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Deformations of degenerate curves on a Segre 3-fold

By

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Abstract

We study the embedded deformations of degenerate curves C on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. Here C is said to be degenerate if it is contained in a hyperplane in $\mathbb{P}^5$. We give a necessary and sufficient condition for each of the following: [i] C is stably degenerate and [ii] the Hilbert scheme of $\mathbb{P}^1 \times \mathbb{P}^2$ is singular at $[C]$.

§ 1. Introduction

Let $V \subset \mathbb{P}^n$ be a projective variety of dimension 3. A curve C on V is said to be *degenerate* if C is contained in a hyperplane section S of V. We say C is *stably degenerate* if every small deformation C' of C in V is contained in a deformation S' of S in V. In [5], given a degenerate curve C on a smooth del Pezzo 3-fold V, the problem of determining whether or not C is stably degenerate has been studied. In this paper we study the same problem when V is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$. Then every smooth hyperplane section $S \subset \mathbb{P}^4$ of V is isomorphic to the rational scroll $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ (i.e. cubic scroll). In this paper $\mathrm{Hilb}^{sc} V$ denotes the Hilbert scheme of smooth connected curves on V. The following is our main theorem.

Theorem 1.1. *Let $V \subset \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ and let C be a smooth connected curve on a smooth hyperplane section S of V. Then:*

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(1) *C is stably degenerate if and only if* $\chi(V, \mathcal{I}_C(1)) \geq 1$.

(2) *Let C_0 and f be the negative section and a fiber of the $\mathbb{P}^1$-bundle $S \to \mathbb{P}^1$, respectively. In particular C_0 is a (-1)-$\mathbb{P}^1$ on S. Then:*

[i] *If $C \sim n(C_0 + f)$ for some integer $n \geq 5$, then* $\mathrm{Hilb}^{sc} V$ *is non-reduced along a neighborhood of $[C]$;*

[ii] *Otherwise* $\mathrm{Hilb}^{sc} V$ *is nonsingular at $[C]$.*

Note that $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown-up at a point. Theorem 1.1 shows that every *obstructed* curve C (i.e. $\mathrm{Hilb}^{sc} V$ is singular at $[C]$) on a smooth hyperplane section S is the pull-back of a plane curve of degree n for $n \geq 5$. The fact that the curve $C \sim 5C_0 + 5f$ has an obstructed deformation in V was first noticed by Akahori and Namba in [1]. They considered a nonsingular plane quintic curve $D \subset \mathbb{P}^2$ and proved that the graph $\Gamma \subset D \times \mathbb{P}^1$ of the projection $\pi_p : D \to \mathbb{P}^1$ with the center $p \in \mathbb{P}^2 \setminus D$ has an obstructed first order infinitesimal deformation by a different method from ours. In Figure 1 we show the region of pairs (a, b) of integers such that $C \sim aC_0 + bf$ is stably degenerate (cf. Lemma 3.8) and also the half-line along which C is obstructed.

We refer to [5] for the deformations of degenerate curves on the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^8$, which is del Pezzo and of degree 6. The organization of this paper is as follows. In §2 we recall a few general results obtained in [4] and [5]. In §3 we study the deformations of degenerate curves on the Segre 3-fold V and prove Theorem 1.1 in §3.3.

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Notation and Conventions We work over an algebraically closed field k of characteristic 0. Given a projective scheme V over k and its closed subscheme X, $\mathcal{I}_X$ and $N_{X/V}$ denote the ideal sheaf of X in V and the normal sheaf $(\mathcal{I}_X/\mathcal{I}_X{}^2)^\vee$ of X, respectively. For a sheaf $\mathcal{F}$ on V, we denote the restriction map $H^i(V, \mathcal{F}) \to H^i(X, \mathcal{F}\big|_X)$ by

