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EULER-MAHONIAN STATISTICS ON ORDERED PARTITIONS AND STEINGRÍMSSON'S CONJECTURE — A SURVEY

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ABSTRACT. An ordered partition with k blocks of $[n] := \{1, 2, \ldots, n\}$ is a sequence of k disjoint and nonempty subsets, called blocks, whose union is $[n]$. In this article, we consider Steingrímsson's conjectures about Euler-Mahonian statistics on ordered partitions dated back to 1997. We encode ordered partitions by walks in some digraphs and then derive their generating functions using the transfer-matrix method. In particular, we prove half of Steingrímsson's conjectures by the computation of the resulting determinants. This article is a very short version of our paper: "Statistics on Ordered Partitions of Sets and q-Stirling Numbers" (`arxiv:math.CO/0605390`), announcing and surveying some of the results in it.

Keywords: ordered partitions, Euler-Mahonian statistics, q-Stirling numbers of the second kind, transfer-matrix method.

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1. INTRODUCTION

The systematic study of statistics on permutations and words has its origins in the work of MacMahon, at the turn of the last century. MacMahon [12] considered four different statistics for a permutation π: The number of descents ($\operatorname{des}\pi$), the number of excedances ($\operatorname{exc}\pi$), the number of inversions ($\operatorname{inv}\pi$), and the major index ($\operatorname{maj}\pi$). These are defined as follows: A descent in a permutation $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > a_{i+1}$, an excedance is an i such that $a_i > i$, an inversion is a pair (i, j) such that $i < j$ and $a_i > a_j$, and the major index of π is the sum of the descents in π.

In fact, MacMahon studied these statistics in greater generality, namely over the rearrangement class of an arbitrary word w with repeated letters. The rearrangement class $R(w)$ of a word $w = a_1 a_2 \cdots a_n$ is the set of all words obtained by permutating the letters of w. All of the statistics mentioned above generalize to words, and in each case, except for that of exc, the generalization is trivial.

MacMahon showed, algebraically, that exc is *equi-distributed* with des, and that inv is equi-distributed with maj, over $R(w)$ for any word w. That is to say,

$$\sum_{z \in R(w)} t^{\operatorname{exc} z} = \sum_{z \in R(w)} t^{\operatorname{des} z} \quad \text{and} \quad \sum_{z \in R(w)} t^{\operatorname{inv} z} = \sum_{z \in R(w)} t^{\operatorname{maj} z}.$$

Any statistic that is equi-distributed with des is said to be *Eulerian*, while any statistic equi-distributed with inv is said to be *Mahonian*. Foata [5] coined the name *Euler-Mahonian* statistic for a bivariate statistic $(\operatorname{eul}\pi, \operatorname{mah}\pi)$, where eul is Eulerian and mah is Mahonian and carried out the first study of such pairs.

In 1997 Steingrímsson [19] introduced the notion of *Euler-Mahonian* statistic on ordered partitions. A partition $\pi = B_1/B_2/\cdots/B_k$ of $[n]$ is a collection of disjoint and nonempty subsets $B_i's$, called *blocks*, whose union is $[n]$, where we write π in the standard way, i.e., the blocks B_i are arranged in increasing order of their minimal elements and separated by /. Let $\mathcal{P}_n^k$ denote the set of partitions of $[n]$ with k blocks. For example, $\pi = 1\,4\,8/2/3/5\,6/7/9$ is a partition of [9] with 6 blocks.

Now, if $\pi = B_1/B_2/\cdots/B_k \in \mathcal{P}_n^k$ and σ is a permutation in S_k, the sequence $\pi_\sigma = B_{\sigma(1)}/B_{\sigma(2)}/\cdots/B_{\sigma(k)}$ is called an *ordered partition* of $[n]$ with k blocks. We set $\sigma = \mathrm{perm}(\pi)$. Let $\mathcal{OP}_n^k$ denote the set of ordered partitions of $[n]$ into k blocks. For example, $\pi = 2/9/3/1\,4\,8/5\,6/7$ is an ordered partition of [9] with 6 blocks.

Define the p,q-integer $[n]_{p,q} = \frac{p^n-q^n}{p-q}$, the p,q-factorial $[n]_{p,q}! = [1]_{p,q}[2]_{p,q}\cdots[n]_{p,q}$ and the *p,q-binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{cases} \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!} & \text{if } 0 \le k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

If $p = 1$, we shall write $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for $[n]_{1,q}$, $[n]_{1,q}!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{1,q}$, and we usually call them *q-integer*, *q-factorial* and *q-binomial coefficient*, respectively.

The Stirling number $S(n,k)$ of the second kind counts the partitions of n-element set into k blocks. A natural q-Stirling numbers $S_q(n,k)$ of the second kind is defined by the recurrence equation

$$S_q(n,\,k) = q^{k-1}S_q(n-1,\,k-1) + [k]_q S_q(n-1,\,k) \qquad (n \ge k \ge 0), \tag{1.1}$$

where $S_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$.

There has been a considerable amount of recent interest in properties and combinatorial interpretations of the q-Stirling numbers and related numbers (see e.g. [1, 2, 3, 11, 13, 14, 15, 16, 17, 19, 22, 21, 23]).

Definition 1.1. *A statistic* STAT *on $\mathcal{OP}_n^k$ is called Euler-Mahonian if its generating function is equal to $[k]_q!\widetilde{S}_q(n,k)$, i.e.,*

$$\sum_{\pi\in\mathcal{OP}_n^k} q^{\mathrm{STAT}\,\pi} = [k]_q!\,\widetilde{S}_q(n,k).$$

An Euler-Mahonian statistic on ordered partitions can be derived from a result of Wachs [13, Theorem 2.1] (see also [19, Theorem 4]). Steingrímsson [19] gave several Euler-Mahonian statistics and conjectured more such statistics.

2. Definitions and Steingrímsson's conjectures

Let π be an ordered partition. The *opener* of a block is its least element and the *closer* is its greatest element. For instance, the ordered partition $\pi = 2/9/3/1\,4\,8/5\,6/7$ has openers 2, 9, 3, 1, 5 and 7 and closers 2, 9, 3, 8, 6 and 7.

