_{j,-k;d}]\!]]\!] = [\![[\![x^+_{i,k;d}, Y_\lambda]\!], z^-_{j,-k;d}]\!] - \delta_{ij}(-1)^{p(\alpha_{j,d})p(\lambda)} [(\alpha_{i,d}, \lambda)]_q Y_\lambda$$

$$(6.28)\quad [\![z^{\pm}_{i,k;d}, z^{\pm}_{j,r;d}]\!] = q^{(\alpha_{i,d},\alpha_{j,d})} [\![x^{\pm}_{i,k;d}, x^{\pm}_{j,r;d}]\!] K^{\pm 1}_{i,d} K^{\pm 1}_{j,d} K^{k+r}_{\hat{\delta}_d;d},$$

$$(6.29)\quad [z^+_{i,k;d}, z^-_{j,r;d}] = q^{-(\alpha_{i,d},\alpha_{j,d})} [x^+_{i,k;d}, x^-_{j,r;d}] K_{i,d} K^{-1}_{j,d} K^{k+r}_{\hat{\delta}_d;d}$$
$$= \begin{cases} q^{-(\alpha_{i,d},\alpha_{j,d})} \dfrac{K_{i,d} K^{k}_{\hat{\delta}_d;d} - K^{-1}_{i,d} K^{-k}_{\hat{\delta}_d;d}}{q-q^{-1}} & \text{if } i=j \text{ and } r=-k, \\ 0 & \text{if } i \neq j. \end{cases}$$

(2) *If $p(\lambda) = 1$ and $(\lambda,\lambda) = 0$, then one has*

$$(6.30)\quad [\![[\![X_\mu, Y_\lambda]\!], Y_\lambda]\!] = [\![Y_\lambda, [\![Y_\lambda, X_\mu]\!]]\!] = 0 \quad \text{if } Y_\lambda^2 = 0,$$

$$(6.31)\quad [\![X_\mu, Y_\lambda]\!]^2 = 0 \quad \text{if } Y_\lambda^2 = 0,\ [\![X_\mu, [\![X_\mu, Y_\lambda]\!]]\!] = 0, \text{ and } q^{(\mu,\mu)} + (-1)^{p(\mu)} \neq 0,$$

$$(6.32)\quad [\![Y_\lambda, X_\mu]\!]^2 = 0 \quad \text{if } Y_\lambda^2 = 0,\ [\![[\![Y_\lambda, X_\mu]\!], X_\mu]\!] = 0, \text{ and } q^{(\mu,\mu)} + (-1)^{p(\mu)} \neq 0.$$

(3) *If $(\mu,\lambda) + (\lambda,\xi) + (\xi,\mu) = 0$ and $p(\mu) = p(\lambda) = p(\xi) = 1$, then one has*

$$\begin{aligned} &[(\mu,\xi)]_q [\![[\![X_\mu, Y_\lambda]\!], Z_\xi]\!] - [(\mu,\lambda)]_q [\![[\![X_\mu, Z_\xi]\!], Y_\lambda]\!] \\ &= [(\mu,\xi)]_q (X_\mu Y_\lambda Z_\xi - Z_\xi Y_\lambda X_\mu) + [(\lambda,\mu)]_q (Y_\lambda Z_\xi X_\mu - X_\mu Z_\xi Y_\lambda) \\ &\quad + [(\xi,\lambda)]_q (Z_\xi X_\mu Y_\lambda - Y_\lambda X_\mu Z_\xi) \\ &= -[(\mu,\xi)]_q [\![Z_\xi, [\![Y_\lambda, X_\mu]\!]]\!] + [(\mu,\lambda)]_q [\![Y_\lambda, [\![Z_\xi, X_\mu]\!]]\!]. \end{aligned}$$