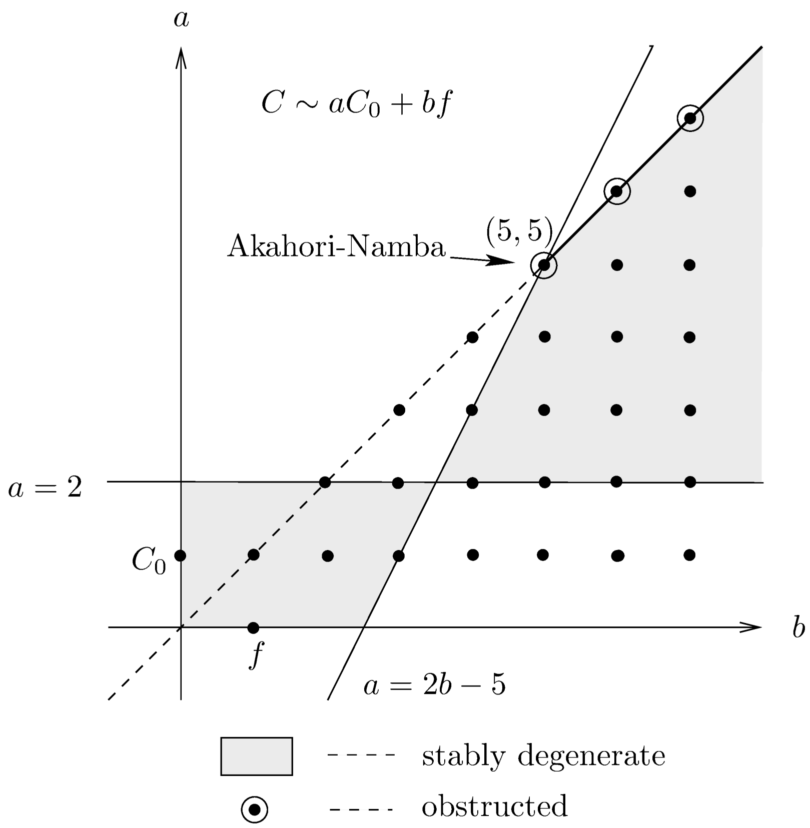

Figure 1. Stably degenerate curves and obstructed curves

$\big|_X$. We denote the Euler-Poincaré characteristic of $\mathcal{F}$ by $\chi(V, \mathcal{F})$ or $\chi(\mathcal{F})$. We denote by $(Z_1 \cdot Z_2)_V$ or $Z_1 \cdot Z_2$, the intersection number of two cycles Z_1 and Z_2 on V if $\dim Z_1 = \operatorname{codim} Z_2$.

§ 2. Generalities

Let $V \subset \mathbb{P}^n$ be a smooth projective 3-fold and C a smooth connected curve on V. Suppose that C is contained in a smooth hyperplane section S of V. We say C is *stably degenerate* if every small deformation of C in V is contained in a deformation of S in V, or more precisely if there exists an open neighborhood $U \subset \mathrm{Hilb}^{sc} V$ of $[C]$ such that for any member $[C'] \in U$, there exists a deformation $S' \subset V$ of S such that $C' \subset S'$. We say C is *S-normal* if the restriction map

$$H^0(S, N_{S/V}) \xrightarrow{|_C} H^0(C, N_{S/V}\big|_C) \tag{2.1}$$

is surjective. If $\operatorname{Hilb} V$ is nonsingular at $[S]$ and $H^1(C, N_{C/S}) = 0$, then the Hilbert-flag scheme $\operatorname{Flag} V$, which parametrizes all pairs (C', S') of curves C' and surfaces S' such that $C' \subset S' \subset V$, is nonsingular at (C, S) (cf. [2]). If C is S-normal as well, then the map

$$\kappa_{C,S} : \mathcal{T}_{\operatorname{Flag} V,(C,S)} \longrightarrow \mathcal{T}_{\operatorname{Hilb} V,[C]} = H^0(N_{C/V}) \tag{2.2}$$

of tangent spaces induced by the projection morphism $pr_1 : \operatorname{Flag} V \to \operatorname{Hilb} V$, $(C', S') \to [C']$, is surjective at (C, S) (cf. [3],[4, Lemma 3.1]). We deduce from these two facts the following:

Theorem 2.1 (cf. [2],[5]). *Suppose that* $\operatorname{Hilb} V$ *is nonsingular at* $[S]$ *and that* $H^1(C, N_{C/S}) = 0$. *If* C *is* S*-normal, then (1)* C *is stably degenerate and (2)* $\operatorname{Hilb}^{sc} V$ *is nonsingular at* $[C]$.

If C is not S-normal, then it is generally difficult to prove that C is stably degenerate. However under some conditions, we can prove this by computing infinitesimal deformations of C and their obstructions.