Definition 2.1. *(Steingrímsson) Given an ordered partition $\pi \in \mathcal{OP}_n^k$, let* $\mathrm{open}\,\pi$ *and* $\mathrm{clos}\,\pi$ *be the set of openers and closers of π, respectively. Let* $\mathrm{block}_\pi(i)$ *denote the index of*

the block (counting from left to right) which contains the letter i. Following Steingrímsson, we introduce the eight coordinate statistics:

$$\begin{aligned}
\mathrm{ros}_i\,\pi &= \sharp\{j \mid i > j,\ j \in \mathrm{open}\,\pi,\ \mathrm{block}_\pi(j) > \mathrm{block}_\pi(i)\},\\
\mathrm{rob}_i\,\pi &= \sharp\{j \mid i < j,\ j \in \mathrm{open}\,\pi,\ \mathrm{block}_\pi(j) > \mathrm{block}_\pi(i)\},\\
\mathrm{rcs}_i\,\pi &= \sharp\{j \mid i > j,\ j \in \mathrm{clos}\,\pi,\ \mathrm{block}_\pi(j) > \mathrm{block}_\pi(i)\},\\
\mathrm{rcb}_i\,\pi &= \sharp\{j \mid i < j,\ j \in \mathrm{clos}\,\pi,\ \mathrm{block}_\pi(j) > \mathrm{block}_\pi(i)\},\\
\mathrm{los}_i\,\pi &= \sharp\{j \mid i > j,\ j \in \mathrm{open}\,\pi,\ \mathrm{block}_\pi(j) < \mathrm{block}_\pi(i)\},\\
\mathrm{lob}_i\,\pi &= \sharp\{j \mid i < j,\ j \in \mathrm{open}\,\pi,\ \mathrm{block}_\pi(j) < \mathrm{block}_\pi(i)\},\\
\mathrm{lcs}_i\,\pi &= \sharp\{j \mid i > j,\ j \in \mathrm{clos}\,\pi,\ \mathrm{block}_\pi(j) < \mathrm{block}_\pi(i)\},\\
\mathrm{lcb}_i\,\pi &= \sharp\{j \mid i < j,\ j \in \mathrm{clos}\,\pi,\ \mathrm{block}_\pi(j) < \mathrm{block}_\pi(i)\}.
\end{aligned}$$

Let rsb_i *be the number of blocks B in π such that B is to the right of the block containing i, the opener of B is smaller than i and the closer of B is greater than i. We also define* lsb_i *analogously with "right" replaced by "left". Then define* ros, rob, rcs, rcb, lob, los, lcs, lcb, lsb *and* rsb *as the sum of their coordinate statistics, e.g.*

$$\mathrm{ros} = \sum_i \mathrm{ros}_i\,.$$

Note that ros is the abbreviation of "right, opener, smaller", while lcb is the abbreviation of "left, closer, bigger", etc. For instance, we give the values of the coordinate statistics computed on the ordered partition $\pi = 6\ 8/5/1\ 4\ 7/3\ 9/2$:

$\pi =$	6 8	/	5	/	1 4 7	/	3 9	/	2
los_i :	0 0	/	0	/	0 0 2	/	1 3	/	1
ros_i :	4 4	/	3	/	0 2 2	/	1 1	/	0
lob_i :	0 0	/	1	/	2 2 0	/	2 0	/	3
rob_i :	0 0	/	0	/	2 0 0	/	0 0	/	0
lcs_i :	0 0	/	0	/	0 0 1	/	0 3	/	0
rcs_i :	2 3	/	1	/	0 1 1	/	1 1	/	0
lcb_i :	0 0	/	1	/	2 2 1	/	3 0	/	4
rcb_i :	2 1	/	2	/	2 1 1	/	0 0	/	0
lsb_i :	0 0	/	0	/	0 0 1	/	1 0	/	1
rsb_i :	2 1	/	2	/	0 1 1	/	0 0	/	0

Thus we have $\mathrm{los}(\pi) = 7$, $\mathrm{ros}(\pi) = 17$, $\mathrm{lob}(\pi) = 10$, $\mathrm{rob}(\pi) = 2$, $\mathrm{lcs}(\pi) = 4$, $\mathrm{rcs}(\pi) = 10$, $\mathrm{lcb}(\pi) = 13$, $\mathrm{rcb}(\pi) = 9$, $\mathrm{lsb}(\pi) = 3$ and $\mathrm{rsb}(\pi) = 7$.

Let $\pi = B_1/B_2/\cdots/B_k$ be an ordered partition in $\mathcal{OP}_n^k$. We define a *partial order* on blocks B_i's as follows : $B_i > B_j$ if each letter of B_i is greater than every letter of B_j; in other words, if the opener of B_i is greater than the closer of B_j.

Definition 2.2. *Let $\pi = B_1/B_2/\cdots/B_k$ be an ordered partition in $\mathcal{OP}_n^k$. We say that i is a block descent in π if $B_i > B_{i+1}$. The block major index of π, denoted by* $\mathrm{bMaj}\,\pi$, *is the sum of the block descents in π. A block inversion in π is a pair (i, j) such that*

$i < j$ and $B_i > B_j$. The block inversion number, denote by $\mathrm{bInv}\,\pi$, *is the number of block inversions in* π. *We also set* $\mathrm{cbMaj} = \binom{k}{2} - \mathrm{bMaj}$ *and* $\mathrm{cbInv} = \binom{k}{2} - \mathrm{bInv}$.

For instance, if $\pi = 6\ 8/5/1\ 4\ 7/3\ 9/2 \in \mathcal{OP}_9^5$, then there are four block inversions: $\{6,8\} > \{5\}$, $\{6,8\} > \{2\}$, $\{5\} > \{2\}$ and $\{3,9\} > \{2\}$, and two block descents at $i = 1$ and 4; thus $\mathrm{bInv}\,\pi = 4$ and $\mathrm{bmaj}\,\pi = 1 + 4 = 5$. Note also that $\mathrm{cbMaj}\,\pi = \binom{4}{2} - 5 = 3$ and $\mathrm{cbInv}\,\pi = \binom{4}{2} - 4 = 4$.

