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<th>Title</th>
<th>Lefschetz properties, Schur polynomials and Jordan canonical forms (Combinatorial Representation Theory and Related Topics)</th>
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<tr>
<td>Author(s)</td>
<td>Iima, Kei-ichiro</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2008), B8: 37-41</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/174303">http://hdl.handle.net/2433/174303</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Lefschetz properties, Schur polynomials and Jordan canonical forms

Kei-ichiro Iima

Graduate School of Natural Science and Technology,
Okayama University.

This report is a survey of the preprint [6] which is a joint work with Ryo Iwamatsu, but partly joint work with Professor Yuji Yoshino. For further details, please refer to it.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \), and \( J(\alpha, m) \) means the Jordan block with eigenvalue \( \alpha \in k \) and size \( m \). We shall consider the problem of finding out a Jordan canonical form of \( J(\alpha, m) \otimes J(\beta, n) \), where \( \otimes \) means \( \otimes_k \) (\( m \leq n \)).

Over an algebraically closed base field of characteristic zero, this problem has been solved by many authors including T. Harima and J. Watanabe [4], and A. Martsinkovsky and A. Vlassov [8] etc. M. Herschend [5] solve it for extended quivers of type \( \tilde{A}_n \), with arbitrary orientation and any \( n \). In this note we solve it for any characteristic \( p \geq 0 \). That is, we obtain two way to determine the Jordan decomposition of the tensored matrix \( J(\alpha, m) \otimes J(\beta, n) \).

In the case of \( \alpha \beta = 0 \), the tensored matrix \( J(\alpha, m) \otimes J(\beta, n) \) has the same direct sum decomposition as in Theorem 2.1 independently of characteristic of the base field \( k \) in Proposition 2.6. In the case of \( \alpha \beta \neq 0 \), our problem is reduced to the problem of finding the indecomposable decomposition of \( R \) as a \( k[Z] \)-module, where \( R \) means the quotient ring \( k[x, y]/(x^m, y^n) \), \( Z = x + y \) and \( k[x, y] \) be a polynomial ring over \( k \). We regard finding the indecomposable decomposition of \( R \) as calculating the partition \( c = (c_1, c_2, \ldots, c_r) \) of \( mn \) in Lemma 2.5. Then, we are able to determine the Jordan decomposition of tensored matrix \( J(\alpha, m) \otimes J(\beta, n) \).

2. Main results

Throughout this section, let \( k \) be an algebraically closed field. For an integer \( m \geq 1 \) and an element \( \alpha \in k \), let

\[
J(\alpha, m) = \begin{pmatrix}
\alpha & 1 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
& & & \alpha \\
\end{pmatrix}
\]

denote the Jordan block of size \( m \times m \) with an eigenvalue \( \alpha \).

**Theorem 2.1.** [8, Theorem 2] Suppose that \( k \) has characteristic zero. Then the following holds for integers \( m \leq n \) and \( \alpha, \beta \in k \):

\[
J(\alpha, m) \otimes J(\beta, n) = \begin{cases}
J(0, m)^{\otimes n-m+1} \oplus \bigoplus_{i=1}^{2m-2} J(0, m - \lfloor \frac{i}{2} \rfloor) & \text{if } \alpha = 0 \neq \beta \\
J(0, m)^{\otimes m} & \text{if } \alpha = 0 \neq \beta \\
J(0, n)^{\otimes n} & \text{if } \alpha \neq 0 = \beta \\
\bigoplus_{i=1}^{n} J(\alpha\beta, m + n + 1 - 2i) & \text{if } \alpha \neq 0 \neq \beta
\end{cases}
\]

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Remark 2.2. If one of the eigenvalues $\alpha$ and $\beta$ equals zero, then the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ has the same direct sum decomposition as in Theorem 2.1 independently of characteristic of the base field $k$ (Proposition 2.6).

**Theorem 2.3.** There is an algorithm to determine the Jordan decomposition of the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$.

**Remark 2.4.** (1) The matrix $J(\alpha, m)$ represents the action of $X$ on $k[X]/(X-\alpha)^{m}$ as a $k[X]$-module.

(2) The tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ is triangular. Therefore its eigenvalue is $\alpha \beta$.

(3) One has an isomorphism

$$k[X]/(X-\alpha)^{m} \otimes k[Y]/(Y-\beta)^{n} \cong k[X, Y]/((X-\alpha)^{m}, (Y-\beta)^{n})$$

of $k$-algebras.

Tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ represents the action of $XY$ on $k[X, Y]/((X-\alpha)^{m}, (Y-\beta)^{n})$ as a $k[XY]$-module.

