On Low Dimensional Ricci Limit Spaces

Shouhei Honda

Abstract

We call a Gromov-Hausdorff limit of complete Riemannian manifolds with a lower bound of Ricci curvature a *Ricci limit space*. In this paper, we prove that any Ricci limit space has integral Hausdorff dimension provided that its Hausdorff dimension is not greater than two. We also classify one-dimensional Ricci limit spaces.

1 Introduction

In this paper, we study a pointed metric space (Y, y) that is a pointed Gromov-Hausdorff limit of a sequence of complete, pointed, connected n-dimensional Riemannian manifolds, $\{(M_i, m_i)\}_i$, with $\mathrm{Ric}_{M_i} \geq -(n-1)$, we call such a pointed metric space (Y, y) a *Ricci limit space*. The structure theory was much developed by Cheeger-Colding, and has many important applications to Riemannian manifolds (see [5, 6, 7]). The main purpose of this paper is to study low dimensional Ricci limit spaces by using their theory and several results of [16]. First, we give the classification of Ricci limit spaces whose Hausdorff dimension is smaller than two:

Theorem 1.1. *Let (Y, y) be a Ricci limit space. Assume that Y is not a single point. Then, the following conditions are equivalent:*

1. $1 \leq \dim_H Y < 2$ *holds.*
2. $\mathcal{R}_i = \emptyset$ *holds for every* $i \geq 2$
3. $\upsilon(\mathcal{R}_i) = 0$ *holds for every* $i \geq 2$
4. Y *is isometric either to* $\mathbf{R}$, *or to* $\mathbf{R}_{\geq 0}$, *or to* $\mathbf{S}^1(r) = \{x \in \mathbf{R}^2 | |x| = r\}$ *for some* $r > 0$, *or to* $[0, l]$ *for some* $l > 0$.

Key words and phrases. Ricci curvature, Gromov-Hausdorff convergence, Geometric measure theory.

Here, $\mathcal{R}_i$ is the i-dimensional regular set of Y, $\dim_H Y$ is the Hausdorff dimension of Y, υ is a limit measure on Y. See Definition 2.4 and Definition 2.6. Remark that $\dim_H Y < 1$ holds if and only if Y is a single point. Therefore, Theorem 1.1 gives the isometric clasification of Ricci limit spaces whose Hausdorff dimension is smaller than two. As a corollary of Theorem 1.1, we have that if $\dim_H Y \le 2$ holds then $\dim_H Y$ is an integer.

We will give an another characterization of low dimensional points under additional assumption. For that, we define the *local Hausdorff dimension* $\dim_H^{loc} x$ around a point $x \in Y$ by

$$\dim_H^{\mathrm{loc}} x = \lim_{r \to 0} \dim_H B_r(x).$$

Put $Y(\alpha) = \{x \in Y | \dim_H^{\mathrm{loc}} x = \alpha\}$ for $\alpha \ge 0$. Remark that if Y is not a single point, then $\dim_H^{\mathrm{loc}} x \ge 1$ holds for every $x \in Y$. Next, we shall define the notion of *Alexandrov point*. For a proper geodesic space X and a point $x \in X$, we say that x is an *Alexandrov point (in X)* if there exist an open neighbourhood U of x, and a negative number $K < 0$ satisfying the following properties: For every $x_1, x_2, x_3 \in U$ and every $x_4 \in X$ with $\overline{x_1, x_4} + \overline{x_4, x_2} = \overline{x_1, x_2}$, there exist points $y_1, y_2, y_3, y_4 \in \mathbb{H}^2(K)$ such that $\overline{x_1, x_2} = \overline{y_1, y_2}$, $\overline{x_2, x_3} = \overline{y_2, y_3}$, $\overline{x_3, x_1} = \overline{y_3, y_1}$, $\overline{x_1, x_4} = \overline{y_1, y_4}$, $\overline{y_1, y_4} + \overline{y_4, y_2} = \overline{y_1, y_2}$ and $\overline{x_3, x_4} \ge \overline{y_3, y_4}$. Here, $\mathbb{H}^2(K)$ is the two-dimensional space form with the sectional curvature $K_{\mathbf{H}^2(K)} \equiv K$, $\overline{x_1, x_2}$ is the distance between x_1 and x_2.

Denote by $\mathrm{Alex}(X)$ the set of Alexandrov points in X. Roughly speaking, an Alexandrov points on a metric space means that there exists a lower bound of sectional curvature around the point in the sense of Alexandrov geometry. Therefore, by the definition, all points in every Alexandrov spaces are Alexandrov points. We shall state an another characterization of low dimensional points in Ricci limit spaces:

THEOREM 1.2. *Let (Y, y) be a Ricci limit space. Assume that $\mathcal{R}_1 \neq \emptyset$. Then, we have* $\mathrm{Alex}(Y) = \bigcup_{\alpha < 2} Y(\alpha) = Y(1)$.

Remark that this theorem is *stronger* than Theorem 1.1. An idea of the proof of $\bigcup_{\alpha<2} Y(\alpha) \subset \mathrm{Alex}(Y)$ is the same as the proof of Theorem 1.1. A main idea of the proof of $\mathrm{Alex}(Y) \subset Y(1)$ is to compare between a measure theoretic property of a point in $\mathcal{R}_1$ and one of an Alexandrov point by using [16, Theorem 1.1]. We give some application to Theorem 1.2 in the following.

Fix a sufficiently small positive number $\epsilon > 0$. Let Z be the completion of the 5-dimensional Riemannian manifold $(\mathbf{R}_{>0} \times \mathbf{S}^4, dr^2 + (r^{1+\epsilon}/2)^2 g_{\mathbf{S}^4})$, where $g_{\mathbf{S}^4}$ is the standard Riemannian metric on a 4-dimensional unit sphere in $\mathbb{R}^5$. It is known that this space is a Ricci limit space (see [5, Example 8.77]). On the other hand, for every $\tau > 0$, let Z_τ be the space obtained by adjoining the segment $[-\tau, 0]$ to Z at their origins. Cheeger-Colding showed that for every $\tau > 0$, Z_τ is *not* a Ricci limit space as a corollary of [6, Theorem 5.1]. This non-existence result also follows from Theorem 1.2 directly. This is a simply alternative proof.

Let Z_1 and Z_2 be copies of Z (namely, Z_1 and Z_2 are isometric to Z, respectively) and $\hat{Z}$ the space obtained by adjoining Z_1 to Z_2 at their origins. It follows directly from Theorem 1.2 that $\hat{Z}$ is *not* a Ricci limit space. Remark that the non-existence of $\hat{Z}$ as a Ricci limit space, does *not* follow from [6, Theorem 3.7] or [6, Theorem 5.1]. See also Proposition 4.7 and [27, Theorem 1.3].

Theorem 1.2 implies that it is very difficult to construct a Ricci limit space whose one dimensional regular set is not empty and whose Hausdorff dimension is not one. In fact, by using the results of this paper, we can prove that if $\mathcal{R}_1 \neq \emptyset$, then $\dim_H Y = 1$ in [17]. As more non-existence results, we will also get that $(M \times Z_\tau, (m, 0))$ is *not* a Ricci limit space for every $\tau > 0$ and every pointed connected complete k-dimensional Riemannian manifold (M, m). See Remark 5.8.

The organization of this paper is as follows. In Section 2, we will introduce several notions on metric spaces needed subsequently. The proof of Theorem 1.1 is based on several results on regular sets due to Cheeger-Colding's works. In Section 3, we will recall them. In Section 4, we will study a local structure around given *'low dimensional'* points. Theorem 1.1 follows directly from the local strucure properties. See Theorem 4.3 and Theorem 4.5. The main idea of the proof is a geometric rescaling argument based on several properties of regular sets from Section 3. We will study that under what condition a limit measure υ is locally equivalent to the one-dimensional Hausdorff measure H^1. Here, for a topological space X, a point $x \in X$ and Borel measures υ, μ on X, we say that υ is *locally equivalent to* μ *at* $x \in X$ if there exist a positive number $C > 1$ and an open neighbourfood U of x such that $C^{-1}\mu(A) \leq \upsilon(A) \leq C\mu(A)$ for every Borel set $A \subset U$. We will give a necessary and sufficient condition that υ is locally equivalent to H^1 at a point. See Theorem 4.8. The proof is based on Theorem 1.1 and [16, Theorem 1.1], essentially. Roughly speaking, Theorem 4.8 implies a characterization of the local structure around a low-dimensional point in a Ricci limit space as a metric measure space. In Section 5, we will study several properties of the Alexandrov set in a Ricci limit space. A main result in Section 5 is Theorem 5.4. As a corollary, we will give a proof of Theorem 1.2. In Section 6 and 7, we will also study the problem whether the Hausdorff dimension of a Ricci limit space is an integer. Especially, under the assumption $2 \leq \dim_H Y < 3$, by using an idea of the proof of Theorem 1.1, we will prove that $\dim_H(Y \setminus C_x) \leq 2$ holds for every $x \in Y$. Here, C_x is the cut locus of x, defined by $C_x = \{z \in X | \ \overline{x,z} + \overline{z,w} - \overline{x,w} > 0$ for every $w \in X \setminus \{z\}$ $\}$ if X is not a single point, $C_x = \emptyset$ if otherwise. See Corollary 6.4. Cheeger-Colding defined the polarity of a Ricci limit space, which is a sufficient condition for a

Ricci limit space to have integral Hausdorff dimension. We can rewrite the condition by using properties of cut locus on iterated tangent cones. Actually, it is easy to check that a Ricci limit space (Y, y) is polar if and only if $C_x = \emptyset$ holds for every iterated tangent cone (X, x) of Y. Menguy showed that there exists a non-polar Ricci limit space whose Hausdorff dimension is an integer. See [19]. We will give an another sufficient condition for a Ricci limit space to have integral Hausdorff dimension that is *weaker* condition than the polarity. Actually, in Section 8, we will prove that if $\dim_H(X \setminus C_x) = \dim_H X$ holds for every iterated tangent cone (X, x) of Y, then $\dim_H B_r(z) \in \mathbf{Z}$ holds for every $z \in Y$ and every $r > 0$. We say that a Ricci limit space is *weakly polar* if the space satisfies the condition. See Theorem 7.2 for the detail. It is unknown whether there exists a non-weakly polar Ricci limit space. In fact, note that the non-polar Ricci limit space in the example in [19] is weakly polar. We also study several properties of a weakly polar limit space. See Corollary 7.7.