(4) *Let* $X_{-\mu} \in DU'_{d,-\mu}$ *and* $Y_{-\lambda} \in DU'_{d,-\lambda}$. *Assume that* $[X_\mu, X_{-\mu}] = \frac{K_{\mu;d}-K_{-\mu;d}}{q-q^{-1}}$, $[Y_\lambda, Y_{-\lambda}] = \frac{K_{\lambda;d}-K_{-\lambda;d}}{q-q^{-1}}$, *and* $[X_{\pm\mu}, Y_{\mp\lambda}] = 0$. *Then one has*

$$[\![X_\mu, Y_\lambda]\!], [\![X_{-\mu}, Y_{-\lambda}]\!]] = (-1)^{p(\lambda)p(\mu)} q^{-(\lambda,\mu)} [(\lambda, \mu)]_q \frac{K_{\mu+\lambda;d} - K_{-\mu-\lambda;d}}{q - q^{-1}},$$

where $K_{\mu;d}$ *and* $K_{\lambda;d}$ *are defined as in* (4.9).

Proof. One needs only the definition of q-super-bracket and Eq. (6.7). □

We also need the following lemma.

Lemma 6.5. *Let* $\{d, i, j\} = \{1, 2, 3\}$ *and* $l, r, m \in \mathbb{Z}$. *Let* $\mathbb{Y}_1^\pm$ *and* $\mathbb{Y}_2^\pm$ *be the elements of* DU'_d *defined by* $\mathbb{Y}_1^\pm := [\![[\![z^\pm_{d,\mp l;d}, z^\pm_{j,\mp r;d}]\!], z^\pm_{k,\mp m;d}]\!]$ *and* $\mathbb{Y}_2^\pm := [\![\mathbb{Y}_1^\pm, z^\pm_{d,\mp(l-1);d}]\!]$.

(1) *One has the following:*

$$(6.33)\qquad (z^-_{d,l;d})^2 = 0, \quad [\![z^-_{j,r;d}, z^-_{k,m;d}]\!] = [\![z^-_{k,m;d}, z^-_{j,r;d}]\!] = 0,$$

$$(6.34)\qquad [\![z^-_{d,l;d}, z^-_{j,r;d}]\!] = -[\![z^-_{j,r+1;d}, z^-_{d,l-1;d}]\!], \quad [\![[\![z^-_{d,l;d}, z^-_{k,m;d}]\!], z^-_{k,m;d}]\!] = 0,$$

$$(6.35)\qquad \mathbb{Y}_1^- = [\![[\![z^-_{d,l;d}, z^-_{k,m;d}]\!], z^-_{j,r;d}]\!], \quad [\![[\![z^-_{j,r;d}, z^-_{d,l;d}]\!], z^-_{d,l;d}]\!] = 0,$$

$$(6.36)\qquad [\![z^-_{j,r;d}, z^-_{d,l;d}]\!]^2 = 0, \quad [\![z^-_{d,l;d}, z^-_{j,r;d}]\!]^2 = 0,$$

$$(6.37)\qquad [\![\mathbb{Y}_1^-, z^-_{j,r;d}]\!] = 0, \quad [\![\mathbb{Y}_1^-, z^-_{k,m;d}]\!] = 0, \quad (\mathbb{Y}_1^-)^2 = 0, \quad [\![\mathbb{Y}_1^-, \mathbb{Y}_2^-]\!] = 0,$$

$$(6.38)\qquad \mathbb{Y}_1^- = -[\![[\![z^-_{k,m+1;d}, z^-_{d,l-1;d}]\!], z^-_{j,r;d}]\!] = -[\![z^-_{k,m+1;d}, [\![z^-_{d,l-1;d}, z^-_{j,r;d}]\!]]\!],$$

$$(6.39)\qquad [\mathbb{Y}_2^-, x^-_{j,r;d}] = 0, \quad [\mathbb{Y}_2^-, x^-_{k,m;d}] = 0.$$

(2) *One has*

$$(6.40)\qquad [\![x^+_{d,-l+1;d}, \mathbb{Y}_1^-]\!] = 0, \quad [\![x^+_{d,-l+1;d}, \mathbb{Y}_2^-]\!] = -[(\alpha_{j,d} + \alpha_{k,d}, \alpha_{d,d})]_q \mathbb{Y}_1^-,$$

$$(6.41)\qquad [\![[\![x^+_{d,-l+1;d}, \mathbb{Y}_2^-]\!], \mathbb{Y}_2^-]\!] = 0, \quad [\![x^+_{j,-r;d}, \mathbb{Y}_2^-]\!] = 0, \quad [\![x^+_{k,-m;d}, \mathbb{Y}_2^-]\!] = 0.$$

