NOTE ON THE COHOMOLOGY OF FINITE CYCLIC COVERINGS

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Abstract. We introduce the height of a normal cyclic p-fold covering and show a cohomological relation between the base and the total spaces of the covering in terms of the height. We also interpret the height in terms of the category weight.

1. Statement of results

The purpose of this note is to show a cohomological property of a normal cyclic p-fold covering with respect to a certain cup-length type invariant of the covering. Let p be a prime and let $E \to B$ be a normal cyclic p-fold covering where B is path connected. Suppose $p = 2$. In [Ko], Kozlov defined the *height* of the covering $\mathrm{h}(E)$ as the maximum n such that $w_1(E)^n \neq 0$, where $w_1(E)$ is the first Stiefel-Whitney class of the covering. By a chain level consideration, he proved

$$H^{\mathrm{h}(E)}(E; \mathbb{Z}/2) \neq 0.$$

This also follows immediately from the Gysin sequence of the double covering $E \to B$. We would like to generalize this result to any prime p. Let p be an arbitrary prime. Let C_p be a cyclic group of order p and let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. The *height* of the covering can be generalized as

$$\mathrm{h}(E) = \max\{n \mid \rho^* : H^n(BC_p; \mathbb{Z}/p) \to H^n(B; \mathbb{Z}/p) \text{ is non-trivial}\}.$$

We remark here that the height of a normal cyclic p-fold covering is closely related with the ideal-valued cohomological index theory of Fadell and Husseini [FH1] and hence the Borsuk-Ulam theorem. We will interpret the height in terms of the category weight introduced by Fadell and Husseini [FH2] and studied further by Rudyak [Ru] and Strom [S]. The most difficult point in generalizing the result of Kozlov is the non-existence of the Gysin sequence for the covering $E \to B$ when p is odd. However, we define the corresponding spectral sequence by which we prove:

Theorem 1.1. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. Then*

$$H^{\mathrm{h}(E)}(E; \mathbb{Z}/p) \neq 0.$$

As an immediate corollary, we have:

1991 *Mathematics Subject Classification.* 55M35.

Key words and phrases. covering, cohomology, spectral sequence, category weight.

The second author is partially supported by the Grant-in-Aidfor Scientific Research (C)(No.25400087) from the Japan Society for Promotion of Sciences.

Corollary 1.2. *Let $E \to B$ be a normal cyclic p-fold covering, where B is path-connected. If* $\mathrm{h}(E) \geq n$ *and* $H^n(E;\mathbb{Z}/p) = 0$*, it holds that* $\mathrm{h}(E) \geq n+1$.

In section 2, we construct a spectral sequence for a normal cyclic p-fold covering which calculate the mod p cohomology of the total space from the base space whose differential is shown to be given as a certain higher Massey product of Kraines [Kr]. Using this spectral sequence, we prove Theorem 1.1. In section 3, we interpret the height of a normal cyclic p-fold covering in terms of the category weight introduced by Fadell and Husseini [FH2] and elaborated by [Ru] and [S].

Acknowledgement. The authors are grateful to the referee for leading them to the proof using the spectral sequence. In the first version of the paper, the proof is done in a quite elementary but lengthy way using the Smith special cohomology. (cf. [B])

2. Proof of Theorem 1.1

Throughout this section, let p be an odd prime and the coefficient of cohomology is $\mathbb{Z}/p$.

2.1. **Spectral sequence.** Let $E \to B$ be a normal p-fold covering where B is path-connected. In this subsection, we introduce a spectral sequence which calculates the mod p cohomology of E from B. Analogous spectral sequences were considered in [F] and [Re]. We first set notation. Let $\rho : B \to BC_p$ be the classifying map of the covering $E \to B$. Recall that the mod p cohomology of BC_p is given as

$$H^*(BC_p) = \Lambda(u) \otimes \mathbb{Z}/p[v], \quad \beta u = v, \quad |u| = 1,$$

where β is the Bockstein operation. We denote the cohomology classes $\rho^*(u)$ and $\rho^*(v)$ of B by $\bar{u}$ and $\bar{v}$, respectively. Let $R[C_p]$ denote the group ring of C_p over a ring R. Note that the singular chain complex $S_*(E)$ is a free $\mathbb{Z}[C_p]$-module. We regard $\mathbb{Z}/p[C_p]$ and $\mathbb{Z}/p$ as $\mathbb{Z}[C_p]$-modules by the modulo p reduction and the trivial C_p-action, respectively. Then there are natural isomorphisms

$$H^*(\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p[C_p])) \cong H^*(E) \quad \text{and} \quad H^*(\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p)) \cong H^*(B). \tag{2.1}$$