Acknowledgement: The author is grateful to Professor Takashi Shioya for helpful discussions and teaching me essential ideas of the proofs of Proposition 4.1 and Theorem 4.3. He would like to express his thanks to Professor Tobias Holck Colding for valuable suggestions about Theorem 5.6 and Remark 5.8. He also thanks to Professor Kenji Fukaya for his numerous suggestions and advices. He wishes to thank the referee for valuable suggestions, comments, advise and for pointing out corrections and misprints of the first draft. He was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, by GCOE 'Fostering top leaders in mathematics', Kyoto University and by Grant-in-Aid for Research Activity Start-up 22840027 from JSPS.

2 Notation

We recall some fundamental notions on metric spaces and Ricci limit spaces.

Definition 2.1. We say that a metric space X is *proper* if every bounded closed subset of X is compact. A metric space X is said to be a *geodesic space* if for every points $x_1, x_2 \in X$, there exists an isometric embedding $\gamma : [0, \overline{x_1, x_2}] \to X$ such that $\gamma(0) = x_1$ and $\gamma(\overline{x_1, x_2}) = x_2$ hold. We say that γ is a *minimal geodesic from x_1 to x_2*.

For a proper geodesic space X, $x \in X$, $A \subset X$, and $r > 0$, put: $B_r(x) = \{z \in X | \overline{x, z} < r\}$, $\overline{B}_r(x) = \{z \in X | \overline{x, z} \le r\}$, $\partial B_r(x) = \{z \in X | \overline{x, z} = r\}, C_x(A) = \{z \in X|$ There exists $w \in A$ such that $\overline{x, z} + \overline{z, w} = \overline{x, w}$ holds.$\}$. *Throughout the paper, we fix a positive integer $n > 0$.*

Definition 2.2. Let (Y, y) be a pointed proper geodesic space and K a real number. We say that (Y, y) is a (n, K)*-Ricci limit space (of $\{(M_i, m_i)\}_i$)* if there exist sequences of real numbers $\{K_i\}_i$ and of pointed, complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \geq K_i(n-1)$, such that K_i converges to K and that (M_i, m_i) converges to (Y, y) as $i \to \infty$ in the sense of pointed Gromov-Hausdorff topology.

We recall the definition of pointed Gromov-Hausdorff convergence. For a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_i$, we say that (X_i, x_i) *converges to a pointed proper geodesic space* (X_∞, x_∞) *in the sense of Gromov-Hausdorff topology* if there exist sequences of positive numbers $\{\epsilon_i\}_i, \{R_i\}_i$ and of maps $\phi_i : (B_{R_i}(x_i), x_i) \to (B_{R_i}(x_\infty), x_\infty)$ such that $\epsilon_i \to 0$, $R_i \to \infty$, $B_{\epsilon_i}(\mathrm{Image}(\phi_i)) \supset B_{R_i}(x_\infty)$ and that $|\overline{z_i, w_i} - \overline{\phi_i(z_i), \phi_i(w_i)}| < \epsilon_i$ for every $z_i, w_i \in B_{R_i}(x_i)$. Denote it by $(X_i, x_i) \to (X_\infty, x_\infty)$ for the sake of simplicity. Moreover for a sequence of points $z_i \in B_{R_i}(x_i)$, we say that z_i *converges to* $z_\infty \in X_\infty$ if $\phi_i(z_i) \to z_\infty$. Denote it by $z_i \to z_\infty$ for the sake of simplicity.

Remark that for every $K \neq 0$ and every (n, K)-Ricci limit space (Y, y), by suitable rescaling, there exists a sequence of complete, connected n-dimensional Riemannian manifolds $\{(M_i, m_i)\}_i$ with $\mathrm{Ric}_{M_i} \geq K(n-1)$, such that $(M_i, m_i) \to (Y, y)$. *Throughout the paper,* (Y, y) *is always a fixed* $(n, -1)$*-Ricci limit space of* $\{(M_i, m_i)\}$ *and not reduced to a single point. We will say that such a* (Y, y) *is a Ricci limit space* for the sake of simplicity.

DEFINITION 2.3. Let (W, w), (Z, z) be pointed proper geodesic spaces. We say that (W, w) is *a tangent cone at* $z \in Z$ if there exists a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $(Z, r_i^{-1} d_Z, z) \to (W, w)$, where, d_Z is the distance function on Z.

Remark that by Gromov's compactness theorem, for every $x \in Y$, there exists a tangent cone $(T_xY, 0_x)$ at x, however, in general, it is *not* unique. See [20] for an example. Note that $(T_xY, 0_x)$ is a $(n, 0)$-Ricci limit space for every tangent cone $(T_xY, 0_x)$ at x.

Next, we shall give several fundamental notions on Ricci limit spaces due to Cheeger-Colding (see [5]). *Throughout this paper, for every metric spaces* X_1, X_2, *the metric on* $X_1 \times X_2$ *is always* $\sqrt{d_{X_1}^2 + d_{X_2}^2}$.

DEFINITION 2.4. Let Z be a proper geodesic space. Assume that for every $\alpha \in Z$, there exists a tangent cone $(T_\alpha Z, 0_\alpha)$ at α. For every $k \geq 0$ and every $\epsilon > 0$, put

1. $\mathcal{WE}_k(Z) = \{x \in Z|$ There exist a tangent cone $(T_xZ, 0_x)$ at x, and a proper geodesic space W such that T_xZ is isometric to $\mathbf{R}^k \times W$. $\}$,

2. $\mathcal{E}_k(Z) = \{x \in Z|$ For every tangent cone $(T_xZ, 0_x)$ at x, there exists a proper geodesic space W such that T_xZ is isometric to $\mathbf{R}^k \times W$. $\}$,

3. $\underline{\mathcal{WE}}_k(Z) = \{x \in Z|$ There exist a tangent cone $(T_xZ, 0_x)$ at x, and a proper geodesic space W such that W is not a single point and that T_xZ is isometric to $\mathbf{R}^k \times W$.$\}$,

4. $\mathcal{R}_k(Z) = \{x \in Z|$Every tangent cone $(T_xZ, 0_x)$ at x is isometric to $(\mathbf{R}^k, 0_k)$.$\}$,

5. $(\mathcal{WE}_k)_\epsilon(Z) = \{x \in Z|$ There exist $0 < r < \epsilon$ and a proper geodesic space (W, w) such that $d_{GH}((\overline{B}_r(x), x), (\overline{B}_r((0_k, w)), (0_k, w))) < \epsilon r$ for $\overline{B}_r((0_k, w)) \subset \mathbf{R}^k \times W$. $\}$,

6. $(\mathcal{E}_k)_\epsilon(Z) = \{x \in Z|$ There exists $r > 0$ such that for every $0 < t < r$, there exists a proper geodesic space (W, w) such that $d_{GH}((\overline{B}_t(x), x), (\overline{B}_t((0_k, w)), (0_k, w))) < \epsilon t$ holds for $\overline{B}_r((0_k, w)) \subset \mathbf{R}^k \times W$. $\}$,

   where d_{GH} is the Gromov-Hausdorff distance between pointed compact metric spaces.

For the sake of simplicity, we use the following notations for (Y, y): $\mathcal{WE}_k = \mathcal{WE}_k(Y)$, $\mathcal{E}_k = \mathcal{E}_k(Y)$, etc. We call the set $\mathcal{R}_k$ the *k-dimensional regular set of Y* and call the set $\mathcal{R} = \bigcup_k \mathcal{R}_k$ *the regular set of Y*.

Remark 2.5. It is easy to check the following:

1. $(\mathcal{WE}_k)_\epsilon$ is open.

2. $\mathcal{WE}_k = \bigcap_{\epsilon>0}(\mathcal{WE}_k)_\epsilon$, $\mathcal{E}_k = \bigcap_{\epsilon>0}(\mathcal{E}_k)_\epsilon$.

3. $\mathcal{WE}_k = \mathcal{E}_k = \mathcal{R}_k = \emptyset$ for every $k \geq n + 1$.

We end this section by giving the definition of *limit measure*. The measure is useful tool to study Ricci limit spaces.

