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On a geometric setting for DAHAs of type $A_1^{(1)}$

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Abstract

We explain how the construction of double affine Hecke algebras of Vasserot [Duke Math. 126 (2005)] carries over the case of unequal parameters by examing the simplest case. In a technical language, we give a realization of double affine Hecke algebras of type $(C_1^\vee, C_1)$ with $(4+1)$ parameters (while Vasserot's construction treats the same case with $(1+1)$ parameters).

Introduction

It is a commonly accepted "fact" that a K-theoretic realization of an algebra attached to a semi-simple algebraic group obtains one extra action of the character group of a torus in a nice way when we take account into a torus action.

When this is applied to the Steinberg variety, which originally gave a geometric realization of (finite) Weyl groups (cf. [Spr76]), then we obtain a geometric realization of an affine Hecke algerbra (cf. [Lus85]). When this is applied to a quiver variety of finite type, then we get a quantum loop algebra (cf. [Nak01]).

When we apply this idea to the affine version of Steinberg varieties, then we should obtain a "loop extension" of an affine Hecke algebra, which is now widely recognized as the double affine Hecke algebra in the sense of Cherednik (cf. [Che95]).

In fact, Vasserot [Vas05] pursues this analogy to obtain his geometric realization of double affine Hecke algebras in terms of certain subvarieties of the "cotangent bundle" of the affine flag varieties.

Here Vasserot obtains a geometric realization of a double affine Hecke algebra with $(1+1)$ parameters since he starts from a geometric realization of an one-parameter affine Hecke algebra due to Kazhdan-Lusztig and Ginzburg (cf. [CG97]). (Here the meaning of $(1+1)$ is

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that the first "1" stands for the parameter which appears in some Hecke (quadratic) relation and the second "1" stands for the parameter which corresponds to the extra parameter introduced in the affinization.)

In our previous paper [K06a], we found a geometric realization of an affine Hecke algebra with 3-parameters of type C_n by modifying the construction of Lusztig and Ginzburg. Therefore, it is natural to consider its counter-part in the setting of double affine Hecke algebra. In view of [Vas05], it is rather straightforward to achieve this and we obtain a geometric realization of the double affine Hecke algebra of type C_n with $(3+1)$ parameters.

However, as Noumi noticed (cf. Sahi [Sah99] and the references therein), a double affine Hecke algebra of type C_n acquires a distinguished two-parameter deformation $\mathcal{H}$ usually called the Cherednik-Noumi-Sahi algebra (the CNS algebra for short).

In this article, we explain how to give a geometric realization of the CNS algebra by adapting its rank one counterpart.

As in [Vas05], the main point is to justify the following story:

Let $\mathcal{G}$ be the Kac-Moody group associated to $G = Sp(2n, \mathbb{C})$ $(n \geq 2)$. Let $\mathcal{B}$ be the flag variety of $\mathcal{G}$. Consider a suitable vector bundle $\mathcal{F}$ over $\mathcal{B}$ with a map ν to a vector space $\mathbb{V}$. Then, we form a fiber product $\mathcal{Z} := \mathcal{F} \times_{\mathbb{V}} \mathcal{F}$. Consider a suitable $\mathcal{G} \times (\mathbb{C}^\times)^5$-action[1] on $\mathcal{F}$. Then, we want to prove

"Theorem" A. *We have an isomorphism*

$$K^{\mathcal{G}\times(\mathbb{C}^\times)^5}(\mathcal{Z}) \cong \mathcal{H},$$

where the LHS is the $\mathcal{G} \times (\mathbb{C}^\times)^5$-equivariant K-theory with complex coefficient, together with the ring structure given by the convolution operation.

For an algebraic group H, let $R(H)$ be the finite-dimensional representation ring of H with complex coefficient.

Proposition B. *We have*

$$R(\mathcal{G} \times (\mathbb{C}^\times)^5) \cong R((\mathbb{C}^\times)^5) \cong Z(\mathcal{H})/(\mathbf{q} = 1),$$

where $\mathbf{q}$ *is some distinguished element of* $\mathcal{H}$.

Notice that the first isomorphism claims $R(\mathcal{G}) \cong \mathbb{C}[\mathsf{triv}]$, where triv is the trivial representation of $\mathcal{G}$. In other words, we have no finite dimensional representation of $\mathcal{G}$ except for the trivial one.

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[1] In the main part of this article, we restrict ourselves to the case $n = 1$. In this case, the possible number of $\mathbb{C}^\times$ becomes 4 since the number of direct summands of $\mathcal{F}$ is strictly smaller (2 summands) than that of $n \geq 2$ case (3 summands). It is expected from the aspect of so-called "elliptic" Hecke algebras studied by Saito-Shiota (cf. [Sho06]).

For each semisimple element $a \in \mathcal{G} \times (\mathbb{C}^\times)^5$, we have a map

$$R(\mathcal{G} \times (\mathbb{C}^\times)^5) \ni [V] \mapsto \mathrm{tr}(a, V) \in \mathbb{C}$$

which we denote by $\mathbb{C}_a$.

"Theorem" C. *Let $a \in \mathcal{G} \times (\mathbb{C}^\times)^5$ be a semi-simple element. Let $\mathcal{Z}^a$ be the set of a-fixed points of $\mathcal{Z}$. Then, we have an isomorphism*[2]

$$K(\mathcal{Z}^a) \cong \mathbb{C}_a \otimes_{Z(\mathcal{H})} \mathcal{H},$$

where the LHS is the K-theory with complex coefficient, equipped with the ring structure given by the convolution operation.