We now fix a generator g of C_p and put $\tau = 1 - g \in \mathbb{Z}/p[C_p]$. Observe that $\mathbb{Z}/p[C_p] = \mathbb{Z}/p[\tau]/(\tau^p)$. Consider the filtration

$$0 \subset \tau^{p-1}\mathbb{Z}/p[C_p] \subset \tau^{p-2}\mathbb{Z}/p[C_p] \subset \cdots \subset \tau\mathbb{Z}/p[C_p] \subset \mathbb{Z}/p[C_p].$$

Then there is a spectral sequence (E_r, d_r) associated with the induced filtration of the cochain complex $\mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \mathbb{Z}/p[C_p])$. By (2.1), we have

$$E_1^{s,t} \cong \begin{cases} H^t(B) & 0 \leq s \leq p-1 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad H^*(E) \tag{2.2}$$

and the degree of the differential d_r is $(-r, 1)$, where the total degree of $E_r^{s,t}$ is t. Let us identify the differential of this spectral sequence. To this end, we calculate the induced coboundary map

$\bar{\delta}$ of the associated graded cochain complex

$$\bigoplus_{i=0}^{p-1} \mathrm{Hom}_{\mathbb{Z}[C_p]}(S_*(E), \tau^i\mathbb{Z}/p[C_p]/\tau^{i-1}\mathbb{Z}/p[C_p]) \cong \bigoplus_{i=0}^{p-1} \tau^i \mathrm{Hom}_{\mathbb{Z}}(S_*(B), \mathbb{Z}/p).$$

In the special case of the universal bundle $EC_p \to BC_p$, we may put

$$\bar{\delta}(1) = \tau u_1 + \cdots + \tau^{p-1}u_{p-1}, \quad u_i \in \mathrm{Hom}_{\mathbb{Z}}(S_1(B), \mathbb{Z}/p)$$

for $1 \in \mathrm{Hom}_{\mathbb{Z}}(S_0(B), \mathbb{Z}/p)$. Consider the map $E \xrightarrow{\tilde{\rho}\times\pi} EC_p \times B$, where $\tilde{\rho}$ is a lift of ρ and π is the projection. Then we see that

$$\bar{\delta}x = \delta x + \tau\rho^*(u_1)x + \cdots + \tau^{p-1}\rho^*(u_{p-1})x. \tag{2.3}$$

for any $x \in \mathrm{Hom}_{\mathbb{Z}}(S_*(B), \mathbb{Z}/p)$ in general. If $[u_1] = 0$, $1 \in E^{1,0}$ becomes a permanent cycle in the spectral sequence (2.2) for the universal bundle $EC_p \to BC_p$, which contradicts to the contractibility of EC_p. Then by normalizing u if necessary, we may assume

$$[u_1] = u. \tag{2.4}$$

Applying (2.3) in turn to $u_1, \ldots, u_{p-1}$, we inductively see from the equality $\bar{\delta}^2 = 0$ that

$$\delta u_i = -\sum_{j<i} u_j u_{i-j} \quad \text{for} \quad i \geq 2. \tag{2.5}$$

Let $\langle x_1, \ldots, x_n\rangle_n$ stand for the n-fold Massey product in the sense of Kraines [Kr], where $\langle x_1, x_2\rangle = \pm x_1x_2$. Then by (2.3), (2.4) and (2.5), we obtain that $d_r x$ is represented by an element of $\pm\langle \bar{u}, \ldots, \bar{u}, x\rangle_{r+1}$ whose defining system $\{x_{ij}\}_{1\leq i\leq j\leq r+1}$ satisfies $x_{ij} = \rho^*(u_{j-i+1})$ for $j \leq r$, where $x_{i,r+1}$ can be an arbitrary cochain satisfying the condition of defining systems. Hence by [Kr], $\{x_{ij}\}_{1\leq i\leq j\leq r}$ is the pullback of a defining system for

$$\langle u, \ldots, u\rangle_k = \begin{cases} \{0\} & k < p \\ \{v\} & k = p. \end{cases} \tag{2.6}$$

Recall the following associativity formula of higher Massey products [May]. Suppose a defining system for $\langle x_1, \ldots, x_{n-1}\rangle_{n-1}$ extends to those of $\langle x_{k+1}, \ldots, x_n\rangle_{n-k}$. Put $\{x'_{ij}\}_{1\leq i\leq j\leq k+1}$

$$x'_{ij} = \pm x_{ij} \quad \text{for} \quad j \leq k \quad \text{and} \quad x'_{i,k+1} = \sum_{l=k+1}^{n-1} \pm x_{il}x_{ln} \quad \text{for} \quad 2 \leq i \leq k+1. \tag{2.7}$$

