COUNTEREXAMPLES TO KODAIRA'S VANISHING AND YAU'S INEQUALITY IN POSITIVE CHARACTERISTICS

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To the memory of Professor Masaki Maruyama

Abstract. We generalize Tango's theorem [T1] on the Frobenius map of the first cohomology groups to higher dimensional algebraic varieties in characteristic $p > 0$. As application we construct counterexamples to Kodaira vanishing in higher dimension, and prove the Ramanujam type vanishing on surfaces which are not of general type when $p \geq 5$.

Let X be a smooth complete algebraic variety over an algebraically closed field of positive characteristic $p > 0$, and let D be an effective divisor on X. In this article we study the kernel of the Frobenius map

$$F^* : H^1(X, \mathcal{O}_X(-D)) \to H^1(X, \mathcal{O}_X(-pD)) \tag{1}$$

of the first cohomology groups of line bundles.

Tango [T1] described the kernel of F^* in terms of the exact differentials in the case of curves. First we generalize this result to varieties of arbitrary dimension, that is, we prove

Theorem 1. *The kernel of the Frobenius map* (1) *is isomorphic to the vector space*

$$\{df \in \Omega_{Q(X)} \mid f \in Q(X),\ (df) \geq pD\},$$

where $Q(X)$ is the function field of X and $(\omega) \geq pD$ means that a rational differential $\omega \in \Omega_{Q(X)}$ belongs to $\Gamma(X, \Omega_X(-pD))$.

Using this description and generalizing Raynaud's method [Ra], we construct pathological varieties of higher dimension which are similar to his surfaces:

Theorem 2. *Let p be a prime number and $n \geq 2$ an integer. Then there exist an n-dimensional smooth projective variety X of characteristic p and an ample line bundle L such that*

(a) $H^1(X, L^{-1}) \neq 0$,

(b) *the canonical divisor class K_X is ample and the intersection number $(c_i(X).K_X^{n-i})$ is negative for every $i \geq 2$, and*

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(c) *there is a finite cover G of X and a sequence of morphisms*

$$G = G_n \to G_{n-1} \to \cdots \to G_2 \to G_1$$

such that $G_{i+1} \to G_i$ is a $\mathbb{P}^1$-bundle for every $i = 1, \cdots, n-1$ and that G_1 is a nonsingular curve. The Euler characteristic $e(X)(:= \deg c_n(X))$ of X is equal to $e(G) = 2^{n-1}e(G_1)$.

Here $c_i(X)$ is the ith Chern class of X.

When $p = 2, 3$, we obtain similar varieties X with quasi-elliptic fibrations $X \to Y$. In this case, the canonical classes K_X are the pull-back of ample divisor classes on Y. By the property (b) and Yau's inequality ([Y1], [Y2]) or by (a) and [DI], we have

Corollary. *The algebraic variety X in the theorem is not liftable to characteristic zero.*

Throughout this article R.V. (Ramanujam vanishing) on an algebraic surface X means the vanishing of $H^1(X, L^{-1})$ for all nef and big line bundles on X. Conversely to the above counterexample, using Theorem 1 and [LM], we prove the following.

Theorem 3. *In the case where X is of dimension two, we have the following:*

(a) *Assume that X is not of general type and that the Iitaka fibration $X \to C$ is not quasi-elliptic when the Kodaira dimension $\kappa(X)$ is 1 and $p = 2, 3$. Then R.V. holds on X.*
(b) *If R.V. does not hold on X, then there exist a birational morphism $X' \to X$ and a morphism $g : X' \to C$ onto a smooth algebraic curve C such that every fiber F of g is connected and singular. Furthermore, the cotangent sheaf Ω_F has nonzero torsion.*

Our counterexamples X in dimension two are sandwiched between two $\mathbb{P}^1$-bundles, and the general fibers F in (b) of Theorem 3 are rational for them. A curve of higher (geometric) genus appears as such a fiber F if we take a sufficiently general separable cover $\pi : \tilde{X} \to X$ with $(\deg \pi, p) = 1$. R.V. does not hold on $\tilde{X}$ either since L^{-1} is a direct summand of $\pi_*\pi^*L^{-1}$.

All results of this article are contained in either [M1] or [M2] except for Proposition 3.2. The report [M1] is an outcome of the author's seminar around 1977 on [Mum] and a preprint of [Ra] with Professor Masaki Maruyama, to whose advice and encouragement the author expresses his sincere gratitude on this occasion. The author is also grateful to the referee for his careful reading and suggestion of useful references.

Convention. In the following we assume the characteristic p is positive and mean by K.V. (Kodaira vanishing) the vanishing of the first cohomology group $H^1(X, L^{-1})$ for all ample line bundles L on X.

1. Tango's theorem

The feature of positive characteristic is the existence of the Frobenius morphisms $F : X \to X$ and the Frobenius maps. Let L be a line bundle on X. The Frobenius morphism induces the Frobenius map

$$F^* : H^1(X, L^{-1}) \to H^1(X, L^{-p}) \tag{2}$$

between the first cohomology groups. When X is normal and $\dim X \geq 2$, we have

Lemma 1.1 (Enriques-Severi-Zariski)**.** $H^1(X, L^{-m}) = 0$ *holds if* L *is ample and* m *is sufficiently large.*

Therefore, by the sequence

$$H^1(X, L^{-1}) \to H^1(X, L^{-p}) \to H^1(X, L^{-p^2}) \to \cdots$$

of Frobenius maps, K.V. holds on X if and only if the following holds:

(*) $F^* : H^1(X, L^{-1}) \to H^1(X, L^{-p})$ is injective for every ample line bundle L on X.

1.1. **Tango-Raynaud curve.** The statement (*) makes sense even when $\dim X = 1$. The following is fundamental for (*) in this case:

Theorem 1.2 (Tango [T1])**.** *Let* D *be an effective divisor on a smooth algebraic curve* X*. Then the kernel of the Frobenius map* (1) *is isomorphic to the space of exact differentials* df *of rational functions* f *on* X *with* $(df) \geq pD$*.*

The following example, which was found by Raynaud [Ra] in the case $e = 1$, shows that (*) does not holds when $\dim X = 1$.