"Theorem" C should be derived from "Theorem" A by means of a localization argument. However, we do not know how to do this in the setting of Kac-Moody groups. As a consequence, the rigorous relationship between "Theorems" A and C is rather unclear to us.

Moreover, as we saw in Proposition B, the semi-simple element of $\mathcal{G}$ in a looks nothing to do since every element of $\mathcal{G}$ acts by 1 for every finite-dimensional representation of $\mathcal{G}$. It means that the LHS can vary significantly with the RHS fixed in the description of "Theorem" C. This is clearly strange.....

The main difficulties preventing us from the proof of "Theorem" A is the following:

1. The variety $\mathcal{F}$ is smooth but infinite-dimensional. It implies that we do not know a good definition of K-groups;
2. The map ν is not proper. Hence, the convolution operation (ring structure of $K^{\mathcal{G} \times (\mathbb{C}^\times)^5}(\mathcal{Z})$) is not necessarily well-defined;
3. The group $\mathcal{G} \times (\mathbb{C}^\times)^5$ cannot act on $\mathcal{F}$ as we want to be.

The difficulties 1. and 2. are in common with [Vas05]. The only new difficulty arising from our setting is 3. The author has no tool to overcome these difficulties in a straightforward way.

Instead, we regard "Theorem" C as a re-incanation of "Theorem" A. However, it also needs a suitable correction in order to deduce non-trivial conclusions.

For this, we introduce an embedding of $\mathbb{C}_a \otimes_{Z(\mathcal{H})} \mathcal{H}$ into a topological algebra $\mathrm{End}\mathfrak{A}_a$, whose structure and the embedding essentially depends on a. Then, we take the closure of $\mathbb{C}_a \otimes_{Z(\mathcal{H})} \mathcal{H}$ in $\mathrm{End}\mathfrak{A}_a$. This makes subtle differences between the completed algebras, which depend on the choice of semi-simple elements of $\mathcal{G}$. This strategy itself is similar to [Vas05] and our new finding is that we can introduce extra torus action *after* "localization" (cf. Lemma 3.4).

Therefore, our main result (Theorem 4.2) in this article is an improvement of "Theorem" C, which depends on the data of semi-simple elements of $\mathcal{G}$.

[2]This isomorphism is consistent with the description of the center of $\mathcal{H}$. See eg. Theorem 1.8.

1 Preliminaries

Throughtout this article, we fix the base field to be $\mathbb{C}$. A variety is a possibly infinitely many disjoint union of noetherian normal schemes of finite type over $\mathbb{C}$. For a variety $\mathcal{X}$, we denote by $K(\mathcal{X})$ the Grothendieck group of the category of coherent sheaves on $\mathcal{X}$ with coefficient $\mathbb{C}$. It coincides with the Grothendicek group of the locally free sheaves on $\mathcal{X}$ (with coefficient $\mathbb{C}$) if $\mathcal{X}$ is smooth.

Let $G := SL(2, \mathbb{C})$. We define two subgroups B and T of G as:

$$T := \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbb{C}^{\times}\} \subset B := \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbb{C}^{\times}, b \in \mathbb{C}\}.$$

Let $X^*(T)$ be the character group of T. Let $\epsilon \in X^*(T)$ be the positive fundamental weight with respect to the choice of B.

1.1 DAHAs of type $A_1^{(1)}$

In the following, a variable written by a boldface (eg. $\mathbf{t}, \mathbf{q}, \ldots$) indicates that we treat it as an indeterminant.

Let $\mathcal{A}_0 := \mathbb{C}[\mathbf{t}_0^{\pm 1}, \mathbf{t}_1^{\pm 1}]$, $\mathcal{A}_1 := \mathcal{A}_0[\mathbf{q}^{\pm 1}]$, and $\mathcal{A} := \mathcal{A}_1[(\mathbf{t}_0^*)^{\pm 1}, (\mathbf{t}_1^*)^{\pm 1}]$.

Definition 1.1. An affine Hecke algebra of type $A_1^{(1)}$ is an $\mathcal{A}_0$-algebra $\mathbb{H}$ generated by T_0 and T_1 subject to:

$$(T_0 + 1)(T_0 - \mathbf{t}_0^2) = 0 \text{ and } (T_1 + 1)(T_1 - \mathbf{t}_1^2) = 0. \tag{1.1}$$

Lemma 1.2. *The group W generated by the images of T_0, T_1 in the residual algebra $\mathbb{H}/(\mathbf{t}_0 = 1 = \mathbf{t}_1)$ is isomorphic to the affine Coxeter group of type $A_1^{(1)}$ with its Coxeter generators as the images s_0, s_1 of T_0, T_1.*

Theorem 1.3 (Bernstein-Lusztig). *We put $Y := T_1 T_0$. Then, we have*

$$\mathbb{H} = (\mathcal{A}_0 \oplus \mathcal{A}_0 T_1) \otimes_{\mathbb{C}} \mathbb{C}[Y^{\pm 1}]$$

as vector spaces.

For a polynomial ring $\mathcal{A}_1[X^{\pm 1}]$, we define an $\mathcal{A}_1$-linear action[3] of W on it as

$$s_0(X) = \mathbf{q}^2 X^{-1}, s_1(X) := X^{-1}. \tag{1.2}$$

[3]This action is supposed to be compatible with the multiplications.