Definition 2.6. Let υ be a Borel measure on Y. We say that υ is the *limit measure of* $\{(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))\}_j$ if

$$\frac{\mathrm{vol}\, B_r(x_j)}{\mathrm{vol}\, B_1(m_j)} \to \upsilon(B_r(x))$$

as $j \to \infty$ for every $r > 0$, every $x \in Y$ and every sequence $x_j \in M_j$ with $x_j \to x$. Then, we say that $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))$ *converges to* (Y, y, υ) *in the sense of measured Gromov-Hausdorff topology, or* (Y, y, υ) *is the Ricci limit space of* $\{(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))\}_j$. Denote it by $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j)) \to (Y, y, \upsilon)$ for the sake of simplicity.

By taking a subsequence $\{(M_{i(j)}, m_{i(j)})\}_j$ of $\{(M_i, m_i)\}_i$, there exists the limit measure on Y of $\{(M_{i(j)}, m_{i(j)}, \mathrm{vol}/\mathrm{vol}\, B_1(m_{i(j)})\}_j$. See for instance [5, Theorem 1.6], [5, Theorem 1.10], [10]. Therefore, *throughout the paper, υ is always the limit measure on Y of* $\{(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j))\}_j$.

3 Some properties of regular set

One of important results on regular set due to Cheeger-Colding, is that $\upsilon(Y \setminus \mathcal{R}) = 0$. See [5, Theorem 2.1]. We need more detailed properties of regular set to study low dimensional

Ricci limit spaces in the following sections. These results are not stated in the form we need for this paper in Cheeger-Colding's papers but are essentially direct consequence of their work. Remark that the following proposition is *not* a direct consequence of $\upsilon(Y \setminus \mathcal{R}) = 0$.

PROPOSITION 3.1. *We have that* $\upsilon\left(B_r(x) \cap \left(\bigcup_{j\geq k} \mathcal{R}_j\right)\right) > 0$ *for every* $x \in \mathcal{WE}_k$ *and every* $r > 0$.

PROOF. By [7, Theorem 3.3], we have that $\upsilon(B_r(x) \cap \mathcal{E}_k) > 0$ for every $r > 0$. If $\upsilon(B_r(x) \cap \mathcal{R}_k) > 0$, then we have the claim. Assume $\upsilon(B_r(x) \cap \mathcal{R}_k) = 0$. Then, since $\upsilon(B_r(x) \cap \mathcal{E}_k) \leq \upsilon(B_r(x) \cap \mathcal{R}_k) + \upsilon(B_r(x) \cap \underline{\mathcal{WE}}_k)$, we have $\upsilon(B_r(x) \cap \underline{\mathcal{WE}}_k) > 0$. By [5, Lemma 2.5] and [5, Lemma 2.6], we have $\upsilon(B_r(x) \cap \mathcal{E}_{k+1}) > 0$. The iteration stops since $\mathcal{E}_l = \emptyset$ for any $l > n$ by Hausdorff dimension argument. By iterating this argument, we have the assertion. □

PROPOSITION 3.2. *We have that* $\upsilon\left(B_r(x) \cap \left(\bigcup_{j\geq k+1} \mathcal{R}_j\right)\right) > 0$ *for every* $x \in \underline{\mathcal{WE}}_k$ *and every* $r > 0$.

PROOF. First, remark that for every $\epsilon > 0, \delta > 0$ and *every* $x \in \underline{\mathcal{WE}}_k$, there exists $s > 0$ with $s < \epsilon$ such that

$$\frac{\upsilon(B_s(x) \setminus (\mathcal{WE}_{k+1})_\delta)}{\upsilon(B_s(x))} < \epsilon.$$

See (2.42) in [5] for the proof. Remark that this statement does *not* follow directly from the result $\upsilon(\underline{\mathcal{WE}}_k \setminus \mathcal{WE}_{k+1}) = 0$. Fix a sequence of positive numbers $\{\epsilon_i\}_i$ with $\epsilon_i \to 0$. Then there exists a sequence $x_i \in (\mathcal{WE}_{k+1})_{\epsilon_i}$ with $x_i \to x$. By [7, Theorem 3.3] and the definition of $(\mathcal{WE}_{k+1})_\epsilon$, there exists a sequence of positive numbers $\{\delta_i\}_i$ with $\delta_i \to 0$ such that $\upsilon(B_{\delta_i}(x_i) \cap \mathcal{E}_{k+1}) > 0$. Since $B_{\delta_i}(x_i) \subset B_r(x)$ for every sufficiently large i, we have $\upsilon(B_r(x) \cap \mathcal{E}_{k+1}) > 0$. By an argument similar to the proof of Proposition 3.1, we have the assertion. □

We will use next corollaries in the following sections, essentially.

COROLLARY 3.3. *We have that* $\underline{\mathcal{WE}}_k \subset \bigcup_{i\geq k+1} \overline{\mathcal{R}}_i$ *for every* $k \geq 1$.

COROLLARY 3.4. *Let* $i \geq 1$.

1. *If* $\upsilon(\mathcal{R}_j) = 0$ *for every* $j \geq i$, *then we have that* $\mathcal{WE}_j = \phi$ *for every* $j \geq i$. *Especially, we have that* $\mathcal{R}_j = \emptyset$ *for every* $j \geq i$.

2. *If* $\upsilon(\mathcal{R}_j) = 0$ *for every* $j \geq i+1$, *then we have that* $\underline{\mathcal{WE}}_j = \emptyset$ *for every* $j \geq i$.

4 Local structure around low dimensional points

In this section, we exhibit a local structure around a low dimensional point in a Ricci limit space. As a corollary, it gives Theorem 1.1.

4.1 Local metric structure around low dimensional points

We say that a point $x \in Y$ is *an interior point* on a minimal geodesic $\gamma : [0, l] \to Y$ $(l > 0)$ if $x \in \gamma((0, l))$ holds.

PROPOSITION 4.1. *Let x be a point in $\mathcal{R}_1$. Then, x is an interior point on a minimal geodesic.*

PROOF. This proof is done by contradiction. Assume that the assertion is false. Let $\{r_i\}_i$ be a sequence of positive numbers with $r_i \to 0$ such that $(Y, r_i^{-1}d_Y, x) \to (\mathbf{R}, 0)$. Then there exist sequences of points $\{x_i^-\}_i, \{x_i^+\}_i \in Y$ and of positive numbers $\{\epsilon_i\}_i$ such that $\epsilon_i \to 0$, $|\overline{x_i^-, x} - r_i| < \epsilon_i r_i$, $|\overline{x_i^+, x} - r_i| < \epsilon_i r_i$ and $\overline{x_i^-, x} + \overline{x_i^+, x} - \overline{x_i^-, x_i^+} < \epsilon_i r_i$. Fix a minimal geodesic $\gamma_i : [0, \overline{x_i^-, x_i^+}] \to Y$ from x_i^- to x_i^+ and put $s_i = \overline{x, \mathrm{Image}(\gamma_i)}$. By the assumption, we have $s_i > 0$. By triangle inequality, we have $s_i \to 0$. By Gromov's compactness theorem, without loss of generality, we can assume that $(Y, x, s_i^{-1}d_Y)$ converges to a tangent cone $(T_xY, 0_x)$ at x. By the construction, there exist $z \in \partial B_1(0_x)$ and an isometric embedding $L : \mathbf{R} \to T_xY$ such that $z \in \mathrm{Image}(L)$ and $0_x \notin \mathrm{Image}(L)$. By applying splitting theorem to (T_xY, z) (see [4, Theorem 6.64]), there exists a proper geodesic space W such that W is not a single point and that T_xY is isometric to $\mathbf{R} \times W$. This contradicts the assumption $x \in \mathcal{R}_1$. $\square$

REMARK 4.2. By the proof of Proposition 4.1, we have that every $x \in \mathcal{R}_1$ is an interior point on a *limit* minimal geodesic. Here we say that a minimal geodesic $\gamma : [0, l] \to Y$ is *a limit minimal geodesic (of $\{(M_i, m_i)\}_i$)* if there exists a sequence of minimal geodesics $\gamma_i : [0, l_i] \to M_i$ such that $l_i \to l$ and $\gamma_i \to \gamma$ in the sense of Gromov-Hausdorff topology. This result is essentially used in [17].

THEOREM 4.3. *Let $x \in Y \setminus \bigcup_{i \ge 2} \overline{\mathcal{R}}_i$. Then, there exists $\epsilon > 0$ such that $(B_\epsilon(x), x)$ is isometric either to $((-\epsilon, \epsilon), 0)$ or to $([0, \epsilon), 0)$.*

PROOF. 1. *The case $x \in \mathcal{R}_1$.*

By Proposition 4.1, there exist $r > 0$, $x_-, x_+ \in Y$ and a minimal geodesic $\gamma : [0, \overline{x_-, x_+}] \to Y$ from x_- to x_+ such that $\overline{x_-, x} = \overline{x_+, x} = 100r$, $x \in \mathrm{Image}(\gamma)$ and $\overline{B}_{100r}(x) \subset Y \setminus \bigcup_{i \ge 2} \overline{\mathcal{R}}_i$. It suffices to check that $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) = \emptyset$. Assume that $\overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. Let $z \in \overline{B}_{10r}(x) \setminus \mathrm{Image}(\gamma)$ and $w \in \mathrm{Image}(\gamma)$

with $\overline{z,w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Remark that $w \in B_{50r}(x)$. Fix a minimal geodesic $\gamma_1 : [0, \overline{z,w}] \to Y$ from z to w. For every $\epsilon > 0$ with $\epsilon << \overline{z, \mathrm{Image}(\gamma)}$, let $w(\epsilon) \in \mathrm{Image}(\gamma_1)$ and $x_-(\epsilon), x_+(\epsilon) \in \mathrm{Image}(\gamma)$ with $\overline{w, w(\epsilon)} = \overline{x_-(\epsilon), w} = \overline{x_+(\epsilon), w} = \epsilon$. Then we have that $\overline{x_-(\epsilon), w(\epsilon)} = \overline{x_-(\epsilon), w(\epsilon)} + \overline{w(\epsilon), z} - \overline{w(\epsilon), z} \geq \overline{z,w} - \overline{w(\epsilon), z} = \epsilon$. Similarly, we have $\overline{x_+(\epsilon), w(\epsilon)} \geq \epsilon$. Therefore, for every tangent cone $(T_wY, 0_w)$ at w, there exists a proper geodesic space W such that W is not a single point and that T_wY is isometric to $\mathbf{R} \times W$. Thus, we have $w \in \underline{\mathcal{WE}}_1$. By Corollary 3.3, we have $w \in \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$. This contradicts the assumption $\mathrm{Image}(\gamma) \subset Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$.