(3) *If* $l = 1$ *and* $r = m = 0$ *then one has*

$$(6.42)\qquad [\mathbb{Y}_2^-, \mathbb{Y}_2^+] = [(\alpha_{d,d}, \alpha_{j,d})]_q [(\alpha_{d,d}, \alpha_{k,d})]_q [(\alpha_{d,d}, \alpha_{0,d})]_q \frac{K_{0,d} - K_{0,d}^{-1}}{q - q^{-1}}.$$

Proof. To Part (1): Eq.s (6.33),(6.34) hold just by (6.11)-(6.12) and (6.28). Each equation in (6.35)-(6.38) follows from the ones in (6.33)-(6.38) before it and Lemma 6.4(1),(2). One obtains the first equation in (6.39) in the following way.

$$\begin{aligned}
&[\mathbb{Y}_2^-, x^-_{j,r;d}] K_{j,d}^{-1} K^r_{\hat{\delta}_{d;d}} = [\![\mathbb{Y}_2^-, z^-_{j,r;d}]\!] \quad \text{(by the equation } [K_{j,d}^{-1} K^r_{\hat{\delta}_{d;d}}, \mathbb{Y}_2^-] = 0) \\
&= [\![[\![\mathbb{Y}_1^-, z^-_{d,l-1;d}]\!], z^-_{j,r;d}]\!] \quad \text{(by the definition of } \mathbb{Y}_2^-) \\
&= [\![\mathbb{Y}_1^-, [\![z^-_{d,l-1;d}, z^-_{j,r;d}]\!]]\!] \quad \text{(by Eq. (6.25) and the first equation in (6.37))} \\
&= -[\![[\![z^-_{k,m+1;d}, [\![z^-_{d,l-1;d}, z^-_{j,r;d}]\!]]\!], [\![z^-_{d,l-1;d}, z^-_{j,r;d}]\!]]\!] \quad \text{(by using (6.38))} \\
&= 0 \quad \text{(by the second equation in (6.36) and by using (6.30))}.
\end{aligned}$$

Similarly one has the second equation in (6.39).

To Part (2): Applying Eq. (6.27) twice, one has the first equation in (6.40) by using the first one in (6.38) and the second one in (6.33). Similarly, the second equation in (6.40) holds by the first one in (6.40). The first equation in (6.41) follows from the second one in (6.40) and the fourth one in (6.37). As for the second equation in (6.41), one has

$$\begin{aligned}\llbracket x^+_{j,-r;d}, \mathbb{Y}^-_2\rrbracket &= -[(\alpha_{d,d},\alpha_{j,d})]_q \llbracket\llbracket z^-_{k,m+1;d}, z^-_{d,l-1;d}\rrbracket, z^-_{d,l-1;d}\rrbracket \\ &\qquad \text{(by using (6.27) and the first equation in (6.38))} \\ &= 0 \text{ (by the second equation in (6.35)).}\end{aligned}$$

Similarly one has the third equation in (6.41).

To Part (3): Using the second equation in (6.24) and applying $D\Psi_d$ to the first one in (6.40) one has $[z^{\pm}_{d,-l+1;d}, Y^{\mp}_2] = 0$. Then applying Lemma 6.4 (4) and Eq. (6.29) repeatedly one obtains (6.42). □

We first give a modified version of the Drinfeld second realization of U'_d.

Theorem 6.6. *For each $d \in \mathcal{D}\setminus\{0\}$ the map $\mathcal{F}_d : DU'_d \to U'_d$ is a $\mathbb{Z}\Pi_d$-graded $\mathbb{C}$-algebra isomorphism.*

Proof. We show the existence of the inverse map $\mathcal{F}_d^{-1}$ directly.

