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Geometrical Formulation of 3D Space-Time Finite Integration Method

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A geometrical formulation of a space-time finite integration (FI) method is studied for application to electromagnetic wave propagation calculations. Based on the Hodge duality and Lorentzian metric, a modified relation is derived between the incidence matrices of space-time primal and dual grids. A systematic method to construct the Maxwell grid equations on the space-time primal and dual grids is developed. The geometrical formulation is implemented on a simple space-time grid, which is proven equivalent to an explicit time-marching scheme of the space-time FI method.

Index Terms—Finite integration method, graph theory, Hodge duality, space-time grid.

I. INTRODUCTION

THE finite integration (FI) method [1]–[5] has been studied to accomplish time-domain electromagnetic field computations using unstructured spatial grids. The FI method derives grid-based Maxwell equations using incidence matrices based on the dual computational-grid geometry. Graph theory enables a systematic construction of the spatial dual grid from the primal grid geometry. However, the geometry description is restricted to the spatial domain. Accordingly, similar to the FDTD method [6], the FI method uses a uniform time-step, which is restricted by the Courant-Friedrichs-Lewy condition [7] based on the smallest spatial grid size.

Previous work [8] introduced a space-time FI method that achieves non-uniform time-steps naturally on the three-dimensional (3D) space-time grid with 2D space. The Hodge dual grid was proposed in [9] to construct the 4D space-time grid for electromagnetic field computation. An application of space-time FI method to a photonic band computation was reported in [12]. However, it is not always a simple task to construct the Maxwell grid equations on these dual space-time grids. To realize a systematic derivation of Maxwell grid equations, a graph-theory-based formulation for the space-time dual grids is required. This paper discusses a geometrical formulation of the 3D space-time FI method that is based on the Hodge duality and the Lorentzian metric but is not a straightforward extension of the conventional spatial FI formulation.

II. FINITE INTEGRATION METHOD ON A SPACE-TIME GRID

A. Electromagnetics in Space-Time

The Maxwell equations are described in the differential form as

$$dF = 0, \quad dG = J. \quad (1)$$

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In the coordinate system $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, F , G and J are written as

$$\begin{aligned} F &= -\sum_{i=1}^3 E_i dx^0 dx^i + \sum_{j=1}^3 B_j dx^k dx^l, \\ G &= \sum_{i=1}^3 H_i dx^0 dx^i + \sum_{j=1}^3 D_j dx^k dx^l, \\ J &= c\rho dx^1 dx^2 dx^3 - \sum_{j=1}^3 cJ_j dx^0 dx^k dx^l \end{aligned} \quad (2)$$

where c is the speed of light, ρ is the electric charge density and (j, k, l) is a cyclic permutation of $(1, 2, 3)$. The integral form of (1) is given as

$$\oint_{\partial\Omega_p} F = 0, \quad \oint_{\partial\Omega_d} G = \int_{\Omega_d} J \quad (3)$$

where Ω_p and Ω_d are hypersurfaces in space-time.

For simplicity, assuming the uniformity along the z -direction, this paper discusses the FI formulation for the electromagnetic field (B_z, E_x, E_y) in the (w, x, y) -3D free space-time [8], where $w = ct$. Accordingly, F and G are written as

$$\begin{aligned} F &= B_z dx dy + \mathcal{E}_y dy dw - \mathcal{E}_x dw dx, \\ G &= \mathcal{H}_z dw - D_y dx + D_x dy \end{aligned} \quad (4)$$

where $(\mathcal{E}_x, \mathcal{E}_y) = (E_x/c, E_y/c)$, $\mathcal{H}_z = H_z/c$. Defining 3D vectors $\mathbf{F}$ and $\mathbf{G}$ as

$$\mathbf{F} = (B_z, \mathcal{E}_y, -\mathcal{E}_x), \quad \mathbf{G} = (\mathcal{H}_z, -D_y, D_x) \quad (5)$$

the integral form is written without source term as

$$\oint_S \mathbf{F} \cdot \mathbf{n} dS = 0, \quad \oint_C \mathbf{G} \cdot \mathbf{t} ds = 0 \quad (6)$$

where $\mathbf{n}$ is the normal vector at each point on the closed surface S and $\mathbf{t}$ the tangent vector at each point on the closed curve C . The Euclidean metric is used for the dot product operation.