Example 1.3. Let $P(Y)$ be a polynomial of degree e in one variable Y and let $C \subset \mathbb{P}^2$ be the plane curve of degree pe defined by

$$P(Y^p) - Y = Z^{pe-1}, \tag{3}$$

where (Y, Z) is a system of inhomogeneous coordinates of $\mathbb{P}^2$. It is easy to check that C is smooth and has exactly one point ∞ on the line of infinity. By the relation

$$-dY = -Z^{pe-2}dZ$$

between the differentials dY and dZ, Ω_C is generated by dZ over $C \cap \mathbb{A}^2$. In other words, dZ has no poles or zeros over $C \cap \mathbb{A}^2$. Since $\deg \Omega_C = 2g(C) - 2 = pe(pe-3)$, we have $(dZ) = pe(pe-3)(\infty)$. Therefore, by the above theorem of Tango, the Frobenius map (1) is not injective for the divisor $D = e(pe-3)(\infty)$.

A curve C of genus ≥ 2 is called *a Tango-Raynaud curve* if C satisfies the following mutually equivalent conditions:

(a) there exists a line bundle L on C such that $L^p \simeq \Omega_C$ and that the Frobenius map (2) is not injective, and

(b) there exists a rational function f on C such that $df \neq 0$ and that the divisor (df) is divisible by p.

The curve C in the above Example is a Tango-Raynaud curve.

1.2. **Higher dimensional generalization.** Following [T1] we denote the cokernel of the natural (pth power) homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ by $\mathcal{B}_X$. For a Cartier divisor D on X we have the exact sequence

$$0 \to \mathcal{O}_X(-D) \to F_*(\mathcal{O}_X(-pD)) \to \mathcal{B}_X(-D) \to 0 \tag{4}$$

and the associated long exact sequence

$$\begin{array}{rlll} 0 & \to & H^0(\mathcal{O}_X(-D)) \xrightarrow{F^*} H^0(\mathcal{O}_X(-pD)) \to & H^0(\mathcal{B}_X(-D)) \\ & \xrightarrow{\delta} & H^1(\mathcal{O}_X(-D)) \xrightarrow{F^*} H^1(\mathcal{O}_X(-pD)) \to & \cdots . \end{array} \tag{5}$$

If D is effective, then $F^* : H^0(\mathcal{O}_X(-D)) \to H^0(\mathcal{O}_X(-pD))$ is surjective. Hence we have the following

Lemma 1.4. *If D is effective, then the coboundary map δ of* (5) *induces the isomorphism*

$$\mathrm{Ker}[F^* : H^1(\mathcal{O}_X(-D)) \to H^1(\mathcal{O}_X(-pD))] \simeq H^0(\mathcal{B}_X(-D)). \tag{6}$$

Assume that X is normal and consider the direct image of the derivation $d : \mathcal{O}_X \to \Omega_X$ by F. By F_*d, $\mathcal{B}_X$ is regarded as a subsheaf of $F_*\Omega_X$. Let $\Omega_{Q(X)}$ be the $Q(X)$-vector space of differentials. We denote the constant sheaf associated with $Q(X)$ or $\Omega_{Q(X)}$ on X by the same symbol, and consider the intersection $dQ(X) \cap \Omega_X$ in the constant sheaf $\Omega_{Q(X)}$. Then, more precisely, $\mathcal{B}_X$ is contained in $F_*(dQ(X) \cap \Omega_X)$. We also have $\mathcal{B}_X(-D) \hookrightarrow F_*(dQ(X) \cap \Omega_X(-pD))$. Therefore, by the exact sequence (5), we have

Proposition 1.5. *If X is normal, then the kernel of the Frobenius map of $H^1(\mathcal{O}_X(-D))$ is isomorphic to a subspace of the vector space*

$$\{df \in \Omega_{Q(X)} \mid f \in Q(X),\, (df) \geq pD\}.$$

Corollary. *If X is normal and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(pD), \Omega_X) = 0$, then the Frobenius map of $H^1(\mathcal{O}_X(-D))$ is injective.*

When X is smooth, $\mathcal{B}_X = F_*(dQ(X) \cap \Omega_X)$ holds, by the existence of a p-basis. Hence $\mathcal{B}_X(-D) = F_*(dQ(X) \cap \Omega_X(-pD))$ holds for a Cartier divisor D and we have Theorem 1.

1.3. **Purely inseparable covering in an $\mathbb{A}^1$-bundle.** When a vector bundle E on X is given, we have the relative Frobenius morphism $\mathbb{P}(E) \to \mathbb{P}(E^{(p)})$ over X. We denote this morphism by φ. We consider the special case where E is an extension of two line bundles:

$$0 \to \mathcal{O}_X(-D) \to E \to \mathcal{O}_X \to 0. \tag{**}$$

Then $E^{(p)}$ is also an extension of line bundles

$$0 \to \mathcal{O}_X(-pD) \to E^{(p)} \to \mathcal{O}_X \to 0. \tag{***}$$

Let $F_\infty \subset \mathbb{P}(E)$ be the section corresponding to the exact sequence (**). Then $\mathbb{P}(E) \setminus F_\infty$ an $\mathbb{A}^1$-bundle and $\mathbb{P}(E)$ is its compactification. Assume that the extension class α of (**) belongs to the kernel of the Frobenius map (1). Then (***) have a splitting, which yields a section G' of $\mathbb{P}(E^{(p)})$ disjoint from $F'_\infty := \varphi(F_\infty)$.

Definition 1.6. Let $G = G(X, D, \alpha)$ be the (scheme-theoretic) inverse image of G' by the relative Frobenius morphism φ. We denote the restriction of the projection $\bar{g} : \mathbb{P}(E) \to X$ to G by τ.