Theorem 1.4 (Noumi). *There exists a two-parameter family $\pi = \pi_{\mathbf{t}_0^*,\mathbf{t}_1^*}$ of faithful representations $\mathbb{H}$ on the Laurant polynomial ring $\mathcal{A}[X^{\pm 1}]$ such that:*

$$\pi(T_0) := \mathbf{t}_0^2 + \frac{(1-\mathbf{c}X^{-1})(1-\mathbf{d}X^{-1})}{1-\mathbf{q}^2X^{-2}}(s_0 - 1)$$
$$\pi(T_1) := \mathbf{t}_1^2 + \frac{(1-\mathbf{a}X)(1-\mathbf{b}X)}{1-X^2}(s_1 - 1),$$

where $\mathbf{a} := \mathbf{t}_1\mathbf{t}_1^*$, $\mathbf{b} := -\mathbf{t}_1(\mathbf{t}_1^*)^{-1}$, $\mathbf{c} := \mathbf{q}\mathbf{t}_0\mathbf{t}_0^*$, *and* $\mathbf{d} := -\mathbf{q}\mathbf{t}_0(\mathbf{t}_0^*)^{-1}$.

Definition 1.5. The CNS algebra of rank one is defined as the subalgebra $\mathcal{H} \subset \mathrm{End}_{\mathcal{A}}\mathcal{A}[X^{\pm 1}]$ generated by $\pi(\mathbb{H})$ and $\mathcal{A}[X^{\pm 1}]$.

Remark 1.6. *The algebra $\mathcal{H}$ can also be presented by its generators and relations as in [Sah99].*

Theorem 1.7 (Poincaré-Birkhoff-Witt type theorem). *We have an isomorphism*

$$\mathcal{H} \cong \bigoplus_{l,m\in\mathbb{Z}} \mathcal{A}X^lY^m \oplus \bigoplus_{l,m\in\mathbb{Z}} \mathcal{A}X^lT_1Y^m$$

of free $\mathcal{A}$-modules.

1.2 Highest weight modules of $\mathcal{H}$

Let M be an irreducible $\mathcal{H}$-module of at most countable dimension. By Diximir's version of Schur's lemma, each element of the center of $\mathcal{H}$ acts on M by some scalar.

Theorem 1.8 (Bernstein-Lusztig). *1. The center of $\mathbb{H}$ is $\mathcal{A}_0[Y+Y^{-1}]$;*

2. The center of $\mathcal{H}$ is $\mathcal{A}$.

Remark 1.9. *It is known that the center of $\mathcal{H}$ becomes pretty large if one specialize* $\mathbf{q}$ *to a root of unity (cf. Oblomkov [Obl04] or Shinkado [Shn07]).*

Corollary 1.10. *The character of the center of $\mathbb{H}$ can be identified with a conjugacy classes of semi-simple elements of $G \times (\mathbb{C}^\times)^2$ via the evaluation map*

$$(s, t_0, t_1) : \mathcal{A}_0[Y+Y^{-1}] \ni f(Y, \mathbf{t}_0, \mathbf{t}_1) \mapsto f(\eta, t_0, t_1) \in \mathbb{C},$$

where $\begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{pmatrix}$ is an element[4] of T in the G-conjugacy class of s.

[4] Every possible η gives the same value.

Definition 1.11 (Highest weight modules). A finitely generated $\mathcal{H}$-module M is said to be a highest weight module iff its restriction to $\mathbb{H}$ is a direct sum of finite dimensional $\mathbb{H}$-modules.

Let q be a non-zero complex number. We put $\widetilde{q} := \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix} \in SL(2, \mathbb{C})$.

Proposition 1.12. *Let M be an irreducible $\mathcal{H}$-module with $\mathbf{q}$ acting by q. Consider a direct sum decomposition $\mathrm{Res}^{\mathcal{H}}_{\mathbb{H}} M = \bigoplus_i M_i$ of M into finite-dimensional $\mathbb{H}$-modules. If $(s, t_0, t_1) \in T \times (\mathbb{C}^\times)^2$ is the (generalized) central character of $M_0 \neq 0$, then every cenral character of M_i is contained in the set $\{(\widetilde{q}^m s, t_0, t_1); m \in \mathbb{Z}\}$.*

Consider an equivalence relation $\sim_q$ on $T \times (\mathbb{C}^\times)^2 = (\mathbb{C}^\times)^3$ generated by:

$$(s, t_0, t_1) \sim_q (s^{-1}, t_0, t_1), \text{ and } (s, t_0, t_1) \sim_q (\widetilde{q}s, t_0, t_1).$$

For each $\lambda \in (\mathbb{C}^\times)^3 / \sim_q$, we denote by $\mathcal{O}^q_\lambda$ the category of finitely generated $\mathcal{H}$-module such that all of its (generalized) central characters as $\mathbb{H}$-modules belongs to λ.

Corollary 1.13. *The category of finitely generated $\mathcal{H}$-modules with $\mathbf{q}$ acting by q decomposes into the direct sum of $\mathcal{O}^q_\lambda$'s.*

1.3 An affine flag variety of $SL(2, \mathbb{C})$

Consider the groups

$$LG := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C}((z)), ad - bc = 1\} \supset LG_{\geq 0} := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}; a, b, c, d \in \mathbb{C}[[z]]\}$$

obtained as some rational points of G, which we regard as algebraic groups over $\mathbb{C}$. We have a canonical surjection

$$LG_{\geq 0} \twoheadrightarrow G$$

obtained by the map given by taking the residue modulo $z\mathbb{C}[[z]]$ of their coordinates. We define I to be the pullback of B via this map. We have an extension

$$1 \longrightarrow LG \longrightarrow \mathcal{G} \longrightarrow \mathbb{C}^\times \longrightarrow 1 \quad (exact),$$

whose multiplication rule is given by

$$\left(LG \times \mathbb{C}^\times\right)^2 \ni ((h_1(z), c_1), (h_2(z), c_2)) \mapsto (h_1(z) h_2(c_1 z), c_1 c_2) \in LG \times \mathbb{C}^\times.$$

By restricting this extension to $I \subset LG$, we obtain a subgroup $\mathcal{I} \subset \mathcal{G}$.