The space-time FI method uses discretized variables as

$$f = \int_p \mathbf{F} \cdot \mathbf{n} dS, \quad g = \int_s \mathbf{G} \cdot \mathbf{t} ds \quad (7)$$

where p is a face of a primal grid and $\tilde{s}$ is an edge of a dual grid. To express the constitutive equation between f and g simply, $\mathbf{n}$ and $\mathbf{t}$ are given as [8]

$$\mathbf{n} = (n_w, n_x, n_y), \quad \mathbf{t} = (n_w, -n_x, -n_y). \quad (8)$$

Fig. 1 illustrates the geometrical relation between $\mathbf{n}$ and $\mathbf{t}$, where $\tilde{s}$ is orthogonal to p in the Lorentzian 3D space-time. The relation between $\mathbf{F} \cdot \mathbf{n}$ and $\mathbf{G} \cdot \mathbf{t}$ is given as

$$\mathbf{F} \cdot \mathbf{n} = Z \mathbf{G} \cdot \mathbf{t} \quad (9)$$

where Z is the impedance of the medium. Thus, f is related to g as

$$f = z g \quad (10)$$

$$z = Z \frac{\Delta S}{\Delta l} \quad (11)$$

where ΔS is the area of p and Δl is the length of $\tilde{s}$.

Ref. [9] extended the dual-grid construction above to the 4D space-time, called the Hodge dual grid. It is based on the Hodge duality with the Lorentzian metric between F and G , where the metric is modified in materials depending on the speed of light.

Fig. 1: Relation of primal face and dual edge in space-time grid when (a) $\mathbf{n} \cdot \mathbf{t} > 0$ and (b) $\mathbf{n} \cdot \mathbf{t} < 0$.

Fig. 2: Edges and faces on (a) the primal grid and (b) the dual grid.

B. Explicit Time-Marching Scheme

Refs. [8] and [9] have shown explicit time-marching schemes for 3D and 4D space-time FI analyses of electromagnetic wave propagation. This subsection presents an explicit

time-marching scheme on a simple 2D space-time grid with 1D space to relate to the geometrical formulation described later.

Fig. 2 illustrates 2D space-time primal and dual grids that have temporal step sizes Δw and $\Delta w/2$ and spatial step size Δx along the x - and y - directions.

Based on (5) and (7), the variables in Fig. 2 have the following meaning; b : magnetic flux, e : electromotive force, f : the composition of b and e , h : magnetomotive force, d : electric flux, and g : the composition of h and d . The arrow directions in Fig. 2 are based on (7) and (9) using the definition (5) and (8). Note that the arrow direction of d is opposite to that of e . These do not correspond directly to the directions of $\mathbf{E}$ and $\mathbf{D}$ in the Euclidean space.

The explicit time-marching scheme is given as follows. According to a numerical examination in [10], the scheme is stable when $(l_a - 1)^2 + (\Delta w)^2/2 < 1$.

Variables $d_{2i-1}^{n+1/4}$ and $e_{2i-1}^{n+1/4}$ are given as

$$d_{2i-1}^{n+1/4} = d_{2i-1}^{n-1/4} - (h_{2i}^n - h_{2i-1}^n) \quad (12)$$

$$e_{2i-1}^{n+1/4} = z_{e1} d_{2i-1}^{n+1/4}, \quad z_{e1} = Z \frac{\Delta w \Delta x}{2l_a}. \quad (13)$$

On the primal grid, $f_{2i-1}^{n+1/4}$, $f_{2i}^{n+1/4}$ and consequently $g_{2i-1}^{n+1/4}$, $g_{2i}^{n+1/4}$ are given as

$$\begin{aligned} f_{2i-1}^{n+1/4} &= -e_{2i-1}^{n+1/4} + b_{2i-1}^n, \\ f_{2i}^{n+1/4} &= e_{2i-1}^{n+1/4} + b_{2i}^n \end{aligned} \quad (14)$$

$$g_k^{n+1/4} = \frac{1}{z_f} f_k^{n+1/4} (k = 2i - 1, 2i), \quad z_f = Z \frac{4\Delta x^2}{\Delta w}. \quad (15)$$