By construction G is embedded in the $\mathbb{A}^1$-bundle $\mathbb{P}(E) \setminus F_\infty$. When α corresponds to $\eta = df \in H^0(\mathcal{B}_X(-D))$ in the way of Theorem 1, that is, when $\alpha = \delta(\eta)$, we denote G by $G(X, D, \eta)$ also. The morphism $\tau : G \to X$ is flat, finite of degree p and ramifies everywhere. If X is normal, then the local equation of G in $\mathbb{P}(E)$ is either irreducible or a pth power. Therefore, if X is normal and if $\eta \neq 0$, then G is a variety and its function field is a purely inseparable extension of $Q(X)$. By construction we have the following linear equivalence:

$$G - pF_\infty \sim -\bar{g}^*(pD). \tag{7}$$

Now we can state a criterion for G to be smooth.

Proposition 1.7. *Assume that X is smooth. Then $G = G(X, D, \eta)$ is smooth if and only if $\eta \in H^0(\mathcal{B}_X(-D))$ is nowhere vanishing. If these equivalent conditions are satisfied, then the natural sequence*

$$0 \to \tau^*\mathcal{O}_X(pD) \xrightarrow{\times\eta} \tau^*\Omega_X \xrightarrow{\tau^*} \Omega_G \to \Omega_{G/X} \to 0. \tag{8}$$

is exact and $\Omega_{G/X}$ is isomorphic to $\tau^*\mathcal{O}_X(D)$. In particular the image of τ^* is a vector bundle of rank $n-1$.

Proof. Assume that D is given by a system $\{g_i\}_{i\in I}$ of local equations for an open covering $\{U_i\}_{i\in I}$ of X. We may assume that η is represented by a 0-cochain $\{b_i\}_{i\in I}$ which satisfies

$$b_i = g_i^p c_i \in \Gamma(U_i, \mathcal{O}_X(-pD)), \quad b_j - b_i = a_{ij}^p \in \Gamma(U_i \cap U_j, \mathcal{O}_X(-pD))$$

for some $c_i \in \Gamma(U_i, \mathcal{O}_X)$ and $a_{ij} \in \Gamma(U_i, \mathcal{O}_X(-D))$. Then $\{a_{ij}\}_{i,j\in I}$ is a 1-cocycle which represents $\alpha = \delta(\eta)$ and the vector bundle E in (**) is defined by the 1-cocycle $\left\{\begin{pmatrix} g_i g_j^{-1} & 0 \\ a_{ij} g_j^{-1} & 1 \end{pmatrix}\right\}$ with coefficients in $GL(2, \mathcal{O}_X)$. Since

$$(c_i \ 1)\begin{pmatrix} g_i^p g_j^{-p} & 0 \\ a_{ij}^p g_j^{-p} & 1 \end{pmatrix} = (c_j \ 1)$$

holds, the 0-cocycle $\{(c_i \ 1)\}_{i\in I}$ defines a splitting $\mathcal{O}_X \to E^{(p)}$ of the extension (***).

On each open set U_i, $G \subset \mathbb{P}(E) \setminus F_\infty$ is defined by the equation $S_i^p = c_i$, where S_i is a fiber coordinate of $U_i \times \mathbb{A}^1$. On their intersection, $S_i^p = c_i$ (over U_i) and $S_j^p = c_j$ (over U_j) are patched by the affine transformation $g_j S_j =$

$g_iS_i + a_{ij}$. Let $\mathcal{O}_X(pD) \xrightarrow{\times\eta} \Omega_X$ be the the multiplication homomorphism by η. Since $\tau^* dc_i = 0$, we have the complex (8).

Let x be a point in U_i. If dc_i vanishes at x, then $S_i^p = c_i$ is singular at x. Assume that dc_i is nonzero at x. Then G is smooth at $\tau^{-1}(x)$. Moreover, the cotangent space of X at x has a basis of the form $\{\gamma_1, \dots, \gamma_{n-1}, dc_i\}$, and $\{\tau^*\gamma_1, \dots, \tau^*\gamma_{n-1}, dS_i\}$ is a basis of the cotangent space of G at $\tau^{-1}(x)$. Therefore, the kernel of τ^* is spanned by dc_i and the cokernel by dS_i. Hence (8) is exact and the image of τ^* is a vector bundle of rank $n-1$. Since $g_j dS_j = g_i dS_i$ holds in $\Omega_{G/X}$, $\Omega_{G/X}$ is isomorphic to $\tau^*\mathcal{O}_X(D)$. □

Corollary 1.8. $\tau^* K_X \sim K_G + (p-1)\tau^* D$.

We define the *Euler number* $e(X)$ of X by the top Chern number $\deg c_{top}(X)$.

Corollary 1.9. $\tau^* c_n(X) \sim pc_n(G)$, *where* $n = \dim X$. *In particular, we have* $e(X) = e(G)$.

Proof. Let B be the image of τ^*. Then by the proposition we have $c_n(G) \sim \tau^*(-D)\cdot c_{n-1}(B^\vee)$ and $\tau^* c_n(X) \sim \tau^*(-pD)\cdot c_{n-1}(B^\vee)$. Hence $\tau^* c_n(X)$ is rationally equivalent to $pc_n(G)$. The second half of Corollary is obtained by taking the degree of these two 0-cycles. □

If X is a Tango-Raynaud curve, then $\tau : G \to X$ is nothing but the Frobenius morphism of X.

Remark 1.10. Purely inseparable coverings such as $G(X, D, \alpha)$ in Definition 1.6 were studied by many authors and by now it is more or less well known. It is worth to mention that Kollár[K, Chap. II.6] proves the vanishing of H^1 in characteristic 0 by this using this construction and mod p reduction.

The morphism $G \to X$ in Proposition 1.7 is a special case of a quotient by 1-foliation and the exact sequence (8) is described in Ekedahl[E1].

2. Construction of counterexamples

By a *Tango-Raynaud triple*, or a *TR-triple* for short, we mean a triple (X, D, f) of a smooth variety X, a divisor D on X and a rational function $f \in Q(X)$ with $(df) \geq pD$. In this section, we shall construct a new TR-triple $(\tilde{X}, \tilde{D}, \tilde{f})$ from (X, D, f) under a certain divisibility assumption.

