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DESCRIPTIONS OF THE CRYSTAL $\mathcal{B}(\infty)$ FOR G_2

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1. Introduction

We study the crystal base of the negative part of a quantum group. We introduce two explicit descriptions of the crystal $\mathcal{B}(\infty)$ for types G_2, namely, those that use Young tableaux [3], extended Nakajima monomials [10]. We also extend Cliff's [1] description of $\mathcal{B}(\infty)$ for classical finite types to the G_2-type. Note that this result was dealt with by Kashiwara in [6] Example 2.2.7 and by Nakashima and Zelevinsky in a more general form [12]. And we observe correspondence between the three descriptions.

The paper is organized as follows. We start by reviewing Young tableau expression of crystal $\mathcal{B}(\infty)$. Also, we cite the notion of extended Nakajima monomials and the crystal structure given on the set of such monomials. We then proceed to give a monomial description of the crystal $\mathcal{B}(\infty)$. In the last section, we deal with Cliff's approach of describing $\mathcal{B}(\infty)$. In the process of obtaining these results, we give explicit correspondences between the three descriptions.

2. Notations

We fix basic notations. Please refer to the references cited in the introduction or books on quantum groups [2, 4] for the basic concepts on quantum groups and crystal bases.

- $I = \{1, 2\}$: index set for G_2-type
- $A = (a_{ij})_{i,j\in I}$: Cartan matrix of type G_2 with $a_{12} = -3$ and $a_{21} = -1$
- α_i, Λ_i $(i \in I)$: simple root, fundamental weight
- $\Pi^\vee = \{h_i \mid i \in I\}$: the set of simple coroots
- $\Pi = \{\alpha_i | i \in I\}$: the set of simple root
- $P^\vee = \bigoplus_{i\in I} \mathbf{Z}h_i$: dual weight lattice
- $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbf{Z}\} = \oplus_{i\in I}\mathbf{Z}\Lambda_i$: weight lattice, where $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$
- $P^+ = \{\lambda \in P | \lambda(h_i) \geq 0 \text{ for all } i \in I\}$: the set of dominant integral weights
- $U_q(G_2)$: quantum group for G_2
- $U_q^-(G_2)$: subalgebra of $U_q(G_2)$ generated by f_i $(i \in I)$
- $\tilde{f}_i, \tilde{e}_i$: Kashiwara operators
- $\mathcal{B}(\lambda)$: irreducible highest weight crystal of highest weight λ
- $\mathcal{B}(\infty)$: crystal base of $U_q^-(G_2)$

Throughout this paper, a $U_q(G_2)$-crystal will refer to a (abstract) crystal associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$. The crystal base $\mathcal{B}(\infty)$ of $U_q^-(G_2)$ is a $U_q(G_2)$-crystal.

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$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&2&3&0&\bar{3}&\bar{2}&\bar{1}\\\hline 2&2&3\\\cline{1-3}\end{array}\qquad\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&3&0&\bar{3}&\bar{2}&\bar{1}\\\hline 2&3\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|c|c|c|}\hline 1&2&0&\bar{3}&\bar{3}&\bar{2}\\\hline 2&3&\bar{2}\\\cline{1-3}\end{array}$$

FIGURE 1. Large (left), marginally large (middle), and non-large (right) tableaux

3. YOUNG TABLEAU DESCRIPTION

In this section, we introduce a Young tableau description for the crystal $\mathcal{B}(\infty)$ over type G_2 [3].

For the G_2-type, we shall take the Young tableau description of highest weight crystal $\mathcal{B}(\lambda)$ given in [5] as the definition of semi-standard tableaux. Since the work is a rather well known result, we refer readers to the original papers and shall not repeat the complicated definition here. The alphabet to be used inside the boxes constituting the Young tableaux will be denoted by J, and it will be equipped with an ordering $\prec$, as given in [5].

$$J = \{1 \prec 2 \prec 3 \prec 0 \prec \bar{3} \prec \bar{2} \prec \bar{1}\}.$$

Definition 3.1.

(1) A semi-standard tableau T of shape $\lambda \in P^+$, equivalently, an element of an irreducible highest weight crystal $\mathcal{B}(\lambda)$ for the G_2 type, is *large* if it consists of 2 non-empty rows, and if the number of 1-boxes in the first row is strictly greater than the number of all boxes in the second row and the second row contains at least one 2-box.

(2) A large tableau T is *marginally* large, if the number of 1-boxes in the first row of T is greater than the number of all boxes in the second row by exactly one and the second row of T contain one 2-box.

In Figure 1, we give examples of semi-standard tableaux. The one on the left is large, the one on the middle is marginally large, and the one on the right is not large.

Definition 3.2. We denote by $\mathcal{T}(\infty)$ the set of all marginally large tableaux. The marginally large tableau whose i-th row consists only of i-boxes ($i \in I$) is denoted by T_∞.

The set $\mathcal{T}(\infty)$ consists of all tableaux of the following form. The unshaded part must exist, whereas the shaded part is optional with variable size.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1\cdots1&1&2\cdots2&3\cdots3&0&\bar{3}\cdots\bar{3}&\bar{2}\cdots\bar{2}&\bar{1}\cdots\bar{1}\\\hline 2&3\cdots3\\\cline{1-2}\end{array}\qquad T_\infty = \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$$

We recall the action of Kashiwara operators $\tilde{f}_i$, $\tilde{e}_i$ ($i \in I$) on marginally large tableaux $T \in \mathcal{T}(\infty)$.

(1) We first read the boxes in the tableau T through the *far eastern reading* and write down the boxes in *tensor product form*. That is, we read through each column from top to bottom starting from the rightmost column, continuing to the left, and lay down the read boxes from left to right in tensor product form.