Inspired by the statistic mak on the permutations in [6], Steingrímsson introduced its analogues on $\mathcal{OP}_n^k$ as follows:

$$\mathrm{mak} = \mathrm{ros} + \mathrm{lcs}, \tag{2.1}$$

$$\mathrm{lmak} = n(k-1) - (\mathrm{los} + \mathrm{rcs}), \tag{2.2}$$

$$\mathrm{mak}' = \mathrm{lob} + \mathrm{rcb}, \tag{2.3}$$

$$\mathrm{lmak}' = n(k-1) - (\mathrm{lcb} + \mathrm{rob}). \tag{2.4}$$

In [19] Steingrímsson conjectured each of the following statistics on $\mathcal{OP}_n^k$ is Euler-Mahonian.

Conjecture 2.3. *(Steingrímsson) Each of the following eight statistics is Euler-Mahonian on* $\mathcal{OP}_n^k$.

$$\begin{array}{llll} \mathrm{mak} + \mathrm{bMaj}, & \mathrm{mak}' + \mathrm{bMaj}, & \mathrm{lmak} + \mathrm{bMaj}, & \mathrm{lmak}' + \mathrm{bMaj}, \\ \mathrm{mak} + \mathrm{bInv}, & \mathrm{mak}' + \mathrm{bInv}, & \mathrm{lmak} + \mathrm{bInv}, & \mathrm{lmak}' + \mathrm{bInv}. \end{array}$$

In this article, we prove that the last half of the above eight statistics are really Euler-Mahonian. Steingrímsson also presented the following conjecture in [19].

Conjecture 2.4. *(Steingrímsson) The statistics*

$$\mathrm{cmajLSB} = \mathrm{lsb} + \mathrm{cbMaj} + \binom{k}{2}$$

$$\mathrm{cinvLSB} = \mathrm{lsb} + \mathrm{cbInv} + \binom{k}{2}$$

are Euler-Mahonian on $\mathcal{OP}_n^k$.

We also prove that cinvLSB is Euler-Mahonian.

3. Main results and a refined conjecture

In this section, we give joint distribution functions of the statistics which are the main purpose of this article (see Theorem 3.3). From the joint distribution functions, we conclude that the half of the statistics defined by Steingrímsson are Euler-Mahonian (see Theorem 3.4). The main ingredients to obtain the generating functions are walk diagrams and the transfer matrix method, which we explain in the next section. At the end of this section we present a new conjecture which generalize the remaining half of Steingrímsson's conjecture in the viewpoint of the joint distribution (see Conjecture 3.7).

Definition 3.1. *For a permutation σ of $[n]$, the pair (i,j) is an inversion if $1 \le i < j \le n$ and $\sigma(i) > \sigma(j)$. Let $\operatorname{inv}\sigma$ be the number of inversions in σ and*

$$\operatorname{cinv}\sigma = \binom{n}{2} - \operatorname{inv}\sigma.$$

By convention, for any ordered partition π, we put $\operatorname{inv}\pi = \operatorname{inv}(\operatorname{perm}(\pi))$ and $\operatorname{cinv}\pi = \operatorname{cinv}(\operatorname{perm}(\pi))$.

The following result due to Ksavrelof and Zeng [11] permits to reduce the original conjectures in [19] almost by half. The reader can also find a straightforward proof in [9].

Proposition 3.2. *On $\mathcal{OP}_n^k$ the following functional identities hold:*

$$\operatorname{mak} = \operatorname{lmak}' \quad \textit{and} \quad \operatorname{mak}' = \operatorname{lmak}.$$

Let $\mathcal{OP}^k$ be the set of all ordered partitions with k blocks. Consider the following two generating functions of ordered partitions with $k \ge 0$ blocks:

$$\phi_k(a;x,y,t,u) := \sum_{\pi\in\mathcal{OP}^k} x^{(\operatorname{mak}+\operatorname{bInv})\pi}\, y^{\operatorname{cinvLSB}\pi}\, t^{\operatorname{inv}\pi}\, u^{\operatorname{cinv}\pi} a^{|\pi|}, \tag{3.1}$$

$$\varphi_k(a;x,y,t,u) := \sum_{\pi\in\mathcal{OP}^k} x^{(\operatorname{lmak}+\operatorname{bInv})\pi}\, y^{\operatorname{cinvLSB}\pi}\, t^{\operatorname{inv}\pi}\, u^{\operatorname{cinv}\pi} a^{|\pi|}, \tag{3.2}$$

where $|\pi| = n$ if π is an ordered partition of $[n]$. The aim of this article is to prove the following theorem.

Theorem 3.3. *We have*

$$\phi_k(a;x,y,t,u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{3.3}$$

$$\varphi_k(a;x,y,t,u) = \frac{a^k (xy)^{\binom{k}{2}} [k]_{tx,uy}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}. \tag{3.4}$$

We first show how to derive Euler-Mahonian statistics on ordered partitions from the above theorem. By (3.1) and (3.2) we have

$$\sum_{\pi\in\mathcal{OP}^k} q^{(\operatorname{mak}+\operatorname{bInv})\pi}\, a^{|\pi|} = \phi_k(a;q,1,1,1),$$

$$\sum_{\pi\in\mathcal{OP}^k} q^{(\operatorname{lmak}+\operatorname{bInv})\pi}\, a^{|\pi|} = \varphi_k(a;q,1,1,1),$$

$$\sum_{\pi\in\mathcal{OP}^k} q^{\operatorname{cinvLSB}\pi}\, a^{|\pi|} = \phi_k(a;1,q,1,1) = \varphi_k(a;1,q,1,1).$$

Theorem 3.3 infers that the right-hand sides of the above three identities are all equal to

$$\frac{a^k\, q^{\binom{k}{2}} [k]_q!}{\prod_{i=1}^k (1 - a[i]_q)} = \sum_{n\ge k} [k]_q! S_q(n,k)\, a^n, \tag{3.5}$$

where the last equality follows directly from (1.1). Thus we can prove the following result, which was conjectured by Steingrímsson [19].

Theorem 3.4. *The following inversion-like statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak}+\mathrm{bInv},\quad \mathrm{mak}'+\mathrm{bInv},\quad \mathrm{lmak}+\mathrm{bInv},\quad \mathrm{lmak}'+\mathrm{bInv},\quad \mathrm{cinvLSB}.$$

In other words, the generating functions of the above statistics over $\mathcal{OP}_n^k$ are all equal to $[k]_q!\widetilde{S}_q(n,k)$.

By a similar argument on the generating functions, we can easily obtain the following result.