**Lemma 2.5.** Put $R = k[X, Y]/((X-\alpha)^{m}, (Y-\beta)^{n})$, which we regard as a $k[Z]$-module through the map $k[Z] \rightarrow R$ given by $Z \mapsto XY$. Then there is a sequence of integers such that $c_{1} \geq c_{2} \geq \cdots \geq c_{r} \geq 1$

$$R \cong \bigoplus_{i=1}^{r} k[Z]/(Z-\alpha \beta)^{c_{i}}$$

of $k[Z]$-modules.

This means that $J(\alpha, m) \otimes J(\beta, n) = \bigoplus_{i=1}^{r} J(\alpha \beta, c_{i})$. We can regard $c = (c_{1}, c_{2}, \ldots, c_{r})$ as a partition of $mn$ in obvious manner. The main problem is to determine the partition $c$. For this purpose let $b = (b_{1}, b_{2}, \ldots, b_{m+n-1})$ be the partition conjugate to $c$. Put $z = Z - \alpha \beta$. Note that $b_{i} = \#\{j|c_{j} \geq i\} = \dim_{k}(z^{i-1}R/z^{i}R)$. Setting $a_{i} = \dim_{k}(R/z^{i}R)$, we have $b_{i} = a_{i} - a_{i-1}$. Therefore, it is sufficient that we calculate the value of $a_{i}$ for each case.

If one of the eigenvalues $\alpha$ and $\beta$ equals zero, then the result is independent of the characteristic of $k$ as we show in the next proposition.

**Proposition 2.6.** We have the following equalities;

$$a_{i} = \begin{cases} (m+n)i - i^{2} & (1 \leq i \leq m) \text{ if } \alpha = 0 = \beta \\ ni & (1 \leq i \leq m) \text{ if } \alpha = 0 \neq \beta \\ mi & (1 \leq i \leq n) \text{ if } \alpha \neq 0 = \beta \end{cases}$$

**Proof.** Put $x = X - \alpha$ and $y = Y - \beta$.

(1) The case $\alpha = 0 = \beta$:

Since $R/z^{i}R = k[x, y]/(x^{m}, y^{n}, (xy)^{i})$, we have $a_{i} = (m+n)i - i^{2}$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0, m)^{\oplus n} \oplus \bigoplus_{i=1}^{2m-2} J(0, m-\lceil \frac{i}{2} \rceil)$.

(2) The case $\alpha = 0 \neq \beta$:

Since $R/z^{i}R = k[x, y]/(x^{m}, y^{n}, x^{i})$ as $y + \beta$ is a unit in $k[x, y]/(x^{m}, y^{n})$, we have $a_{i} = ni$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0, m)^{\oplus n}$.

(3) The case $\alpha \neq 0 = \beta$:

Since $R/z^{i}R = k[x, y]/(x^{m}, y^{n}, y^{i})$ as $x + \alpha$ is a unit in $k[x, y]/(x^{m}, y^{n})$, we have $a_{i} = mi$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0, n)^{\oplus m}$. 

$\square$
In the case of \( \alpha \neq 0 \neq \beta \), then we have the following isomorphism of \( k \)-algebras, given by \( X - \alpha \mapsto x \), \( Y - \beta \mapsto y' \) and \( \frac{\alpha y'}{y+\beta} \mapsto y \):

\[
k[X, Y]/((X-\alpha)^m, (Y-\beta)^n, (XY-\alpha\beta)^{\iota}) \cong k[x, y]/(x^{rn}, y^{n}, (x+y)^{l}).
\]

Using this isomorphism together with [3, Proposition 4.4][4, Proposition 8], we have the following proposition in the case of characteristic zero.

**Proposition 2.7.** Suppose that \( \alpha \neq 0 \neq \beta \) and that \( k \) has characteristic zero. Then we have

\[
b = \left( \begin{array}{c} m, m, \ldots, m, m-1, m-1, m-2, m-2, \ldots, 1, 1 \end{array} \right).
\]

**Proof.** It is easy to show by using \( x + y \in k[x, y]/(x^{m}, y^{n}) \) is a Lefschetz element [4]. Therefore we get \( J(\alpha, m) \otimes J(\beta, n) = \bigoplus_{i=1}^{m} J(\alpha \beta, m+n+1-2i) \).

We consider in the rest the case where \( \alpha \neq 0 \neq \beta \) and that \( k \) is of positive characteristic \( p \). Put \( S = k[x, y] \), \( R = k[x, y]/(x^{m}, y^{n}) \) and \( A^{(l)} = R/(x+y)^{l}R \). To determine \( a_{l} = \dim_{k}(A^{(l)}) \). We have the following isomorphism:

\[
A^{(l)} \cong k[x, y, z]/(x^{m}, y^{n}, z^{t}, x+y+z).
\]

Therefore we may assume that \( m \leq n \leq l \) without loss of generality. For each integer \( l \) satisfying \( m \leq n \leq l \leq m+n-1 \), we describe

\[
(x+y)^{l} \equiv \sum_{i=0}^{l} \binom{l}{m-n+i} x^{m-n+i} y^{l-m-n+i} \pmod{(x^{m}, y^{n})}.
\]

We set \( q_{1} = \binom{l}{m-1}, q_{2} = \binom{l}{m-2}, \ldots, q_{r} = \binom{l}{l-n+1} \) and \( r = m+n-1-l \).