Let $C \subset V$ be as above. An (embedded) *first order infinitesimal deformation* of C in V is a closed subscheme $\tilde{C} \subset V \times \operatorname{Spec} k[t]/(t^2)$ which is flat over $\operatorname{Spec} k[t]/(t^2)$ and whose central fiber is C. It is well known that there exists a natural one-to-one correspondence between the global sections α of the normal bundle $N_{C/V}$ and the first order infinitesimal deformations $\tilde{C}$ of C in V. The tangent space of the Hilbert scheme $\operatorname{Hilb} V$ at $[C]$ is isomorphic to $H^0(N_{C/V})$. Let $\alpha \in H^0(N_{C/V})$ be the global section corresponding to $\tilde{C}$. Then there exists an element $\operatorname{ob}(\alpha)$ of $H^1(N_{C/V})$ determined from α such that $\tilde{C}$ lifts to a deformation over $\operatorname{Spec} k[t]/(t^3)$ if and only if $\operatorname{ob}(\alpha) = 0$. In [4] Mukai and Nasu have given a sufficient condition for $\operatorname{ob}(\alpha)$ to be nonzero in terms of the *exterior component*. Here the exterior component of α is the image $\pi_{C/S}(\alpha)$ by the natural projection $\pi_{C/S} : N_{C/V} \to N_{S/V}\big|_C$.

Theorem 2.2 ([4, Theorem 1.1]). *Let* $C \subset S \subset V$ *be as above and let* E *be a* (-1)-$\mathbb{P}^1$ *on* S. *A first order infinitesimal deformation* $\alpha \in H^0(N_{C/V})$ *is (primarily) obstructed if its exterior component lifts to a global section* $v \in H^0(N_{S/V}(E)) \setminus H^0(N_{S/V})$, i.e., $\pi_{C/S}(\alpha) = v\big|_C$ *in* $H^0(N_{S/V}(E)\big|_C)$, *and if the following conditions are satisfied:*

(a) $(\Delta \cdot E)_S = 0$, *where we put a divisor* $\Delta := C - 2E + K_V\big|_S$ *on* S,

(b) $v\big|_E$ *does not belong to the image of*

$$\pi_{E/S}(E) : H^0(E, N_{E/V}(E)) \longrightarrow H^0(E, N_{S/V}(E)\big|_E),$$

(c) *The restriction map* $H^0(S, \Delta) \to H^0(E, \Delta\big|_E) \overset{(a)}{\simeq} H^0(E, \mathcal{O}_E) = k$ *is surjective.*

The following diagram illustrates the relation between α and $v\big|_E$ in (b).

$$\begin{array}{ccccc}
H^0(N_{C/V}) & \ni \alpha & & & H^0(N_{E/V}(E)) \\
{\scriptstyle \pi_{C/S}}\big\downarrow & \big\downarrow & & & {\scriptstyle \pi_{E/S}(E)}\big\downarrow \\
H^0(N_{S/V}\big|_C) & \ni v\big|_C \overset{res}{\longleftarrow\!\shortmid} & v & \overset{res}{\longmapsto} v\big|_E \in & H^0(N_{S/V}(E)\big|_E) \\
\bigcap & & \Cap & & \\
H^0(N_{S/V}(E)\big|_C) & \overset{res}{\longleftarrow} & H^0(N_{S/V}(E)) & &
\end{array}$$

Now we explain how to prove that C is stably degenerate when C is not S-normal. As we have seen, the projection pr_1 : $\operatorname{Flag} V \to \operatorname{Hilb}^{sc} V$, $(C', S') \to [C']$ induces the map $\kappa_{C,S}$ (2.2) of the tangent spaces. Then the kernel and the cokernel of the restriction map (2.1) are isomorphic to those of $\kappa_{C,S}$, respectively by [4, Lemma 3.1]. Thus if C is not S-normal, then there exists a first order infinitesimal deformation $\tilde{C}$ of C in V which is not contained in any first order infinitesimal deformation $\tilde{S}$ of S in V. When C is not S-normal, the following theorem is useful.

Theorem 2.3 (cf. [5]). *Suppose that* $\operatorname{Hilb} V$ *is nonsingular at* $[S]$ *and that* $H^1(N_{C/S}) = 0$. *Suppose further that* C *is not* S*-normal. If the obstruction* $\operatorname{ob}(\alpha)$ *is nonzero for any* $\alpha \in H^0(N_{C/V}) \setminus \operatorname{im} \kappa_{C,S}$, *then (1)* C *is stably degenerate and (2)* $\operatorname{Hilb}^{sc} V$ *is singular at* $[C]$.