(2) Under each tensor component x of T, write down $\varepsilon_i(x)$-many 1s followed by $\varphi_i(x)$-many 0s. Then, from the long sequence of mixed 0s and 1s, successively cancel out every occurrence of (0,1) pair until we arrive at a sequence

of 1s followed by 0s, reading from left to right. This is called the i-signature of T.

(3) Denote by T', the tableau obtained from T, by replacing the box x corresponding to the leftmost 0 in the i-signature of T with the box $\tilde{f}_i x$.
  - If T' is a large tableau, it is automatically marginally large. We define $\tilde{f}_i T$ to be T'.
  - If T' is not large, then we define $\tilde{f}_i T$ to be the large tableau obtained by inserting one column consisting of i rows to the left of the box $\tilde{f}_i$ acted upon. The added column should have a k-box at the k-th row for $1 \leq k \leq i$.

(4) Denote by T', the tableau obtained from T, by replacing the box x corresponding to the rightmost 1 in the i-signature of T with the box $\tilde{e}_i x$.
  - If T' is a marginally large tableau, then we define $\tilde{e}_i T$ to be T'.
  - If T' is large but not marginally large, then we define $\tilde{e}_i T$ to be the large tableau obtained by removing the column containing the changed box. It will be of i rows and have a k-box at the k-th row for $1 \leq k \leq i$.

(5) If there is no 1 in the i-signature of T, we define $\tilde{e}_i T = 0$.

Remark 3.3. The condition *large* imposed on the tableau T ensures that its i-signature always contains 0's.

Let T be a tableau in $\mathcal{T}(\infty)$ with the second row consisting of $b^2_{\bar{3}}$-many 3 s, one 2 and the first row consisting of b^1_j-many j s $(1 \prec j \preceq \bar{1})$, $(b^2_{\bar{3}} + 2)$-many 1 s. We define the maps $\mathrm{wt} : \mathcal{T}(\infty) \to P$, $\varphi_i, \varepsilon_i : \mathcal{T}(\infty) \to \mathbf{Z}$ by setting

$$\begin{aligned} \mathrm{wt}(T) &= (-b^1_2 - b^1_3 - 2b^1_0 - 3b^1_{\bar{3}} - 3b^1_{\bar{2}} - 4b^1_{\bar{1}})\alpha_1 \\ &\quad + (-b^1_3 - b^1_0 - b^1_{\bar{3}} - 2b^1_{\bar{2}} - 2b^1_{\bar{1}} - b^2_{\bar{3}})\alpha_2, \end{aligned} \tag{3.1}$$

$$\varepsilon_i(T) = \text{the number of 1s in the } i\text{-signature of } T, \tag{3.2}$$

$$\varphi_i(T) = \varepsilon_i(T) + \langle h_i, \mathrm{wt}(T)\rangle. \tag{3.3}$$

Theorem 3.4. ([3]) *The maps given above, together with Kashiwara operators define a crystal structure on $\mathcal{T}(\infty)$. The crystal $\mathcal{T}(\infty)$ is isomorphic to $\mathcal{B}(\infty)$ as a $U_q(G_2)$-crystal.*

4. Extended Nakajima monomial description

It was Nakajima [11] that first introduced a crystal structure to a certain set of monomials. A modified crystal structure was given to the same set by Kashiwara [7] and an extension was introduced in [8]. The later two constructions were defined for all symmetrizable Kac-Moody algebras, but we shall restrict ourselves to the G_2 case in this paper.

Let $\mathcal{M}^{\mathcal{E}}$ be a certain set of formal monomials in the variables $Y_i(m)^{(1,0)}$ and $Y_i(m)^{(0,1)}$ $(i \in I,\, m \in \mathbf{Z})$ given by

$$\mathcal{M}^{\mathcal{E}} \doteq \left\{ \prod_{(i,m)\in I\times\mathbf{Z}} Y_i(m)^{y_i(m)} \;\middle|\; \begin{array}{l} y_i(m) = (y^0_i(m), y^1_i(m)) \in \mathbf{Z}\times\mathbf{Z} \\ \text{vanish except at finitely many } (i,m) \end{array} \right\}. \tag{4.1}$$

The product of monomials $Y_i(m)^{(u,v)}$ and $Y_i(m)^{(u',v')}$ are set to $Y_i(m)^{(u+u',v+v')}$, for (u,v), $(u',v') \in \mathbf{Z}\times\mathbf{Z}$. We give the lexicographic order to the set $\mathbf{Z}\times\mathbf{Z}$ of variable exponents.

Fix any set of integers $c = (c_{ij})_{i \neq j \in I}$ such that $c_{ij} + c_{ji} = 1$, and set

$$A_i(m)^{\pm 1} = Y_i(m)^{(0,\pm 1)} Y_i(m+1)^{(0,\pm 1)} \prod_{j \neq i} Y_j(m + c_{ji})^{(0, \pm \langle h_j, \alpha_i \rangle)}. \tag{4.2}$$

The crystal structure on $\mathcal{M}^{\mathcal{E}}$ is defined as follows. For each monomial $M = \prod_{(i,m) \in I \times \mathbf{Z}} Y_i(m)^{y_i(m)} \in \mathcal{M}^{\mathcal{E}}$, we set

$$\tilde{\mathrm{wt}}(M) = \sum_i \Big(\sum_m y_i(m) \Big) \Lambda_i = \sum_i \Big(\sum_m (y_i^0(m), y_i^1(m)) \Big) \Lambda_i, \tag{4.3}$$

$$\tilde{\varphi}_i(M) = \max \Big\{ \sum_{k \leq m} y_i(k) \ \Big| \ m \in \mathbf{Z} \Big\}, \tag{4.4}$$

$$\tilde{\varepsilon}_i(M) = \max \Big\{ - \sum_{k > m} y_i(k) \ \Big| \ m \in \mathbf{Z} \Big\}. \tag{4.5}$$