2.1. **New triple of higher dimension.** Let (X, D, f) be a TR-triple. We assume that $D = kD'$ for a divisor D' and an integer $k \geq 2$ which is prime to p, and construct a new TR-triple $(\tilde{X}, \tilde{D}, \tilde{f})$ with $\dim \tilde{X} = \dim X + 1$.

Under the same setting as in the proof of Proposition 1.7, we choose and fix a non-empty open subset $U \subset X$ among U_i's, $i \in I$. We shrink U and replace f with f' satisfying $df' = df$ if necessary so that f is regular over U. We take a fiber coordinate S of $\mathbb{P}(E) \to X$ over U such that the section of infinity F_∞ is defined by $S = \infty$ and $G = G(X, D, df)$ is defined by $S^p - f = 0$. Our new variety $\tilde{X}$ is a model of the function field

$Q(X)(S, \sqrt[k]{S^p - f})$. We construct it in two steps. Let m be a positive integer such that $p + m$ is divisible by k. By the linear equivalence (7), we have

$$G + mF_\infty \sim k(\frac{p+m}{k}F_\infty - \bar{g}^*(pD')),$$

that is, $G + mF_\infty$ is the zero locus of a global section of M^{-k}, where $M = \mathcal{O}_{\mathbb{P}}(-\frac{p+m}{k}F_\infty + g^*(pD'))$. First in the usual way we take the global k-fold cyclic covering

$$Spec\left(\bigoplus_{i=0}^{k-1} M^i\right) \to \mathbb{P}(E) \tag{9}$$

with algebra structure given by $M^k \simeq \mathcal{O}_{\mathbb{P}(E)}(-G - mF_\infty) \hookrightarrow \mathcal{O}_{\mathbb{P}(E)}$. Then we take the *relative normalization* of this covering over a neighborhood of F_∞.

Definition 2.1. We put

$$\tilde{X} = Spec\left(\bigoplus_{i=0}^{k-1} M^i([im/k]F_\infty)\right) \tag{10}$$

with natural algebra structure induced by (9), where [ ] is the Gauss symbol. The composite of this k-fold cyclic covering $\pi : \tilde{X} \to \mathbb{P}(E)$ and the structure morphism $\mathbb{P}(E) \to X$ is denoted by $g : \tilde{X} \to X$. Furthermore, we set

$$\tilde{D} := (k-1)F_\infty + g^*D' \quad \text{and} \quad \tilde{f} = \sqrt[k]{S^p - f} \in Q(\tilde{X}),$$

where the unique section of g lying over F_∞ is denoted by the same symbol.

The complete linear system $|mF_\infty|$ defines an embedding outside G for sufficiently large m. Hence we have

Lemma 2.2. *If D is ample, so is $\tilde{D}$.*

Now we assume further that $\eta := df \in H^0(\mathcal{B}_X(-D))$ is nowhere vanishing. Then G is smooth by Proposition 1.7 and $\tilde{X}$ is smooth since the branch locus $F_\infty \sqcup G$ is smooth. Since $\tilde{X}$ is defined by the equation $T^k = S^p - f$ on $g^{-1}(U)$, taking differential, we have $kT^{k-1}dT = -df$. Hence dT has no zero along G. The differential dT vanishes along the infinity section F_∞ with order $p(k-1)$. Therefore, dT defines a nonzero global section of $\Omega_{\tilde{X}}(-p(k-1)F_\infty - ph^*D')$. It is easily checked that $d\tilde{f} \in H^0(\mathcal{B}_{\tilde{X}}(-\tilde{D}))$ is nowhere vanishing. Thus we have

Proposition 2.3. *If X is smooth and (X, D, f) is a TR-triple with ample D and nowhere vanishing $\eta = df$, then $\tilde{X}$ is smooth and $(\tilde{X}, \tilde{D}, \tilde{f})$ is also a TR-triple with ample $\tilde{D}$ and nowhere vanishing $\tilde{\eta} := d\tilde{f}$.*

Every fiber of g is a rational curve with the unique singular point at the intersection with $\pi^{-1}G$. The singularity is the cusp of the form $T^k = S^p$.

Let $\tilde{\tau} : \tilde{G} \to \tilde{X}$ be the everywhere ramified covering constructed from $(\tilde{X}, \tilde{D}, \tilde{f})$ (Definition 1.6). Since $\sqrt[p]{\tilde{f}} = \sqrt[k]{S - \sqrt[p]{f}}$, the composite $g \circ \tilde{\tau}$ factors through τ and we have the commutative diagram

$$\begin{array}{ccccc} & & h & & \\ & \tilde{G} & \longrightarrow & G & \\ \tilde{\tau} & \downarrow & & \downarrow & \tau \\ & \tilde{X} & \longrightarrow & X. & \\ & & g & & \end{array}$$

Moreover this morphism $h : \tilde{G} \to G$ is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_G \oplus \mathcal{O}_G(\tau^* D'))$ over G. Let U and V be the infinity and zero sections of the $\mathbb{P}^1$-bundle h, respectively. They are disjoint and we have

$$U - V \sim h^* \tau^* D'. \tag{11}$$

The pull-backs $\tilde{\tau}^* F_\infty$ and $\tilde{\tau}^* G$ are U and pV, respectively. In particular, we have

$$\tilde{\tau}^* \tilde{D} \sim (k-1)U + h^* \tau^* D' \sim kU - V. \tag{12}$$

Proposition 2.4. *Assume that X is smooth and (X, D, f) is a TR-triple with $(df) = pD$. Then $\tilde{G}$ is a $\mathbb{P}^1$-bundle over G and the Euler number $e(\tilde{X})$ of $\tilde{X}$ is equal to $2e(X)$.*

Proof. The first half is already shown above. This implies $e(\tilde{G}) = e(\mathbb{P}^1)e(G) = 2e(G)$. Hence the second half follows from Corollary 1.9. $\square$

Remark 2.5. The expression (10) of the cyclic covering $\pi : \tilde{X} \to \mathbb{P}(E)$ was not given in the original [M1] though it is now standard. See e.g., Hesnault-Viehweg[HV, Section 3].