Finally we recall the definition of S*-maximal family* introduced in [4]. Let $\mathcal{W}_{S,C}$ be the irreducible component of $\operatorname{Flag} V$ passing through (C, S). Then the image $W_{S,C} \subset \operatorname{Hilb}^{sc} V$ of $\mathcal{W}_{S,C}$ by pr_1 is called the S*-maximal family of curves* containing C.

§ 3. Stably degenerate curves

In this section, we prove Theorem 1.1 by applying Theorems 2.1, 2.2 and 2.3.

§ 3.1. Curves on a cubic scroll

First we recall cubic scrolls of dimension 2 and 3. Let $V \subset \mathbb{P}^5$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. We denote the two projections to $\mathbb{P}^1$ and $\mathbb{P}^2$ by p and q, respectively.

Put $L := p^*\mathcal{O}_{\mathbb{P}^1}(1)$ and $M := q^*\mathcal{O}_{\mathbb{P}^2}(1)$, the generators of $\operatorname{Pic} V \simeq \mathbb{Z}^{\oplus 2}$. Here and later, the same symbols L and M represent the class of Cartier divisors on V corresponding to these invertible sheaves. The degree of V equals three by $(L+M)^3 = 3L \cdot M^2 = 3$. Let $\mathbb{F}_1$ denote the rational scroll $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. As is well known, $\mathbb{F}_1$ is isomorphic to the blown-up of $\mathbb{P}^2$ at a point. We define C_0 and f as in the introduction. The complete linear system $|C_0+f|$ defines the blow-up, which contracts C_0 to the point. Similarly $|f|$ defines a morphism $\mathbb{F}_1 \to \mathbb{P}^1$, which is the $\mathbb{P}^1$-bundle structure on $\mathbb{F}_1$. Since the multiplication map $H^0(f) \otimes H^0(C_0+f) \to H^0(C_0+2f)$ is surjective and C_0+2f is very ample, we have a closed embedding $\mathbb{F}_1 \hookrightarrow V \subset \mathbb{P}^5$. Moreover the image S of $\mathbb{F}_1$ is contained in a hyperplane of $\mathbb{P}^5$, because we have $h^0(C_0+2f) = h^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(2)) = 5$. By degree reason, S is isomorphic to a hyperplane section of V. Conversely every smooth hyperplane section S of V is obtained in this way and isomorphic to $\mathbb{F}_1$. Here S and V are so-called *varieties of minimal degrees.*

Let C be a smooth connected curve on V. We say C is of *bidegree* (a,b) $(a,b \in \mathbb{Z}_{\geq 0})$ if $C \cdot L = a$ and $C \cdot M = b$. The degree d of C on V equals $a+b$ because $S \sim L+M$. In what follows, we assume that C is of bidegree (a,b). The expected dimension of the Hilbert scheme $\operatorname{Hilb}^{sc} V$ at $[C]$ is equal to $2a+3b$ because $\chi(N_{C/V}) = -K_V \cdot C = (2L+3M) \cdot C$.

Lemma 3.1. *Suppose that C is contained in a smooth hyperplane section S of V. Then $C \sim aC_0 + bf$. In particular, we have either $b \geq a \geq 1$, $C = C_0$, or $C = f$. The genus g of C equals $(a-1)(2b-a-2)/2$.*

Proof. Since $L\big|_S \sim f$ and $M\big|_S \sim C_0 + f$, we have $C \cdot f = a$ and $C \cdot (C_0+f) = b$. Then we deduce the divisor class of C from the fact that $C_0^2 = -1$, $C_0 \cdot f = 1$ and $f^2 = 0$. Since $-K_S \sim 2C_0 + 3f$, we obtain the genus of C by the adjunction theorem on S. □

§ 3.2. Deformation of degenerate curves

In what follows, we assume that C is contained in a smooth hyperplane section S of V. Since $-K_V = 2L+3M$ and $-K_S = 2C_0+3f$ are ample, we have $H^1(N_{S/V}) \simeq H^1(-K_V\big|_S + K_S) = 0$ and $H^1(N_{C/S}) \simeq H^1(-K_S\big|_C + K_C) = 0$. In particular $\operatorname{Hilb} V$ and $\operatorname{Hilb} S$ are nonsingular at $[S]$ and $[C]$, respectively. Therefore by Theorem 2.1 if C is S-normal then C is stably degenerate and $\operatorname{Hilb}^{sc} V$ is nonsingular at $[C]$. In Lemma 3.2 and Lemma 3.3, we give a few sufficient conditions for C to be S-normal.