Then $\{x'_{ij}\}_{1\leq i\leq j\leq k+1}$ is a defining system for $\langle x_1, \ldots, x_k, \langle x_{k+1}, \ldots, x_n\rangle_{n-k}\rangle_{k+1}$ and the resulting element x satisfies

$$x = \pm y x_n$$

for some $y \in \langle x_1, \ldots, x_{n-1}\rangle_{n-1}$. Consider the defining system of $\langle \bar{u}, \ldots, \bar{u}\rangle_{r+r'}$ given by $\rho^*(u_i)$ for $r + r' \leq p$. By the above observation on $d_{r'}x$, we can extend this defining system to that for

$\langle \bar{u}, \ldots, \bar{u}, x\rangle_{r'+1}$ as (2.7) so that the resulting element x' represents $d_{r'}x$. Moreover, by (2.6) and the above associativity formula, we have

$$d_r x' = \begin{cases} 0 & r + r' < p \\ \pm \bar{v}x & r + r' = p. \end{cases} \tag{2.8}$$

2.2. **Proof of Theorem 1.1.** We prove the result by calculating the spectral sequence (2.2). We first consider the case $\mathrm{h}(E) = 2m + 1$. We can easily see that in the spectral sequence for the universal bundle $EC_p \to BC_p$, it holds that $d_r^{p-1,2m+1}uv^m = 0$ and av^{m+1} according as $r < p - 1$ and $r = p - 1$, where $a \in (\mathbb{Z}/p)^\times$. Then it follows from naturality of the spectral sequence that

$$d_r^{p-1,2m+1}\bar{u}\bar{v}^m = \rho^*(d_r^{p-1,2m+1}uv^m) = \begin{cases} 0 & r < p - 1 \\ \rho^*(av^{m+1}) = 0 & r = p - 1, \end{cases}$$

implying that $H^{2m+1}(E) \neq 0$.

We next consider the case $\mathrm{h}(E) = 2m$. Let r be the maximum integer such that $\bar{v}^m \in E_1^{s,2m}$ survives at the E_r-term for all $0 \le s \le p-1$. Suppose that $d_r^{s,2m}\bar{v}^m \neq 0$ for some s. Then we have

$$d_r^{r,2m}\bar{v}^m \neq 0. \tag{2.9}$$

If $\bar{v}^m \in E_1^{r-1,2m}$ survives at the $E_{r'}$-term for $r \le r'$ and satisfies $d_{r'}^{r+r'-1,2m-1}x = \bar{v}^m$ for some x, we have

$$d_r^{r,2m}\bar{v}^m \in \pm\langle \bar{u}, \ldots, \bar{u}, \bar{v}^m\rangle_{r+1}, \quad \bar{v}^m \in \pm\langle \bar{u}, \ldots, \bar{u}, x\rangle_{r'+1} \quad \text{and} \quad r + r' \le p,$$

where defining systems for both higher Massey products are described above. Then it follows from (2.8) that

$$d_r^{r,2m}\bar{v}^m = \begin{cases} 0 & r + r' < p \\ \pm \bar{v}x & r + r' = p \end{cases}$$

in the E_r-term. The upper case contradicts to (2.9). Let us consider the lower case. If $r' = 1$, $\bar{u}x = \bar{v}$ and then $\beta(\bar{u}x) = 0$. If $r' \ge 2$, $\bar{u}x = 0$ and so $\beta(\bar{u}x) = 0$. Then in both cases, we have $\bar{v}x = \bar{u}(\beta x)$, and so $\bar{v}x$ turns out to be trivial in the E_r-term, which contradicts to (2.9). Therefore we obtain that $\bar{v}^m \in E_1^{r-1,2m}$ is a permanent cycle, implying that $H^{2m}(E) \neq 0$. Suppose next that $d_r^{s,2m-1}x = \bar{v}^m$ for some s. Then for any $r + r' \le p$, we can choose a representative of $d_{r'}^{r+1,2m}\bar{v}^m$ as above, and hence by an argument similar to the above case, we see that $\bar{v}^m \in E_1^{r+1,2m}$ is a permanent cycle, implying that $H^{2m}(E) \neq 0$. Therefore the proof of Theorem 1.1 is completed.