On the dual grid, d_{2i} and e_{2i} are updated using

$$\begin{aligned} d_{2i}^{n+1/2} &= d_{2i}^{n-1/2} + h_{2i}^n - h_{2i+1}^n \\ &\quad + g_{2i}^{n-1/4} - g_{2i+1}^{n-1/4} + g_{2i}^{n+1/4} - g_{2i+1}^{n+1/4} \end{aligned} \quad (16)$$

$$e_{2i}^{n+1/2} = z_{e2} d_{2i}^{n+1/2}, \quad z_{e2} = Z \frac{\Delta w \Delta x}{2 - l_a + (\Delta w)^2/4}. \quad (17)$$

Similarly, $f_{2i-1}^{n+3/4}$, $f_{2i}^{n+3/4}$ and $g_{2i-1}^{n+3/4}$, $g_{2i}^{n+3/4}$ are given as

$$\begin{aligned} f_{2i-1}^{n+3/4} &= f_{2i-1}^{n+1/4} + e_{2i-2}^{n+1/2}, \\ f_{2i}^{n+3/4} &= f_{2i}^{n+1/4} - e_{2i}^{n+1/2} \end{aligned} \quad (18)$$

$$g_k^{n+3/4} = \frac{1}{z_f} f_k^{n+3/4} (k = 2i - 1, 2i). \quad (19)$$

On the dual grid, $d_{2i-1}^{n+3/4}$ and $e_{2i-1}^{n+3/4}$ are obtained from

$$\begin{aligned} d_{2i-1}^{n+3/4} &= d_{2i-1}^{n+1/4} + g_{2i-1}^{n+1/4} - g_{2i}^{n+1/4} \\ &\quad + g_{2i-1}^{n+3/4} - g_{2i}^{n+3/4} \end{aligned} \quad (20)$$

$$e_{2i-1}^{n+3/4} = z_{e1} d_{2i-1}^{n+3/4}. \quad (21)$$

Hence, b_{2i-1}^{n+1} , b_{2i}^{n+1} and h_{2i-1}^{n+1} , h_{2i}^{n+1} are given by

$$\begin{aligned} b_{2i-1}^{n+1} &= f_{2i-1}^{n+3/4} - e_{2i-1}^{n+3/4}, \\ b_{2i}^{n+1} &= f_{2i}^{n+3/4} + e_{2i-1}^{n+3/4} \end{aligned} \quad (22)$$

$$h_k^n = \frac{1}{z_b} b_k^n (k = 2i - 1, 2i), \quad z_b = Z \frac{2\Delta x^2}{\Delta w}. \quad (23)$$

C. Incidence Matrices on 3D Euclidean Space

The FI method is generally formulated with the Maxwell grid equations using the incidence matrices from graph theory.

Let arrays $\{n\}$, $\{s\}$, $\{p\}$ and $\{v\}$ denote the sets of nodes, edges, faces, and volumes in the primal grid, respectively. These are related by incidence matrices $[G]$, $[C]$ and $[D]$ [1], [2], [11] as

$$\partial\{s\} = [G]\{n\}, \partial\{p\} = [C]\{s\}, \partial\{v\} = [D]\{p\} \quad (24)$$

where ∂ denotes restriction to the boundary. Similarly, the sets of nodes, edges, faces and volumes in the dual grid are related as

$$\partial\{\tilde{s}\} = [\tilde{G}]\{\tilde{n}\}, \partial\{\tilde{p}\} = [\tilde{C}]\{\tilde{s}\}, \partial\{\tilde{v}\} = [\tilde{D}]\{\tilde{p}\}. \quad (25)$$

In the Euclidean space, the dual grid is generally constructed so the incidence matrices satisfy

$$[\tilde{C}] = [C]^T, [\tilde{D}] = -[G]^T, [\tilde{G}] = -[D]^T. \quad (26)$$

The relation above derives the Maxwell grid equations systematically from the primal grid geometry.

The space-time primal and dual grids based on (8) have a similar property to that described by (26), which gives the one-to-one correspondence between the faces $\{p\}$ on the primal grid and the edges $\{\tilde{s}\}$ on the dual grid. However, the directional relation between $\{p\}$ and $\{\tilde{s}\}$ determined by (8) differs from that given by (26). The following subsection derives the matrix relation for the space-time primal and dual grids based on (8).