Assume that $d \in \{1,2,3\}$, $\{d,j,k\} = \{1,2,3\}$, and $j<k$. Let $\tilde{\mathbb{X}}^+_d = -(r_{d,j;d}r_{d,k;d}r_{d,0;d})^{-1}\mathbb{X}^+_d$. One has

$$\tilde{\mathbb{X}}^+_d = \llbracket\llbracket\llbracket z^-_{d,1;d}, z^-_{j,0;d}\rrbracket, z^-_{k,0;d}\rrbracket, z^-_{d,0;d}\rrbracket = \llbracket\llbracket\llbracket z^-_{d,1;d}, z^-_{k,0;d}\rrbracket, z^-_{j,0;d}\rrbracket, z^-_{d,0;d}\rrbracket, \tag{6.43}$$

where we used the first part of (6.35) for the second equation. By the first equation in (6.43) and Lemmas 6.4(1), 6.5(1),(2), one has

$$\begin{aligned}\llbracket\llbracket x^+_{d,0;d}, x^+_{j,0;d}\rrbracket, \llbracket x^+_{d,0;d}, \tilde{\mathbb{X}}^+_d\rrbracket\rrbracket &= \llbracket x^+_{d,0;d}, \llbracket x^+_{j,0;d}, \llbracket x^+_{d,0;d}, \tilde{\mathbb{X}}^+_d\rrbracket\rrbracket\rrbracket \\ &\qquad \text{(by Eq.s (6.25),(6.30) and the equation } (x^+_{j,0;d})^2=0) \\ &= [(\alpha_{d,d},\alpha_{0,d})]_q \llbracket x^+_{d,0;d}, \llbracket x^+_{j,0;d}, \llbracket\llbracket z^-_{d,1;d}, z^-_{j,0;d}\rrbracket, z^-_{k,0;d}\rrbracket\rrbracket\rrbracket \\ &\qquad \text{(by the second equation in Eq.s (6.40))} \\ &= -[(\alpha_{d,d},\alpha_{0,d})]_q \llbracket x^+_{d,0;d}, \llbracket x^+_{j,0;d}, \llbracket\llbracket z^-_{k,1;d}, z^-_{d,0;d}\rrbracket, z^-_{j,0;d}\rrbracket\rrbracket\rrbracket \\ &\qquad \text{(by the first equation in (6.38))} \\ &= -[(\alpha_{d,d},\alpha_{0,d})]_q[(\alpha_{d,d},\alpha_{j,d})]_q[(\alpha_{d,d},\alpha_{k,d})]_q z^-_{k,1;d} \text{ (by using (6.27)).}\end{aligned} \tag{6.44}$$

By Eq.s (6.11),(6.27) one has

$$\llbracket\llbracket x^+_{d,0;d}, x^+_{k,0;d}\rrbracket, z^-_{k,1;d}\rrbracket = -\llbracket\llbracket x^+_{k,-1;d}, x^+_{d,1;d}\rrbracket, z^-_{k,1;d}\rrbracket = -[(\alpha_{d,d},\alpha_{k,d})]_q x^+_{d,1;d}, \tag{6.45}$$

By the second equation in (6.43), using the same argument as above, one also has the equations obtained from (6.44),(6.45) by changing j and k. Hence one conclude that the following equation holds:

$$\begin{aligned}(6.46)\qquad &[(\alpha_{d,d},\alpha_{j,d})]_q[\![[\![x^+_{d,0;d},x^+_{k,0;d}]\!],[\![[\![x^+_{d,0;d},x^+_{j,0;d}]\!],[\![x^+_{d,0;d},\tilde{\mathbb{X}}^+_d]\!]]\!]]\!] \\ &= [(\alpha_{d,d},\alpha_{0,d})]_q[(\alpha_{d,d},\alpha_{j,d})]^2_q[(\alpha_{d,d},\alpha_{k,d})]^2_q x^+_{d,1;d}, \\ &= [(\alpha_{d,d},\alpha_{k,d})]_q[\![[\![x^+_{d,0;d},x^+_{j,0;d}]\!],[\![[\![x^+_{d,0;d},x^+_{k,0;d}]\!],[\![x^+_{d,0;d},\tilde{\mathbb{X}}^+_d]\!]]\!]]\!].\end{aligned}$$