Notice that the coefficients of $\tilde{\mathrm{wt}}(M)$ are pairs of integers. In this setting, we have $\tilde{\varphi}_i(M) \geq (0,0)$, $\tilde{\varepsilon}_i(M) \geq (0,0)$, and $\tilde{\mathrm{wt}}(M) = \sum_i \big(\tilde{\varphi}_i(M) - \tilde{\varepsilon}_i(M) \big) \Lambda_i$. Set

$$\mathrm{wt}(M) = \sum_i \Big(\sum_m y_i^1(m) \Big) \Lambda_i, \tag{4.6}$$

$$\varphi_i(M) = \sum_{k \leq m} y_i^1(k) \quad \text{where } \tilde{\varphi}_i(M) = \sum_{k \leq m} \big(y_i^0(k), y_i^1(k) \big), \tag{4.7}$$

$$\varepsilon_i(M) = - \sum_{k > m} y_i^1(k) \quad \text{where } \tilde{\varepsilon}_i(M) = - \sum_{k > m} \big(y_i^0(k), y_i^1(k) \big). \tag{4.8}$$

For the monomial M, we trivially have $\mathrm{wt}(M) = \sum_i \big(\varphi_i(M) - \varepsilon_i(M) \big) \Lambda_i$. From the above definition, $Y_i(m)^{(0,1)}$ has the weight Λ_i, and so $A_i(m)$ has the weight α_i. We define the action of Kashiwara operators by

$$\tilde{f}_i(M) = \begin{cases} 0 & \text{if } \tilde{\varphi}_i(M) = (0,0), \\ A_i(m_f)^{-1} M & \text{if } \tilde{\varphi}_i(M) > (0,0), \end{cases} \tag{4.9}$$

$$\tilde{e}_i(M) = \begin{cases} 0 & \text{if } \tilde{\varepsilon}_i(M) = (0,0), \\ A_i(m_e) M & \text{if } \tilde{\varepsilon}_i(M) > (0,0). \end{cases} \tag{4.10}$$

Here,

$$m_f = \min \Big\{ m \,\Big|\, \tilde{\varphi}_i(M) = \sum_{k \leq m} y_i(k) \Big\}, \tag{4.11}$$

$$m_e = \max \Big\{ m \,\Big|\, \tilde{\varepsilon}_i(M) = - \sum_{k > m} y_i(k) \Big\}. \tag{4.12}$$

Note that $y_i(m_f) > (0,0)$, $y_i(m_f + 1) \leq (0,0)$, $y_i(m_e + 1) < (0,0)$, and $y_i(m_e) \geq (0,0)$.

For any fixed set of integers $c = (c_{ij})_{i \neq j \in I}$ such that $c_{ij} + c_{ji} = 1$, the Kashiwara operators defined in (4.9) and (4.10), together with the maps φ_i, ε_i ($i \in I$), and wt of (4.6) to (4.8), define a crystal structure on the set $\mathcal{M}^{\mathcal{E}}$ [8]. We refer to an element of the set $\mathcal{M}^{\mathcal{E}}$ as an *extended Nakajima monomial* and denote by $\mathcal{M}^{\mathcal{E}}_c$ the set $\mathcal{M}^{\mathcal{E}}$ subject to the crystal structure depending on the set c, as given above.

Remark 4.1. Now, we may give many different crystal structures to the set of extended Nakajima monomials through the choice of the set c. For G_2 type Lie algebras, all the different crystals induced from the set of extended Nakajima monomials through different choices of the set c, are isomorphic (see [7] or Proposition 3.2 of [8]).

Unless there is possibility of confusion, we shall omit c and use the notation $\mathcal{M}^{\mathcal{E}}$ instead of $\mathcal{M}^{\mathcal{E}}_c$.

We now give a description of the crystal $\mathcal{B}(\infty)$ in terms of extended monomials. For simplicity, from now on, we take the set $C = (c_{ij})_{i\neq j\in I}$ to be $c_{12} = 1$ and $c_{21} = 0$. Then for $m \in \mathbf{Z}$, we have

$$\begin{cases} A_1(m) = Y_1(m)^{(0,1)}Y_1(m+1)^{(0,1)}Y_2(m)^{(0,-1)}, \\ A_2(m) = Y_2(m)^{(0,1)}Y_2(m+1)^{(0,1)}Y_1(m+1)^{(0,-3)}. \end{cases} \tag{4.13}$$

Consider elements of $\mathcal{M}^{\mathcal{E}}$ having the form

$$\begin{aligned} M =& Y_1(-1)^{(1,a_1^{-1})}Y_1(0)^{(0,a_1^0)}Y_1(1)^{(0,a_1^1)}Y_1(2)^{(0,a_1^2)} \\ &\cdot Y_2(-2)^{(1,a_2^{-2})}Y_2(-1)^{(0,a_2^{-1})}Y_2(0)^{(0,a_2^0)}Y_2(1)^{(0,a_2^1)} \end{aligned} \tag{4.14}$$

with conditions

(1) $(a_2^{-2} - a_2^{-1})$, a_2^1, a_1^2, $a_2^{-2} \leq 0$,
(2) $(a_1^{-1} - a_1^1 - a_1^2) + (2a_2^{-2} + a_2^{-1} - a_2^0 - 2a_2^1) = 0$ and
$(a_1^{-1} + a_1^0 - a_1^2) + (a_2^{-2} + 2a_2^{-1} + a_2^0 - a_2^1) = 0$,
(3) $(a_1^0 + a_2^{-1} - a_2^{-2}), (-a_1^1 - a_2^1) \in 2\mathbf{Z}_{\geq 0}$ or
$(a_1^0 + a_2^{-1} - a_2^{-2}), (-a_1^1 - a_2^1) \in \mathbf{Z}_{\geq 0}$ and odd.