2.2. **The canonical classes of $\tilde{G}$ and $\tilde{X}$.** Let (X, D, f) be a TR-triple with an ample divisor D and nowhere vanishing $(df) \in H^0(\mathcal{B}_X(-D))$. We compute the canonical classes of $\tilde{G}$ and $\tilde{X}$. Since $\tilde{G}$ is a $\mathbb{P}^1$-bundle over G with two disjoint sections U and V, the relative cotangent bundle $\Omega_{\tilde{G}/G}$ is isomorphic to $\mathcal{O}_{\tilde{G}}(-U - V)$. Hence we have

$$K_{\tilde{G}} \sim -U - V + h^* K_G \sim -2U + h^*(K_G + \tau^* D'). \tag{13}$$

by (11). By Corollary 1.8 and (12), we have

$$\begin{aligned} \tau^* K_X \sim & K_{\tilde{G}} + (p-1)\tau^* \tilde{D} \\ \sim & -2U + h^*(K_G + \tau^* D') + (p-1)\left\{(k-1)U + h^* \tau^* D'\right\} \\ \sim & (pk - p - k - 1)U + h^*\left(K_G + p\tau^* D'\right). \end{aligned} \tag{14}$$

We note that $pk - p - k - 1 \geq 0$ and the equality holds if and only if $\{p, k\} = \{2, 3\}$.

In the following we denote by $\sim_{\mathbb{Q}}$ the $\mathbb{Q}$-linear (or $\mathbb{Q}$-rational) equivalence of $\mathbb{Q}$-divisors (or $\mathbb{Q}$-cycles). For the later use we put

$$J := K_G + \frac{1}{k-1}\tau^* D \quad \text{and} \quad \tilde{J} := K_{\tilde{G}} + \frac{1}{\tilde{k}-1}\tilde{\tau}^*\tilde{D} \tag{15}$$

for an integer $\tilde{k}$. Since $D' \sim_{\mathbb{Q}} D/k$, we have

$$\begin{aligned}
\tilde{J} &\sim_{\mathbb{Q}} -U - V + h^*K_G + \frac{1}{\tilde{k}-1}\left\{(k-1)U + \frac{1}{k}h^*\tau^*D\right\}\\
&\sim_{\mathbb{Q}} \left(\frac{k-1}{\tilde{k}-1} - 2\right)U + h^*\left\{K_G + \left(\frac{1}{k} + \frac{1}{k(\tilde{k}-1)}\right)\tau^*D\right\}\\
&\sim_{\mathbb{Q}} \left(\frac{k-1}{\tilde{k}-1} - 2\right)U + h^*\left\{J + \frac{1}{k}\left(\frac{1}{\tilde{k}-1} - \frac{1}{k-1}\right)\tau^*D\right\}
\end{aligned} \tag{16}$$

by (13).

2.3. **Chern numbers of $\tilde{X}$.** For the same reason as (13), we have

$$\begin{aligned}
c(\tilde{G}) &\sim (1 + U + V)\cdot h^*c(G)\\
c_i(\tilde{G}) &\sim h^*c_i(G) + (U+V)\cdot h^*c_{i-1}(G).
\end{aligned} \tag{17}$$

Since $U \cap V = \emptyset$, we have

$$\begin{aligned}
&U \cdot V \sim 0,\\
&U^2 \sim ((U-V)+V)\cdot U \sim k^{-1}h^*\tau^*D\cdot U,\\
&V^2 \sim (U-(U-V))\cdot V \sim -k^{-1}h^*\tau^*D\cdot V
\end{aligned} \tag{18}$$

by (11). More generally, we have

$$\begin{aligned}
U^m &\sim k^{-m+1}h^*\tau^*D^{m-1}\cdot U \quad \text{and}\\
V^m &\sim (-k)^{-m+1}h^*\tau^*D^{m-1}\cdot V
\end{aligned} \tag{19}$$

for every integer $m \geq 1$.

Proposition 2.6. *Let λ and μ be nonnegative integers such that $\lambda + i + \mu = \dim \widetilde{G}$. Then we have*

$$\begin{aligned}
&(c_1(\widetilde{G})^\lambda . c_i(\widetilde{G}) . \tilde{\tau}^*\widetilde{D}^\mu)\\
&= \sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(c_1(G)^{\lambda-\alpha}.c_i(G).\tau^*D^{\mu+\alpha-1})(k^{1-\alpha} + (-1)^\mu k^{1-\alpha-\mu})\\
&+ \sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(c_1(G)^{\lambda-\alpha}.c_{i-1}(G).\tau^*D^{\mu+\alpha})(k^{-\alpha} + (-1)^\mu k^{-\alpha-\mu})
\end{aligned}$$

Proof. By (12) and (17), we have

$$
\begin{aligned}
&(c_1(\widetilde{G})^{\lambda}.c_i(\widetilde{G}).\tilde{\tau}^*\widetilde{D}^{\mu})\\
&=(c_1(\widetilde{G})^{\lambda}.h^*c_i(G).\tilde{\tau}^*\widetilde{D}^{\mu}) + (c_1(\widetilde{G})^{\lambda}.h^*c_{i-1}(\widetilde{G}).(U+V).\tilde{\tau}^*\widetilde{D}^{\mu})\\
&=\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_i(G).(U+V)^{\alpha}.(kU-V)^{\mu})\\
&+\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_{i-1}(G).(U+V)^{\alpha+1}.(kU-V)^{\mu})\\
&=\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_i(G).(k^{\mu}U^{\alpha+\mu}+(-1)^{\mu}V^{\alpha+\mu}))\\
&+\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_{i-1}(G).(k^{\mu}U^{\alpha+\mu+1}+(-1)^{\mu}V^{\alpha+\mu+1}))\\
&=\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_i(G).h^*\tau^*D^{\alpha+\mu-1}.(k^{1-\alpha}U+(-1)^{\mu}k^{1-\alpha-\mu}V))\\
&+\sum_{\alpha=0}^{\lambda}\binom{\lambda}{\alpha}(h^*c_1(G)^{\lambda-\alpha}.h^*c_{i-1}(G).h^*\tau^*D^{\alpha+\mu}.(k^{-\alpha}U+(-1)^{\mu}k^{-\alpha-\mu}V)).
\end{aligned}
$$