By (6.46) and Lemma 6.4(3), one obtains

$$\begin{aligned}(6.47)\qquad &[(\alpha_{d,d}+\alpha_{0,d},\alpha_{j,d}+\alpha_{d,d})]_q[\![[\![[\![x^+_{d,0;d},\mathbb{X}^+_d]\!],[\![x^+_{d,0;d},x^+_{k,0;d}]\!]]\!],[\![x^+_{d,0;d},x^+_{j,0;d}]\!]]\!] \\ &= [(\alpha_{d,d}+\alpha_{0,d},\alpha_{k,d}+\alpha_{d,d})]_q[\![[\![[\![x^+_{d,0;d},\mathbb{X}^+_d]\!],[\![x^+_{d,0;d},x^+_{j,0;d}]\!]]\!],[\![x^+_{d,0;d},x^+_{k,0;d}]\!]]\!]\end{aligned}$$

Denote by $(U'_d)^{cl}$ the $\mathbb{C}$-subalgebra of U'_d generated by the elements σ_d, $K^{\pm\frac{1}{2}}_{l,d}$ $(l\in I)$ and $E_{r,d}$, $F_{r,d}$ $(r\in I\setminus\{0\})$. By Theorem 4.5(2), one conclude that $(U'_d)^{cl}$ admits the presentation with these generators and the relations formed by Eq.s (4.3)-(4.7) and the relations $X=0$ for all elements X in (4.13)-(4.18). Note that one also has the same fact with U'_d and I in place of $(U'_d)^{cl}$ and $I\setminus\{0\}$ respectively. Clearly, by Definition 6.1, one has a unique $\mathbb{C}$-algebra homomorphism $(\mathcal{F}'_d)^{cl}:(U'_d)^{cl}\to DU'_d$ such that $(\mathcal{F}'_d)^{cl}(\sigma_d)=\sigma_d$, $(\mathcal{F}'_d)^{cl}(K^{\pm\frac{1}{2}}_{l,d})=K^{\pm\frac{1}{2}}_{l,d}$ $(l\in I)$, $(\mathcal{F}'_d)^{cl}(E_{r,d})=x^+_{r,0;d}$, $(\mathcal{F}'_d)^{cl}(F_{r,d})=x^-_{r,0;d}$ $(r\in I\setminus\{0\})$. Then, by Eq.s (6.47), (6.39),(6.41),(6.42), using $D\Psi_d$, one has a unique $\mathbb{C}$-algebra homomorphism $\mathcal{F}'_d: U'_d\to DU'_d$ such that $\mathcal{F}'_d(Y)=(\mathcal{F}'_d)^{cl}(Y)$ $(Y\in(U'_d)^{cl})$ and $\mathcal{F}'_d(E_{0,d})=\mathbb{X}^+_d$, $\mathcal{F}'_d(F_{0,d})=\mathbb{X}^-_d$. By the equations in Definition 6.1, one concludes that as a $\mathbb{C}$-algebra, DU'_d is generated by the elements σ_d, $K^{\pm\frac{1}{2}}_{l,d}$ $(l\in\{0,1,2,3\})$, $x^\pm_{r,0;d}$ $(r\in\{1,2,3\})$ and $x^\pm_{k,\mp1;d}$. Hence Eq.s (6.44) and (6.19) imply that the homomorphism $\mathcal{F}'_d$ is surjective. By Lemma 6.3, one obtains the equation $\mathcal{F}_d\mathcal{F}'_d=\mathrm{id}_{DU'_d}$. Hence $\mathcal{F}'_d$ is injective. Thus one gets this theorem for $d\in\{1,2,3\}$.

The theorem for $d=4$ can be proved similarly, or more easily. $\square$

The following lemma implies that Eq.s (6.8)-(6.10) are equivalent to the ones of the original Drinfeld second realization.