Specifically, in case of $a_i^j = 0$ for all i, j, we have

$$M = Y_1(-1)^{(1,0)}Y_2(-2)^{(1,0)}. \tag{4.15}$$

We denote by $\mathcal{M}(\infty)$ the set of all monomials of these form and by M_∞ the monomial of (4.15).

This set was originally obtained by applying Kashiwara actions $\tilde{f}_i$ continuously on the single element $Y_1(-1)^{(1,0)}Y_2(-2)^{(1,0)} \in \mathcal{M}^{\mathcal{E}}$. This choice of starting monomial will allow us to relate monomials of the set defined below to tableaux in $\mathcal{T}(\infty)$ naturally.

We now introduce new expressions for elements of $\mathcal{M}(\infty)$. First, we introduce the following notation.

Definition 4.2. For $u \in \mathbf{Z}_{\geq 0}$, $v \in \mathbf{Z}$, and $m \in \mathbf{Z}$, we use the notation

$$X_j(m)^{(u,v)} = \begin{cases} Y_j(m)^{(u,v)}Y_{j-1}(m+1)^{(-u,-v)} & \text{for } j = 1, 2, \\ Y_1(m+1)^{(2u,2v)}Y_2(m+1)^{(-u,-v)} & \text{for } j = 3, \end{cases} \tag{4.16}$$

$$X_0(m)^{(u,v)} = Y_1(m+1)^{(u,v)}Y_1(m+2)^{(-u,-v)}, \tag{4.17}$$

$$X_{\bar{j}}(m)^{(u,v)} = \begin{cases} Y_{j-1}(m+(4-j))^{(u,v)}Y_j(m+(4-j))^{(-u,-v)} & \text{for } j = 1, 2, \\ Y_2(m+1)^{(u,v)}Y_1(m+2)^{(-2u,-2v)} & \text{for } j = 3. \end{cases} \tag{4.18}$$

Here, we set $Y_0(k)^{(u,v)} = 1$.

Remark 4.3. Using the above notation, we may write

$$\begin{aligned} A_1(m) &= X_1(m)^{(0,1)}X_2(m)^{(0,-1)} \\ &= X_3(m-1)^{(0,1)}X_0(m-1)^{(0,-1)} \\ &= X_0(m-1)^{(0,1)}X_{\bar{3}}(m-1)^{(0,-1)} \\ &= X_{\bar{2}}(m-2)^{(0,1)}X_{\bar{1}}(m-2)^{(0,-1)}, \\ A_2(m) &= X_2(m)^{(0,1)}X_3(m)^{(0,-1)} \\ &= X_{\bar{3}}(m-1)^{(0,1)}X_{\bar{2}}(m-1)^{(0,-1)}. \end{aligned}$$

This is very useful when computing Kashiwara action on monomials written in terms of $X_j(m)^{(u,v)}$ or $X_{\bar{j}}(m)^{(u,v)}$.

Proposition 4.4. *Each element of $\mathcal{M}(\infty)$ may be written uniquely in the form*

$$\begin{aligned} M =& X_1(-1)^{(2,-b_2^{-1}-b_3^{-1}-b_0^{-1}-b_{\bar{3}}^{-1}-b_{\bar{2}}^{-1}-b_{\bar{1}}^{-1})}X_2(-1)^{(0,b_2^{-1})}X_3(-1)^{(0,b_3^{-1})} \\ &\cdot X_0(-1)^{(0,b_0^{-1})}X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}X_{\bar{1}}(-1)^{(0,b_{\bar{1}}^{-1})} \\ &\cdot X_2(-2)^{(1,-b_3^{-2})}X_3(-2)^{(0,b_3^{-2})} \end{aligned} \tag{4.19}$$

where $b_i^j \geq 0$ for all i,j and $b_0^{-1} \leq 1$. Conversely, any element in $\mathcal{M}^{\mathcal{E}}$ of this form is an element of $\mathcal{M}(\infty)$.

The Kashiwara operator action on $\mathcal{M}^{\mathcal{E}}$ may be rewritten as given below for elements of $\mathcal{M}(\infty)$ of the form (4.19). Elements of the above form constitutes $\mathcal{M}(\infty)$ and this set is closed under Kashiwara operator actions.

(1) Kashiwara actions $\tilde{f}_1$ and $\tilde{e}_1$:

- Consider the following ordered sequence of some components of M.

$$X_{\bar{1}}(-1)^{(0,b_{\bar{1}}^{-1})}X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}X_0(-1)^{(0,b_0^{-1})}X_3(-1)^{(0,b_3^{-1})}X_2(-1)^{(0,b_2^{-1})}.$$

