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Lefschetz properties, Schur polynomials and Jordan canonical forms

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This report is a survey of the preprint [6] which is a joint work with Ryo Iwamatsu, but partly joint work with Professor Yuji Yoshino. For further details, please refer to it.

1. Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$, and $J(\alpha, m)$ means the Jordan block with eigenvalue $\alpha \in k$ and size m. We shall consider the problem of finding out a Jordan canonical form of $J(\alpha, m) \otimes J(\beta, n)$, where $\otimes$ means $\otimes_k$ $(m \leq n)$.

Over an algebraically closed base field of characteristic zero, this problem has been solved by many authors including T. Harima and J. Watanabe [4], and A. Martsinkovsky and A. Vlassov [8] etc. M. Herschend [5] solve it for extended Dynkin quivers of type $\tilde{\mathbb{A}}_n$, with arbitrary orientation and any n. In this note we solve it for any characteristic $p \geq 0$. That is, we obtain two way to determine the Jordan decomposition of the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$.

In the case of $\alpha\beta = 0$, the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ has the same direct sum decomposition as in Theorem 2.1 independently of characteristic of the base field k in Proposition 2.6. In the case of $\alpha\beta \neq 0$, our problem is reduced to the problem of finding the indecomposable decomposition of R as a $k[Z]$-module, where R means the quotient ring $k[x, y]/(x^m, y^n)$, $Z = x + y$ and $k[x, y]$ be a polynomial ring over k . We regard finding the indecomposable decomposition of R as calculating the partition $\mathbf{c} = (c_1, c_2, \dots, c_r)$ of mn in Lemma 2.5. Then, we are able to determine the Jordan decomposition of tensored matrix $J(\alpha, m) \otimes J(\beta, n)$.

2. Main results

Throughout this section, let k be an algebraically closed field. For an integer $m \geq 1$ and an element $\alpha \in k$, let

$$J(\alpha, m) = \begin{pmatrix} \alpha & 1 & & \\ & \ddots & \ddots & \\ & & \alpha & 1 \\ & & & \alpha \end{pmatrix}$$

denote the Jordan block of size $m \times m$ with an eigenvalue α.

Theorem 2.1. [8, Theorem 2] *Suppose that k has characteristic zero. Then the following holds for integers $m \leq n$ and $\alpha, \beta \in k$:*

$$J(\alpha, m) \otimes J(\beta, n) = \begin{cases} J(0, m)^{\oplus n-m+1} \oplus \bigoplus_{i=1}^{2m-2} J(0, m - \lceil \frac{i}{2} \rceil) & \text{if } \alpha = 0 = \beta \\ J(0, m)^{\oplus n} & \text{if } \alpha = 0 \neq \beta \\ J(0, n)^{\oplus m} & \text{if } \alpha \neq 0 = \beta \\ \bigoplus_{i=1}^{m} J(\alpha\beta, m + n + 1 - 2i) & \text{if } \alpha \neq 0 \neq \beta \end{cases}$$

Remark 2.2. If one of the eigenvalues α and β equals zero, then the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ has the same direct sum decomposition as in Theorem 2.1 independently of characteristic of the base field k (Proposition 2.6).

Theorem 2.3. *There is an algorithm to determine the Jordan decomposition of the tensored matrix $J(\alpha, m) \otimes J(\beta, n)$.*

Remark 2.4. (1)The matrix $J(\alpha, m)$ represents the action of X on $k[X]/(X-\alpha)^m$ as a $k[X]$-module.

(2)The tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ is triangular. Therefore its eigenvalue is $\alpha\beta$.

(3)One has an isomorphism

$$k[X]/(X-\alpha)^m \otimes k[Y]/(Y-\beta)^n \cong k[X,Y]/((X-\alpha)^m, (Y-\beta)^n)$$

of k-algebras.

Tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ represents the action of XY on $k[X,Y]/((X-\alpha)^m, (Y-\beta)^n)$ as a $k[XY]$-module.

Lemma 2.5. *Put $R = k[X,Y]/((X-\alpha)^m, (Y-\beta)^n)$, which we regard as a $k[Z]$-module through the map $k[Z] \to R$ given by $Z \mapsto XY$. Then there is a sequence of integers such that $c_1 \geq c_2 \geq \cdots \geq c_r \geq 1$*

$$R \cong \bigoplus_{i=1}^{r} k[Z]/(Z-\alpha\beta)^{c_i}$$

of $k[Z]$-modules.