Lemma 3.2. *If C is irrational and $C \cdot C_0 \geq 1$, then C is S-normal.*

Proof. By the exact sequence

$$0 \longrightarrow N_{S/V}(-C) \longrightarrow N_{S/V} \longrightarrow N_{S/V}\big|_C \longrightarrow 0, \tag{3.1}$$

it suffices to show that $H^1(N_{S/V}(-C)) = 0$. By Lemma 3.1, we have $C \sim aC_0 + bf$. Moreover since $C \cdot C_0 \geq 1$ and C is irrational, we have $b > a \geq 2$. Note that $N_{S/V} \sim C_0 + 2f$. Then the dual invertible sheaf $\{N_{S/V}(-C)\}^{-1}$ is represented by a smooth connected curve D on S, i.e., $N_{S/V}(-C) \simeq \mathcal{O}_S(-D)$, because $\{N_{S/V}(-C)\}^{-1} \sim (a-1)C_0+(b-2)f$ and $b-2 \geq a-1 \geq 1$. Therefore we have $H^1(N_{S/V}(-C)) \simeq H^1(-D) = 0$. □

Lemma 3.3. *Assume that $\chi(N_{S/V}(-C)) \geq 0$. If $H^1(N_{S/V}\big|_C) = 0$ then C is S-normal.*

Proof. Though the proof is very similar to that of a lemma in [5, §4.3], we repeat it here for the reader's convenience. It suffices to show that $H^1(N_{S/V}(-C)) = 0$. Since $H^2(N_{S/V}) = H^1(N_{S/V}\big|_C) = 0$, we have $H^2(N_{S/V}(-C)) = 0$ by (3.1). Then by assumption, we have $0 \leq \chi(N_{S/V}(-C)) = h^0(N_{S/V}(-C)) - h^1(N_{S/V}(-C))$. Thus if $H^0(N_{S/V}(-C)) = 0$ then the lemma has been proved. Otherwise there exists an effective divisor D on S such that $N_{S/V}(-C) \simeq \mathcal{O}_S(D)$. If $D = 0$ then the proof has been finished. If $D \neq 0$ then we have $D \cdot h > 0$ for a smooth hyperplane section h of S. Since $h \simeq \mathbb{P}^1$ (a twisted cubic curve), we have $H^1(h, \mathcal{O}_h(D)) = 0$. Since C is connected, it follows from the exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ that $H^1(-C) = 0$, and hence $H^1(D-h) = 0$. Therefore we conclude that $H^1(D) = 0$ by the exact sequence $[0 \to \mathcal{O}_S(-h) \to \mathcal{O}_S \to \mathcal{O}_h \to 0] \otimes \mathcal{O}_S(D)$. □

A standard exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/V} \overset{\pi_{C/S}}{\longrightarrow} N_{S/V}\big|_C \longrightarrow 0$$

induces an isomorphism $H^1(N_{C/V}) \simeq H^1(N_{S/V}\big|_C)$. Thus if $H^1(N_{S/V}\big|_C) = 0$ then $\operatorname{Hilb} V$ is nonsingular of expected dimension $2a+3b$. For example, if C is rational then we have $H^1(N_{S/V}\big|_C) = 0$, because $N_{S/V}$ is ample.

Lemma 3.4. *If C is not S-normal and $H^1(N_{S/V}\big|_C) \neq 0$, then $C \sim n(C_0 + f)$ for some integer $n \geq 5$.*

Proof. By the latter assumption C is irrational. Then by Lemma 3.2, we have $C \cdot C_0 = 0$. Hence $C \sim n(C_0 + f)$ for some $n \in \mathbb{Z}_{>0}$. Note that $H^i(N_{S/V}) = 0$ for $i = 1$ and 2.

Then it follows from the exact sequence (3.1) and the Serre duality that

$$\begin{aligned} H^1(C, N_{S/V}\big|_C) &\simeq H^2(S, N_{S/V}(-C)) \\ &\simeq H^0(S, {N_{S/V}}^{-1}(C+K_S))^\vee \\ &\simeq H^0(S, (n-3)C_0 + (n-5)f)^\vee, \end{aligned}$$

which concludes that $n \geq 5$. □

Proposition 3.5. *Suppose that $C \sim n(C_0 + f)$ for $n \geq 5$. Then C is stably degenerate and* $\mathrm{Hilb}^{sc} V$ *is singular at* $[C]$.