Theorem 3.5. *The following statistics are Euler-Mahonian on $\mathcal{OP}_n^k$:*

$$\mathrm{mak}+\mathrm{bInv}-(\mathrm{inv}-\mathrm{cinv}),\quad \mathrm{lmak}+\mathrm{bInv}-(\mathrm{inv}-\mathrm{cinv}),\quad \mathrm{cinvLSB}+(\mathrm{inv}-\mathrm{cinv}).$$

As a further consequence of Theorem 3.3, we can give an alternative proof of the following "hard" combinatorial interpretations for q-Stirling numbers of the second kind, where the first two interpretations were proved by Ksavrelof and Zeng [11] and the third interpretation was first proved by Stanton (see [21]).

Corollary 3.6. *We have*

$$S_q(n,k)=\sum_{\pi\in\mathcal{P}_n^k} q^{\mathrm{mak}\,\pi}=\sum_{\pi\in\mathcal{P}_n^k} q^{\mathrm{lmak}\,\pi}=\sum_{\pi\in\mathcal{P}_n^k} q^{\mathrm{lsb}\,\pi+\binom{k}{2}}.$$

As the reader may notice, the half of Steingrímsson's conjectures are still open, i.e. the remaining five statistics $\mathrm{mak}+\mathrm{bMaj}$, $\mathrm{mak}'+\mathrm{bMaj}$, $\mathrm{lmak}+\mathrm{bMaj}$, $\mathrm{lmak}'+\mathrm{bMaj}$ and cmajLSB are conjectured to be Euler-Mahonian. From the reasoning on the symmetry and numerical experiments we can expect more than Conjecture 2.3 and Conjecture 2.4. Consider the following two generating functions of ordered partitions with $k\geq 0$ blocks:

$$\xi_k(a;x,y):=\sum_{\pi\in\mathcal{OP}^k} x^{(\mathrm{mak}+\mathrm{bMaj})\pi}\,y^{\mathrm{cmajLSB}\,\pi}\,a^{|\pi|}, \tag{3.6}$$

$$\eta_k(a;x,y):=\sum_{\pi\in\mathcal{OP}^k} x^{(\mathrm{lmak}+\mathrm{bMaj})\pi}\,y^{\mathrm{cmajLSB}\,\pi}\,a^{|\pi|}. \tag{3.7}$$

Then we expect the following more general conjecture would hold:

Conjecture 3.7. *For $k\geq 0$, the following identities would hold:*

$$\xi_k(a;x,y)=\frac{a^k\,(xy)^{\binom{k}{2}}[k]_{x,y}!}{\prod_{i=1}^k(1-a[i]_{x,y})}, \tag{3.8}$$

$$\eta_k(a;x,y)=\frac{a^k(xy)^{\binom{k}{2}}\,[k]_{x,y}!}{\prod_{i=1}^k(1-a[i]_{x,y})}. \tag{3.9}$$

Comparing Conjecture 3.7 with Theorem 3.3, one may notice that (3.9) lacks the statistics corresponding to inv and cinv. At this point we don't have a guess on the statistics.

4. ORDERED PARTITIONS AND WALKS IN DIGRAPHS

Let $\pi \in \mathcal{OP}_n^k$ be an ordered partition. We classify each element of $[n]$ into the following four classes. Let $\mathcal{S}(\pi)$ denote the set of $i \in [n]$ which is in a block composed of a single element. We call the elements of $\mathcal{S}(\pi)$ the *singletons* of π. Let $\mathcal{O}(\pi)$ (resp. $\mathcal{C}(\pi)$) denote the set of the openers (resp. closers) which are not singletons. Let $\mathcal{T}(\pi)$ denotes the set of $i \in [n]$ which is not an opener nor a closer. We call the elements of $\mathcal{T}(\pi)$ the *transients* of π. Thus we easily see that $\mathcal{O}(\pi) \cup \mathcal{C}(\pi) \cup \mathcal{S}(\pi) \cup \mathcal{T}(\pi) = [n]$ is a disjoint union. We call the 4-tuple $(\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$ the *type* of π. For instance, if $\pi = 35/246/1/78$, then we have

$$\mathcal{O}(\pi) = \{2,3,7\}, \quad \mathcal{C}(\pi) = \{5,6,8\}, \quad \mathcal{S}(\pi) = \{1\} \quad \text{and} \quad \mathcal{T}(\pi) = \{4\}.$$

Let $\boldsymbol{t} = (t_1, t_2, t_3, t_4, t_5, t_6, t_7)$ be 7-tuple of variables. We define the generating function $Q_k(a; \boldsymbol{t})$ of the set $\mathcal{OP}^k$ of ordered partitions as

$$Q_k(a;\boldsymbol{t}) = \sum_{\pi \in \mathcal{OP}^k} t_1^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{O}\cup\mathcal{S})\pi}\, t_2^{(\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C})\pi}\, t_3^{\mathrm{rsb}(\mathcal{T}\cup\mathcal{C})\pi} \times t_4^{\mathrm{lsb}(\mathcal{T}\cup\mathcal{C})\pi}\, t_5^{\mathrm{ros}(\mathcal{O}\cup\mathcal{S})\pi}\, t_6^{\mathrm{los}(\mathcal{O}\cup\mathcal{S})\pi}\, t_7^{(\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})\pi}\, a^{|\pi|}. \tag{4.1}$$

In this section we restate ordered partitions in terms of walk diagrams, then use the transfer matrix method to obtain a determinantal expression for the generating function $Q_k(a; \boldsymbol{t})$.

Let $D = (V, E)$ be the digraph on $V = \mathbb{N}^2$ with edge set E defined by

$$E = \{e = (u,v) \in V \times V \mid u = v = (x,y) \text{ with } y > 0 \quad \text{or} \quad u - v = (0,1),\ (1,0),\ (1,-1)\}.$$

For any integer $k \geq 0$, let $V_k = \{(i,j) \in V | i+j \leq k\}$ and $D_k = (V_k, E_k)$ be the restriction of the digraph D on V_k. Thus the number of vertices in D_k is equal to

$$\widehat{k} := 1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.$$

An illustration of D_k is given in Figure 1. For each type of edges in D, we call an edge $e = (u, v)$

- *North* if $v = u + (0,1)$;
- *East* if $v = u + (1,0)$;
- *South-East* if $v = u + (1,-1)$;
- *Null* if $v = u$.