Let V_1 be the vector representation of G with its T-eigenvectors v_+ (of weight ϵ) and v_- (of weight $-\epsilon$). We equip $V_1 \otimes \mathbb{C}((z))$ with an action of $\mathcal{G}$ via an extension of a natural inclusion

$$LG \hookrightarrow GL(V_1 \otimes \mathbb{C}((z))) \subset \mathrm{End}_{\mathbb{C}}(V_1 \otimes \mathbb{C}((z)))$$

to $\mathcal{G}$ by letting $\mathbb{C}^\times$ act by degree m-character on z^m. Since $\wedge^2 V_1$ is isomorphic to a trivial representation of G, we have a natural identification

$$\wedge^2 (V_1 \otimes \mathbb{C}((z))) \cong \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}((z))$$

of $\mathcal{G}$-modules, where $\wedge^2$ is taken as $\mathbb{C}((z))$-vector spaces. Consider the set

$$\mathcal{B} := \{E_0 \subset E_1 \subset z^{-1}E_0 \subset V_1 \otimes \mathbb{C}((z)); \dim_{\mathbb{C}} E_1/E_0 = 1, \wedge^2_{[[t]]} E_0 = \mathbb{C}[[z]] \subset \mathbb{C}((z))\},$$

where $\wedge^2_{[[t]]}$ denotes the wedge product as $\mathbb{C}[[t]]$-modules. For the exhaustive sequence $\{F_i\}_i$ of $\mathbb{C}$-vector subspaces

$$F_0 := V_1 \otimes \mathbb{C}[[z]] \subset F_1 := V_1 \otimes z^{-1}\mathbb{C}[[z]] \subset \cdots \subset V_1 \otimes \mathbb{C}((z)),$$

we define $\mathcal{B}_i := \{(E_0, E_1) \in \mathcal{B}; E_0 \subset F_i\}$ as a subset of $\mathcal{B}_i$. It is straightforward to see that each $\mathcal{B}_i$ admits a scheme structure by regarding it as a subscheme of a suitable Grassmanian. Clearly, each embedding $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ defines an embedding of schemes. It follows that $\varinjlim_i \mathcal{B}_i$ equip $\mathcal{B}$ with a structure of ind-schemes on $\mathcal{B}$, which we denote by the same letter. We refer this as an affine flag variety of $\widehat{\mathfrak{sl}}_2$. We have a natural $\mathcal{G}$-action on $\mathcal{B}$ extending that of LG at the level of points. This makes $\mathcal{B}$ into a homogeneous space of $\mathcal{G}$ with its stabilizer $\mathcal{I}$. We have two-dimensional torus $\mathbb{T} \subset \mathcal{G}$ such that $\mathbb{T} \cap G = T$.

Lemma 1.14. *For each pair of integers $l, m \in \mathbb{Z}$, we define a $\mathbb{C}[[z]]$-lattice*

$$E(l, m) := z^l\mathbb{C}[[z]]v_+ \oplus z^m\mathbb{C}[[z]]v_- \subset V_1 \otimes \mathbb{C}((z)).$$

Then, we have

$$\mathcal{B}^{\mathbb{T}} = \{E_0 = E(m, -m) \subset E_1 = E(m-1, -m) \text{ or } E(m, -m-1); m \in \mathbb{Z}\}$$

as sets.

Corollary 1.15. *Define a projective action of W on $V_1 \otimes \mathbb{C}((z))$ as*

$$s_0(z^m v_\sigma) = \begin{cases} z^{m-1}v_- & (\sigma = +) \\ -z^{m+1}v_+ & (\sigma = -) \end{cases}, s_1(z^m v_\sigma) = \begin{cases} z^m v_- & (\sigma = +) \\ -z^m v_+ & (\sigma = -) \end{cases},$$

for each $m \in \mathbb{Z}$. We have

$$\mathcal{B}^{\mathbb{T}} = W \left(\mathbb{C}[[z]]v_+ \oplus \mathbb{C}[[z]]v_- \subset z^{-1}\mathbb{C}[[z]]v_+ \oplus \mathbb{C}[[z]]v_-\right).$$

In particular, the group $W \cong N_{\mathcal{G}}(\mathbb{T})/\mathbb{T}$ acts transitively on the set $\mathcal{B}^{\mathbb{T}}$.

1.4 Exotic affine Springer map

Consider the linear subspace

$$\mathcal{V} := (\mathbb{C}^2) \boxtimes (\mathbb{C}v_+ \oplus V_1 \otimes z\mathbb{C}[[z]]) \subset (\mathbb{C}^2) \boxtimes (V_1 \otimes \mathbb{C}((z))) =: \mathbb{V},$$

where $GL(2,\mathbb{C})$ acts on the first component as a vector representation and $\mathcal{G}$ acts on the second component of $\mathbb{V}$ (of the external tensor products). The inclusion $\mathcal{V} \subset \mathbb{V}$ is clearly $\mathcal{I}$-stable. Moreover, the $GL(2,\mathbb{C})$-action and the $\mathcal{I}$-action (or the $\mathcal{G}$-action) on $\mathcal{V}$ (or $\mathbb{V}$) commutes with each other. As a consequence, we have a vector bundle[5] $\mathcal{F}$ over $\mathcal{B}$ whose fiber over $\mathcal{I}$ is $\mathcal{V}$. Using this, we form a $\mathcal{G} \times GL(2,\mathbb{C})$-equivariant composition map of sets[6]

$$\begin{tikzcd} \mathcal{F} \cong \mathcal{G} \times^{\mathcal{I}} \mathcal{V} \arrow[r, hook] \arrow[drr, "\nu"'] & \mathcal{G} \times^{\mathcal{I}} \mathbb{V} \arrow[r] & \mathcal{B} \times \mathbb{V} . \arrow[d] \\ & & \mathbb{V} \end{tikzcd}$$

1.5 Assumption on parameter

In the following, we fix an element

$$a = (s, \tau, q_0, q_1, r_0, r_1) \in (\mathbb{C}^\times)^5$$

satisfying:

($\star$) τ, r_0, r_1 are not roots of unity.