D. Incidence Matrices on 3D Space-Time

A simple space-time primal grid illustrated in Fig. 3 is examined. Assuming spatial periodicity in the grid geometry, the edges s_i and faces p_i are periodically numbered for notational simplicity. Moreover, the edges and faces perpendicular to the y -axis is omitted. The direction of edges $s_1, s_2, s_3, s'_1, s'_2$ is along the $+y$ -direction. The edge $\tilde{s}_i$ on the dual grid corresponds to the face p_i on the primal grid, where their directions satisfy (8).

Fig. 3: Edges and faces on (a) the primal grid and (b) the dual grid.

The geometrical relation between edges and faces on the primal grid is represented by the following equations.

$$\begin{aligned} \partial p_1 &= s_1 - s_2, \partial p_2 = -s_1 + s_2, \partial p_3 = s_1 - s_3, \\ \partial p_4 &= -s_2 + s_3, \partial p_5 = s_2 - s_3, \partial p_6 = s_2 - s'_2, \\ \partial p_7 &= s_3 - s'_2, \partial p_8 = -s_3 + s'_2, \partial p_9 = s_3 - s'_1. \end{aligned} \quad (27)$$

To examine the relation between $[C]$ and $[\tilde{C}]$, a subset of $\{s\}$ and a subset of $\{p\}$ are defined as

$$\{s\}_{\text{sb}} = [s_1 \ s_2 \ s_3]^T, \{p\}_{\text{sb}} = [p_1 \ p_2 \ \cdots \ p_9]^T. \quad (28)$$

Omitting s'_1, s'_2 , relation (27) is written $\{p\}_{\text{sb}} = [C]_{\text{sb}}\{s\}_{\text{sb}}$ where $[C]_{\text{sb}}$ is a submatrix of $[C]$ and given as

$$[C]_{\text{sb}}^T = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 \end{bmatrix}. \quad (29)$$

On the dual grid, the relation between edges and faces is found to be:

$$\begin{aligned} \partial \tilde{p}_1 &= \tilde{s}_1 - \tilde{s}_2 - \tilde{s}_3 + \tilde{s}'_9, \\ \partial \tilde{p}_2 &= -\tilde{s}_1 + \tilde{s}_2 - \tilde{s}_4 + \tilde{s}_5 - \tilde{s}_6 + \tilde{s}'_6 - \tilde{s}'_7 + \tilde{s}'_8, \\ \partial \tilde{p}_3 &= \tilde{s}_3 + \tilde{s}_4 - \tilde{s}_5 + \tilde{s}_7 - \tilde{s}_8 - \tilde{s}_9. \end{aligned} \quad (30)$$

Corresponding to $\{s\}_{\text{sb}}$ and $\{p\}_{\text{sb}}$, subsets of $\{\tilde{p}\}$ and $\{\tilde{s}\}$ are defined as

$$\{\tilde{p}\}_{\text{sb}} = [\tilde{p}_1 \ \tilde{p}_2 \ \tilde{p}_3]^T, \{\tilde{s}\}_{\text{sb}} = [\tilde{s}_1 \ \tilde{s}_2 \ \cdots \ \tilde{s}_9]^T. \quad (31)$$

Omitting $\tilde{s}'_6, \tilde{s}'_7, \tilde{s}'_8, \tilde{s}'_9$, relation (30) is written as $\{\tilde{p}\}_{\text{sb}} = [\tilde{C}]_{\text{sb}}\{\tilde{s}\}_{\text{sb}}$ where $[\tilde{C}]_{\text{sb}}$ is a submatrix of $[\tilde{C}]$ and given as

$$[\tilde{C}]_{\text{sb}} = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & -1 & -1 \end{bmatrix}. \quad (32)$$

Comparing $[\tilde{C}]_{\text{sb}}$ with $[C]_{\text{sb}}^T$ shows that the elements of $[\tilde{C}]_{\text{sb}}$ at the 3rd, 6th, and 9th columns have opposite signs to the corresponding elements of $[C]_{\text{sb}}^T$. This sign inversion caused by (8) is illustrated in Fig. 1. If $\mathbf{n}$ and $\mathbf{t}$ given by (8) satisfy $\mathbf{n} \cdot \mathbf{t} < 0$, the direction of the edge is opposite to the direction of the face as depicted in Fig. 1(b).