Proof. Note that $H^i(N_{S/V}(C_0 - C)) \simeq H^i(-(n-2)(C_0 + f)) = 0$ for $i = 0$ and 1. Since $C \cap C_0 = \emptyset$, it follows from the exact sequence (3.1)$\otimes\mathcal{O}_S(C_0)$ that the restriction map $H^0(N_{S/V}(C_0)) \xrightarrow{|_C} H^0(N_{S/V}\big|_C)$ is an isomorphism. Since $h^0(N_{S/V}(C_0)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = 6$, we have $h^0(N_{S/V}\big|_C) = h^0(N_{S/V}(C_0)) > h^0(N_{S/V}) = 5$. Therefore C is not S-normal and we have the following commutative diagram:

$$\begin{array}{ccccccc} H^0(N_{S/V}) & \xrightarrow{|_C} & H^0(N_{S/V}\big|_C) & \longrightarrow & H^1(N_{S/V}(-C)) \simeq k & \longrightarrow & 0 \\ \cap & & \| & & \downarrow & & \\ H^0(N_{S/V}(C_0)) & \xrightarrow[\simeq]{|_C} & H^0(N_{S/V}\big|_C) & \longrightarrow & H^1(N_{S/V}(C_0 - C)) = 0. & & \end{array}$$

Let $\kappa_{C,S}$ be the tangential map (2.2) and let α be a global section of $N_{C/V}$ not contained in the image of $\kappa_{C,S}$. Then the exterior component $\pi_{C/S}(\alpha) \in H^0(N_{S/V}\big|_C)$ is not contained in the image of (2.1). By the diagram above, there exists a global section v of $N_{S/V}(C_0)$ such that $v\big|_C = \pi_{C/S}(\alpha)$. Now we check that the three conditions (a), (b) and (c) in Theorem 2.2 are satisfied. Put $\Delta := C - 2C_0 + K_V\big|_S$, a divisor on S as in the theorem. Since $-K_V\big|_S = 2L\big|_S + 3M\big|_S = 3C_0 + 5f$, we have $\Delta = (n-5)(C_0+f)$ and hence $\Delta \cdot C_0 = 0$, which is (a). Since C_0 is a fiber of the $\mathbb{P}^1$-bundle $V \to \mathbb{P}^2$, C_0 is a good line on V, i.e. $N_{C_0/V}$ is trivial. Then $H^0(N_{C_0/V}(C_0)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}) = 0$, while $v\big|_{C_0}$ is nonzero in $H^0(N_{S/V}(C_0)\big|_{C_0})$ because $v \in H^0(N_{S/V}(C_0)) \setminus H^0(N_{S/V})$. Therefore we have (b). Finally we check (c). Let $\varepsilon : S \to \mathbb{P}^2$ be the blow-down contracting C_0 to a point $P \in \mathbb{P}^2$. Since $\Delta \sim (n-5)(C_0 + f)$, we have a commutative diagram

$$\begin{array}{ccc} H^0(S, \Delta) & \xrightarrow{|_{C_0}} & H^0(C_0, \Delta\big|_{C_0}) \simeq k \\ \simeq\downarrow & & \simeq\downarrow \\ H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n-5)) & \xrightarrow{\mathrm{ev}_P} & H^0(P, k(P)), \end{array}$$

where $k(P)$ is the residue field at P and ev_P is the evaluation map at P. Since ev_P is surjective, we obtain (c). Consequently by Theorem 2.2, the obstruction $\mathrm{ob}(\alpha)$ is nonzero. By Theorem 2.3, we have proved the proposition. $\square$

We compute the dimension of the S-maximal family $W_{S,C}$ of curves containing C. Notation is same as in §2.

Lemma 3.6.

(1) *The Hilbert-flag scheme* $\mathrm{Flag}\, V$ *is of dimension* $g+2d-a+4$ *at* (C,S).

(2) *If* $d>3$ *then the projection morphism* pr_1 *is a closed embedding in a neighborhood of* (C,S) *and* $\dim W_{S,C} = g+2d-a+4$.