We also regard a as an element $\mathcal{G} \times GL(2,\mathbb{C}) \times (\mathbb{C}^\times)^2$ as:

$$s \in T \subset G, (s,\tau) \in \mathbb{T} \subset \mathcal{G}, \begin{pmatrix} q_0 & 0 \\ 0 & q_1 \end{pmatrix} \in GL(2,\mathbb{C}), (r_0, r_1) \in (\mathbb{C}^\times)^2.$$

2 Some completions of DAHA of type $A_1^{(1)}$

We work in the same setting as in the previous section. By definition, $\mathcal{H}$ acts on $\mathcal{A}[X]$ faithfully. Let Λ be a countable subset of maximal ideals of $\mathcal{A}[X^{\pm 1}]$. For each $N > 0$, we have a map

$$\mathsf{ex}_\Lambda^N : \mathcal{A}[X^{\pm 1}] \to \mathcal{A}[X^{\pm 1}]_\Lambda^N := \prod_{\mathfrak{m} \in \Lambda} \mathcal{A}[X^{\pm 1}]/\mathfrak{m}^N$$

[5]Notice that $\mathcal{F}$ is *not* a vector bundle in some sense since it has uncountable rank (and hence not even quasi-coherent). Here we naively claim so because we do not need to justify this in the later part of this paper.

[6]The map ν makes sense as a map of pro-ind-varieties since $\mathbb{C}[[z]]$ is written as the limit of an inverse system of finite-dimensional vector spaces. The fiber of a point on $\mathbb{V}$ along ν makes sense as an ind-variety. However, we avoid to count this type of difficulty in this article.

given by the degree[7] ($< N$)-terms of the expansion of a polynomial in $\mathcal{A}[X^{\pm 1}]$ along the all points of Λ.

Proposition 2.1. *The subset $\mathcal{H} \subset \mathrm{End}_{\mathbb{C}}\mathcal{A}[X^{\pm 1}]$ defines a subset of $\mathrm{End}_{\mathbb{C}}\mathcal{A}[X^{\pm 1}]^N_\Lambda$ if and only if Λ is stable under the W-action on $\mathcal{A}[X^{\pm 1}]$ given as the scalar extention of* (1.2).

Proof. A direct consequence of Corollary 1.13 and Remark 2.2. □

Remark 2.2. *An automorphism θ of $\mathcal{A}[X^{\pm 1}]$ induces an automorphism of $\mathcal{A}[X^{\pm 1}]^N_\Lambda$ if and only if $\theta(\Lambda) = \Lambda$.*

Assume that W acts transitively on Λ. Let $\mathbf{0} \in \Lambda$ be a point. Then, we define a sequence of sets $\{\Lambda_m\}_m$ inductively as:

$$\Lambda_0 := \{\mathbf{0}\}, \text{ and } \Lambda_m := s_0\Lambda_{m-1} \cup s_1\Lambda_{m-1} \subset \Lambda.$$

Then, we form an inverse system

$$\mathcal{A}[X^{\pm 1}]^N_{\Lambda_l} \twoheadrightarrow \mathcal{A}[X^{\pm 1}]^N_{\Lambda_m}$$

if $l \geq m$. Let $\mathcal{A}((X))^N_\Lambda$ denote the limit of this inverse system. We introduce a topology[8] on $\mathcal{A}((X))^N_\Lambda$ by setting its open sets as

$$\ker[\mathcal{A}((X))^N_\Lambda \to \mathcal{A}[X^{\pm 1}]^N_{\Lambda_m}] \text{ for all } m.$$

Lemma 2.3. *Keep the settings as above. For each $N > 0$, the natural map $\mathcal{A}[X^{\pm 1}] \longrightarrow \mathcal{A}((X))^N_\Lambda$ is a dense open embedding.*

Remark 2.4. *Notice that if τ is a root of unity, then every W-orbit is finite. In particular, the above completion procedure gives a finite direct sum of finite dimensional vector spaces and we have no chance to obtain an embedding as in Lemma 2.3.*

Proposition 2.5. *The elements $T_0, T_1 \in \mathcal{H}$ acts on $\mathcal{A}((X))^N_\Lambda$ continuously.*

The summary of the construction of this section is:

Theorem 2.6. *Fix an element $a \in (\mathbb{C}^\times)^5$. Let $\Lambda := Wa$. For each $N > 0$, we have an embedding*

$$\mathcal{H}/(\mathbf{q} = \tau, \mathbf{t}_0 = t_0, \ldots) \hookrightarrow \mathrm{End}\mathcal{A}((X))^N_\Lambda,$$

where $q, t_0, t_1, t_0^*, t_1^* \in \mathbb{C}^\times$ are some numbers satisfying

$$q^2 = \tau, (t_1)^2 = -q_0q_1, (t_1^*)^2 = -q_0/q_1, (t_0)^2 := -q_0q_1r_0r_1\tau, (t_0^*)^2 = -\frac{q_0r_0}{q_1r_1}.$$

Moreover, the closure of this embedding contains the ring multiplication of $\mathcal{A}((X))^N_\Lambda$ on itself.