Since both U and V are sections of $h : \widetilde{G} \to G$, we have $(h^*Z.U) = \deg Z$ for every 0-cycle Z on G. Therefore the proposition follows from the last expression. □

Corollary. $(c_1(\widetilde{G})^{\lambda}.c_i(\widetilde{G}).\tilde{\tau}^*\widetilde{D}^{\mu})$ *is of degree* ≤ 1 *as a Laurent polynomial in the variable* k. *Moreover, the coefficient of* k *is equal to* $(c_1(G)^{\lambda}.c_i(G).\tau^*D^{\mu-1})$ *if* $\mu \geq 1$ *and* 0 *otherwise.*

2.4. **Proof of Theorem 2.** Now we are ready to construct an n-dimensional TR-triple (X_n, D_n, df_n). We define two sequences $\{k_i\}_{1\leq i\leq n-1}$ and $\{e_i\}_{1\leq i\leq n-1}$ of positive integers inductively by the rule

$$k_i = 1 + c_i e_{i-1} \quad \text{and} \quad e_i = e_{i-1}k_i$$

for $2 \leq i \leq n-1$, where $\{c_i\}_i$ is a non-decreasing sequence of integers $c_i \geq 2$ such that k_i's are not divisible by p. (The simplest choice is $c_i := p$ for every i.) We start with an arbitrary positive integer $k_1 \geq 2$ prime to p and $e_1 := k_1$.

The first TR-triple (X_1, D_1, df_1) consists of a Tango-Raynaud curve X_1, a divisor D_1 and an exact differential df_1 with $(df_1) = pD_1$ such that D_1 is divisible by e_{n-1}. Then we apply the construction in §2.1 by taking k_{n-1}-fold covering of the $\mathbb{P}^1$-bundle $\mathbb{P}(E_1)$ over X_1, and put $(X_2, D_2, df_2) = (\tilde{X}_1, \tilde{D}_1, d\tilde{f}_1)$. This is a TR-triple of dimension 2. We repeat this process $n-1$ times. We note that the divisor $D_2 = (k_{n-1}-1)F_{\infty} + D_1/k_{n-1}$ is divisible by e_{n-2}. In particular, D_2 is divisible by k_{n-2}. Hence, taking k_{n-2}-fold covering of $\mathbb{P}(E_2)$ over X_2, we obtain $(X_3, D_3, df_3) = (\tilde{X}_2, \tilde{D}_2, d\tilde{f}_2)$, which is a TR-triple of dimension 3 such that D_3 is divisible by e_{n-3}, and so on. In the final $(n-1)$st step we take the k_1-fold covering of $\mathbb{P}(E_{n-1})$ since D_{n-1} is divisible by $e_1 = k_1$. We obtain a new TR-triple (X_n, D_n, df_n), which is an n-dimensional counterexample to Kodaira's vanishing by Proposition 2.3.

The first half of (b) of Theorem 2 is a consequence of the following

Proposition 2.7. *The canonical class K_{X_n} is ample if $\{p, k_1\} \neq \{2, 3\}$, and the pull-back of an ample divisor on X_{n-1} if $\{p, k_1\} = \{2, 3\}$.*

Proof. Since $\tau_n : G_n \to X_n$ is finite, it suffices to show that $K_{G_{n-1}} + \frac{p}{k_1}\tau_{n-1}^* D_{n-1}$ is ample by (14). We put

$$J_i := K_{G_i} + \frac{1}{k_{n-i} - 1}\tau_i^* D_i$$

for every $1 \leq i \leq n-1$ after (15). Since $p/k_1 \geq 1/(k_1 - 1)$, it suffices to show the following:

claim 1. *J_i is ample.*

We prove it by induction on i. In the case $i = 1$, both K_{G_1} and D_1 are ample. Hence J_1 is ample. Assume that $i \geq 2$. We have

$$\frac{k_{n-i+1} - 1}{k_{n-i} - 1} = \frac{c_{n-i+1}e_{n-i}}{c_{n-i}e_{n-i-1}} \geq k_{n-i} \geq 2$$

if $n - i \geq 2$, and $(k_2 - 1)/(k_1 - 1) = c_2 k_1/(k_1 - 1) > 2$. By the formula (16), J_i is ample since so is J_{i-1} and since $k_{n-i+1} > k_{n-i}$. □

Now we consider the sequence of the morphisms

$$G_n \xrightarrow{h_{n-1}} G_{n-1} \xrightarrow{h_{n-2}} \cdots \xrightarrow{h_2} G_2 \xrightarrow{h_1} G_1,$$

in order to investigate the asymptotic behavior of certain Chern numbers of X_n as $k_1, \cdots, k_{n-1}$ go to ∞, where $G_j := \widetilde{G}_{j-1}$ for $j = 2, \cdots, n$. Since G_1 is a curve, we have $-\deg c_1(G_1) = \deg \tau_1^* D_1 = 2g - 2$, where g is the genus of the Tango-Raynaud curve $G_1 \simeq X_1$. Applying Proposition 2.6 (or its Corollary) successively to the above morphisms h_i, we have the following

Proposition 2.8. *The intersection number $(c_1(G_n)^\lambda . c_i(G_n) . \tau_n^* D_n^\mu)$ is a Laurent polynomial in the variables $k_1, \ldots, k_{n-1}$ whose coefficients are integers independent of X_1 and D_1. The degree of the Laurent polynomial is at most 1 with respect to every variable. Moreover, the coefficient of $k_1 \cdots k_{n-1}$ is equal to*

$$\begin{cases} 2g - 2 & \text{if } (\lambda, i, \mu) = (0, 0, n), \\ -(2g - 2) & \text{if } (\lambda, i, \mu) = (1, 0, n-1), (0, 1, n-1), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore we have