- Under each of the components

$$X_{\bar{1}}(-1)^{(0,b_{\bar{1}}^{-1})}, X_0(-1)^{(0,b_0^{-1})}, X_2(-1)^{(0,b_2^{-1})},$$

  given in the above sequence, write b_j^{-1}-many 1's and under $X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}$, write $(2b_{\bar{3}}^{-1})$-many 1's. Also, under each of the components

$$X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}, X_0(-1)^{(0,b_0^{-1})},$$

  write b_j^{-1}-many 0's and under $X_3(-1)^{(0,b_3^{-1})}$, write $(2b_3^{-1})$-many 0's.
- From this sequence of 1's and 0's, successively cancel out each $(0,1)$-pair to obtain a sequence of 1's followed by 0's (reading from left to right). This remaining 1 and 0 sequence is called the ***1-signature of M***.
- Depending on the component X corresponding to the leftmost 0 of the 1-signature of M, we define $\tilde{f}_1 M$ as follows:

$$\tilde{f}_1 M = \begin{cases} MX_{\bar{2}}(-1)^{(0,-1)}X_{\bar{1}}(-1)^{(0,1)} = MA_1(1)^{-1} & \text{if } X = X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}, \\ MX_0(-1)^{(0,-1)}X_{\bar{3}}(-1)^{(0,1)} = MA_1(0)^{-1} & \text{if } X = X_0(-1)^{(0,b_0^{-1})}, \\ MX_3(-1)^{(0,-1)}X_0(-1)^{(0,1)} = MA_1(0)^{-1} & \text{if } X = X_3(-1)^{(0,b_3^{-1})}. \end{cases} \tag{4.20}$$

  We define

$$\tilde{f}_1 M = MX_1(-1)^{(0,-1)}X_2(-1)^{(0,1)} = MA_1(-1)^{-1} \tag{4.21}$$

if no 0 remains.

- Depending on the component X corresponding to the rightmost 1 of the 1-signature of M, we define $\tilde{e}_1 M$ as follows:

$$\tilde{e}_1 M = \begin{cases} MX_{\bar{2}}(-1)^{(0,1)}X_{\bar{1}}(-1)^{(0,-1)} = MA_1(1) & \text{if } X = X_{\bar{1}}(-1)^{(0,b_{\bar{1}}^{-1})}, \\ MX_0(-1)^{(0,1)}X_{\bar{3}}(-1)^{(0,-1)} = MA_1(0) & \text{if } X = X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}, \\ MX_3(-1)^{(0,1)}X_0(-1)^{(0,-1)} = MA_1(0) & \text{if } X = X_0(-1)^{(0,b_0^{-1})}, \\ MX_1(-1)^{(0,1)}X_2(-1)^{(0,-1)} = MA_1(-1) & \text{if } X = X_2(-1)^{(0,b_2^{-1})}. \end{cases}$$

We define $\tilde{e}_1 M = 0$ if no 1 remains.

(2) Kashiwara actions $\tilde{f}_2$ and $\tilde{e}_2$:

- Consider the following finite ordered sequence of some components of M.

$$X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}X_3(-1)^{(0,b_3^{-1})}X_2(-1)^{(0,b_2^{-1})}X_3(-2)^{(0,b_3^{-2})}.$$

- Under each of the components

$$X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}, X_3(-1)^{(0,b_3^{-1})}, X_3(-2)^{(0,b_3^{-2})},$$

from the above sequence, write b_j^k-many 1's, and under each

$$X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}, X_2(-1)^{(0,b_2^{-1})},$$

write b_j^{-1}-many 0's.

- From this sequence of 1's and 0's, successively cancel out each $(0,1)$-pair to obtain a sequence of 1's followed by 0's. This remaining 1 and 0 sequence is called the ***2-signature of** M*.
- Depending on the component X corresponding to the leftmost 0 of the 2-signature of M, we define $\tilde{f}_2 M$ as follows:

$$\tilde{f}_2 M = \begin{cases} MX_{\bar{3}}(-1)^{(0,-1)}X_{\bar{2}}(-1)^{(0,1)} = MA_2(0)^{-1} & \text{if } X = X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^{-1})}, \\ MX_2(-1)^{(0,-1)}X_3(-1)^{(0,1)} = MA_2(-1)^{-1} & \text{if } X = X_2(-1)^{(0,b_2^{-1})}. \end{cases} \tag{4.22}$$

We define

$$\tilde{f}_2 M = MX_2(-2)^{(0,-1)}X_3(-2)^{(0,1)} = MA_2(-2)^{-1} \tag{4.23}$$

if no 0 remains.

- Depending on the component X corresponding to the rightmost 1 of the 2-signature of M, we define $\tilde{e}_2 M$ as follows:

$$\tilde{e}_2 M = \begin{cases} MX_{\bar{3}}(-1)^{(0,1)}X_{\bar{2}}(-1)^{(0,-1)} = MA_2(0) & \text{if } X = X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^{-1})}, \\ MX_2(-1)^{(0,1)}X_3(-1)^{(0,-1)} = MA_2(-1) & \text{if } X = X_3(-1)^{(0,b_3^{-1})}, \\ MX_2(-2)^{(0,1)}X_3(-2)^{(0,-1)} = MA_2(-2) & \text{if } X = X_3(-2)^{(0,b_3^{-2})}. \end{cases}$$

We define $\tilde{e}_2 M = 0$ if no 1 remains.