Lemma 6.7. *Let $d\in\mathcal{D}\setminus\{0\}$. Let*

$$(6.48)\qquad \hat{\psi}^\pm_{i,\pm k;d}:=K^{\pm1}_{i,d}(\delta_{k0}+\Theta(k)(q-q^{-1})K^{\pm\frac{k}{2}}_{\hat{\delta}_{d};d}\psi_{i,\pm k;d}),\ (i\in I\setminus\{0\},\ k\in\mathbb{Z})$$

$$(6.49)\qquad \hat{h}_{i,r;d}:=K^{-\frac{r}{2}}_{\hat{\delta}_{d};d}h_{i,r;d}.\ (i\in I\setminus\{0\},\ r\in\mathbb{Z}\setminus\{0\})$$

Then one has

$$(6.50)\qquad K^{\pm1}_{i,d}\exp(\pm(q-q^{-1})\sum_{r=1}^{\infty}z^r\hat{h}_{i,\pm r;d})=1+(q-q^{-1})\sum_{k=-\infty}^{\infty}z^k\hat{\psi}_{i,\pm k;d},$$

(as equations of generating functions in z)

$$(6.51)\qquad [x^+_{i,k;d}, x^-_{j,l;d}] = \delta_{ij}\frac{K^{\frac{k-l}{2}}_{\hat{\delta};d}\hat{\psi}^+_{i,k+l;d} - K^{\frac{l-k}{2}}_{\hat{\delta}_d;d}\hat{\psi}^-_{i,k+l;d}}{q-q^{-1}}$$

$$(6.52)\qquad [\hat{h}_{i,k;d}, \hat{h}_{j,l;d}] = \delta_{k,-l}\frac{[k(\alpha_{i,d},\alpha_{j,d})]_q}{k}\frac{K^k_{\hat{\delta}_d;d} - K^{-k}_{\hat{\delta}_d;d}}{q-q^{-1}}$$

$$(6.53)\qquad [\hat{h}_{i,k;d}, x^\pm_{j,l;d}] = \pm\frac{[k(\alpha_{i,d},\alpha_{j,d})]_q}{k}K^{\mp\frac{|k|}{2}}_{\hat{\delta}_d;d}x^\pm_{j,k+l;d}$$

Proof. This follows from (6.8)-(6.10) by using the definition of the elements in (6.48)-(6.49). □

Now we give the Drinfeld second realization of U'_d.

Theorem 6.8. *Let $d \in \mathcal{D}\setminus\{0\} = \{1,2,3,4\}$.*

(1) *As a $\mathbb{Z}\Pi_d$-graded $\mathbb{C}$-algebra, DU'_d admits the presentation with the generators σ_d, $K^{\pm\frac{1}{2}}_{u,d}$, $x^\pm_{i,k;d} \in DU'_{d,\pm\alpha_{i,d}+k\hat{\delta}_d}$, and $\hat{h}_{i,r;d} \in DU'_{d,r\hat{\delta}_d}$, where $u \in I$, $i \in I\setminus\{0\}$, $k \in \mathbb{Z}$ and $r \in \mathbb{Z}\setminus\{0\}$, and the defining relations obtained from Eq.s* (6.4), (6.5), (6.6), (6.11), (6.12), (6.13), (6.51), (6.52), *and* (6.53) *by determining the elements $\hat{\psi}^\pm_{i,k;d}$ for all $i \in I\setminus\{0\}$ and all $k \in \mathbb{Z}$ by Eq.s* (6.50).

(2) *There exists a unique $\mathbb{Z}\Pi_d$-graded $\mathbb{C}$-algebra isomorphism $\widehat{\mathcal{F}}_d : DU'_d \to U'_d$ satisfying the equations obtained from Eq.s* (6.15), (6.16), (6.17) *by replacing $\mathcal{F}_d$ with $\widehat{\mathcal{F}}_d$, and the equations $\widehat{\mathcal{F}}_d(\hat{h}_{i,r;d}) = \epsilon^r_{i,i;d}K^{-\frac{r}{2}}_{\hat{\delta}_d;d}\bar{h}_{i,r;d}$ for all $i \in I\setminus\{0\}$ and all $r \in \mathbb{Z}\setminus\{0\}$. Further, $\widehat{\mathcal{F}}_d$ coincides with $\mathcal{F}_d$ as a map and one has $\widehat{\mathcal{F}}_d(\hat{\psi}^\pm_{i,\pm l;d}) = \epsilon^l_{i,i;d}(q-q^{-1})K^{\pm1}_{i,d}K^{\pm\frac{l}{2}}_{\hat{\delta}_d;d}\bar{\psi}_{i,\pm l;d}$ for all $i \in I\setminus\{0\}$ and all $l \in \mathbb{N}$.*