Proof. Note that the algebra structure on the RHS does not depend on the value of $t_0, t_1, \ldots$. Hence, the result is immediate from the discussion of this section and Definition 1.5. □

Remark 2.7. *The algebra $\mathcal{A}((X))^N_\Lambda$ in Theorem 2.6 is the algebra $\mathfrak{A}_a$ mentioned in the introduction.*

[7]This degree is counted via the local uniformizers along $\mathfrak{m}$.

[8]This is a natural topology coming from the inverse system.

3 Geometry of basic representations

We work under the same setting as in the previous section.

Theorem 3.1. *We have an isomorphism*

$$X^*(\mathbb{T}) \ni \gamma \mapsto [\mathcal{G} \times^{\mathcal{I}} \gamma^{-1}] \in \mathrm{Pic}\mathcal{B}.$$

Let a be an element of $\mathbb{T} \times GL(2,\mathbb{C}) \times (\mathbb{C}^\times)^2$. We have $Z_{\mathcal{G}}(s,\tau), G \times \mathbb{C}^\times \subset \mathcal{G}$, where the second embedding arises from the natural embedding of G into the constant coefficient part of LG.

Lemma 3.2. *We have $Z_{\mathcal{G}}(s,\tau) \subset G \times \mathbb{C}^\times$ as algebraic subgroups of LG.*

Definition 3.3. Let us introduce a $(\mathbb{C}^\times)^2$-action on $V_1 \otimes \mathbb{C}((z))$ by letting the first $\mathbb{C}^\times$ act as the scalar multiplication on the whole vector space and letting the second $\mathbb{C}^\times$ act trivially on V_1 but dilates z^m as $\mathbb{C}^\times \times \mathbb{C}((z)) \ni (c, z^m) \mapsto c^m z^m \in \mathbb{C}((z))$. We denote this action by $\kappa_0 : (\mathbb{C}^\times)^2 \circlearrowright V_1 \otimes \mathbb{C}((z))$.

By duplicating the κ_0-action using the isomorphism $\mathbb{V} \cong V_1 \otimes \mathbb{C}((z)) \oplus V_1 \otimes \mathbb{C}((z))$, we define $(\mathbb{C}^\times)^4$-action on $\mathbb{V}$ as

$$(q_i, r_i) \in (\mathbb{C}^\times)^2 \text{ acts on the } i\text{-th component of } \mathbb{V} \text{ by } \kappa_0 \text{ for } i = 1, 2.$$

We refer this $(\mathbb{C}^\times)^4$-action as the κ-action. Here we naturally regard $(\mathbb{C}^\times)^4$ as a subgroup of $\mathbb{T} \times GL(2,\mathbb{C}) \times (\mathbb{C}^\times)^2$ as in §1.5. By letting it act on the first factor trivially, we define a $(\mathbb{C}^\times)^4$-action on $\mathcal{B} \times \mathbb{V}$ or $\mathcal{F}$.

Lemma 3.4. *The κ-action on $\mathcal{B} \times \mathbb{V}$ preserves $\mathcal{F}^a$ and commutes with the $Z_{\mathcal{G}}(s,\tau)$-action.*

Remark 3.5. *At a first glance, the assumption of Lemma 3.4 may look restrictive. However, all choices of (s,τ) in a single W-orbit give equivalent completions of $\mathcal{H}$. In other words, we can freely assume the setting of Lemmas 3.2 and 3.4.*

We have

$$\mathcal{F} = \{(g\mathcal{I}, X) \in \mathcal{G}/\mathcal{I} \times \mathbb{V}; X \in g\mathcal{V}\} \text{ as sets.}$$

By $(\star)$, it is straightforward to see that the set of a-fixed points $\mathbb{V}^a$ of $\mathbb{V}$ is a finite-dimensional vector space. Together with Lemma 3.2, the set of a-fixed points $\mathcal{F}^a$ of $\mathcal{F}$ is a disjoint union of infinitely many finite rank vector bundles on flag varieties[9] of $Z_{\mathcal{G}}(s,\tau)$.

Therefore, we regard $\mathcal{F}^a$ as an infinite disjoint union of flag varieties in the below.

Lemma 3.6. *We have a natural isomorphism $W \cong N_{\mathcal{G}}(\mathbb{T})/\mathbb{T}$.*

[9] Thanks to Lemma 3.4, we know that it is either $\mathbb{P}^1$ or one point.

For each $w \in W$, we denote its lift in $\mathcal{G}$ by $\dot{w} \in N_{\mathcal{G}}(\mathbb{T})$. We put

$$\mathcal{F}_w^a := \{(g\dot{w}\mathcal{I}, X) \in \mathcal{F}; g \in Z_{\mathcal{G}}(s,\tau), X \in \mathbb{V}^a\}.$$

We define the restriction map[10]

$$\mathsf{res}_w^a : \mathrm{Pic}^{\mathcal{G}}\mathcal{B} \ni [\mathcal{G} \times^{\mathcal{I}} \gamma^{-1}] \mapsto [Z_{\mathcal{G}}(s,\tau) \times^{Z_{\mathcal{G}}(s,\tau)\cap \mathrm{Ad}(\dot{w})\mathcal{I}} w\gamma^{-1}] \in K^{\mathbb{T}}(\mathcal{F}_w^a)$$

and the specialization map

$$\mathsf{sp}_w^a : K^{\mathbb{T}}(\mathcal{F}_w^a) \longrightarrow R(\mathbb{T})/\mathfrak{m}(w(s,\tau)) \otimes K(\mathcal{F}_w^a) \cong K(\mathcal{F}_w^a),$$

where $\mathfrak{m}(w(s,\tau))$ is the maximal ideal of $R(\mathbb{T})$ corresponding to the point $\mathrm{Ad}(\dot{w})(s,\tau) \in \mathbb{T}$.