Proposition 2.9. *The intersection number $(K_{X_n}^{n-i}.c_i(X_n))$ is a Laurent polynomial in the variables $k_1, \dots, k_{n-1}$ and the degree is at most 1 with respect to each variable. If $i \geq 2$, then the coefficient of the highest monomial $k_1 \cdots k_{n-1}$ in the Laurent expression of $(K_{X_n}^{n-i}.c_i(X_n))$ is equal to $-p^{-n}(p-1)^{n-i}(n-i)(2g-2)$.*

Proof. By (17), $\tau_n^* c_i(X_n)$ is rationally equivalent to

$$\begin{aligned}
&c_i(G_n) + (1-p)\sum_{j=1}^{i} c_{i-j}(G_n).\tau_n^* D_n^j \\
&\sim (1-p)\tau_n^* D_n^i + (1-p)c_1(G_n)\tau_n^* D_n^{i-1} + (\text{lower terms in } D_n).
\end{aligned}$$

Since $\tau_n^* c_1(X_n) \sim (1-p)\tau_n^* D_n + c_1(G_n)$, we have

$$\begin{aligned}
&p^n(c_1(X_n)^{n-i}.c_i(X_n)) \\
=&(\tau_n^* c_1(X_n)^{n-i}.\tau_n^* c_i(X_n)) \\
=&(1-p)^{n-i+1}(\tau_n^* D_n^n) + (1-p)^{n-i}(n-i)(c_1(G_n).\tau_n^* D_n^{n-1}) \\
&+ (1-p)^{n-i+1}(c_1(G_n).\tau_n^* D_n^{n-1}) + (\text{lower terms in } D_n).
\end{aligned}$$

Hence our assertion follows from Proposition 2.8. □

By the proposition, $(K_X^{n-i}.c_i(X_n))$ is negative for sufficiently large choice of $k_1, \cdots, k_{n-1}$ for $i \geq 2$. This shows (b) of Theorem 2. (c) is a direct consequence of Proposition 2.4.

2.5. **Properties of** (X_2, D_2, df_2)**.** Here we remark a few properties of 2-dimensional counterexamples $(X, D, df) := (X_2, D_2, df_2)$, which is a k-fold covering of a $\mathbb{P}^1$-bundle over a Tango-Raynaud curve C. By Proposition 1.7, the cokernel of the multiplication map by df is locally free. In our case, the cokernel is a line bundle. Hence we have the exact sequence

$$0 \to \mathcal{O}_X(pD) \xrightarrow{\times df} \Omega_X \longrightarrow \mathcal{O}_X(K_X - pD) \to 0. \tag{21}$$

Proposition 2.10. (a) *The complete linear system $|p(pD - K_X)|$ is non-empty.*

(b) *If $k \equiv -1\,(p)$, then X has a nonzero vector field, that is, $H^0(T_X) \neq 0$.*

(c) *When $\{p, k\} \neq \{2, 3\}$, the canonical class K_X is ample and K.V. holds for K_X, that is, $H^1(\mathcal{O}_X(-K_X)) = 0$.*

Proof. First we compute the canonical class K_X more rigorously than in §2.2. Since $K_{\mathbb{P}(E)/C} = -2F_\infty + D_1$ and since the k-fold cyclic covering $\pi : X \to \mathbb{P}(E)$ has branch locus $G \sqcup F_\infty$, we have

$$K_{X/C} = \pi^* K_{\mathbb{P}(E)/C} + (k-1)G + (k-1)F_\infty \sim -(k+1)F_\infty + (k-1)G + g^* D_1.$$

The rational function $S^p - f$ gives the linear equivalence $G \sim p(F_\infty - D_1)$ on $\mathbb{P}(E)$, which is (7). Hence its kth root $\sqrt[k]{S^p - f} \in Q(X)$ gives the equivalence $G \sim p(F_\infty - D_1/k)$ on X. Therefore, we have

$$K_X \sim K_{X/C} + pD \sim (pk - p - k - 1)F_\infty + (p + k)D_1/k \tag{22}$$

and $pD - K_X \sim (k+1)F_\infty - D_1$. Now we are ready to prove our assertions.

(a) $|p(pD - K_X)|$ is non-empty since $p(pD - K_X)$ is linearly equivalent to $(k+1)G + pD_1/k$.

(b) Put $k = ap - 1$ for a nonnegative integer a. Then we have $pD - K_X \sim apF_\infty - D_1 \sim aG + D_1/k$. Since T_X contains $\mathcal{O}_X(pD - K_X)$ as a line subbundle, we have $H^0(T_X) \neq 0$.

(c) K_X is ample by Proposition 2.7. Since $p(pk - p - k - 1) > k - 1$, we have $\mathrm{Hom}(\mathcal{O}_X(p^m K_X), \mathcal{O}_X(D)) = 0$ for every $m \geq 1$. Hence we have $\mathrm{Hom}(\mathcal{O}_X(p^m K_X), \Omega_X) = 0$ by (21) and (a). Therefore, we have $H^1(\mathcal{O}_X(-K_X)) = 0$ by the corollary of Proposition 1.5 and Lemma 1.1. □

By (a) of the proposition the cotangent bundle Ω_X is not slope stable with respect to any ample line bundle. Since any positive dimensional algebraic group does not act on a surface of general type, the group scheme $\mathrm{Aut}\, X$ is not reduced by (b). See [Ru] and [La] for alternative treatment of (generalized) Raynaud's surface from this viewpoint. We refer to [E2] and [SB] for the pluricanonical maps of surfaces of general type in positive characteristic.

3. Surfaces on which R.V. does not hold

In this section we prove Theorem 3. By virtue of the following result, Ramanujam's vanishing (R.V.) on a (smooth complete) surface X is equivalent to the injectivity of the Frobenius map (2) for all nef and big line bundle L.