Proof. (1) Let $\mathcal{W}_{S,C}$ be the irreducible component of the Hilbert-flag scheme $\mathrm{Flag}\, V$ passing through (C,S). Note that $H^i(S,C)=0$ for $i=1,2$. Then by the Riemann-Roch theorem, we compute that

$$\begin{aligned} h^0(S,C) &= \frac{1}{2}(C+K_S)\cdot C - K_S\cdot C + \chi(\mathcal{O}_S) \\ &= \frac{1}{2}\deg K_C - (-2h+f)\cdot C + 1 \\ &= g+2d-a, \end{aligned}$$

where h is the class of hyperplane sections of S. Then $\mathcal{W}_{S,C}$ is birationally equivalent to $\mathbb{P}^{g+2d-a-1}$-bundle over an open subset of the projective space $|\mathcal{O}_V(1)| \simeq \mathbb{P}^5$. Hence we have $\dim \mathcal{W}_{S,C} = g+2d-a+4$.

(2) Since $d=a+b>3$, we have $(h-C)\cdot f = 1-a<0$ or $(h-C)\cdot(C_0+f) = 2-b<0$. Since both f and C_0+f are nef, we have $H^0(N_{S/V}(-C)) = H^0(h-C) = 0$. Then the restriction map (2.1) is injective, and hence so is the tangential map $\kappa_{C,S}$ (2.2). This implies the first assertion. Hence we have $\dim W_{S,C} = \dim \mathcal{W}_{S,C}$. $\square$

The dimension of every irreducible component of $\mathrm{Hilb}^{sc} V$ passing through $[C]$ is greater than or equal to the expected dimension $\chi(N_{C/V}) = 2a+3b$. By this fact we have the following.

Proposition 3.7. *If* $\chi(V,\mathcal{I}_C(1))<1$ *then* C *is not stably degenerate, i.e., there exists a global deformation* C' *of* C *in* V *which is not contained in any deformation* S' *of* S *in* V.

Proof. By assumption, the lemma below shows that $g < d-4$. Then we have $\dim W_{S,C} \le \dim \mathcal{W}_{S,C} = g + 2d - a + 4 < 2a + 3b$. Hence there exists an irreducible component $W' \supset W_{S,C}$ of $\mathrm{Hilb}^{sc} V$ such that $\dim W' > \dim W_{S,C}$. Every member C' of $W' \setminus W_{S,C}$ is a required deformation of C in V. $\square$

Lemma 3.8. *The following conditions are equivalent: (i) $\chi(S, N_{S/V}(-C)) \ge 0$, (ii) $\chi(V, \mathcal{I}_C(1)) \ge 1$, (iii) $g \ge d-4$ and (iv) $2 \le a \le 2b-5$ or $2b-5 \le a \le 2$.*

Proof. By the exact sequence $[0 \to \mathcal{I}_S \to \mathcal{I}_C \to \mathcal{O}_S(-C) \to 0] \otimes \mathcal{O}_V(S)$, we have $\chi(N_{S/V}(-C)) = \chi(\mathcal{I}_C(S)) - \chi(\mathcal{I}_S(S)) = \chi(\mathcal{I}_C(1)) - 1$. Hence we obtain $(i) \Longleftrightarrow (ii)$. Note that $\chi(\mathcal{O}_C(1)) = d + 1 - g$ and $\chi(\mathcal{O}_V(1)) = \chi(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1,1)) = 6$. By the exact sequence $[0 \to \mathcal{I}_C \to \mathcal{O}_V \to \mathcal{O}_C \to 0] \otimes \mathcal{O}(1)$, we have $\chi(\mathcal{I}_C(1)) = g - d + 5$. Hence we obtain $(ii) \Longleftrightarrow (iii)$. Finally we prove $(iii) \Longleftrightarrow (iv)$. By Lemma 3.1, an easy calculation shows that $d - 4 - g = (a-2)(a-2b+5)/2$. Hence the proof is complete. $\square$

§ 3.3. Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. Notation is same as in the preceding sections. First we prove the first statement of Theorem 1.1. We have already proved the "only if" part of the statement (Proposition 3.7). Suppose now that $\chi(V, \mathcal{I}_C(1)) \ge 1$, i.e., $\chi(S, N_{S/V}(-C)) \ge 0$ by Lemma 3.8. Then C is clearly stably degenerate if C is S-normal by Theorem 2.1, or if $H^1(N_{S/V}\big|_C) = 0$ by Lemma 3.3. Assume that C is not S-normal and $H^1(N_{S/V}\big|_C) \ne 0$. Then $C \sim n(C_0 + f)$ for some $n \ge 5$ by Lemma 3.4. Then by Proposition 3.5, C is stably degenerate.