2. *The case $x \in Y \setminus \mathcal{R}_1$.*

   There exist $r > 0$, $x_+ \in Y$ and a minimal geodesic $\gamma : [0, \overline{x, x_+}] \to Y$ from x to x_+ such that $\overline{x, x_+} = 100r$ and $B_{100r}(x) \subset Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$. It suffices to check that $B_{10r}(x) \setminus \mathrm{Image}(\gamma) = \emptyset$. Assume that $B_{10r}(x) \setminus \mathrm{Image}(\gamma) \neq \emptyset$. Let $z \in B_{10r}(x) \setminus \mathrm{Image}(\gamma)$ and $w \in \mathrm{Image}(\gamma)$ with $\overline{z,w} = \overline{z, \mathrm{Image}(\gamma)} > 0$. Remark that $w \in B_{50r}(x)$. If $w \neq x$, then, by the case 1, there exists $\epsilon > 0$ such that $(B_\epsilon(w), w)$ is isometric to $((-\epsilon, \epsilon), 0)$. This contradicts the fact $\overline{z,w} = \overline{z, \mathrm{Image}(\gamma)}$. Thus, we have $w = x$. Fix $\epsilon > 0$ with $\epsilon << 100r$, $x_+(\epsilon) \in \mathrm{Image}(\gamma)$ with $\overline{x, x_+(\epsilon)} = \epsilon$ and a minimal geodesic $\gamma_\epsilon : [0, \overline{z, x_+(\epsilon)}] \to Y$ from z to $x_+(\epsilon)$.

   Claim 4.4. $x \in \mathrm{Image}(\gamma_\epsilon)$.

   This proof is done by contradiction. Assume that the assertion is false. Put $t = \inf\{\overline{z,m} \mid m \in \mathrm{Image}(\gamma_\epsilon) \cap \mathrm{Image}(\gamma)\} > 0$. By the definition, we have that $\gamma_\epsilon(t) \in \mathrm{Image}(\gamma)$ and that $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s < t$. On the other hand, by the assumption, we have $\gamma_\epsilon(t) \in \mathcal{E}_1$. Since $\gamma_\epsilon(t) \notin \underline{\mathcal{WE}}_1$, we have $\gamma_\epsilon(t) \in \mathcal{R}_1$. By the case 1, there exists $\tau > 0$ such that $(B_\tau(\gamma_\epsilon(t)), \gamma_\epsilon(t))$ is isometric to $((-\tau, \tau), 0)$. This contradicts the fact that $\gamma_\epsilon(s) \notin \mathrm{Image}(\gamma)$ for every $s < t$. Therefore we have Claim 4.4.

   By Claim 4.4, we have $x \in \mathcal{E}_1$. Since $x \notin \underline{\mathcal{WE}}_1$, we have $x \in \mathcal{R}_1$. This contradicts the assumption $x \in Y \setminus \mathcal{R}_1$.

□

Theorem 4.5. *Let x be a point in Y. Then, $1 \leq \dim_H^{\mathrm{loc}} x < 2$ holds if and only if $x \in Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$ holds.*

Proof. By Theorem 4.3, if $x \in Y \setminus \bigcup_{i \geq 2} \overline{\mathcal{R}}_i$, then $1 \leq \dim_H^{\mathrm{loc}} x < 2$. Let $i \geq 2$ and $x \in \overline{\mathcal{R}}_i$. For every $s > 0$, take $z_s \in B_s(x) \cap \mathcal{R}_i$. By [6, Corollary 1.36], we have that $\dim_H B_t(z_s) \geq 2$ for every $s, t > 0$. Especially, we have that $\dim_H B_s(x) \geq i \geq 2$ for every $s > 0$. Therefore, we have $\dim_H^{\mathrm{loc}} x \geq i \geq 2$. □

Theorem 1.1 follows directly from Corollary 3.4, Theorem 4.3 and Theorem 4.5. Put $A_Y(1) = \{x \in Y | \liminf_{r\to 0} \upsilon(B_r(x))/r > 0\}$ (is called the *Ahlfors one regular set of* (Y, y, υ)). See Section 6 in [16] for the definition of the set $A_Y(\alpha)$ for a real number $1 \le \alpha \le n$. Remark that the subset $A_Y(1)$ is *one dimension* in some sense. Actually, υ and the one dimensional Hausdorff measure H^1 are mutually absolutely continuous on $A_Y(1)$. We end this section by giving the following corollary:

COROLLARY 4.6. *Assume* $\upsilon(Y \setminus A_Y(1)) = 0$. *Then we have* $\dim_H Y = 1$.

PROOF. By [7, Theorem 3.23] and [7, Theorem 4.6], we have that $\upsilon(\mathcal{R}_i \setminus (\mathcal{R}_i \cap A_Y(i))) = 0$ for every i. Therefore, by the assumption, we have that $\upsilon(\mathcal{R}_i) = 0$ for every $i \ge 2$. Thus, the assertion follows directly from Theorem 1.1. □

4.2 Local measure structure around low dimensional points

In this subsection, we will study locally equivalence between a limit measure υ and the one-dimensional Hausdorff measure H^1. Remark that it follows from Bishop-Gromov inequality for υ that $\upsilon_{-1}(\{x\}) \le \liminf_{r\to 0} \upsilon(B_r(x))/r \le C(n)\upsilon_{-1}(\{x\})$ for every $x \in Y$ (see [6], [16] for the definition of the measure υ_{-1} on Y).

PROPOSITION 4.7. *Let* x *be a point in* $\mathcal{R}_1$. *Then we have* $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$.

PROOF. The proof is done by contradiction. Assume that the assertion is false. Hence we have $\upsilon_{-1}(\{x\}) = 0$. Then, by [6, Theorem 3.7], for every $x_1, x_2 \in Y \setminus \{x\}$ and every $\epsilon > 0$, there exist $y_1, y_2 \in Y$ and a minimal geodesic $\gamma : [0, \overline{y_1, y_2}] \to Y$ from y_1 to y_2 such that $\overline{x_1, y_1} \le \epsilon$, $\overline{x_2, y_2} \le \epsilon$ and $x \notin \mathrm{Image}(\gamma)$. Then, by an argument similar to the proof of Proposition 4.1, there exist a tangent cone $(T_x Y, 0_x)$ at x and a proper geodesic space W such that W is not a single point and that $T_x Y$ is isometric to $\mathbf{R} \times W$. This contradicts the assumption $x \in \mathcal{R}_1$. □

The next theorem is the main result in this section. This is a characterization of local equivalence between a limit measure and H^1.

THEOREM 4.8. *Let* x *be a point in* Y. *The following conditions are equivalent:*

1. *A limit measure* υ *and the one dimensional Hausdorff measure* H^1 *are locally equivalent at* x.
2. $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$ *and* $1 \le \dim_H^{\mathrm{loc}} x < 2$ *hold.*

PROOF. If υ is locally equivalent to H^1 at x, then it follows from Theorem 4.3 and Theorem 4.5 that $\dim_H^{\mathrm{loc}} x = 1$ and $\liminf_{r\to 0} \upsilon(B_r(x))/r > 0$. Assume that $\liminf_{r\to 0} \upsilon(B_r(x))/r >$

0 and $1 \leq \dim_H^{\mathrm{loc}} x < 2$. Then, by Theorem 4.3 and Theorem 4.5, there exists $\epsilon > 0$ such that $(B_{2\epsilon}(x), x)$ is isometric either to $((-2\epsilon, 2\epsilon), 0)$ or to $([0, 2\epsilon), 0)$. It follows from [16, Theorem 1.1] that there exists $d \geq 1$ such that $d^{-1} \leq \liminf \upsilon(B_r(y))/r \leq \limsup \upsilon(B_r(y))/r \leq d$ for every $y \in B_\epsilon(x)$. For every $a \in B_\epsilon(x)$, there exists $r_a > 0$ such that $d^{-1}/2 \leq \upsilon(B_r(a))/r \leq 2d$ for every $r < r_0$. It follows from standard covering lemma (see Chapter 1 in [24]) that there exists $C(d,n) \geq 1$ such that $C(d,n)^{-1}H^1(A) \leq \upsilon(A) \leq C(d,n)H^1(A)$ for every Borel subset A of $B_\epsilon(x)$. $\square$

Remark that there exist two limit measures υ_1, υ_2 on a $(2,0)$-Ricci limit space $[0,1]$ such that υ_1 is locally equivalent to H^1 at 0 and that υ_2 is *not* locally equivalent to H^1 at 0. See [5, Example 1.24].