We obtain the representation matrix of \( R \xrightarrow{(x+y)^{l}} R \) with respect to the natural base \( \{1, x, y, x^{2}, xy, y^{2}, \ldots, x^{m-1}y^{n-1}\} \) as follows;

\[
\begin{pmatrix}
H_{0} & \vdots \\
H_{1} & \ddots \\
& \ddots \\
& & H_{r-2} \\
& & & H_{r-1}
\end{pmatrix},
\]

where

\[
H_{i} = \begin{pmatrix}
q_{i+1} & q_{i} & \cdots & q_{1} \\
q_{i+2} & q_{i+1} & \cdots & q_{2} \\
& \vdots & \ddots & \vdots \\
& & & q_{r} \\
& & & q_{r-1} & \cdots & q_{r-i}
\end{pmatrix}.
\]

For each \( 0 \leq i \leq r-1 \) the matrix \( H_{i} \) is an \((r-i) \times (i+1)\) matrix whose entries are integers. We denote by \( I_{i+1}(H_{i}) \) the ideal of \( \mathbb{Z} \) generated by \((i+1)\)-minors of \( H_{i} \) for \( 0 \leq i \leq r-1 \). Obviously there exists an integer \( \delta_{i} \geq 0 \) such that \( I_{i+1}(H_{i}) = \delta_{i}\mathbb{Z} \). From the argument in the case of characteristic zero in [3, Proposition 4.4], we have \( I_{i+1}(H_{i}) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0 \), particularly \( \delta_{i} \neq 0 \), for any \( 0 \leq i \leq [(r-1)/2] \).
Proposition 2.8. Under the same notation as above, for each $l$ satisfying $1 \leq m \leq n \leq l \leq m + n - 1$, and for each $i$ satisfying $0 \leq i \leq \lfloor (r - 1)/2 \rfloor (r = m + n - 1 - l)$, the following equalities hold;
\[
\delta_i = \gcd(S_{\lambda_i}(1, 1, \ldots, 1) | j = (j_1, j_2, \ldots, j_{i+1}), 1 \leq j_1 < j_2 < \ldots < j_{i+1} \leq r - i},
\]
where $\lambda^j$ is the partition conjugate to $\mu^j = (m - j_1, m - j_2 - 1, \ldots, m - j_{i+1} - i)$, and $S_{\lambda}$ is the Schur polynomial.

Proof. Computation using Jacobi-Trudi formula [2],[7].

Let
\[
0 \rightarrow S(-a) \oplus S(-b) \rightarrow S(-m) \oplus S(-n) \oplus S(-l) \rightarrow (x^{m}, y^{n}, (x+y)^l) \rightarrow A^{(l)} \rightarrow 0
\]
be a minimal graded $S$-free resolution of $A^{(l)}$, where $1 \leq m \leq n \leq l \leq a \leq b$. The Hilbert-Burch theorem implies that $a + b = m + n + l$, and the Hilbert series of $A^{(l)}$ is given as
\[
H_{A^{(l)}}(t) = \frac{1-t^{m}-t^{n}-t^{l}+t^{a}+t^{b}}{(1-t)^{2}}.
\]
It follows from this that $\dim_k(A^{(l)}) = mn+ml+nl-ab$. Letting $i_0 = \min\{i | \delta_i \equiv 0 \pmod{p}\}$, we get $a = l+i_0$ and $b = m+n-i_0$, since $a$ is the least value of degrees of relations of $(x^{m}, y^{n}, (x+y)^l)$. Thus, we can calculate the dimension of the $k$-vector space $A^{(l)}$, and hence the indecomposable decomposition of $J(\alpha, m) \otimes J(\beta, n)$.

Theorem 2.9. We are able to compute a Jordan canonical form of $J(\alpha, m) \otimes J(\beta, n)$ by taking the following steps:

(1) Every $\delta_i$ is determined.
(2) For each $1 \leq l \leq m + n - 1$, $a_l$ is determined.
(3) The partition $b$ is determined.
(4) The partition $c$ is determined.
(5) The Jordan decomposition of tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ is determined.

From the discussion in Theorem 2.9, one immediately obtains the following.

Theorem 2.10. The tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ has the same direct sum decomposition as in Theorem 2.1 if $\text{char}(k) \geq m + n - 1$ or $I_{i+1}(H_i) \otimes_k k \neq 0$ for any $0 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$.

Proof. It is easy to show that if $\text{char}(k) \geq m + n - 1$ then $I_{i+1}(H_i) \otimes_k k \neq 0$ for any $0 \leq i \leq \lfloor \frac{r-1}{2} \rfloor$.

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