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A geometrical formulation of a space-time finite integration (FI) method is studied for application to electromagnetic wave propagation calculations. Based on the Hodge duality and Lorentzian metric, a modified relation is derived between the incidence matrices of space-time primal and dual grids. A systematic method to construct the Maxwell grid equations on the space-time primal and dual grids is developed. The geometrical formulation is implemented on a simple space-time grid, which is proven equivalent to an explicit time-marching scheme of the space-time FI method.

Index Terms—Finite integration method, graph theory, Hodge duality, space-time grid.

I. INTRODUCTION

THE finite integration (FI) method [1]-[5] has been studied to accomplish time-domain electromagnetic field computations using unstructured spatial grids. The FI method derives grid-based Maxwell equations using incidence matrices based on the dual computational-grid geometry. Graph theory enables a systematic construction of the spatial dual grid from the primal grid geometry. However, the geometry description is restricted to the spatial domain. Accordingly, similar to the FDTD method [6], the FI method uses a uniform time-step, which is restricted by the Courant-Friedrichs-Lewy condition [7] based on the smallest spatial grid size.

Previous work [8] introduced a space-time FI method that achieves non-uniform time-steps naturally on the three-dimensional (3D) space-time grid with 2D space. The Hodge dual grid was proposed in [9] to construct the 4D space-time FI method to a photonic band computation was reported in [12]. However, it is not always a simple task to construct the Maxwell grid equations on these dual space-time grids. To realize a systematic derivation of Maxwell grid equations, a graph-theory-based formulation for the space-time dual grids is required. This paper discusses a geometrical formulation of the 3D space-time FI method that is based on the Hodge duality and the Lorentzian metric but is not a straightforward extension of the conventional spatial FI formulation.

II. FINITE INTEGRATION METHOD ON A SPACE-TIME GRID

A. Electromagnetics in Space-Time

The Maxwell equations are described in the differential form as

$$\begin{align*}
\text{d}F &= 0, \\
\text{d}G &= J.
\end{align*}$$

(1)

In the coordinate system $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, $F, G$ and $J$ are written as

$$\begin{align*}
F &= -\Sigma^3_{i=1} E_i \text{d}x^0 \text{d}x^i + \Sigma^3_{j=1} B_j \text{d}x^k \text{d}x^l, \\
G &= \Sigma^3_{i=1} H_i \text{d}x^0 \text{d}x^i + \Sigma^3_{j=1} D_j \text{d}x^k \text{d}x^l, \\
J &= \rho \text{d}t \text{d}x^1 \text{d}x^2 \text{d}x^3 - \Sigma^3_{j=1} c J_j \text{d}x^0 \text{d}x^k \text{d}x^l
\end{align*}$$

(2)

where $c$ is the speed of light, $\rho$ is the electric charge density and $(j, k, l)$ is a cyclic permutation of $(1, 2, 3)$. The integral form of (1) is given as

$$\begin{align*}
\oint_{\partial \Omega_y} F &= 0, \\
\oint_{\partial \Omega_d} G &= \oint_{\Omega_d} J
\end{align*}$$

(3)

where $\Omega_y$ and $\Omega_d$ are hypersurfaces in space-time.

For simplicity, assuming the uniformity along the $z$-direction, this paper discusses the FI formulation for the electromagnetic field $(B_z, E_x, E_y)$ in the $(w, x, y)$-3D free space-time [8], where $w = ct$. Accordingly, $F$ and $G$ are written as

$$\begin{align*}
F &= B_z \text{d}x \text{d}y + E_x \text{d}y \text{d}w - E_y \text{d}w \text{d}x, \\
G &= H_z \text{d}w - D_y \text{d}x + D_x \text{d}y
\end{align*}$$

(4)

where $(E_x, E_y) = (E_x/c, E_y/c)$, $H_z = H_z/c$. Defining 3D vectors $F$ and $G$ as

$$\begin{align*}
F &= (B_z, E_y, -E_x), \\
G &= (H_z, -D_y, D_x)
\end{align*}$$

(5)

the integral form is written without source term as

$$\begin{align*}
\int_S F \cdot n \text{d}S &= 0, \\
\int_C G \cdot t \text{d}s &= 0
\end{align*}$$

(6)

where $n$ is the normal vector at each point on the closed surface $S$ and $t$ the tangent vector at each point on the closed curve $C$. The Euclidean metric is used for the dot product operation.