5 Alexandrov set

In this section, we study the Alexandrov set in a Ricci limit space (Y, y). Especially, we will give a proof of Theorem 1.2 and show several non-existence results for a metric spaces as a Ricci limit space (e.g. $Z_\tau, \hat{Z}$ in Section 1).

5.1 A proof of Theorem 1.2

Remark that the next proposition is a direct consequence of the facts that the rescaled pointed proper geodesic space $(Y, r^{-1}d_Y, x)$ is a Ricci limit space for every $0 < r \leq 1$ and every $x \in Y$ and that the measure $\upsilon_r = \upsilon / \upsilon(B_r(x))$ is a limit measure of it.

Proposition 5.1. *For every $0 < r < 1$ and every $x \in Y$, there exists a limit measure υ_r on $(Y, r^{-1}d_Y, x)$ such that $\upsilon_r(B_{s_1}^{r^{-1}d_Y}(x_1))\upsilon(B_{s_2 r}(x_2)) = \upsilon_r(B_{s_2}^{r^{-1}d_Y}(x_2))\upsilon(B_{s_1 r}(x_1))$ for every $x_1, x_2 \in Y$ and every $s_1, s_2 > 0$. Especially, for every tangent cone $(T_xY, 0_x)$ at x, there exist a limit measure υ_∞ on $(T_xY, 0_x)$ and a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $\upsilon(B_{sr_i}(x))/\upsilon(B_{r_i}(x)) \to \upsilon_\infty(B_s(0_x))$ for every $s > 0$.*

We will give a proof of the next proposition in appendix.

Proposition 5.2. *Let (W, w) be a pointed proper geodesic space and $d \geq 1$ with $d^{-1} \leq \mathrm{diam}\, W \leq d$. Assume that $(\mathbf{R}^k \times W, (0_k, w))$ is a $(n,0)$-Ricci limit space. Then, for every limit measure υ on $\mathbf{R}^k \times W$, there exists a Borel measure υ_W on W such that $\upsilon = H^k \times \upsilon_W$ and that $\limsup_{\delta \to 0} \upsilon_W(B_\delta(z))/\delta \leq C(n,d,R) < \infty$ for every $R > 0$ and every $z \in B_R(w)$.*

Compare the following proposition and Proposition 4.7:

Proposition 5.3. *Let x be a point in $\underline{\mathcal{WE}}_1$. Then we have $\liminf_{r \to 0} \upsilon(B_r(x))/r = 0$.*

Proof. The proof is done by contradiction. Assume that the assertion is false. There exist a tangent cone $(T_xY, 0_x)$ at x and a proper geodesic space W such that W is not a single point and that T_xY is isometric to $\mathbf{R} \times W$. Let υ_∞ be a limit measure on T_xY as in Proposition 5.1. Then it follows from [16, Proposition 4.3] that $(\upsilon_\infty)_{-1}(\{0_x\}) > 0$. This contradicts Proposition 5.2. $\square$

The following theorem is the main result in this subsection.

Theorem 5.4. *Let x be a point in Y and w, z points in $Y \setminus \{x\}$. Assume that $\overline{x,w} + \overline{w,z} = \overline{x,z}$, $\upsilon(C_w(\{z\})) > 0$ and $\dim_H^{\mathrm{loc}} x > 1$. Then, x is not an Alexandrov point.*

Proof. This proof is done by contradiction. Assume that x is an Alexandrov point. Fix a sufficiently small $r > 0$ and a minimal geodesic $\gamma : [0, \overline{x,z}] \to Y$ from x to z. Without loss of generality, we can assume that $B_r(x) \subset \mathrm{Alex}(Y)$. Put $\alpha = \gamma(r)$ and $w = \gamma(r/2)$.

Claim 5.5. *Let $\hat{\gamma} : [0, \overline{w,z}] \to Y$ be a minimal geodesic from w to z. Then, we have $\alpha \in \mathrm{Image}(\hat{\gamma})$.*

The proof is done by contradiction. Assume that the assertion is false. Then there exists $s \in [0, \overline{w,z}]$ such that $\hat{\gamma}(s) \in \partial B_r(x)$ and $\hat{\gamma}(s) \neq \alpha$. Put $\hat{\alpha} = \hat{\gamma}(s)$. Then, we have that $0 \le \overline{x,w} + \overline{w,\hat{\alpha}} - \overline{x,\hat{\alpha}} = \overline{x,w} + (\overline{w,\hat{\alpha}} + \overline{\hat{\alpha},z}) - (\overline{x,\hat{\alpha}} + \overline{\hat{\alpha},z}) \le \overline{x,w} + \overline{w,z} - \overline{x,z} = 0$. Therefore, there exists a minimal geodesic $\Gamma : [0, \overline{x,\hat{\alpha}}] \to Y$ from x to $\hat{\alpha}$ such that $w \in \mathrm{Image}(\Gamma)$. This contradicts the assumption $B_r(x) \subset \mathrm{Alex}(Y)$. Thus, we have the assertion.

By Claim 5.5, for every sufficiently small $t > 0$, there exists $\alpha_t \in Y$ such that $\partial B_t(w) \cap C_w(\{z\}) = \{\alpha_t\}$. By the assumption of $\upsilon(C_w(\{z\})) > 0$ and [16, Theorem 4.6], we have $\upsilon_{-1}(\{\alpha_t\}) > 0$. On the other hand, for the tangent cone $(T_{\alpha_t}Y, 0_{\alpha_t})$ at α_t, there exists a proper geodesic space W such that $T_{\alpha_t}Y$ is isometric to $\mathbf{R} \times W$. By the assumption of $\dim_H^{\mathrm{loc}} x > 1$ and $\alpha_t \in \mathrm{Alex}(Y)$, we have that W is not a single point. Therefore, by Proposition 5.3, we have $\upsilon_{-1}(\{\alpha_t\}) = 0$. This is a contradiction. $\square$

We end this subsection by giving a proof of Theorem 1.2.

A proof of Theorem 1.2. It suffices to check that $\mathrm{Alex}(Y) \subset Y(1)$. Let $x \in \mathrm{Alex}(Y)$ and $z \in \mathcal{R}_1$. If $z = x$, then, it follows from the fact $x \in \mathrm{Alex}(Y)$ that there exists $\epsilon > 0$ such that $(B_\epsilon(x), x)$ is isometric to $((-\epsilon, \epsilon), 0)$. Especially, we have $\dim_H^{\mathrm{loc}} x = 1$. Hence, assume $x \neq z$ below. Let r be a sufficiently small positive number and $w \in B_r(x) \setminus \{x\} \subset \mathrm{Alex}(Y)$ with $\overline{x,w} + \overline{w,z} = \overline{x,z}$. By Proposition 4.7 and [16, Corollary 5.7], we have $\upsilon(C_w(\{z\})) > 0$. Thus, by Theorem 5.4, we have $\dim_H^{\mathrm{loc}} x = 1$. Therefore we have Theorem 1.2. $\square$

5.2 Alexandrov set in tangent cones

In this subsection, we will give an analogous statement to Theorem 1.2 for tangent cones by using measure contraction argument. See for instance Appendix 2 in [5] or [23] for the measure contraction property.

THEOREM 5.6. *Let (X, x) be a proper geodesic space and k a non-negative integer. Assume that $(\mathbf{R}^k \times X, (0_k, x))$ is a $(n, 0)$-Ricci limit space and $X(1) \neq \emptyset$. Then we have* $\mathrm{Alex}(X) = X(1)$.

PROOF. Let $w \in \mathrm{Alex}(X)$ and $z \in X(1)$. Assume that $\dim_H^{\mathrm{loc}} w > 1$ holds. By Corollary 3.3 and an argument similar to the proof of Theorem 4.5, there exists an open neighbourhood U of z such that $U \cap \underline{\mathcal{WE}}_1(X) = \emptyset$. Then, by an argument similar to the proof of Theorem 4.3, there exists a sufficiently small $\epsilon > 0$ such that $(B_\epsilon(z), z)$ is isometric either to $((-\epsilon, \epsilon), 0)$ or to $([0, \epsilon), 0)$. Fix $\tau > 0$ with $\tau << \epsilon$ and a minimal geodesic $\gamma : [0, \overline{z, w}] \to X$ from z to w. Put $\hat{z} = \gamma(\epsilon/2)$, $\hat{w} = \gamma(\overline{z, w} - \epsilon)$ and $\alpha = \gamma(\overline{z, w} - 2\epsilon)$.

CLAIM 5.7. $C_{(0_k, \hat{w})}(B_\tau(0_k, \hat{z})) \cap (B_{\epsilon+\tau}(0_k, \hat{w}) \setminus B_\epsilon(0_k, \hat{w})) \subset B_{3\tau}(0_k, \alpha)$.