Proof. This theorem holds by Definition 6.1, Theorem 6.6, and Lemma 6.7. □

6.2 Extension of the $\mathbb{C}$-algebras

Recall the definitions of $\hat{\mathfrak{g}}$ and $D^{(1)}(2,1;x)$ from Section 2. Assume $\hat{\mathfrak{g}}$ to be $D^{(1)}(2,1;x)$. Strictly U'_d is the quantum superalgebra of $[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$. Here we treat the quantum superalgebra of $\hat{\mathfrak{g}}$.

Definition 6.9. (1) Let $d \in \mathcal{D}$. Define the additive group map $\chi_d : \mathbb{Z}\Pi_d \to \mathbb{Z}$ by $\chi_d(\alpha_{i,d}) = \delta_{i0}$ for $i \in I$. The group ring of $\mathbb{Z}$ is the commutative and cocommutative Hopf algebra $\mathbb{C}[K^{\frac{1}{2}}_{\Lambda_0;d}, K^{-\frac{1}{2}}_{\Lambda_0;d}]$, where

$$K^{\frac{1}{2}}_{\Lambda_0;d}K^{-\frac{1}{2}}_{\Lambda_0;d} = 1,\quad \Delta(K^{\frac{1}{2}}_{\Lambda_0;d}) = K^{\frac{1}{2}}_{\Lambda_0;d}\otimes K^{\frac{1}{2}}_{\Lambda_0;d},\quad \Delta(K^{-\frac{1}{2}}_{\Lambda_0;d}) = K^{-\frac{1}{2}}_{\Lambda_0;d}\otimes K^{-\frac{1}{2}}_{\Lambda_0;d}.$$

Then U'_d is a left $\mathbb{C}[K^{\frac{1}{2}}_{\Lambda_0;d}, K^{-\frac{1}{2}}_{\Lambda_0;d}]$-module algebra [Mo, Sect. 4.1] with left action $\cdot : \mathbb{C}[K^{\frac{1}{2}}_{\Lambda_0;d}, K^{-\frac{1}{2}}_{\Lambda_0;d}] \times U'_d \to U'_d$ defined by

$$K^{\frac{1}{2}}_{\Lambda_0;d} \cdot X_\mu = q^{\frac{\chi_d(\mu)}{2}} X_\mu, \qquad \mu \in \mathbb{Z}\Pi_d, X_\mu \in U'_{d,\mu}.$$

Let U_d be the smash product algebra [Mo, Def. 4.1.3] $U_d := U'_d \# \mathbb{C}[K^{\frac{1}{2}}_{\Lambda_0;d}, K^{-\frac{1}{2}}_{\Lambda_0;d}]$.

(2) Let $d \in \mathcal{D} \setminus \{0\}$. Similarly to the construction in Part (1) define the smash product algebra $DU_d := DU'_d \# \mathbb{C}[K^{\frac{1}{2}}_{\Lambda_0;d}, K^{-\frac{1}{2}}_{\Lambda_0;d}]$.

We extend Theorem 6.8 to that for U_d.

Theorem 6.10. *The map $\widehat{\mathcal{F}}_d$ can be extended to a $\mathbb{C}$-algebra isomorphism from DU_d to U_d by letting $\widehat{\mathcal{F}}_d(K^{\frac{m}{2}}_{\Lambda_0,d}) = K^{\frac{m}{2}}_{\Lambda_0,d}$ for all $m \in \mathbb{Z}$.*

Proof. This theorem follows from Theorem 6.8 and Definition 6.9. □

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