The space-time FI method uses discretized variables as

$$f = \int_{\partial S} F \cdot n \text{d}S, \\
g = \int_C G \cdot t \text{d}s$$

(7)
where \( p \) is a face of a primal grid and \( s \) is an edge of a dual grid. To express the constitutive equation between \( f \) and \( g \) simply, \( n \) and \( t \) are given as [8]

\[
\mathbf{n} = (n_w, n_x, n_y), \quad \mathbf{t} = (n_w, -n_x, -n_y),
\]

(8)

Fig. 1 illustrates the geometrical relation between \( n \) and \( t \), where \( s \) is orthogonal to \( p \) in the Lorentzian 3D space-time. The relation between \( F \cdot \mathbf{n} \) and \( G \cdot \mathbf{t} \) is given as

\[
F \cdot \mathbf{n} = ZG \cdot \mathbf{t}
\]

(9)

where \( Z \) is the impedance of the medium. Thus, \( f \) is related to \( g \) as

\[
f = zg
\]

(10)

\[
z = Z \frac{\Delta S}{\Delta t}
\]

(11)

where \( \Delta S \) is the area of \( p \) and \( \Delta t \) is the length of \( \mathbf{s} \).

Ref. [9] extended the dual-grid construction above to the 4D space-time, called the Hodge dual grid. It is based on the Hodge duality with the Lorentzian metric between \( F \) and \( G \), where the metric is modified in materials depending on the speed of light.

![Fig. 1: Relation of primal face and dual edge in space-time grid when (a) \( \mathbf{n} \cdot \mathbf{t} > 0 \) and (b) \( \mathbf{n} \cdot \mathbf{t} < 0 \).](image)

![Fig. 2: Edges and faces on (a) the primal grid and (b) the dual grid.](image)

### B. Explicit Time-Marching Scheme

Refs. [8] and [9] have shown explicit time-marching schemes for 3D and 4D space-time FI analyses of electromagnetic wave propagation. This subsection presents an explicit time-marching scheme on a simple 2D space-time grid with 1D space to relate to the geometrical formulation described later.

Fig. 2 illustrates 2D space-time primal and dual grids that have temporal step sizes \( \Delta w \) and \( \Delta w/2 \) and spatial step size \( \Delta x \) along the \( x \)- and \( y \)- directions.

Based on (5) and (7), the variables in Fig. 2 have the following meaning: \( b \): magnetic flux, \( c \): electromotive force, \( f \): the composition of \( b \) and \( e \), \( h \): magnetomotive force, \( d \): electric flux, and \( g \): the composition of \( h \) and \( d \). The arrow directions in Fig. 2 are based on (7) and (9) using the definition (5) and (8). Note that the arrow direction of \( d \) is opposite to that of \( e \). These do not correspond directly to the directions of \( E \) and \( D \) in the Euclidean space.

The explicit time-marching scheme is given as follows. According to a numerical examination in [10], the scheme is stable when \((l_a - 1)^2 + (\Delta w)^2/2 < 1\).

Variables \( d_{2i-1}^{n+1/4} \) and \( e_{2i-1}^{n+1/4} \) are given as

\[
d_{2i-1}^{n+1/4} = d_{2i-1}^{n-1/4} - (h_{2i-1}^n - h_{2i-1}^{n+1}) \quad (12)
\]

\[
e_{2i-1}^{n+1/4} = e_{2i-1}^{n+1/4} - \frac{\Delta t}{4} \Delta w \Delta x
\]

(13)

On the primal grid, \( f_{2i-1}^{n+1/4} \) and \( g_{2i-1}^{n+1/4} \) are given as

\[
f_{2i-1}^{n+1/4} = e_{2i-1}^{n+1/4} + b_{2i-1}^n \quad (14)
\]

\[
g_{2i-1}^{n+1/4} = \frac{1}{z_f} f_{2i}^{n+1/4} - z_{e_1} Z \frac{\Delta w \Delta x}{2l_a}
\]

(15)

On the dual grid, \( d_{2i} \) and \( e_{2i} \) are updated using

\[
d_{2i}^{n+1/2} = d_{2i}^{n-1/2} + h_{2i}^n - h_{2i}^{n+1} + g_{2i}^{n+1/2} - g_{2i}^{n-1/2} \quad (16)
\]

\[
e_{2i}^{n+1/2} = e_{2i}^{n+1/2} + \frac{\Delta t}{4} \Delta w \Delta x
\]