Next we prove the second statement of Theorem 1.1. If $C \not\sim n(C_0 + f)$ for any $n \ge 5$, then C is S-normal or $H^1(N_{S/V}\big|_C) = 0$ by Lemma 3.4. Then $\mathrm{Hilb}^{sc} V$ is nonsingular at $[C]$. If $C \sim n(C_0 + f)$ for some $n \ge 5$, then $\mathrm{Hilb}^{sc} V$ is singular at $[C]$ by Proposition 3.5. In fact $\mathrm{Hilb}^{sc} V$ is non-reduced along a neighborhood of $[C]$ by Theorem 3.9 below.

We denote by $\mathrm{Hilb}^{sc}_{d,g} V$ the open and closed subscheme of $\mathrm{Hilb}^{sc} V$ of curves of degree d and genus g. If $C \sim n(C_0 + f)$, then we have $d = 2n$ and $g = (n-1)(n-2)/2$. Given an integer n, let us define a locally closed subset

$$W_n := \left\{ C \in \mathrm{Hilb}^{sc} V \;\middle|\; \begin{array}{c} C \text{ is contained in a smooth hyperplane section } S \text{ of } V \\ \text{and } C \sim n(C_0 + f) \text{ on } S \end{array} \right\}$$
$$\subset \mathrm{Hilb}^{sc}_{2n, \frac{1}{2}(n-1)(n-2)} V.$$

Theorem 3.9. *If $n \geq 5$ then the closure $\overline{W}_n$ of W_n in $\mathrm{Hilb}^{sc} V$ is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$ of dimension $(n^2+3n+10)/2$ and $\mathrm{Hilb}^{sc} V$ is generically non-reduced along W_n.*

Proof. Since $d = 2n \geq 10$, by Lemma 3.6 (2) every member C of W_n is contained in a unique hyperplane section S of V. Every smooth hyperplane section of V is parametrized by an open subset U of the projective space $|\mathcal{O}_V(1)| \simeq \mathbb{P}^5$. Since $\dim |\mathcal{O}_S(C)| = n(n+3)/2$, W_n is isomorphic to an open subset of a $\mathbb{P}^{n(n+3)/2}$-bundle over U, and hence $\overline{W}_n$ is irreducible and of dimension $(n^2+3n+10)/2$. Since C is stably degenerate, $\overline{W}_n$ is a maximal closed subset of $\mathrm{Hilb}^{sc} V$ and hence an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$. Since $\mathrm{Hilb}^{sc} V$ is singular at the generic point of W_n, the proof is complete. □

Thus the proof of Theorem 1.1 has been completed.

Remark. Among the non-reduced components obtained from Theorem 3.9, the component $\overline{W}_5 \subset \mathrm{Hilb}^{sc}_{10,6} V$ is the only one of expected dimension, i.e., $\dim W_5 = 25$.

Given a pair (a,b) of integers such that either $b \geq a > 0$ or $(a,b) = (1,0), (0,1)$, we define a locally closed subset $W_{a,b}$ of $\mathrm{Hilb}^{sc} V$ by

$$W_{a,b} := \left\{ C \in \mathrm{Hilb}^{sc} V \;\middle|\; \begin{array}{c} C \text{ is contained in a smooth hyperplane section } S \text{ of } V \\ \text{and } C \sim aC_0 + bf \text{ on } S \end{array} \right\}.$$

Then an argument similar to Theorem 3.9 shows the following:

(1) If $2 \leq a \leq 2b-5$ or $2b-5 \leq a \leq 2$, then $\overline{W}_{a,b}$ is an irreducible component of $(\mathrm{Hilb}^{sc}_{d,g} V)_{\mathrm{red}}$, where $d = a+b$ and $g = (a-1)(2b-a-2)/2$;

(2) $\dim W_{a,b} = \begin{cases} g + 2d - a + 4 & \text{if } d > 3, \\ 2a + 3b & \text{if } d \leq 3 \end{cases}$; and

(3) If $(a,b) \neq (n,n)$ for any integer $n \geq 5$, then $\mathrm{Hilb}^{sc} V$ is generically smooth along $W_{a,b}$.

In particular $\overline{W}_{a,b}$ is an irreducible component of $(\mathrm{Hilb}^{sc} V)_{\mathrm{red}}$ of expected dimension $2a+3b$ if and only if either $d \leq 3$ or $g = d - 4$.

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