By construction, we have a map

$$\mu_w^a : \mathcal{F}_w^a \ni (g\dot{w}\mathcal{I}, X) \mapsto X \in \mathbb{V}^a.$$

Each μ_w^a is projective. By sending all of $\mathcal{F}_w^a$'s to the same target $\mathbb{V}^a$, we obtain a map[11]

$$\mu^a : \mathcal{F}^a \ni (g\mathcal{I}, X) \mapsto X \in \mathbb{V}^a.$$

We form a variety $\mathcal{Z}^a := \mathcal{F}^a \times_{\mathbb{V}^a} \mathcal{F}^a$.

Lemma 3.7. *We have* $\mathcal{F}^a = \bigcup_{w\in W} \mathcal{F}_w^a$.

Proof. Since $(s,\tau) \in \mathbb{T}$, the variety $\mathcal{F}^a$ is stable under the $\mathbb{T}$-action. By construction, every $\mathbb{T}$-fixed point of $\mathcal{F}$ is concentrated in $\mathcal{B}$, regarded as the zero section of $\mathcal{F}$. Here each connected component of $\mathcal{B}^a$ is projective. Thanks to Corollary 1.15, the RHS meets every connected component of $\mathcal{F}^a$. It is easy to see that the connected component of $\mathcal{F}^a$ containing $\dot{w}\mathcal{I} \in \mathcal{B}$ must be the form $\mathcal{F}_w^a$. □

Let $W_m \subset W$ be the set consisting of elements of W written by at most m-compositions of s_0 and s_1. We put

$$\mathcal{F}_m^a := \bigcup_{w\in W_m} \mathcal{F}_w^a.$$

We have a pullback map

$$\sum_{w\in W_l} K(\mathcal{F}_w^a) = K(\mathcal{F}_l^a) \longrightarrow K(\mathcal{F}_m^a) = \sum_{w\in W_m} K(\mathcal{F}_w^a)$$

if $l \geq m$ holds. This gives an inverse system and we put

$$\widehat{K(\mathcal{F}^a)} := \varprojlim_m \bigoplus_{w\in W_m} K(\mathcal{F}_w^a).$$

This vector space also admits a linear topology coming from the inverse system.

[10]This construction of restriction map is wrong from the point of view of localization theorem. Since we absorb the difference by invertible factors, we do not take care of them.

[11]This map fails to be projective since the fiber is a disjoint union of infinitely many compact varieties.

Lemma 3.8. *We have a natural embedding*

$$\prod_w \mathsf{sp}^a_w \circ \mathsf{res}^a_w : X^*(\mathbb{T}) \hookrightarrow \widehat{K(\mathcal{F}^a)}.$$

Proof. This follows from the fact that different characters are sent to different values by generic two sp^a_w's since $(\star)$ holds. □

Lemma 3.9. *We have an isomorphism as topological vector spaces:*

$$\widehat{K(\mathcal{F}^a)} \longrightarrow \mathcal{A}((X^{\pm 1}))^N_\Lambda/(\mathbf{q} = q, \mathbf{t}_0 = t_0, \mathbf{t}_1 = t_1, \ldots),$$

where $q, t_0, t_2, \ldots$ are as in Theorem 2.6, $\Lambda = Wa$, and $N = 1$ ($Z_G(s,\tau)$ is torus) or 2 (if $Z_G(s,\tau)$ contains G).

Proof. First of all, the topological structure of the RHS does not depend on $t_0, t_1 \ldots$. Hence, we can neglect it at this stage. This lemma follows from the following two observations. **1)** The set of connected components of $\mathcal{F}^a$ is in one-to-one correspondence with the set Wa. **2)** For each $w \in W$, we have an isomorphism

$$K(\mathcal{F}^a_w) \cong \mathcal{A}((X^{\pm 1}))^N_{\{wa\}}$$

as algebras. □

Remark 3.10. *For G general semi-simple algebraic group, the algebra $\mathcal{A}((X^{\pm 1}))^N_{\Lambda \le 0}$ should be replaced by the total cohomology ring of the flag variety of the commutator subgroup of some semi-simple element of G.*

4 Main theorem

We work under the same setting as in the previous section. Let $l, m \geq 0$ be integers. We consider a subvariety

$$\mathcal{Z}^a_{l,m} := \mathcal{F}^a_l \times_{\mathbb{V}^a} \mathcal{F}^a_m \subset \mathcal{Z}^a.$$

We define the completion of $K(\mathcal{Z}^a)$ as:

$$\widehat{K(\mathcal{Z}^a)} = \varprojlim_l \varinjlim_m K(\mathcal{Z}^a_{l,m}).$$

We have projections $p_1 : \mathcal{Z}^a_{l,m} \longrightarrow \mathcal{F}^a_l$ and $p_2 : \mathcal{Z}^a_{l,m} \longrightarrow \mathcal{F}^a_m$. Hence, we have a map

$$\star_{l,m} : K(\mathcal{Z}^a_{l,m}) \times K(\mathcal{F}^a_m) \ni ([\mathcal{E}], [\mathcal{F}]) \mapsto \sum (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{E} \otimes^{\mathbb{L}} p_2^* \mathcal{F})] \in K(\mathcal{F}^a_l).$$

Lemma 4.1. *As the limit of $\star_{l,m}$, we have an action*

$$\star : \widehat{K(\mathcal{Z}^a)} \otimes \widehat{K(\mathcal{F}^a)} \mapsto \widehat{K(\mathcal{F}^a)}.$$

Moreover, $\star$ makes $\widehat{K(\mathcal{Z}^a)}$ a unital[12] *subalgebra of* $\mathrm{End}\widehat{K(\mathcal{F}^a)}$.