(17)

Similarly, \( f_{2i-1}^{n+3/4} \) and \( g_{2i-1}^{n+3/4} \) are given as

\[
f_{2i-1}^{n+3/4} = f_{2i-1}^{n+1/4} + e_{2i-1}^{n+1/2} \quad (18)
\]

\[
g_{2i-1}^{n+3/4} = \frac{1}{z_f} f_{2i}^{n+3/4} - z_{e_2} Z \frac{\Delta w \Delta x}{2l_a}
\]

(19)

On the dual grid, \( d_{2i}^{n+3/4} \) and \( e_{2i}^{n+3/4} \) are obtained from

\[
d_{2i}^{n+3/4} = e_{2i}^{n+3/4} + g_{2i}^{n+1/4} - g_{2i}^{n-1/4} \quad (20)
\]

\[
e_{2i}^{n+3/4} = e_{2i}^{n+3/4} + \frac{\Delta t}{4} \Delta w \Delta x
\]

(21)

Hence, \( b_{2i-1}^{n+1} \), \( b_{2i}^{n+1} \) and \( h_{2i-1}^{n+1} \), \( h_{2i}^{n+1} \) are given by

\[
b_{2i-1}^{n+1} = b_{2i-1}^{n+3/4} - e_{2i-1}^{n+1/2} \quad (22)
\]

\[
b_{2i}^{n+1} = b_{2i}^{n+3/4} + e_{2i}^{n+1/2}
\]

\[
h_{2i-1}^{n+1} = h_{2i-1}^{n+3/4} - e_{2i-1}^{n+1/2}
\]

\[
h_{2i}^{n+1} = h_{2i}^{n+3/4} + e_{2i}^{n+1/2}
\]

(23)
C. Incidence Matrices on 3D Euclidean Space

The FL method is generally formulated with the Maxwell grid equations using the incidence matrices from graph theory.

Let arrays \( \{n\} \), \( \{s\} \), \( \{p\} \) and \( \{v\} \) denote the sets of nodes, edges, faces, and volumes in the primal grid, respectively. These are related by incidence matrices \( G \), \( C \) and \( D \) \([1], [2], [11]\) as

\[
\partial\{s\} = [G]\{n\}, \quad \partial\{p\} = [C]\{s\}, \quad \partial\{v\} = [D]\{p\} \tag{24}
\]

where \( \partial \) denotes restriction to the boundary. Similarly, the sets of nodes, edges, faces and volumes in the dual grid are related as

\[
\partial\{\tilde{s}\} = [\tilde{G}]\{\tilde{n}\}, \quad \partial\{\tilde{p}\} = [\tilde{C}]\{\tilde{s}\}, \quad \partial\{\tilde{v}\} = [\tilde{D}]\{\tilde{p}\}. \tag{25}
\]

In the Euclidean space, the dual grid is generally constructed so the incidence matrices satisfy

\[
[\tilde{C}] = [C]^T, \quad [\tilde{D}] = -[G]^T, \quad [\tilde{G}] = -[D]^T. \tag{26}
\]

The relation above derives the Maxwell grid equations systematically from the primal grid geometry.

The space-time primal and dual grids based on (8) have a similar property to that described by (26), which gives the one-to-one correspondence between the faces \( \{s\} \) on the primal grid and the edges \( \{\tilde{s}\} \) on the dual grid. However, the directional relation between \( \{p\} \) and \( \{\tilde{s}\} \) determined by (8) differs from that given by (26). The following subsection derives the matrix relation for the space-time primal and dual grids based on (8).

D. Incidence Matrices on 3D Space-Time

A simple space-time primal grid illustrated in Fig. 3 is examined. Assuming spatial periodicity in the grid geometry, the edges \( s_i \) and faces \( p_j \) are periodically numbered for notational simplicity. Moreover, the edges and faces perpendicular to the \( y \)-axis is omitted. The direction of edges \( s_1, s_2, s_3, s_1', s_2' \) is along the \( +y \)-direction. The edge \( s_i \) on the dual grid corresponds to the face \( p_i \) on the primal grid, where their directions satisfy (8).