The proof is as follows. Let $g \in C_{(0_k, \hat{w})}(B_\tau(0_k, \hat{z})) \cap (B_{\epsilon+\tau}(0_k, \hat{w}) \setminus B_\epsilon(0_k, \hat{w}))$. There exist $(v, \hat{x}) \in B_\tau(0_k, \hat{z})$ and a minimal geodesic Γ from $(v, \hat{x})$ to $(0_k, \hat{w})$ such that $\Gamma(t_0) = g$ for some t_0. Denote $\Gamma(t) = (a(t), \hat{\gamma}(t))$ and put $\Phi(s) = \hat{\gamma}(\overline{(v, \hat{x}), (0_k, \hat{w})} s / \overline{\hat{x}, \hat{w}})$ for $0 \leq s \leq \overline{\hat{x}, \hat{w}}$. Remark that $|a(t)| \leq \tau$ for every t and that $\Phi(s)$ is a minimal geodesic from $\hat{x}$ to $\hat{w}$. By an argument similar to the proof of Claim 5.5, we have $\alpha \in \mathrm{Image}(\hat{\gamma})$. On the other hand, since $g \in B_{\epsilon+\tau}(0_k, \hat{w}) \setminus B_\epsilon(0_k, \hat{w})$, we have $\hat{\gamma}(t_0) \in B_{\epsilon+\tau}(\hat{w}) \setminus B_{\epsilon-\tau}(\hat{w})$. Since $\alpha \in \mathrm{Image}(\hat{\gamma}) \cap B_{\epsilon+\tau}(\hat{w}) \setminus B_{\epsilon-\tau}(\hat{w})$, we have $\overline{\hat{\gamma}(t_0), \alpha} \leq 2\tau$. Therefore we have $\overline{g, (0_k, \alpha)} \leq |a(t_0)| + \overline{\hat{\gamma}(t_0), \alpha} \leq 3\tau$.

Therefore, by Bishop-Gromov inequality for υ, we have $\upsilon(B_\tau(0_k, \hat{z})) \leq C(\epsilon, n, \overline{z, x})\upsilon(B_{2\tau}(0_k, \alpha))$. Since the ball $B_\tau(0_k, \hat{z})$ is Euclidean (or half a Euclidean ball), by [7, Theorem 4.6], we have $\liminf_{\tau \to 0} \upsilon(B_\tau(0_k, \hat{z}))/\tau^{k+1} > 0$. Therefore, we have $\liminf_{\tau \to 0} \upsilon(B_\tau(0_k, \alpha))/\tau^{k+1} > 0$. Thus, by Proposition 5.1 and Proposition 5.2, there exists $C > 1$ such that $C^{-1}\tau^{k+1} \leq \upsilon(B_\tau(0_k, \alpha)) \leq C\tau^{k+1}$ for every $0 < \tau < 1$. Therefore, there exist a pointed proper geodesic space (Z_1, z_1), a tangent cone $T_{(0_k, \alpha)}(\mathbf{R}^k \times X)$, a limit measure $\hat{\upsilon}$ on $T_{(0_k, \alpha)}(\mathbf{R}^k \times X)$, and a Borel measure υ_{Z_1} on Z_1 such that $T_{(0_k, \alpha)}(\mathbf{R}^k \times X)$ is isometric to $\mathbf{R}^{k+1} \times Z_1$, $\hat{\upsilon} = H^{k+1} \times \upsilon_{Z_1}$ and $\liminf_{\tau \to 0} \hat{\upsilon}(B_\tau(0_k, z_1))/\tau^{k+1} > 0$. On the other hand, since $\alpha \in \mathrm{Alex}(X)$ and $\dim_H^{\mathrm{loc}} w > 1$, we have that Z_1 is not a single point. Therefore, by Proposition 5.2, we have $\liminf_{\tau \to 0} \hat{\upsilon}(B_\tau(0_k, z_1))/\tau^{k+1} = 0$. This is a contradiction. Therefore, we have $\mathrm{Alex}(X) \subset X(1)$.

Let $\beta \in X(1)$ and $\delta > 0$ with $\dim_H B_\delta(\beta) < 2$. By Corollary 3.3 and an argument similar to the proof of Theorem 4.5, we have $B_\delta(\beta) \cap \underline{\mathcal{WE}}_1(X) = \emptyset$. Thus, by an argument similar to the proof of Theorem 4.3, there exists $r > 0$ such that $(B_r(\beta), \beta)$ is isometric either to $((-r, r), 0)$ or to $([0, r), 0)$. Especially, we have $\beta \in \mathrm{Alex}(X)$. $\square$

REMARK 5.8. Let (X, x) be a pointed proper geodesic space. For an open subset U of X, we say that U has *k-dimensional C^∞-Riemannian structure* if for every $x \in U$, there exist an open neighbourhood V of x and a k-dimensional (not necessary complete) Riemannian manifold N such that V is isometric to N. Assume that there exist open sets U_1, U_2 of X such that U_1 has one-dimensional C^∞-Riemannian structure and that U_2 has $k(\geq 2)$-dimensional C^∞-Riemannian structure. Let (M, m) be a pointed l-dimensional complete C^∞-Riemannian manifold. Then, by an argument similar to the proof of Theorem 5.6, we have that $(M \times X, (m, x))$ is *not* a Ricci limit space, especially, $(M \times Z_\tau, (m, 0))$ is not a Ricci limit space.

We say that a proper geodesic space X is *non-branching* if for every $x \in X$ and every $y \in X \setminus C_x$, there exists a unique minimal geodesic from x to y.

THEOREM 5.9. *Assume that $\mathcal{R}_1 \neq \emptyset$ and that Y is non-branching. Then we have* $\dim_H Y = 1$.

PROOF. Let $x \in \mathcal{R}_1$. First, we will show that $Y \setminus C_x \subset A_Y(1)$. Let $z \in Y \setminus C_x$. There exists $w \in Y \setminus C_x$ such that $z \neq w$ and $\overline{x, z} + \overline{z, w} = \overline{x, w}$ hold. By the assumption of non-branching, there exists a unique minimal geodesic $\gamma : [0, \overline{x, w}] \to Y$ from x to w and it satisfies $z \in \mathrm{Image}(\gamma)$. By Proposition 4.7 and [16, Theorem 1.1], we have $\upsilon_{-1}(\{z\}) > 0$. Therefore, we have $Y \setminus C_x \subset A_Y(1)$. It follows from [16, Theorem 3.2] that $\upsilon(Y \setminus A_Y(1)) = 0$. By Corollary 4.6, we have the assertion. $\square$

Remark that it is unknown whether there exists a branching Ricci limit space. However, if we drop the non-branching assumption in the theorem above, then we can get the same conclusion. See [17].

6 The case $2 \leq \dim_H Y < 3$

In this section, we will study the Hausdorff dimension of a Ricci limit space (Y, y) with $2 \leq \dim_H Y < 3$. The main result in this section is Corollary 6.4.

PROPOSITION 6.1. *Let $s \geq 1$, U be an open subset of Y with $\dim_H U \leq s$, $x \in U$, and $(T_x Y, 0_x)$ a tangent cone at x. Assume that there exists a proper geodesic space W such that $T_x Y$ is isometric to $\mathbf{R}^{[s]-1} \times W$. Then, W is isometric either to a single point, or to $\mathbf{R}$, or to $\mathbf{R}_{\geq 0}$, or to $\mathbf{S}^1(r)$ for some $r > 0$, or to $[0, l]$ for some $l > 0$, where* $[s] = \max\{k \in \mathbf{Z} | k \leq s\}$.

PROOF. By an argument similar to the proof of Theorem 4.3, it suffices to check $\underline{\mathcal{WE}_1}(W) = \emptyset$. Assume $\underline{\mathcal{WE}_1}(W) \neq \emptyset$. Then we have $\underline{\mathcal{WE}_{[s]}}(T_xY) \neq \emptyset$. Thus, by Corollary 3.3, we have $\mathcal{WE}_{[s]+1}(T_xY) \neq \emptyset$. Hence, we have that $(\mathcal{WE}_{[s]+1})_\epsilon \cap U \neq \emptyset$ for every $\epsilon > 0$. Thus, by [7, Theorem 3.3] and Corollary 3.3, there exists $i \geq [s]+1$ such that $\mathcal{R}_i \cap U \neq \emptyset$. Therefore, by [6, Corollary 1.36], we have that $\dim_H U \geq i \geq [s]+1 > s$. This is a contradiction. Therefore we have $\underline{\mathcal{WE}_1}(W) = \emptyset$. □

COROLLARY 6.2. *Let $s \geq 1$ and U be an open subset of Y with $\dim_H U \leq s$. Then, we have $\dim_H(\mathcal{E}_{[s]-1} \cap U) \leq [s]$.*

PROOF. First, we will show the following:

CLAIM 6.3. *Let X be a proper geodesic space, $A \subset X$ and $s > 0$. Assume that the following hold:*

1. *For every $x \in X$ and every sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$, there exist a subsequence $\{r_{i(j)}\}_j$ and a tangent cone $(T_xX, 0_x)$ at x such that $(X, r_{i(j)}^{-1}d_X, x) \to (T_xX, 0_x)$.*
2. *$\dim_H T_\alpha X \leq s$ holds for every $\alpha \in A$ and for every tangent cone $(T_\alpha X, 0_\alpha)$ at α.*

Then, we have $\dim_H A \leq s$.

This proof is done by contradiction. Assume $\dim_H A > s$. Fix $\epsilon > 0$ with $\dim_H A > s + \epsilon$. Then it is not difficult to check that there exist $\alpha \in A$ and a sequence of positive numbers $\{r_i\}_i$ with $r_i \to 0$ such that $\lim_{i\to\infty}(H_\infty^{s+\epsilon}(A \cap \overline{B}_{r_i}(\alpha))/{r_i}^{s+\epsilon}) > 0$ (see (1.39) in [6] for the definition of the $(s+\epsilon)$-dimensional spherical Hausdorff content $H_\infty^{s+\epsilon}$). By the first assumption, without loss of generality, we can assume that there exists a tangent cone $(T_\alpha X, 0_\alpha)$ at α such that $(X, r_i^{-1}d_X, \alpha) \to (T_\alpha X, 0_a)$. By the construction, it is not difficult to see that $H^{s+\epsilon}(\overline{B}_1(0_\alpha)) > 0$. Especially, we have that $\dim_H T_\alpha X \geq s + \epsilon > s$. This is a contradiction. Therefore, we have Claim 6.3.