Proposition 4.5. *The set $\mathcal{M}(\infty)$ forms a $U_q(G_2)$-subcrystal of $\mathcal{M}^{\mathcal{E}}$.*

Recall from Theorem 3.4 that the set $\mathcal{T}(\infty)$ gives a description of the crystal $\mathcal{B}(\infty)$. We define a canonical map $\Theta : \mathcal{T}(\infty) \to \mathcal{M}(\infty)$ by setting, for each tableau $T \in \mathcal{T}(\infty)$ with second row consists of b_3^2-many 3-boxes and just one 2-box, and with first row consists of b_j^1-many j-boxes, for each $j \succ 1$, and (b_3^2+2)-many 1-boxes, $\Theta(T) = M$, where

$$\begin{aligned} M =& X_1(-1)^{(2,-b_2^1-b_3^1-b_0^1-b_{\bar{3}}^1-b_{\bar{2}}^1-b_{\bar{1}}^1)} X_2(-1)^{(0,b_2^1)} X_3(-1)^{(0,b_3^1)} \\ & \cdot X_0(-1)^{(0,b_0^1)} X_{\bar{3}}(-1)^{(0,b_{\bar{3}}^1)} X_{\bar{2}}(-1)^{(0,b_{\bar{2}}^1)} X_{\bar{1}}(-1)^{(0,b_{\bar{1}}^1)} \\ & \cdot X_2(-2)^{(1,-b_3^2)} X_3(-2)^{(0,b_3^2)} \in \mathcal{M}(\infty). \end{aligned}$$

It is obvious that this map Θ is well-defined and that it is actually bijective.

The new expression of the action of Kashiwara operators on $\mathcal{M}(\infty)$ follows the process for defining it on $\mathcal{T}(\infty)$. Hence, the map Θ naturally commutes with the Kashiwara operators $\tilde{f}_i$ and $\tilde{e}_i$.

Theorem 4.6. ([10]) *There exists a $U_q(G_2)$-crystal isomorphism*

$$\mathcal{T}(\infty) \xrightarrow{\sim} \mathcal{M}(\infty) \tag{4.24}$$

which maps T_∞ to M_∞.

5. Cliff's description

Let us recall the abstract crystal $\mathcal{B}_i = \{b_i(k) | k \in \mathbf{Z}\}$ introduced in [6] for each $i \in I$. It has the following maps defining the crystal structure.

$$\begin{aligned} &\mathrm{wt}\, b_i(k) = k\alpha_i, && && \\ &\varphi_i(b_i(k)) = k, && \varepsilon_i(b_i(k)) = -k, && \\ &\varphi_i(b_j(k)) = -\infty, && \varepsilon_i(b_j(k)) = -\infty, && \text{for } i \neq j, \\ &\tilde{f}_i(b_i(k)) = b_i(k-1), && \tilde{e}_i(b_i(k)) = b_i(k+1), && \\ &\tilde{f}_i(b_j(k)) = 0, && \tilde{e}_i(b_j(k)) = 0, && \text{for } i \neq j. \end{aligned}$$

From now on, we will denote the element $b_i(0)$ by b_i. We next cite the tensor product rule on crystals.

Proposition 5.1. ([6]) *Let $\mathcal{B}^k (1 \le k \le n)$ be crystals with $b^k \in \mathcal{B}^k$. We set*

$$a_k = \varepsilon_i(b^k) - \sum_{1 \le v < k} \langle h_i, \mathrm{wt}(b^v) \rangle. \tag{5.1}$$

Then we have

(1) $\tilde{e}_i(b^1 \otimes \cdots \otimes b^n) = b^1 \otimes \cdots \otimes b^{k-1} \otimes \tilde{e}_i b^k \otimes b^{k+1} \otimes \cdots \otimes b^n$ *if $a_k > a_v$ for $1 \le v < k$ and $a_k \ge a_v$ for $k < v \le n$,*

(2) $\tilde{f}_i(b^1 \otimes \cdots \otimes b^n) = b^1 \otimes \cdots \otimes b^{k-1} \otimes \tilde{f}_i b^k \otimes b^{k+1} \otimes \cdots \otimes b^n$ *if $a_k \ge a_v$ for $1 \le v < k$ and $a_k > a_v$ for $k < v \le n$.*

Kashiwara has shown [6] the existence of an injective strict crystal morphism

$$\Psi : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \otimes \mathcal{B}_{i_k} \otimes \mathcal{B}_{i_{k-1}} \otimes \cdots \otimes \mathcal{B}_{i_1} \tag{5.2}$$

which sends the highest weight element u_∞ to $u_\infty \otimes b_{i_k} \otimes \cdots \otimes b_{i_1}$, for any sequence $S = i_1, i_2, \cdots, i_k$ of numbers in the index set I of simple roots. In [1], Cliff uses this to give a description of $\mathcal{B}(\infty)$ for all finite classical types, with a specific choice of sequence S. It is our goal to do this for type G_2. This was also dealt with in [6] Example 2.2.7 by Kashiwara and in [12] by Nakashima and Zelevinsky.

Proposition 5.2. *We define*

$$\mathcal{B}(1) = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \text{ and } \mathcal{B}(2) = \mathcal{B}_2.$$

Consider the subset of crystal $\mathcal{B}(\infty) \otimes \mathcal{B}(1) \otimes \mathcal{B}(2)$ *given by*

$$\mathcal{I}(\infty) = \{u_\infty \otimes \beta_1 \otimes \beta_2\},$$

where

(5.3) $\beta_1 = b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \in \mathcal{B}(1)$,

(5.4) $\beta_2 = b_2(-k_{2,2}) \in \mathcal{B}(2)$,

and where $k_{u,v}$ are any nonnegative integers such that

$$0 \le k_{1,\bar{2}} \le k_{1,\bar{3}} \le k_{1,3}/2 \le k_{1,2} \le k_{1,1}.$$

The set $\mathcal{I}(\infty)$ forms a $U_q(G_2)$-subcrystal of $\mathcal{B}(\infty) \otimes \mathcal{B}(1) \otimes \mathcal{B}(2)$.

Proof. It suffices to show that the action of Kashiwara operators satisfy the following properties :

$$\tilde{f}_i\mathcal{I}(\infty) \subset \mathcal{I}(\infty), \quad \tilde{e}_i\mathcal{I}(\infty) \subset \mathcal{I}(\infty) \cup \{0\},$$

for all $i \in I$.