The geometrical relation between edges and faces on the primal grid is represented by the following equations.

\[
\begin{align*}
\partial p_1 &= s_1 - s_2, \quad \partial p_2 = -s_1 + s_2, \quad \partial p_3 = s_1 - s_3, \\
\partial p_4 &= -s_2 + s_3, \quad \partial p_5 = s_2 - s_3, \quad \partial p_6 = s_2 - s_2', \\
\partial p_7 &= s_3 - s_3', \quad \partial p_8 = -s_3 + s_3', \quad \partial p_9 = s_3 - s_1'. \quad \tag{27}
\end{align*}
\]

To examine the relation between \( \{C\} \) and \( \{\tilde{C}\} \), a subset of \( \{s\} \) and a subset of \( \{p\} \) are defined as

\[
\{s\}_{ab} = [s_1 \  s_2 \  s_3]^T, \quad \{p\}_{ab} = [p_1 \ p_2 \ \cdots \ p_9]^T. \tag{28}
\]

Omitting \( s_1', s_2' \), relation (27) is written \( \{p\}_{ab} = [C]_{ab}\{s\}_{ab} \) where \( [C]_{ab} \) is a submatrix of \( \{C\} \) and given as

\[
[C]_{ab}^T = \begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1
\end{bmatrix}. \tag{29}
\]

On the other hand, the relation between edges and faces is found to be:

\[
\begin{align*}
\partial \tilde{p}_1 &= \tilde{s}_1 - \tilde{s}_2 - \tilde{s}_3 + \tilde{s}_6', \\
\partial \tilde{p}_2 &= -\tilde{s}_1 + \tilde{s}_2 - \tilde{s}_4 + \tilde{s}_5 - \tilde{s}_6 + \tilde{s}_6' - \tilde{s}_7 + \tilde{s}_8', \\
\partial \tilde{p}_3 &= \tilde{s}_3 - \tilde{s}_4 - \tilde{s}_5 + \tilde{s}_6 - \tilde{s}_8 - \tilde{s}_9.
\end{align*} \tag{30}
\]

Corresponding to \( \{s\}_{ab} \) and \( \{p\}_{ab} \), subsets of \( \{\tilde{p}\} \) and \( \{\tilde{s}\} \) are defined as

\[
\{\tilde{p}\}_{ab} = [\tilde{p}_1 \ \tilde{p}_2 \ \tilde{p}_3]^T, \quad \{\tilde{s}\}_{ab} = [\tilde{s}_1 \ \tilde{s}_2 \ \cdots \ \tilde{s}_9]^T. \tag{31}
\]

Omitting \( \tilde{s}_6', \tilde{s}_7', \tilde{s}_8', \tilde{s}_9' \), relation (30) is written as \( \{\tilde{p}\}_{ab} = [\tilde{C}]_{ab}\{\tilde{s}\}_{ab} \) where \( [\tilde{C}]_{ab} \) is a submatrix of \( \{\tilde{C}\} \) and given as

\[
[\tilde{C}]_{ab} = \begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & 1
\end{bmatrix}. \tag{32}
\]

Comparing \( [\tilde{C}]_{ab} \) with \( [C]_{ab}^T \) shows that the elements of \( [\tilde{C}]_{ab} \) at the 3rd, 6th, and 9th columns have opposite signs to the corresponding elements of \( [C]_{ab}^T \). This sign inversion caused by (8) is illustrated in Fig. 1. If \( n \) and \( t \) given by (8) satisfy \( n \cdot t < 0 \), the direction of the edge is opposite to the direction of the face as depicted in Fig. 1(b).

Consequently, the incidence matrix \( \{\tilde{C}\} \) of dual grid based on (8) is given as

\[
[\tilde{C}] = [C]^{*T} \tag{33}
\]

where the operator \( *T \) is determined by the mapping

\[
\tilde{c}_{ij} = \begin{cases} 
  c_{ji}, & (c_{ji} \neq 0 \text{ and } n \cdot t > 0) \\
  -c_{ji}, & (c_{ji} \neq 0 \text{ and } n \cdot t < 0) \\
  0, & (c_{ji} = 0)
\end{cases} \tag{34}
\]

This relation is a consequence of the Hodge duality between \( F \) and \( G \) based on the Lorentzian metric in the 3D space-time. Using \( n \cdot t = n_x^2 - n_y^2 - n_z^2 \), the matrix \( [C]^{*T} \) can be obtained without the need for the dual grid.