Proposition 3.1 (Szpiro[Sz], Lewin-Ménégaux [LM])**.** $H^1(X, L^{-m}) = 0$ *holds for* $m >> 0$ *if* L *is nef and big.*

The following is inspired by a similar statement [T2, Corollary 8]. This is not absolutely necessary for our proof but makes it more transparent.

Proposition 3.2. *Let* X' *be the blow-up of a surface* X *at a point. The R.V. holds on* X' *if and only if so does on* X.

Proof. Let $x \in X$ be the center of the blowing up $\pi : X' \to X$. If L is a nef and big line bundle on X, then so is the pull-back $\pi^* L$. If R.V. holds on X', then $H^1(X', \pi^* L^{-1})$ vanishes. Since $H^1(X, L^{-1})$ is isomorphic to $H^1(X', \pi^* L^{-1})$, R.V. holds also on X.

Conversely assume that R.V. holds on X and let D' be a nef and big divisor on X'. Then $D := \pi_* D'$ is also nef and big. By Theorem 1, the vector space $\{df \in \Omega_{Q(X)} \mid f \in Q(X), (df) \geq pD\}$ is zero. The space $\{df \in \Omega_{Q(X')} \mid f \in Q(X'), (df) \geq pD'\}$ is also zero since $(df) \geq pD$ is a divisorial condition. Therefore, R.V. holds on X'. □

We first prove (b). Let X be a surface on which R.V. does not hold. By Proposition 1.5, there exist a rational function f and a nef and big divisor D with $(df) \geq pD$. f gives a rational map from X to the projective line $\mathbb{P}^1$. By taking suitable blowing-ups $X' \to X$ and the Stein factorization, we have the morphism $g : X' \to C$ with $g_*\mathcal{O}_{X'} = \mathcal{O}_C$. C is smooth since so is X. Every fiber of g is connected. Let L be the image of the multiplication homomorphism $\mathcal{O}_{X'}(pD) \longrightarrow \Omega_{X'}$ by df. The relative cotangent sheaf $\Omega_{X'/C} = \Omega_{X'}/g^*\Omega_C$ contains $T := L/[L \cap g^*\Omega_C]$ as a subsheaf. On a non-empty subset of C, Ω_C contains df as its global section. Hence, $L \cap g^*\Omega_C \neq 0$ and T is a torsion sheaf. There exists an effective divisor A with $\operatorname{Supp} A = \operatorname{Supp} T$ which is linearly equivalent to $c_1(L) - c_1(L \cap g^*\Omega_C)$. $c_1(L) = pD$ is a nef and big divisor on X' and $c_1(L \cap g^*\Omega_C) \leq g^*K_C$ holds. Hence A contains a component G different from fibers of g. Then for every fibers B of g, Ω_B has nonzero torsion at the intersection $B \cap G$. In particular, B is singular at $B \cap G$.

Now we prove (a). By Proposition 3.2, we may assume that X is a (relatively) minimal model.

Proposition 3.3. *If X is a ruled surface or an elliptic surface, then R.V. holds on X.*

Proof. Let $h : X \to C$ be a $\mathbb{P}^1$-bundle or an elliptic fibration of X. Then there exists an exact sequence

$$0 \to h^*\Omega_C \to \Omega_X \to \Omega_{X/C} \to 0$$

of torsion free sheaves on X. Let L be a nef and big line bundle on X. The degree of L, $h^*\Omega_C$ and $\Omega_{X/C}$ restricted to general fibers of h are positive, zero and nonpositive, respectively. Therefore, we have $\operatorname{Hom}_{\mathcal{O}_X}(L, h^*\Omega_C) = \operatorname{Hom}_{\mathcal{O}_X}(L, \Omega_{X/C}) = 0$. By the exact sequence, we have $\operatorname{Hom}_{\mathcal{O}_X}(L, \Omega_X) = 0$. Hence R.V. holds on X. (This argument is taken from [T2, Corollary 6].) $\square$

case 1. If $\kappa(X) = -\infty$, then we can take a $\mathbb{P}^1$-bundle as a relatively minimal model. Hence R.V. holds by the proposition.

case 2. If $\kappa(X) = 1$, then the minimal model X is an elliptic surface by our assumption. Hence R.V. holds by the proposition.

case 3. Assume that $\kappa(X) = 0$. By the classification of Bombieri-Mumford [BM1], X and the second Betti number $B_2(X)$ satisfy one of the following:

(a) $B_2(X) = 6$ and X is an abelian surface.

(b) $B_2(X) = 22$ and X is a K3 surface.

(c) $B_2(X) = 10$ and X is either a classical, singular or supersingular Enriques surface. The last two types occurs only when $p = 2$.

(d) $B_2(X) = 6$ and X is either hyperelliptic or quasi-hyperelliptic. The latter appears only when $p = 2, 3$.

In the case (a), R.V. holds by the corollary of Proposition 1.5 since $\Omega_X \simeq \mathcal{O}_X^{\oplus 2}$. In the case (d), R.V. holds by Proposition 3.3 since X has an elliptic fibration also (over $\mathbb{P}^1$) by [BM1, Theorem 3]. Our proof of Theorem 3 is completed by the following

Proposition 3.4. *R.V. holds on a K3 and an Enriques surfaces.*

Proof. It suffices to show injectivity of (1) for all nef and big divisor D on X. Assume the contrary. Then, by Lemma 1.4, $H^0(\mathcal{B}_X(-D))$ is nonzero. By the multiplication map

$$H^0(\mathcal{O}_X(D)) \times H^0(\mathcal{B}_X(-D)) \longrightarrow H^0(\mathcal{B}_X), \tag{23}$$

and by the Riemann-Roch inequality

$$\dim H^0(\mathcal{O}_X(D)) \geq \frac{1}{2}(D^2) + \chi(\mathcal{O}_X) \geq 2, \tag{24}$$

we have

$$\dim \operatorname{Ker}[F^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)] = \dim H^0(\mathcal{B}_X) \geq 2.$$

This is a contradiction since $H^1(\mathcal{O}_X)$ is at most 1-dimensional by [BM1] and [BM2, Lemma 1]. □

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