We will compute the value $\tilde{f}_i$ on each element of $\mathcal{I}(\infty)$, using the tensor product rule given in Proposition 5.1. First, we compute the finite sequence $\{a_k\}$ set by (5.1) for

$$\begin{aligned} b &= u_\infty \otimes \beta_1 \otimes \beta_2 \\ &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \otimes b_2(-k_{2,2}). \end{aligned}$$

In the $i = 1$ case, we have

$$\begin{aligned} &a_1 = 0, \quad a_3 = a_5 = a_7 = -\infty, \\ &a_2 = k_{1,\bar{2}}, \quad a_4 = k_{1,3} + 2k_{1,\bar{2}} - 3k_{1,\bar{3}}, \\ &a_6 = k_{1,1} + 2k_{1,\bar{2}} - 3k_{1,\bar{3}} + 2k_{1,3} - 3k_{1,2}, \end{aligned}$$

and for $i = 2$ case,

$$\begin{aligned} &a_1 = 0, \quad a_2 = a_4 = a_6 = -\infty, \\ &a_3 = k_{1,\bar{3}} - k_{1,\bar{2}}, \quad a_5 = k_{1,2} - k_{1,\bar{2}} + 2k_{1,\bar{3}} - k_{1,3}, \\ &a_7 = k_{2,2} - k_{1,\bar{2}} + 2k_{1,\bar{3}} - k_{1,3} + 2k_{1,2} - k_{1,1}. \end{aligned}$$

By Proposition 5.1, we obtain the following three candidates of $\tilde{f}_i(b)$ for each i:

$$\begin{aligned} \tilde{f}_1(b) &= u_\infty \otimes \tilde{f}_1(b_1(-k_{1,\bar{2}})) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \\ &\qquad \otimes b_2(-k_{2,2}) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}} - 1) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})\big) \\ &\qquad \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2), \end{aligned}$$

when $a_2 \ge a_k$ for $1 \le k < 2$ and $a_2 > a_k$ for $2 < k \le 7$,

$$\begin{aligned} \tilde{f}_1(b) &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes \tilde{f}_1(b_1(-k_{1,3})) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \\ &\qquad \otimes b_2(-k_{2,2}) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3} - 1) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})\big) \\ &\qquad \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2), \end{aligned}$$

when $a_4 \geq a_k$ for $1 \leq k < 4$ and $a_4 > a_k$ for $4 < k \leq 7$,

$$\begin{aligned}\tilde{f}_1(b) &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes \tilde{f}_1(b_1(-k_{1,1})) \\ &\qquad \otimes b_2(-k_{2,2}) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}-1)\big) \\ &\qquad \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2),\end{aligned}$$

when $a_6 \geq a_k$ for $1 \leq k < 6$ and $a_6 > a_k$ for $6 < k \leq 7$,

$$\begin{aligned}\tilde{f}_2(b) &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes \tilde{f}_2(b_2(-k_{1,\bar{3}})) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \\ &\qquad \otimes b_2(-k_{2,2}) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}-1) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})\big) \\ &\qquad \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2),\end{aligned}$$

when $a_3 \geq a_k$ for $1 \leq k < 3$ and $a_3 > a_k$ for $3 < k \leq 7$,

$$\begin{aligned}\tilde{f}_2(b) &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes \tilde{f}_2(b_2(-k_{1,2})) \otimes b_1(-k_{1,1}) \\ &\qquad \otimes b_2(-k_{2,2}) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}-1) \otimes b_1(-k_{1,1})\big) \\ &\qquad \otimes b_2(-k_{2,2}) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2),\end{aligned}$$

when $a_5 \geq a_k$ for $1 \leq k < 5$ and $a_5 > a_k$ for $5 < k \leq 7$,

$$\begin{aligned}\tilde{f}_2(b) &= u_\infty \otimes b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1}) \\ &\qquad \otimes \tilde{f}_2(b_2(-k_{2,2})) \\ &= u_\infty \otimes \big(b_1(-k_{1,\bar{2}}) \otimes b_2(-k_{1,\bar{3}}) \otimes b_1(-k_{1,3}) \otimes b_2(-k_{1,2}) \otimes b_1(-k_{1,1})\big) \\ &\qquad \otimes b_2(-k_{2,2}-1) \in u_\infty \otimes \mathcal{B}(1) \otimes \mathcal{B}(2),\end{aligned}$$

when $a_7 \geq a_k$ for $1 \leq k < 7$. And for each case given above, we obtain the following result from conditions for the sequence a_k. In the $i = 1$ case, $k_{u,v}$ values, appearing in the above expression for $\tilde{f}_i b$, are nonnegative integers satisfying

- $k_{1,\bar{2}} + 1 \leq k_{1,\bar{3}} \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1}$,
  when $a_2 \geq a_k$ for $1 \leq k < 2$ and $a_2 > a_k$ for $2 < k \leq 7$,
- $k_{1,\bar{2}} \leq k_{1,\bar{3}} \leq (k_{1,3}+1)/2 \leq k_{1,2} \leq k_{1,1}$,
  when $a_4 \geq a_k$ for $1 \leq k < 4$ and $a_4 > a_k$ for $4 < k \leq 7$,
- $k_{1,\bar{2}} \leq k_{1,\bar{3}} \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1} + 1$,
  when $a_6 \geq a_k$ for $1 \leq k < 6$ and $a_6 > a_k$ for $6 < k \leq 7$,

and in the $i = 2$ case,

- $0 \leq k_{1,\bar{2}} \leq k_{1,\bar{3}} + 1 \leq k_{1,3}/2 \leq k_{1,2} \leq k_{1,1}$,
  when $a_3 \geq a_k$ for $1 \leq k < 3$ and $a_3 > a_k$ for $3 < k \leq 7$,
- $0 \leq k_{1,\bar{2}} \leq k_{1,\bar{3}} \leq k_{1,3}/2 \leq k_{1,2} + 1 \leq k_{1,1}$,
  when $a_5 \geq a_k$ for $1 \leq k < 5$ and $a_5 > a_k$ for $5 < k \leq 7$,
- $0 \leq k_{2,2} + 1$,
  when $a_7 \geq a_k$ for $1 \leq k < 7$.