By Proposition 6.1, for every $x \in \mathcal{E}_{[s]-1} \cap U$ and every tangent cone $(T_xY, 0_x)$ at x, we have $\dim_H T_xY \leq [s]$. Therefore Corollary 6.2 follows directly from Claim 6.3. □

We end this section by giving the following:

COROLLARY 6.4. *Assume $2 \leq \dim_H Y < 3$. Then we have that $\dim_H(Y \setminus C_x) \leq 2$ for every $x \in Y$.*

PROOF. By $Y \setminus C_x \subset \mathcal{E}_1$ and Corollary 6.2. □

REMARK 6.5. It seems that $\dim_H(Z \setminus C_z) = \dim_H Z$ holds for every Ricci limit space (Y, y), every tangent cones (Z, z) at every $x \in Y$. If it is true, then we can prove that $\dim_H Y \in \mathbf{Z}$ holds for every Ricci limit space (Y, y). See the next section.

7 Hausdorff dimension of Ricci limit spaces

In this section, we will study a weakly polar Ricci limit space (Y, y).

Definition 7.1. A pointed proper geodesic space (X, x) is called by an *iterated tangent cone of* Y if there exists a sequence of pointed proper geodesic spaces $\{(X_i, x_i)\}_{i=0}^N$ such that $X_0 = Y$, $(X_N, x_N) = (X, x)$ and that (X_{i+1}, x_{i+1}) is a tangent cone at a point in X_i for every i.

Recall that a Ricci limit space (Y, y) is weakly polar if $\dim_H X = \dim_H (X \setminus C_x)$ holds for every iterated tangent cone (X, x) of Y.

Theorem 7.2. *Assume that Y is weakly polar. Then we have that $\dim_H B_R(z) \in \mathbf{Z}$ for every $z \in Y$ and every $R > 0$. Especially, we have that $\dim_H Y \in \mathbf{Z}$ and $\dim_H^{\mathrm{loc}} z \in \mathbf{Z}$.*

Proof. Fix an integer $k > 0$ with $\dim_H B_R(z) < k + 1$. It suffices to check that $\dim_H B_R(z) \le k$. By Claim 6.3, it suffices to see that $\dim_H T_z Y \le k$ holds for every $z \in Y$ and every tangent cone $(T_z Y, 0_z)$ at z. Fix a tangent cone $(T_z Y, 0_z)$ and put $(Y_1, y_1) = (T_z Y, 0_z)$. By the assumption and Claim 6.3, it suffices to see that $\dim_H T_{z_1} Y_1 \le k$ holds for every $z_1 \in Y_1 \setminus C_{y_1}$ and every tangent cone $(T_{z_1} Y_1, 0_{z_1})$ at z_1. We also fix a tangent cone $(T_{z_1} Y_1, 0_{z_1})$ and put $(Y_2, y_2) = (T_{z_1} Y_1, 0_{z_1})$. By the construction, there exists a pointed proper geodesic space (W_2, w_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times W_2, (0, w_2))$. Without loss of generality, we can assume that W_2 is not a single point. Remark the following:

Claim 7.3. *We have that $C_{(0_k, w)} = \mathbf{R}^k \times C_w$ in $\mathbf{R}^k \times W$ for every $k \ge 1$ and every pointed proper geodesic space (W, w).*

This claim is a direct consequence of the fact that every minimal geodesic in a product of geodesic spaces is a product of minimal geodesics of the factors (see for instance [1]).

By the assumption of weakly polar, Claim 7.3 and [15, Corollary 5.4], we have $\dim_H (W_2 \setminus C_{w_2}) \ge \dim_H C_{w_2}$. Thus, it suffices to see that $\dim_H T_{\hat{w}_2} W_2 \le k - 1$ for every $\hat{w}_2 \in W_2 \setminus C_{w_2}$ and every tangent cone $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$ at $\hat{w}_2$. Fix a tangent cone $(T_{\hat{w}_2} W_2, 0_{\hat{w}})$ and put $(W_3, w_3) = (T_{\hat{w}_2} W_2, 0_{\hat{w}_2})$. By the construction, there exists a pointed proper geodesic space (W_4, w_4) such that (W_3, w_3) is isometric to $(\mathbf{R} \times W_4, (0, w_4))$. By Claim 6.3, without loss of generality, we can assume that W_4 is not a single point. Since $(\mathbf{R}^2 \times W_4, (0_2, w_4))$ is an iterated tangent cone of Y, by the assumption of weakly polar and Claim 7.3, we have $\dim_H (W_4 \setminus C_{w_4}) \ge \dim_H C_{w_4}$. Therefore, it suffices to see that $\dim_H T_{\hat{w}_4} W_4 \le k - 2$ for every $\hat{w}_4 \in W_4 \setminus C_{w_4}$ and every tangent cone $(T_{\hat{w}_4} W_4, 0_{\hat{w}_4})$ at $\hat{w}_4$.

Continue this argument and construct a pointed proper geodesic space (W_{2k}, w_{2k}) as above. Then, it suffices to see that $\dim_H W_{2k} \le 0$, i.e. W_{2k} is a single point. Assume that W_{2k} is not a single point. Then, by the construction, there exist an iterated tangent cone (X, x) of $B_R(z)$ and a proper geodesic space L such that X is isometric to $\mathbf{R}^{k+1} \times L$. Therefore, we have that $(\mathcal{WE}_{k+1})_\epsilon \cap B_R(z) \neq \emptyset$ for every $\epsilon > 0$. Thus, by Corollary 3.3 and [7, Theorem 3.3], there exists $i \geq k+1$ such that $\mathcal{R}_i \cap B_R(z) \neq \emptyset$. Therefore, by [6, Corollary 1.36], we have that $\dim_H B_R(z) \geq i \geq k+1$. This is a contradiction. Therefore, we have $\dim_H B_R(z) \leq k$. □

REMARK 7.4. By an argument similar to the proof of Theorem 7.2, if $\dim_H(X \setminus \mathcal{WD}_0(x)) \geq \dim_H \mathcal{WD}_0(x)$ holds for every iterated tangent cone (X, x) of Y, then we have the same conclusion to Theorem 7.2 (see [5, Definition 2.10] for the definition of $\mathcal{WD}_0(x)$).

REMARK 7.5. Recall that we say that Y is polar if for every iterated tangent cone (X, x) of Y and every $z \in Z \setminus \{x\}$, there exists an isometric embedding γ from $\mathbf{R}_{\geq 0}$ to X such that $\gamma(0) = x$ and $\gamma(\overline{x,z}) = z$ (see [5]). It is not difficult to see that Y is polar if and only if $C_x = \emptyset$ for every iterated tangent cone (X, x) of Y.

THEOREM 7.6. *Let $R > 0$, $k \geq 1$ and $z \in Y$. Assume that Y is weakly polar and that $\dim_H B_R(z) \geq k$ holds. Then, we have $\upsilon(B_R(z) \cap (\bigcup_{i \geq k} \mathcal{R}_i)) > 0$.*

PROOF. Fix a sufficiently small $\epsilon > 0$. By the assumption, we have $H^{k-\epsilon}(B_R(z)) = \infty$. Hence, by an argument similar to the proof of Claim 6.3, there exist $x \in B_R(z)$ and a tangent cone $(T_x Y, 0_x)$ at x such that $H^{k-\epsilon}(T_x Y) > 0$ holds. Fix a tangent cone $(T_x Y, 0_x)$ and put $(Y_1, y_1) = (T_x Y, 0_x)$. Since $\dim_H Y_1 \geq k - \epsilon > k - 2\epsilon > 0$ and $\dim_H (Y_1 \setminus C_{y_1}) = \dim_H Y_1$, we have $H^{k-2\epsilon}(Y_1 \setminus C_{y_1}) = \infty$. Similarly, there exist $x_1 \in Y_1 \setminus C_{y_1}$ and a tangent cone $(T_{x_1} Y_1, 0_{x_1})$ at x_1 such that $H^{k-2\epsilon}(T_{x_1} Y_1) > 0$ holds. Put $(Y_2, y_2) = (T_{x_1} Y_1, 0_{x_1})$. By the construction, there exists a pointed proper geodesic space (X_2, x_2) such that (Y_2, y_2) is isometric to $(\mathbf{R} \times X_2, (0, x_2))$. Thus, we have that $\dim_H X_2 \geq k - 1 - 2\epsilon > k - 1 - 3\epsilon > 0$. Therefore, since $\dim_H X_2 = \dim_{\mathcal{H}}(X_2 \setminus C_{x_2})$, we have $H^{k-1-3\epsilon}(X_2 \setminus C_{x_2}) = \infty$. By an argument similar to that above, there exist $\hat{x}_2 \in X_2$ and a tangent cone $(T_{\hat{x}_2} X_2, 0_{\hat{x}_2})$ at $\hat{x}_2$ such that $H^{k-1-3\epsilon}(T_{\hat{x}_2} X_2) > 0$. Put $(X_3, x_3) = (T_{\hat{x}_2} X_2, 0_{\hat{x}_2})$. By the construction, there exists a pointed proper geodesic space (X_4, x_4) such that (X_3, x_3) is isometric to $(\mathbf{R} \times X_4, (0, x_4))$. Since $(\mathbf{R}^2 \times X_4, (0_2, x_4))$ is an iterated tangent cone of $B_R(z)$, by the assumption, we have that $\dim_H X_4 = \dim_H (X_4 \setminus C_{x_4})$ and $\dim_H X_4 \geq k - 2 - 3\epsilon > k - 2 - 4\epsilon$.