Consequently, the incidence matrix $[\tilde{C}]$ of dual grid based on (8) is given as

$$[\tilde{C}] = [C]^*{}^T \quad (33)$$

where the operator $*^T$ is determined by the mapping

$$\tilde{c}_{ij} = \begin{cases} c_{ji}, & (c_{ji} \neq 0 \text{ and } \mathbf{n} \cdot \mathbf{t} > 0) \\ -c_{ji}, & (c_{ji} \neq 0 \text{ and } \mathbf{n} \cdot \mathbf{t} < 0) \\ 0, & (c_{ji} = 0) \end{cases}. \quad (34)$$

This relation is a consequence of the Hodge duality between $\mathbf{F}$ and $\mathbf{G}$ based on the Lorentzian metric in the 3D space-time. Using $\mathbf{n} \cdot \mathbf{t} = n_w^2 - n_x^2 - n_y^2$, the matrix $[C]^*{}^T$ can be obtained without the need for the dual grid.

Using $[\tilde{C}]$, the electromagnetic field equations on the dual grid such as (12), (16) and (20) are expressed as

$$[\tilde{C}]\{g\} = 0 \quad (35)$$

where $\{g\}$ consists of the variables defined by the second equation of (7) on the edges corresponding to $\{\tilde{s}\}$.

The relation between faces and volumes on the primal grid is represented similarly. For instance, the volume surrounded by p_1 , p_3 , and p_4 is written:

$$\partial v_1 = -p_1 + p_3 + p_4. \quad (36)$$

These relations are represented by matrix $[D]$ in the form given by the third equation of (24). Using $[D]$, the electromagnetic field equations on the primal grid such as (14), (18) and (22) are expressed as

$$[D]\{f\} = 0. \quad (37)$$

where $\{f\}$ consists of the variables defined by the first equation of (7) on the faces corresponding to $\{p\}$.

Fig. 4 summarizes the geometric relation above. In the 3D Euclidean space, $\mathbf{E}$ and $\mathbf{B}$ are assigned separately to the edges and faces, respectively whereas these are unified into $\mathbf{F}$ and assigned to the faces in the 3D space-time.

Fig. 4: Duality and matrix relations.

E. Maxwell Grid Equations

From (10), $\{g\}$ is related to $\{f\}$ as

$$\{f\} = [z]\{g\} \quad (38)$$

where $[z]$ is a diagonal matrix of which elements are given by (11).

Equations (33), (35), (37), and (38) derive the space-time Maxwell grid equations systematically

$$\begin{bmatrix} [D] \\ [C]^*T[z]^{-1} \end{bmatrix} \{f\} = 0. \quad (39)$$

By modifying the impedance matrix, another formulation is possible, where relation $[\tilde{C}] = [C]^T$ holds. The modified impedance matrix $[z^*]$ is defined by replacing the elements of $[z]$ by $-Z\Delta S/\Delta l$ when $\mathbf{n} \cdot \mathbf{t} < 0$. Thereby, $[C]^*T[z]^{-1} = [C]^T[z^*]^{-1}$ holds.

F. Application Example in 2D Space-Time Grid

The FI method formulated by (39) is implemented and compared with the FI scheme explained in Subsection II.B. The propagation of an electromagnetic wave with components (E_y, B_z) is computed on the periodic 2D space-time grid shown in Fig. 2 with $\Delta x = 1, i = 1, \dots, 50, \Delta w = 0.5, n = 0, \dots, 80$, and $l_a = 1$. The initial condition at $w = 0$ is given as $B_z = \exp(-x^2/25)$ and $E_y = 0$. The impedance matrix $[z]$ consists of z_{e1}, z_{e2}, z_f , and z_b that are given as (13), (15),

Fig. 5: Magnetic flux distribution at $w = 40$.

(17) and (23). The spatially periodic boundary condition is imposed where $e_0^n = e_{80}^n$ and $h_{81}^n = h_1^n$.

Fig. 5 shows the distribution of B_z at $w = 80\Delta w$, where the simulation result given by the FDTD method is also shown for comparison. The FI formulation (39) is equivalent to the FI scheme given in II.B.

III. CONCLUDING REMARKS

A geometrical formulation of the 3D space-time FI method was presented that provides a systematic method to construct the Maxwell grid equations on the space-time primal and dual grids. The relation between the incidence matrices of these space-time grids was derived based on the Hodge duality with Lorentzian metric.

Practically, the systematic formulation is used to derive or confirm the explicit time-marching scheme. The extension to the 4D space-time and its practical application will be addressed in the near future.

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