This means that $J(\alpha, m) \otimes J(\beta, n) = \bigoplus_{i=1}^{r} J(\alpha\beta, c_i)$. We can regard $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ as a partition of mn in obvious manner. The main problem is to determine the partition $\mathbf{c}$. For this purpose let $\mathbf{b} = (b_1, b_2, \ldots, b_{m+n-1})$ be the partition conjugate to $\mathbf{c}$. Put $z = Z - \alpha\beta$. Note that $b_i = \#\{j | c_j \geq i\} = \dim_k(z^{i-1}R/z^iR)$. Setting $a_i = \dim_k(R/z^iR)$, we have $b_i = a_i - a_{i-1}$. Therefore, it is sufficient that we calculate the value of a_i for each case.

If one of the eigenvalues α and β equals zero, then the result is independent of the characteristic of k as we show in the next proposition.

Proposition 2.6. *We have the following equalities;*

$$a_i = \begin{cases} (m+n)i - i^2 \ \ (1 \leq i \leq m) & \textit{if} \ \ \alpha = 0 = \beta \\ ni \ \ (1 \leq i \leq m) & \textit{if} \ \ \alpha = 0 \neq \beta \\ mi \ \ (1 \leq i \leq n) & \textit{if} \ \ \alpha \neq 0 = \beta \end{cases}$$

Proof. Put $x = X - \alpha$ and $y = Y - \beta$.

(1) The case $\alpha = 0 = \beta$:
Since $R/z^iR = k[x,y]/(x^m, y^n, (xy)^i)$, we have $a_i = (m+n)i - i^2$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0,m)^{\oplus n-m+1} \oplus \bigoplus_{i=1}^{2m-2} J(0, m - \lceil \frac{i}{2} \rceil)$.

(2) The case $\alpha = 0 \neq \beta$:
Since $R/z^iR = k[x,y]/(x^m, y^n, x^i)$ as $y + \beta$ is a unit in $k[x,y]/(x^m, y^n)$, we have $a_i = ni$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0,m)^{\oplus n}$.

(3) The case $\alpha \neq 0 = \beta$:
Sinse $R/z^iR = k[x,y]/(x^m, y^n, y^i)$ as $x + \alpha$ is a unit in $k[x,y]/(x^m, y^n)$, we have $a_i = mi$. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = J(0,n)^{\oplus m}$. □

In the case of $\alpha \neq 0 \neq \beta$, then we have the following isomorphism of k-algebras, given by $X - \alpha \mapsto x$, $Y - \beta \mapsto y'$ and $\frac{\alpha y'}{y'+\beta} \mapsto y$:

$$k[X,Y]/((X-\alpha)^m, (Y-\beta)^n, (XY-\alpha\beta)^l) \cong k[x,y]/(x^m, y^n, (x+y)^l).$$

Using this isomorphism together with [3, Proposition 4.4][4, Proposition 8], we have the following proposition in the case of characteristic zero.

Proposition 2.7. *Suppose that $\alpha \neq 0 \neq \beta$ and that k has characteristic zero. Then we have*

$$\mathbf{b} = (\underbrace{m, m, \ldots, m}_{n-m+1}, m-1, m-1, m-2, m-2, \ldots, 1, 1).$$

Proof. It is easy to show by using $x + y \in k[x,y]/(x^m, y^n)$ is a Lefschetz element [4]. Therefore we get $J(\alpha, m) \otimes J(\beta, n) = \bigoplus_{i=1}^{m} J(\alpha\beta, m+n+1-2i)$. □

We consider in the rest the case where $\alpha \neq 0 \neq \beta$ and that k is of positive characteristic p. Put $S = k[x,y]$, $R = k[x,y]/(x^m, y^n)$ and $A^{(l)} = R/(x+y)^l R$. To determine $a_l = \dim_k(A^{(l)})$. We have the following isomorphism:

$$A^{(l)} \cong k[x,y,z]/(x^m, y^n, z^l, x+y+z).$$

Therefore we may assume that $m \leq n \leq l$ without loss of generality. For each integer l satisfying $m \leq n \leq l \leq m+n-1$, we describe

$$(x+y)^l \equiv \binom{l}{m-1} x^{m-1}y^{l-m+1} + \binom{l}{m-2} x^{m-2}y^{l-m+2} + \cdots + \binom{l}{l-n+1} x^{l-n+1}y^{n-1} \pmod{(x^m, y^n)}.$$

We set $q_1 = \binom{l}{m-1}$, $q_2 = \binom{l}{m-2}, \cdots$, $q_r = \binom{l}{l-n+1}$ and $r = m+n-1-l$.