Continue this argument and construct a pointed proper geodesic space $(X_{2(k-1)}, x_{2(k-1)})$ as above. By the construction, $(\mathbf{R}^{k-1} \times X_{2(k-1)}, (0_k, x_{2(k-1)}))$ is an iterated tangent cone of $B_R(z)$. We have $\dim_H X_{2(k-1)} \geq k - (k-1) - 2(k-2)\epsilon > 1 - 2(k-1)\epsilon > 0$. Since $X_{2(k-1)}$ is a geodesic space, we have $\dim_H X_{2(k-1)} \geq 1$. Therefore, there exists a pointed proper geodesic space (W, w) such that $(\mathbf{R}^k \times W, (0_k, w))$ is an iterated tangent cone of

$B_R(z)$. Thus, we have that $(\mathcal{WE}_k)_\epsilon \cap B_R(z) \neq \emptyset$ holds for every $\epsilon > 0$. Therefore, by Corollary 3.3 and [7, Theorem 3.3], we have $\upsilon(B_R(z) \cap (\bigcup_{i\geq k} \mathcal{R}_i)) > 0$. $\square$

The main result in this section is the following:

COROLLARY 7.7. *Assume that Y is weakly polar. Let $k \geq 1$ satisfying that $\mathcal{R}_k \neq \emptyset$ and that $\mathcal{R}_i = \emptyset$ for every $i > k$. Then we have that $\dim_H Y = k$, $H^k(\mathcal{R}^k) > 0$ and $\upsilon(\mathcal{R}^k) > 0$.*

PROOF. By [6, Corollary 1.36], we have $\dim_H Y \geq k$. Assume $\dim_H Y \geq k+1$. Then, by Theorem 7.6, there exists $i \geq k+1$ such that $\mathcal{R}_i \neq \emptyset$. This contradicts the assumption. Thus we have $\dim_H Y < k+1$. By Theorem 7.2, we have $\dim_H Y = k$. Next, assume $\upsilon(\mathcal{R}_k) = 0$. Then we have that $\upsilon(\bigcup_{i\geq k} \mathcal{R}_i) = \upsilon(\mathcal{R}_k) = 0$. This contradicts Proposition 3.1. Thus, we have $\upsilon(\mathcal{R}_k) > 0$. By [7, Theorem 3.23] and [7, Theorem 4.6], we have $H^k(\mathcal{R}^k) > 0$. $\square$

8 Appendix: A proof of Proposition 5.2

In this section, we will give a proof of Proposition 5.2. First, we give the following lemma without the proof because it follows directly from easy calculation:

LEMMA 8.1. *Let (X, x) be a pointed metric space, $R \geq 1$, $\delta, \epsilon > 0$, $v_\alpha, v_\beta \in \overline{B}_1(0_k) \subset \mathbf{R}^k$ and $x_\alpha, x_\beta \in \overline{B}_R(x) \setminus B_{R^{-1}}(x)$. Assume that $\overline{x_\alpha, x_\beta} \leq \delta$ and that $\overline{(0_k, x), (v_\alpha, x_\alpha)} + \overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} - \overline{(0_k, x), (v_\beta, x)} \leq \epsilon$ holds in $\mathbf{R}^k \times X$. Then, we have that $\overline{(v_\alpha, x_\alpha), (v_\beta, x_\beta)} \leq C(r, R)(\delta + \epsilon)$.*

A proof of Proposition 5.2 Without loss of generality, we can assume that $z \in \overline{B}_R(w) \setminus B_{d^{-1}}(w)$. By the assumption, there exist a sequence of pointed complete connected n-dimensional Riemannian manifolds $\{(M_j, m_j)\}_j$ and a sequence of positive numbers $\{\epsilon_j\}_j$ with $\epsilon_j \to 0$ such that $\mathrm{Ric}_{M_j} \geq -\epsilon_j$ and $(M_j, m_j, \mathrm{vol}/\mathrm{vol}\, B_1(m_j)) \to (\mathbf{R}^k \times W, (0_k, w), \upsilon)$. Fix a sufficiently small $\delta > 0$. Let $\{(t_i, x_i)\}_{i=1}^N$ be a maximal δ-separated subset of $[0,1]^k \times \overline{B}_\delta(z)$, $z \in \overline{B}_R(w) \setminus B_r(w)$ and $y_j^i \in M_j$ with $y_j^i \to (t_i, x_i)$ as $j \to \infty$. Remark that $\{\overline{B}_{\delta/3}(y_j^i)\}_i$ is pairwise disoint for every sufficiently large j. Put $r = d^{-1}$, $X_j = \bigcup_i \overline{B}_{\delta/3}(y_j^i)$, $S_{m_j}M_j = \{u \in T_{m_j}M_j | |u| = 1\}$, $t(u) = \sup\{t \in \mathbf{R}_{>0} | \exp_{m_j} su \in M_j \setminus C_{m_j}$ for every $0 < s < t\}$ for $u \in S_{m_j}M_j$, $\hat{S}_{m_j}M_j = \{u \in S_{m_j}M_j |$ There exists $0 < t < t(u)$ such that $\exp_{m_j} tu \in X_j$ holds.$\}$ and $A_j(u) = \{t \in (0, t(u)) | \exp_{m_j} tu \in X_j\}$ for $u \in \hat{S}_{m_j}M_j$ and $\theta(t, u) = t^{n-1}\sqrt{\det(g_{ij}|_{\exp_{m_j} tu})}$, where $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$ for a normal coordinate

$(x_1, x_2, \ldots, x_n)$ around m_j. Then, by Laplacian comparison theorem, we have

$$\begin{aligned}
\operatorname{vol} X_j &= \int_{\hat{S}_{m_j} M_j} \int_{A_j(u)} \theta(t,u) dt du \\
&\leq \int_{\hat{S}_{m_j}} \int_{A_j(u)} \sinh^{n-1}(t) \frac{\theta\left(\frac{r}{2}, u\right)}{\sinh^{n-1}(\frac{r}{2})} dt du \\
&\leq \int_{\hat{S}_{m_j} M_j} \frac{\theta\left(\frac{r}{2}, u\right)}{\sinh^{n-1}\left(\frac{r}{2}\right)} \int_{A_j(u)} \sinh^{n-1}(2R+10) dt du \\
&\leq C(n,r,R) \int_{\hat{S}_{m_j} M_j} \theta\left(\frac{r}{2}, u\right) H^1(A_j(u)) du.
\end{aligned}$$

Put $a_j(u) = \inf A_j(u)$ and $b_j(u) = \sup A_j(u)$ for $u \in \hat{S}_{m_j} M_j$. Then, by Lemma 8.1, we have that $b_j(u) - a_j(u) \leq C(r,R)\delta$ for every sufficiently large j. Thus $\underline{\operatorname{vol}}\, X_j \leq C(r,R)\delta \underline{\operatorname{vol}}(\partial B_{\frac{r}{2}}(m_j) \setminus C_{m_j})$, where $\underline{\operatorname{vol}} = \operatorname{vol}/\operatorname{vol} B_1(m_j)$. By Bishop-Gromov inequality, we have $\operatorname{vol}(\partial B_{\frac{r}{2}}(m_j) \setminus C_{m_j})/\operatorname{vol} B_{\frac{r}{2}}(m_j) \leq \operatorname{vol} \partial B_{\frac{r}{2}}(\underline{p})/\operatorname{vol} B_{\frac{r}{2}}(\underline{p})$, where $\underline{p}$ is a point in the n-dimensional space form whose sectional curvature is equal to -1. Thus, we have

$$\sum_{i=1}^{N} \upsilon(B_{\frac{\delta}{3}}(t_i, x_i)) \leq C(n,r,R)\delta.$$

By [5, Proposition 1.35], there exists a Borel measure υ_W on W such that $\upsilon = H^k \times \upsilon_W$. Therefore, by Bishop-Gromov inequality for υ, we have

$$\begin{aligned}
\upsilon_W(B_\delta(w)) = \upsilon([0,1]^k \times B_\delta(w)) &\leq \sum_{i=1}^{N} \upsilon(B_\delta(t_i, x_i)) \\
&\leq C(n) \sum_{i=1}^{N} \upsilon(B_{\frac{\delta}{3}}(t_i, x_i)) \\
&\leq C(n,r,R)\delta.
\end{aligned}$$

Therefore, we have Proposition 5.2.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES,
KYOTO UNIVERSITY,
KYOTO 606-8502
JAPAN

E-mail address: honda@kurims.kyoto-u.ac.jp

CURRENT ADRESS:
FACULTY OF MATHMATICS,
KYUSHU UNIVERSITY,
744, MOTOOKA, NISHI-KU,
FUKUOKA 819-0395,
JAPAN.

E-mail address: honda@math.kyushu-u.ac.jp