Let $e = (u, v)$ be an edge in D_k where the coordinates of u is (p, q). Assign the weight of the edge e to be

$$\mathrm{v}(e) = \begin{cases} a t_1^p\, t_7^q\, [p+q+1]_{t_5,t_6} & \text{if } e \text{ is North or East;} \\ a t_2^p\, [q]_{t_3,t_4} & \text{if } e \text{ is Null or South-East.} \end{cases} \tag{4.2}$$

Let $A_k = (A_k(i,j))_{1 \leq i,j \leq \widehat{k}}$ be the *adjacency matrix* of D_k relative the above weight v. We label the vertices of D_k by their ranks in the following total ordering: $(i,j) \leq (i',j')$ if and only if $i+j < i'+j'$ or $(i+j = i'+j'$ and $j \geq j')$. Thus $v_1 = (0,0)$, $v_2 = (0,1)$, $v_3 =$

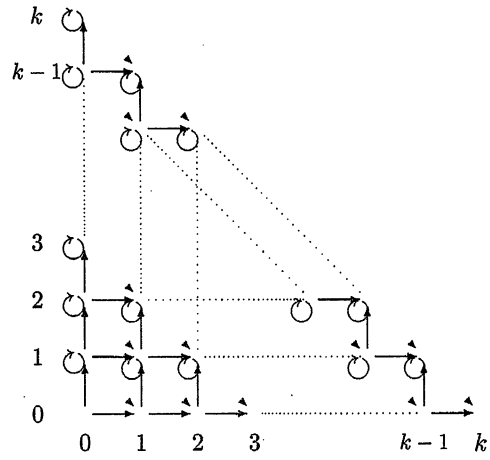

FIGURE 1. The digraph D_k

$(1,0)$, $v_4 = (0,2)$, $v_5 = (1,1)$, $v_6 = (2,0), \cdots, v_{\widehat{k}} = (k,0)$. From this labeling, one easily sees that, for $k \geq 1$, the $\hat{k} \times \hat{k}$ matrix A_k is defined by the following recursive equation:

$$A_0 = (1)\,, \quad A_k = \left(\begin{array}{c|c} A_{k-1} & \overline{A}_{k-1} \\ \hline O_{k+1,\widehat{k-1}} & \widehat{A}_{k-1} \end{array}\right), \tag{4.3}$$

where $\widehat{A}_{k-1}$ is the $(k+1) \times (k+1)$ matrix

$$\widehat{A}_{k-1} = \left(at_2^{i-1}[k+1-i]_{t_3,t_4}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i,j \leq k+1} \tag{4.4}$$

and $\overline{A}_{k-1}$ is the $\widehat{k-1} \times (k+1)$ matrix

$$\overline{A}_{k-1} = \left(\begin{array}{c} O_{\widehat{k-2},k+1} \\ \hline \check{A}_{k-1} \end{array}\right)$$

with the $k \times (k+1)$ matrix

$$\check{A}_{k-1} = \left(at_1^{i-1}t_7^{k-i}[k]_{t_5,t_6}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i \leq k,\, 1 \leq j \leq k+1}. \tag{4.5}$$

Here δ_{ij} stands for the Kronecker delta and $O_{m,n}$ denotes the $m \times n$ zero matrix. For instance, when $k = 2$, we have

$$A_2 = \left(\begin{array}{ccc|ccc} 0 & a & a & 0 & 0 & 0 \\ \hline 0 & a & a & at_7\,[2]_{t_5,t_6} & at_7\,[2]_{t_5,t_6} & 0 \\ 0 & 0 & 0 & 0 & at_1\,[2]_{t_5,t_6} & at_1\,[2]_{t_5,t_6} \\ \hline 0 & 0 & 0 & a[2]_{t_3,t_4} & a[2]_{t_3,t_4} & 0 \\ 0 & 0 & 0 & 0 & at_2 & at_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

4.1. **Encoding ordered partitions by walk diagrams.** Let $\pi = B_1/B_2/\cdots/B_k$ be an ordered partition of $[n]$ and i an integer in $[n]$. The restriction $B_j \cap [i]$ of a block B_j on $[i]$ is said to be *active* if $B_j \neq [i]$ and $B_j \cap [i] \neq \emptyset$, *complete* if $B_j \subseteq [i]$. We define the *trace* of the ordered partition π on $[i]$ as an ordered partition $T_i(\pi)$ on $[i]$ with two kinds of blocks, active or complete, i.e.,

$$T_i(\pi) = B_1(\leq i)/B_2(\leq i)/\cdots/B_k(\leq i),$$

where $B_j(\leq i)$ is the complete or active block $B_j \cap [i]$, while empty sets are omitted. The sequence $(T_i(\pi))_{1\leq i\leq n}$ is called the *trace* of the ordered partition π. Let $\omega_i = (p_i, q_i)$ be the pair such that p_i and q_i are the numbers of complete and active blocks of $T_i(\pi)$, respectively.

A *walk* of depth k and length n in D is a sequence $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of vertices in D such that $\omega_0 = (0,0)$, $\omega_n = (k,0)$ and (ω_i, ω_{i+1}) is an edge of D for $i = 0, \ldots, n-1$. Moreover, the first coordinate p and the second coordinate q of ω_i are called the *abscissa* and *height* of the step (ω_i, ω_{i+1}), respectively. Let Ω_n^k be the set of walks of depth k and length n and Ω^k be the set of walks of depth k. We can visualize a walk ω by drawing a segment from ω_i to ω_{i+1} in the plane. For instance, if

$$\omega = ((0,0), (0,1), (0,2), (0,3), (0,3), (0,3), (1,3), (2,2), (3,1), (4,1), (5,0)),$$

then the illustration is given in Figure 2.

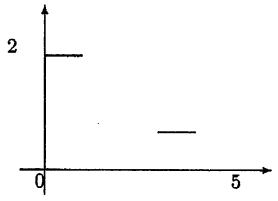

FIGURE 2. A walk in Ω_{10}^5 with two successive Null steps from $(0,3)$ to $(0,3)$.