We obtain the representation matrix of $R \xrightarrow{(x+y)^l} R$ with respect to the natural base $\{1, x, y, x^2, xy, y^2, \ldots, x^{m-1}y^{n-1}\}$ as follows;

$$\begin{pmatrix} H_0 & & & & & \\ & H_1 & & & & \\ & & H_2 & & & \\ & & & \ddots & & \\ & & & & H_{r-2} & \\ & & & & & H_{r-1} \end{pmatrix},$$

where

$$H_i = \begin{pmatrix} q_{i+1} & q_i & \cdots & q_1 \\ q_{i+2} & q_{i+1} & \cdots & q_2 \\ \vdots & \vdots & \ddots & \vdots \\ q_r & q_{r-1} & \cdots & q_{r-i} \end{pmatrix}.$$

For each $0 \leq i \leq r-1$ the matrix H_i is an $(r-i) \times (i+1)$ matrix whose entries are integers. We denote by $I_{i+1}(H_i)$ the ideal of $\mathbb{Z}$ generated by $(i+1)$-minors of H_i for $0 \leq i \leq r-1$. Obviously there exists an integer $\delta_i \geq 0$ such that $I_{i+1}(H_i) = \delta_i\mathbb{Z}$. From the argument in the case of characteristic zero in [3, Proposition 4.4], we have $I_{i+1}(H_i) \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$, particularly $\delta_i \neq 0$, for any $0 \leq i \leq \lfloor (r-1)/2 \rfloor$.

Proposition 2.8. *Under the same notation as above, for each l satisfying $1 \le m \le n \le l \le m+n-1$, and for each i satisfying $0 \le i \le \lfloor (r-1)/2 \rfloor (r = m+n-1-l)$, the following equalities hold;*

$$\delta_i = \gcd\{S_{\lambda^j}(\underbrace{1,1,\ldots,1}_{l}) | j = (j_1, j_2, \ldots, j_{i+1}), 1 \le j_1 < j_2 < \ldots < j_{i+1} \le r-i\},$$

where λ^j is the partition conjugate to $\mu^j = (m-j_1, m-j_2-1, \ldots, m-j_{i+1}-i)$, and S_{λ^j} is the Schur polynomial.

Proof. Computation using Jacobi-Trudi formula [2] ,[7]. □

Let

$$0 \to S(-a) \oplus S(-b) \to S(-m) \oplus S(-n) \oplus S(-l) \xrightarrow{(x^m, y^n, (x+y)^l)} S \to A^{(l)} \to 0$$

be a minimal graded S-free resolution of $A^{(l)}$, where $1 \le m \le n \le l \le a \le b$. The Hilbert-Burch theorem implies that $a+b = m+n+l$, and the Hilbert series of $A^{(l)}$ is given as

$$H_{A^{(l)}}(t) = \frac{1 - t^m - t^n - t^l + t^a + t^b}{(1-t)^2}.$$

It follows from this that $\dim_k(A^{(l)}) = mn + ml + nl - ab$. Letting $i_0 = \min\{i | \delta_i \equiv 0 \pmod p\}$, we get $a = l + i_0$ and $b = m+n-i_0$, since a is the least value of degrees of relations of $(x^m, y^n, (x+y)^l)$. Thus, we can calculate the dimension of the k-vector space $A^{(l)}$, and hence the indecomposable decomposition of $J(\alpha, m) \otimes J(\beta, n)$.

Theorem 2.9. *We are able to compute a Jordan canonical form of $J(\alpha, m) \otimes J(\beta, n)$ by taking the following steps:*

(1) *Every $\delta_\bullet$ is determined.*
(2) *For each $1 \le l \le m+n-1$, a_l is determined.*
(3) *The partition $\mathbf{b}$ is determined.*
(4) *The partition $\mathbf{c}$ is determined.*
(5) *The Jordan decomposition of tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ is determined.*

From the discussion in Theorem 2.9, one immediately obtains the following.

Theorem 2.10. *The tensored matrix $J(\alpha, m) \otimes J(\beta, n)$ has the same direct sum decomposition as in Theorem 2.1 if $\mathrm{char}(k) \ge m+n-1$ or $I_{i+1}(H_i) \otimes_{\mathbb{Z}} k \ne 0$ for any $0 \le i \le \lfloor \frac{r-1}{2} \rfloor$.*

Proof. It is easy to show that if $\mathrm{char}(k) \ge m+n-1$ then $I_{i+1}(H_i) \otimes_{\mathbb{Z}} k \ne 0$ for any $0 \le i \le \lfloor \frac{r-1}{2} \rfloor$. □

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