Using \( \{\tilde{C}\} \), the electromagnetic field equations on the dual grid such as (12), (16) and (20) are expressed as

\[
\{\tilde{C}\}\{g\} = 0 \tag{35}
\]

where \( \{g\} \) consists of the variables defined by the second equation of (7) on the edges corresponding to \( \{\tilde{s}\} \).
The relation between faces and volumes on the primal grid is represented similarly. For instance, the volume surrounded by \( p_1, p_3, \) and \( p_4 \) is written:

\[
\partial v_1 = -p_1 + p_3 + p_4. \tag{36}
\]

These relations are represented by matrix \([D]\) in the form given by the third equation of (24). Using \([D]\), the electromagnetic field equations on the primal grid such as (14), (18) and (22) are expressed as

\[
[D\{f\} = 0. \tag{37}
\]

where \( \{f\} \) consists of the variables defined by the first equation of (7) on the faces corresponding to \( \{p\} \).

Fig. 4 summarizes the geometric relation above. In the 3D Euclidean space, \( E \) and \( B \) are assigned separately to the edges and faces, respectively whereas they are unified into \( F \) and assigned to the faces in the 3D space-time.

\[
\begin{align*}
\{n\} & \rightarrow \{g\} \rightarrow \{p\} \rightarrow \{v\} \rightarrow \{f\} \rightarrow \{\tilde{f}\} \rightarrow \{\tilde{f}^T\} \\
(E) & \rightarrow (p) \rightarrow (v) \rightarrow (F) \rightarrow (\tilde{f}) \rightarrow (\tilde{f}^T)
\end{align*}
\]

(a) 3D Euclidean space (b) 3D space-time

Fig. 4: Duality and matrix relations.

E. Maxwell Grid Equations

From (10), \( \{g\} \) is related to \( \{f\} \) as

\[
\{f\} = \{z\}\{g\} \tag{38}
\]

where \( \{z\} \) is a diagonal matrix of which elements are given by (11).

Equations (33), (35), (37), and (38) derive the space-time Maxwell grid equations systematically

\[
\begin{bmatrix}
[D] \\
\{C\}^T \{z\}^{-1}
\end{bmatrix}\{f\} = 0. \tag{39}
\]

By modifying the impedance matrix, another formulation is possible, where relation \( \{ \tilde{C} = \{ C\}^T \) holds. The modified impedance matrix \( \{ \tilde{z}^+ \} \) is defined by replacing the elements of \( \{z\} \) by \(-Z\Delta S/\Delta t\) when \( \mathbf{n} \cdot \mathbf{t} < 0 \). Thereby, \( \{C\}^T \{z\}^{-1} = \{C\}^T \{\tilde{z}^+\}^{-1} \)

F. Application Example in 2D Space-Time Grid

The FI method formulated by (39) is implemented and compared with the FI scheme explained in Subsection II.B. The propagation of an electromagnetic wave with components \( (E_y, B_z) \) is computed on the periodic 2D space-time grid shown in Fig. 2 with \( \Delta x = 1, \Delta t = 1, \cdots, 50, \Delta w = 0.5, n = 0, \cdots, 80 \), and \( l_a = 1 \). The initial condition at \( w = 0 \) is given as \( B_z = \exp(-x^2/25) \) and \( E_y = 0 \). The impedance matrix \( \{z\} \) consists of \( z_e, z_\epsilon, z_f \), and \( z_b \) that are given as (13), (15), (17) and (23). The spatially periodic boundary condition is imposed where \( e_{1n} = e_{80n} \) and \( h_{81n} = h_{1n} \).

Fig. 5 shows the distribution of \( B_z \) at \( w = 80\Delta w \), where the simulation result given by the FDTD method is also shown for comparison. The FI formulation (39) is equivalent to the FI scheme given in II.B.

III. CONCLUDING REMARKS

A geometrical formulation of the 3D space-time FI method was presented that provides a systematic method to construct the Maxwell grid equations on the space-time primal and dual grids. The relation between the incidence matrices of these space-time grids was derived based on the Hodge duality with Lorentzian metric.

Practically, the systematic formulation is used to derive or confirm the explicit time-marching scheme. The extension to the 4D space-time and its practical application will be addressed in the near future.

REFERENCES