Definition 4.1. *A walk diagram of depth k and length n is a pair (ω, ξ), where $\omega = ((p_i, q_i))_{0\leq i\leq n}$ is a walk in Ω_n^k and $\xi = (\xi_i)_{1\leq i\leq n}$ is a sequence of integers such that*

- $1 \leq \xi_i \leq q_{i-1}$ *if the i-th step of ω is Null or South-East,*
- $1 \leq \xi_i \leq p_{i-1} + q_{i-1} + 1$ *if the i-th step of ω is North or East.*

Denote by Δ_n^k the set of walk diagrams of depth k and length n and by $\Delta^k = \bigcup_{n\geq 0} \Delta_n^k$ the set of walk diagrams of depth k. The following is the main result of this section.

Theorem 4.2. *For each $n \geq k \geq 1$, the above construction gives a bijection $\psi : \Delta_n^k \to \mathcal{OP}_n^k$ such that if $\psi((\omega, \xi)) = \pi$ for $(\omega, \xi) \in \Delta_n^k$, then*

(a) *if the i-th step of ω is North (resp. East), then $i \in \mathcal{O}(\pi)$ (resp. $i \in \mathcal{S}(\pi)$) and*

$$(\mathrm{lcs} + \mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{los}_i(\pi) = \xi_i - 1,$$
$$(\mathrm{lsb} + \mathrm{rsb})_i(\pi) = q_{i-1}, \quad \mathrm{ros}_i(\pi) = p_{i-1} + q_{i-1} + 1 - \xi_i;$$

(b) *if the i-th step of ω is South-East (resp. Null), then $i \in \mathcal{C}(\pi)$ (resp. $i \in \mathcal{T}(\pi)$) and*

$$(\mathrm{lcs}+\mathrm{rcs})_i(\pi) = p_{i-1}, \quad \mathrm{lsb}_i(\pi) = \xi_i - 1,$$
$$(\mathrm{lsb}+\mathrm{rsb})_i(\pi) = q_{i-1} - 1, \quad \mathrm{rsb}_i(\pi) = q_{i-1} - \xi_i.$$

4.2. **Generating functions of walks.** Given a walk ω of finite length in D_k, define the weight of a step (ω_i, ω_{i+1}) of abscissa p and height q by (4.2). The valuation $\mathrm{v}(\omega)$ of ω is the product of the weights of all its steps. From the transfer-matrix method (see e.g. [18, Theorem 4.7.2]), it is easy to see that

$$Q_k(a;\boldsymbol{t}) = \sum_{w\in\Omega^k} \mathrm{v}(w) = \frac{(-1)^{1+\widehat{k}}\det(I_k - A_k;\widehat{k},1)}{\det(I_k - A_k)}, \tag{4.6}$$

where $(B;i,j)$ denotes the matrix obtained by removing the i-th row and j-th column of B and I_k is the $\widehat{k}\times\widehat{k}$ identity matrix. For instance, we have

$$Q_2(a;\boldsymbol{t}) = -\frac{\det(I_2 - aA_2;6,1)}{\det(I_2 - aA_2)} = \frac{a^2[2]_{t_5,t_6}(at_2t_7 + t_1(1-a[2]_{t_3,t_4}))}{(1-a)(1-a[2]_{t_3,t_4})(1-at_2)}.$$

It seems that $Q_k(a;\boldsymbol{t})$ are messy rational functions in general, but, to prove Theorem 3.3, it is sufficient to evaluate the following special cases of $Q_k(a;\boldsymbol{t})$:

$$f_k(a;x,y,t,u) = Q_k(a;x,x,x,y,t,u,y), \tag{4.7}$$
$$g_k(a;x,y,t,u) = Q_k(a;1,x,1,xy,t,u,y). \tag{4.8}$$

From (4.1), we obtain

$$f_k(a;x,y,t,u) = \sum_{\pi\in\mathcal{OP}^k} x^{(\mathrm{lcs}+\mathrm{rcs}+\mathrm{rsb})(\mathcal{T}\cup\mathcal{C})\,\pi} y^{((\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})+\mathrm{lsb}(\mathcal{T}\cup\mathcal{C}))\,\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|},$$
$$g_k(a;x,y,t,u) = \sum_{\pi\in\mathcal{OP}^k} x^{(\mathrm{lcs}+\mathrm{rcs}+\mathrm{lsb})(\mathcal{T}\cup\mathcal{C})\pi} y^{((\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S})+\mathrm{lsb}(\mathcal{T}\cup\mathcal{C}))\,\pi} t^{\mathrm{inv}\,\pi} u^{\mathrm{cinv}\,\pi} a^{|\pi|},$$

and these identities and the following proposition immediately combines the evaluation of $f_k(a;x,y,t,u)$ and $g_k(a;x,y,t,u)$ to the proof of Theorem 3.3.

Proposition 4.3. *The following functional identities hold on $\mathcal{OP}_n^k$:*

$$\mathrm{mak}+\mathrm{bInv} = (\mathrm{lcs}+\mathrm{rcs}) + \mathrm{rsb}(\mathcal{T}\cup\mathcal{C}) + \mathrm{inv},$$
$$\mathrm{lmak}+\mathrm{bInv} = n(k-1) - (\mathrm{lcs}+\mathrm{rcs})(\mathcal{T}\cup\mathcal{C}) - \mathrm{lsb}(\mathcal{T}\cup\mathcal{C}) - \mathrm{cinv},$$
$$\mathrm{cinvLSB} = (\mathrm{lsb}+\mathrm{rsb})(\mathcal{O}\cup\mathcal{S}) + \mathrm{lsb}(\mathcal{T}\cup\mathcal{C}) + \mathrm{inv} + 2\,\mathrm{cinv}.$$

5. Determinant evaluations

The aim of this section is to prove the following theorem.

Theorem 5.1. *For $k \geq 1$, we have*

$$f_k(a;x,y,t,u) = \frac{a^k x^{\binom{k}{2}}[k]_{t,u}!}{\prod_{i=1}^k (1 - a[i]_{x,y})}, \tag{5.1}$$

$$g_k(a;x,y,t,u) = \frac{a^k\,[k]_{t,u}!}{\prod_{i=1}^k (1 - ax^{k-i}[i]_{xy})}. \tag{5.2}$$

Let A'_k and A''_k be the matrices obtained from A_k by making the substitutions corresponding to (4.7) and (4.8), respectively. Now, for each $k \geq 0$, let

$$M_k = I_k - aA'_k \quad \text{and} \quad N_k = I_k - aA''_k.$$

Then we derive from (4.6), (4.7) and (4.8) that for each $k \geq 1$,

$$f_k(a;x,y,t,u) = \frac{(-1)^{1+\widehat{k}} \det(M_k;\widehat{k},1)}{\det M_k}, \tag{5.3}$$

$$g_k(a;x,y,t,u) = \frac{(-1)^{1+\widehat{k}} \det(N_k;\widehat{k},1)}{\det N_k}. \tag{5.4}$$

Since M_k and N_k are upper triangular, we have

$$\det M_k = \prod_{m=1}^{k} \prod_{i=0}^{m} (1 - ax^i[m-i]_{x,y}),$$

$$\det N_k = \prod_{m=1}^{k} \prod_{i=0}^{n-m} (1 - ax^i[m]_{xy}),$$

for each $k \geq 1$. The evaluations of $\det(M_k;\widehat{k},1)$ and $\det(N_k;\widehat{k},1)$ are far from trivial.