Theorem 4.2 (Main theorem). *We have a dense open embedding*

$$\mathcal{H}/(\mathbf{q}-q, \mathbf{t}_0 - t_0, \ldots) \hookrightarrow \widehat{K(\mathcal{Z}^a)}$$

compatible with the isomorphism of Lemma 3.9. Here the correspondence between parameters $(q, t_0, t_0^*, t_1, t_1^*)$ and $a = (s, \tau, q_0, q_1, r_0, r_1)$ is given as:

$$q^2 = \tau, (t_1)^2 = -q_0q_1, (t_1^*)^2 = -q_0/q_1, (t_0)^2 := -q_0q_1r_0r_1\tau, (t_0^*)^2 = -\frac{q_0r_0}{q_1r_1}.$$

The rest of this section is devoted to give a sketch of the proof of Theorem 4.2. Consider $\mathcal{G}$-orbits

$$\mathbb{O}_i := \mathcal{G}(\mathcal{I} \times \dot{s}_i\mathcal{I}) \in \mathcal{B} \times \mathcal{B}$$

for $i = 0, 1$. Then, we define locally closed subsets $\mathbf{O}_i \subset \mathcal{Z}^a$ to be the closures of the pullbacks of $\mathbb{O}_i$ via the composition map

$$\mathcal{Z}^a \longrightarrow \mathcal{B}^a \times \mathcal{B}^a \hookrightarrow \mathcal{B} \times \mathcal{B}.$$

By means of the push-forward by the diagonal embedding $\mathcal{F}^a \hookrightarrow \mathcal{Z}^a$, we have an inclusion $X^*(\mathcal{I}) \subset \widehat{K(\mathcal{F}^a)} \subset \widehat{K(\mathcal{Z}^a)}$ realized by line bundles on the diagonal subset of $\mathcal{Z}^a$. By Theorem 1.7, it suffices to realize T_0 and T_1 as a generator set of the dense subalgebra of $\widehat{K(\mathcal{Z}^a)}$.

Remark 4.3. *The inclusion $\mathcal{F}^a_w \subset \mathcal{F}^a$ induces a projection map*

$$p_w : \widehat{K(\mathcal{F}^a)} \longrightarrow K(\mathcal{F}^a_w)$$

for each $w \in W$. By construction, each p_w admits a splitting $K(\mathcal{F}^a_w) \hookrightarrow \widehat{K(\mathcal{F}^a)}$ and thus defines an element of $\mathrm{End}(\widehat{K(\mathcal{F}^a)})$. *Here the diagonal embedding enables us to regard $p_w \in \widehat{K(\mathcal{Z}^a)} \subset \mathrm{End}(\widehat{K(\mathcal{F}^a)})$.*

We define $T_i^{geom} := [\mathcal{O}_{\mathbf{O}_i}]$ for $i = 0, 1$. Then, we have:

Lemma 4.4. *The algebra $\widehat{K(\mathcal{Z}^a)}$ is presented as the closure of the algebra generated by the diagonal embedding of $\widehat{K(\mathcal{F}^a)}$, T_0^{geom}, and T_1^{geom}.*

---

[12] Our choice of completion is aimed to satisfy this condition.

Proof. We have

$$\mathcal{V}/\mathcal{V} \cap \dot{s}_1\mathcal{V} \cong \mathbb{C}^2 \boxtimes \mathbb{C}v_+ \text{ and } \mathcal{V}/\mathcal{V} \cap \dot{s}_0\mathcal{V} \cong \mathbb{C}^2 \boxtimes \mathbb{C}zv_-.$$

It follows that $\dim \mathcal{V}/\mathcal{V} \cap \dot{w}\mathcal{V}$ $(w \in W)$ is equal to the twice of the length of w with respect to s_0, s_1. In particular, the closure of the pullback of each orbit closure $\mathcal{G}(\mathcal{I} \times \dot{w}\mathcal{I})$ is given by the $\widehat{K(\mathcal{F}^a)}$-linear combination of products of T_0, T_1 with its lenght at most that of w. Taking account into the existence of p_w, the result follows. □

Since we have no nice localization map $K(\mathcal{F}^a) \longrightarrow K(\mathcal{F})$ associated to the direct image map, we cannot expect to write the generators T_i of $\mathbb{H}$ in a simple fashion[13]. Instead, we adjust the difference by using $\widehat{K(\mathcal{F}^a)}$-action to deduce:

Proposition 4.5. *If we regard $\mathcal{A}[X^{\pm 1}]/(\mathbf{q} = q, \mathbf{t}_0 = t_0, \ldots) \subset \widehat{K(\mathcal{F}^a)}$, then we have*

$$\pi(T_i) \in \widehat{K(\mathcal{F}^a)} T_i^{geom} \widehat{K(\mathcal{F}^a)} + \widehat{K(\mathcal{F}^a)}$$

for each $i = 0, 1$.

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[13] If we do this in a naive fashion, then we obtain an infinite product coming from Koszul resolutions of infinite length.

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