3. Height and category weight

In this section, we interpret the height of a normal cyclic p-fold covering in terms of the category weight introduced by Fadell and Husseini [FH2] and studied further by Rudyak [Ru] and Strom [S]. As a consequence, the relation between the height of a normal cyclic p-fold covering and the Lusternik-Schnirelmann (L-S, for short) category of the classifying map of the covering becomes clear. Recall that the L-S category of a space X, denoted by $\mathrm{cat}(X)$, is the minimum n such that there is a cover of X by $(n+1)$-open sets each of which is contractible in X. In [BG], the L-S category of a space was generalized to a map: The L-S category of a map $f : X \to Y$, denoted by $\mathrm{cat}(f)$, is the minimum integer n such that there exists an open cover $X = U_0 \cup \cdots \cup U_n$ where the restriction of f to U_i is null-homotopic for all i. Observe that

$$\mathrm{cat}(f) \leq \mathrm{cat}(1_X) = \mathrm{cat}(X).$$

It is useful to evaluate $\mathrm{cat}(f)$ by the so-called Ganea spaces. Let $G_n(Y)$ be the n^{th} Ganea space of Y and let $\pi_n : G_n(Y) \to Y$ be the projection. See [CLOT] for definition. We know that $\mathrm{cat}(f) \leq n$ if and only if there is a map $g : X \to G_n(Y)$ satisfying $\pi_n \circ g \simeq f$. The homotopy invariant version of the category weight of a space X due to Rudyak [Ru] and Strom [S] is a lower bound for the L-S category of X which refines the cup-length. As in [?], cohomologically, the idea of the homotopy invariant version of the category weight due to Rudyak and Strom is summarized as

$$\mathrm{wgt}(X; R) = \max\{n \,|\, \pi_n^* : \overline{H}^*(X; R) \to \overline{H}^*(G_n(X); R) \text{ is injective}\},$$

where R is a ring and $\overline{H}^*$ denotes the reduced cohomology. By definition, $\mathrm{wgt}(X; R)$ is bounded above by $\mathrm{cat}(X)$. Given a map $f : X \to Y$, we can easily generalize the above definition for a space to a map as

$$\mathrm{wgt}(f; R) = \max\{n \,|\, \text{there exists } y \in \overline{H}^*(Y; R) \text{ satisfying } f^*(y) \neq 0, \\ \text{and } \pi_n^*(z) \neq 0 \text{ whenever } f^*(z) \neq 0 \text{ for } z \in \overline{H}^*(Y; R)\}.$$

Notice that $\mathrm{wgt}(1_X; R) = \mathrm{wgt}(X; R)$ analogously to the L-S category. Obviously, we have

$$\mathrm{cat}(f) \geq \mathrm{wgt}(f; R).$$

Let us consider the relation between the height of a normal cyclic covering and the category weight. Suppose a space Y is path-connected. In general, since the homotopy fiber of the projection $\pi_n : G_n(Y) \to Y$ has the homotopy type of the join of $(n+1)$-copies of ΩY which is n-connected, the induced map $\pi_n^* : H^k(Y; R) \to H^k(G_n(Y); R)$ is an isomorphism for $k < n$ and is injective for $k = n$. See [CLOT]. We specialize to the case $Y = BC_p$. Recall that $G_n(BC_p)$ has the homotopy type of the quotient of the join of the $(n+1)$-copies of C_p by the diagonal free C_p-action, implying that $G_n(BC_p)$ has the homotopy type of an n-dimensional CW-complex. Then the induced map $\pi_n^* : H^k(BC_p; R) \to H^k(G_n(BC_p); R)$ is the zero map for $k > n$. Summarizing, the induced map $\pi_n^* : H^k(BC_p; \mathbb{Z}/p) \to H^k(G_n(BC_p); \mathbb{Z}/p)$ is injective for $k \leq n$ and is the zero map for $k > n$, and hence for a map $f : X \to BC_p$, we have

$$\mathrm{wgt}(f; \mathbb{Z}/p) = \min\{n \,|\, f^* : H^n(BC_p; \mathbb{Z}/p) \to H^n(X; \mathbb{Z}/p) \text{ is non-trivial}\}.$$

Therefore we obtain:

Proposition 3.1. *Let $E \to B$ be a normal cyclic p-fold covering with the classifying map $\rho : B \to BC_p$, where B is path-connected. Then*

$$\mathrm{h}(E) = \mathrm{wgt}(\rho; \mathbb{Z}/p) \leq \mathrm{cat}(\rho) \leq \mathrm{cat}(B).$$

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