Theorem 5.2. *Let $k \geq 1$ be a positive integer. Then*

$$\det(M_k;\widehat{k},1) = (-1)^{\binom{k}{2}} a^k x^{\binom{k}{2}} [k]_{t,u}! \prod_{m=1}^{k-1} \prod_{i=1}^{m} (1 - ax^i[m-i+1]_{x,y}), \tag{5.5}$$

$$\det(N_k;\widehat{k},1) = (-1)^{\binom{k}{2}} a^k [k]_{t,u}! \prod_{m=1}^{k-1} \prod_{k=1}^{k-m} (1 - ax^{k-1}[m]_{xy}). \tag{5.6}$$

5.1. **Proof of** (5.5). By the specialization, the matrix M_k is defined recursively by

$$M_0 = (1), \quad M_k = \left(\begin{array}{c|c} M_{k-1} & \overline{M}_{k-1} \\ \hline O_{k+1,\widehat{k-1}} & \widehat{M}_{k-1} \end{array} \right), \tag{5.7}$$

where

$$\widehat{M}_{k-1} = \left(\delta_{ij} - ax^{i-1}[k+1-i]_{x,y}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i,j \leq k+1} \tag{5.8}$$

and $\overline{M}_{k-1}$ is the $\widehat{k-1} \times (k+1)$ matrix

$$\overline{M}_{k-1} = \left(\begin{array}{c} O_{\widehat{k-2},k+1} \\ \hline \check{M}_{k-1} \end{array} \right)$$

with the $k \times (k+1)$ matrix

$$\check{M}_{k-1} = \left(-ax^{i-1}y^{k-i}[k]_{t,u}(\delta_{ij} + \delta_{i+1,j})\right)_{1 \leq i \leq k,\, 1 \leq j \leq k+1}. \tag{5.9}$$

Let

$$\mathrm{K}_k = \widehat{k} - 1 = \frac{k(k+3)}{2},$$

and let $P_k = (M_k; \widehat{k}, 1)$. In general we can define P_k as follows:

$$P_k = \left(\begin{array}{c|c} P_{k-1} & \overline{P}_{k-1} \\ \hline X_{k-1} & \widehat{P}_{k-1} \end{array} \right),$$

where $\overline{P}_{k-1}$ is a $\mathrm{K}_{k-1} \times (k+1)$ matrix, X_{k-1} is a $(k+1) \times \mathrm{K}_{k-1}$ matrix, and $\widehat{P}_{k-1}$ is a $(k+1) \times (k+1)$ matrix. We shall compute $\det P_k$ by the following well-known formula for any block matrix with an invertible square matrix A,

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det A \cdot \det \left(D - CA^{-1}B \right).$$

Since $\overline{P}_k$ is an $\mathrm{K}_k \times (k+2)$ matrix, we can write

$$\overline{P}_k = \left(\begin{array}{c} O_{\mathrm{K}_{k-1},k+2} \\ \hline U_k \end{array} \right),$$

where U_k is the $(k+1) \times (k+2)$ matrix composed of the last $(k+1)$ rows of $\overline{P}_k$. For $1 \le j \le k+2$, let

$$P_k^j = \left(\begin{array}{c|c} P_{k-1} & \overline{P}_{k-1} \\ \hline X_{k-1} & \widehat{P}_{k-1}^j \end{array} \right)$$

denote the $\mathrm{K}_k \times \mathrm{K}_k$ matrix obtained from P_k by replacing the right-most column with the jth column of $\overline{P}_k$. Here $\widehat{P}_{k-1}^j$ is the $(k+1) \times (k+1)$ matrix obtained from $\widehat{P}_{k-1}$ by replacing the right-most column with the jth column of U_k. Here we don't have space to give the detailed proof of (5.5), but the proof essentially reduces to the following theorem.

Theorem 5.3. *Let $k \ge 1$ be a positive integer. Then we have*

$$\frac{\det P_k}{\det P_{k-1}} = (-1)^{k-1} a x^{k-1} [k]_{t,u} \prod_{i=1}^{k-1} (1 - a x^i [k-i]_{x,y}), \tag{5.10}$$

and

$$\frac{\det P_k^j}{\det P_k} = a\, x^{\frac{(j-1)(j-2)}{2} - \frac{k(k-1)}{2}}\, y^{\frac{(k+1-j)(k+2-j)}{2}}\, [k+1]_{t,u} \begin{bmatrix} k+1 \\ j-1 \end{bmatrix}_{x,y} \tag{5.11}$$

for $1 \le j \le k$,

$$\frac{\det P_k^{k+1}}{\det P_k} = a\, y\, [k+1]_{t,u}\, [k]_{x,y} \tag{5.12}$$

and $\det P_k^{k+2} = 0$.

Lastly we shall note that the following identity is very useful in the proof of Theorem 5.3.