Thus the action of Kashiwara operator $\tilde{f}_i$ is closed on $\mathcal{I}(\infty)$.

Proof for the statements concerning $\tilde{e}_i$ may be done in a similar manner. □

The notation β_1 and β_2 appearing in this proposition will be used a few more times in this section.

Theorem 5.3. *There exists a $U_q(G_2)$-crystal isomorphism*

$$\mathcal{T}(\infty) \xrightarrow{\sim} \mathcal{I}(\infty) \subset \mathcal{B}(\infty) \otimes \mathcal{B}(1) \otimes \mathcal{B}(2), \tag{5.5}$$

which maps T_∞ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_1) \otimes (b_2)$.

Proof. With the help of tensor product rules, it is easy to check the compatibility of this map with Kashiwara operators. Other parts of the proof are similar or easy. Hence we shall only write out the maps and give no proofs.

For each tableau with the second row consisting of b_3^2-many 3-boxes and just one 2-box, and with the first row consisting of b_j^1-many j-boxes, for each $j \succ 1$, and $(b_3^2 + 2)$-many 1-boxes, we may map it to the element $u_\infty \otimes \beta_1 \otimes \beta_2$ where

$$k_{1,1} = \sum_{j=2}^{\bar{1}} b_j^1, \quad k_{1,2} = \sum_{j=3}^{\bar{1}} b_j^1, \quad k_{1,3} = 2(\sum_{j=\bar{3}}^{\bar{1}} b_j^1) + b_0^1,$$

$$k_{1,\bar{3}} = b_{\bar{2}}^1 + b_{\bar{1}}^1, \quad k_{1,\bar{2}} = b_{\bar{1}}^1, \quad k_{2,2} = b_3^2.$$

Conversely, an element $u_\infty \otimes \beta_1 \otimes \beta_2$ is sent to the tableau whose shape we describe below row-by-row.

- The first row consists of

  $(k_{1,\bar{2}})$-many $\bar{1}$s, $(k_{1,\bar{3}} - k_{1,\bar{2}})$-many $\bar{2}$s,
  $\lfloor k_{1,3}/2 - k_{1,\bar{3}} \rfloor$-many $\bar{3}$s, $((A + B) - (A' + B'))$-many 0s,
  $(k_{1,2} - k_{1,3}/2)$-many 3s, $(k_{1,1} - k_{1,2})$-many 2s, and
  $(k_{2,2} + 2)$-many 1s.

- The second row consists of

  $(k_{2,2})$-many 3s and one 2.

Here, $A = k_{1,2} - k_{1,3}/2$, $B = k_{1,3}/2 - k_{1,\bar{3}}$, $A' = \lfloor k_{1,2} - k_{1,3}/2 \rceil$, and $B' = \lfloor k_{1,3}/2 - k_{1,\bar{3}} \rfloor$. □

Since the above theorem has shown $\mathcal{B}(\infty) \cong \mathcal{I}(\infty)$ as crystals, image of the injective crystal morphism

$$\Psi : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \otimes \mathcal{B}(1) \otimes \mathcal{B}(2) = \mathcal{B}(\infty) \otimes (\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1) \otimes (\mathcal{B}_2),$$

which maps u_∞ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_1) \otimes (b_2)$ is $\mathcal{I}(\infty)$.

In the following corollary, a description of $\mathcal{B}(\infty)$ for G_2-type is given following Cliff's method. A specific choice for the index sequence of crystals $S = (1, 2, 1, 2, 1, 2)$ corresponding to a longest word $w_0 = s_1 s_2 s_1 s_2 s_1 s_2$ of the Weyl group is used.

Corollary 5.4. *Image of the injective strict crystal morphism*

$$\Psi : \mathcal{B}(\infty) \to \mathcal{B}(\infty) \otimes (\mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{B}_1) \otimes (\mathcal{B}_2),$$

which maps u_∞ to $u_\infty \otimes (b_1 \otimes b_2 \otimes b_1 \otimes b_2 \otimes b_1) \otimes (b_2)$ is given by

$$\Psi(\mathcal{B}(\infty)) = \mathcal{I}(\infty) = \{u_\infty \otimes \beta_1 \otimes \beta_2\}.$$

We illustrate the correspondence between $\mathcal{T}(\infty)$ and $\Psi(\mathcal{B}(\infty))$ for type G_2.

Example 5.5. The marginally large tableau

1	1	1	1	2	0	$\bar{3}$	$\bar{3}$	$\bar{1}$
2	3	3						

of $\mathcal{T}(\infty)$ corresponds to the element

$$u_\infty \otimes b_1(-1) \otimes b_2(-1) \otimes b_1(-7) \otimes b_2(-4) \otimes b_1(-5) \otimes b_2(-2)$$

of $\Psi(\mathcal{B}(\infty))$ under the map given in Theorem 5.3.

Remark 5.6. We can provide maps between the two giving crystal isomorphisms

$$\mathcal{M}(\infty) \xrightarrow{\sim} \Psi(\mathcal{B}(\infty))$$

in both directions. The maps can easily be drawn from Theorem 4.6 and Theorem 5.3.

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