Lemma 5.4. *For $0 \le m \le n$,*

$$\sum_{k=0}^{m}(-1)^{m-k}x^{\binom{k}{2}}y^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{x,y}\prod_{i=0}^{k-1}\{1-ax^i[n-i]_{x,y}\}\prod_{i=k}^{m-1}\{-ax^i[n-i]_{x,y}\}$$
$$= x^{\binom{m}{2}}y^{\binom{n-m}{2}}\begin{bmatrix}n\\m\end{bmatrix}_{x,y}\prod_{i=1}^{m}\{1-ax^i[n-i]_{x,y}\}. \tag{5.13}$$

5.2. **Proof of** (5.6). We first introduce two generalizations of q-binomial coefficients. For any sequence of non-zero functions $F = \{F_n\}_{n=1}^{\infty}$ in finitely many variables $t_1, t_2, \dots$ we define $F_n! = \prod_{k=1}^{n} F_k$ and for any positive integers n and k,

$$\begin{bmatrix}n\\k\end{bmatrix}_F = \begin{cases}\frac{F_n!}{F_k!F_{n-k}!}, & \text{if } 0 \le k \le n,\\ 0, & \text{otherwise.}\end{cases}$$

Next, for any positive integers n and k, let

$$]n,k[_{x,y} = [n]_{xy} - x^{n-k}[k]_{xy},$$

and

$$\Big]\begin{matrix}n\\k\end{matrix}\Big[_{x,y} = \begin{cases}\frac{\prod_{i=0}^{k-1}]n,i[_{x,y}}{[k]_{xy}!} & \text{if } 0 \le k \le n,\\ 0 & \text{otherwise.}\end{cases}$$

For instance, we have $]3,1[_{x,y} = 1 + xy + x^2y^2 - x^2$ and

$$\Big]\begin{matrix}3\\2\end{matrix}\Big[_{x,y} = \frac{(1+xy+x^2y^2)(1+xy+x^2y^2-x^2)}{(1+xy)}.$$

Note that $]n,k[_{x,1} = [n-k]_x$ and $\Big]\begin{matrix}n\\k\end{matrix}\Big[_{x,1} = \begin{bmatrix}n\\k\end{bmatrix}_x$.

We prove (5.6) by considering the following matrix $N_k(\lambda, a)$, which reduces to the matrix N_k by setting $\lambda = 1$ and $F_n = [n]_{t,u}$. Let $N_k(\lambda, a)$ be the matrix defined inductively as follows:

$$N_0(\lambda, a) = (\lambda)$$

and

$$N_k(\lambda, a) = \left(\begin{array}{c|c} N_{k-1}(\lambda,a) & \overline{N}_{k-1}(\lambda,a)\\ \hline O_{k+1,\widehat{k-1}} & \widehat{N}_{k-1}(\lambda,a)\end{array}\right) \tag{5.14}$$

where $\widehat{N}_{k-1}(\lambda, a)$ is the $(k+1)\times(k+1)$ matrix defined by

$$\widehat{N}_{k-1}(\lambda,a) = \big(\lambda\delta_{ij} - ax^{i-1}[k+1-i]_{xy}(\delta_{ij}+\delta_{i+1,j})\big)_{1\le i,j\le k+1} \tag{5.15}$$

and $\overline{N}_{k-1}(\lambda, a)$ is the $\widehat{k-1}\times(k+1)$ matrix

$$\left(\begin{array}{c} O_{\widehat{k-2},k+1}\\ \hline \check{N}_{k-1}\end{array}\right)$$

with the $k\times(k+1)$ matrix

$$\check{N}_{k-1} = \big(-ay^{k-i}F_k\cdot(\delta_{ij}+\delta_{i+1,j})\big)_{1\le i\le k,\,1\le j\le k+1}. \tag{5.16}$$

For positive integers m and l, define the row vector $\boldsymbol{X}_n^{m,l}$ of degree $\widehat{k}$ as follows: For $1 \le i \le k+1$ and $1 \le j \le i$, the $\left(\frac{i(i-1)}{2}+j\right)$th entry of $\boldsymbol{X}_k^{m,l}$ is equal to

$$X_{i,j}^{m,l} = (-1)^{i+m+l} x^{-(m+l-1)(i-m-l)+\binom{j-l}{2}} y^{\binom{i-m-l}{2}} \times \frac{F_{i-m-l}!}{[i-m-l]_{xy}!} \begin{bmatrix} i-1 \\ m+l-1 \end{bmatrix}_F \begin{bmatrix} m \\ m+l-j \end{bmatrix}_{x,y}. \tag{5.17}$$

Here we use the convention that $F_n! = [n]_{xy}! = 1$ if $n \le 0$. For example, if $k = 3$, $m = l = 1$, then

$$\boldsymbol{X}_3^{1,1} = \left(0, 1, 1, -\frac{F_2}{x}, -\frac{F_2}{x}, 0, \frac{F_2 F_3\, y}{x^2[2]_{xy}!}, \frac{F_2 F_3\, y}{x^2[2]_{xy}!}, 0, 0\right).$$

Then the following lemma shows that the vectors $\boldsymbol{X}_k^{m,l}$ $(0 \le m \le k-1,\ 1 \le l \le k-m)$ are eigenvectors of the matrix $N_k(\lambda, a)$, and plays essential role to prove (5.6).

Lemma 5.5. *Let k be a positive integer. Let m and l be positive integers such that $0 \le m \le k-1$ and $1 \le l \le k-m$. Then we have*

$$\boldsymbol{X}_k^{m,l}\, N_k(\lambda, a) = (\lambda - a x^{l-1}[m]_{xy})\, \boldsymbol{X}_k^{m,l}. \tag{5.18}$$

Let $\dot{N}_k(\lambda, a)$ denote the matrix obtained from $N_k(\lambda, a)$ by deleting the $\widehat{k}$th row and the first column. Then the following corollary easily follows from Lemma 5.5 and the fact that the last entry of $\boldsymbol{X}_k^{m,l}$ is always zero.

Corollary 5.6. *Let k be a positive integer. Then there exists a polynomial $\varphi(\lambda)$ such that*

$$\det \dot{N}_k(\lambda, a) = \varphi(\lambda) \lambda^k \prod_{m=1}^{k-1} \prod_{l=1}^{k-m} (\lambda - a x^{l-1}[m]_{xy}). \tag{5.19}$$

We regard $\det N_k(\lambda, a)$ as a polynomial in λ and check the degree and the leading coefficient in the both sides of (5.19). Then it is not hard to see the following theorem holds. We omit the detail.

Theorem 5.7. *We have*

$$\det \dot{N}_k(\lambda, a) = (-1)^{k(k-1)/2} a^k\, F_k!\, \lambda^k \prod_{m=1}^{k-1} \prod_{l=1}^{k-m} (\lambda - a x^{l-1}[m]_{xy}). \tag{5.20}$$

Finally, by setting $\lambda = 1$ and $F_n = [n]_{t,u}$, we obtain (5.6).

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