RIMS Kôkyûroku Bessatsu
B9 (2008), 153–161

Existence of stable bundles on Calabi-Yau manifolds

By

Tohru Nakashima*

Abstract

We discuss the existence problem of μ-stable bundles on Calabi-Yau threefolds arising from superstring theory. The possibility of strengthening the classical Bogomolov inequality will be examined.

§ 1. Introduction

Let X be a smooth projective variety of dimension $n \geq 2$ defined over $\mathbb{C}$ and let H be an ample line bundle on X. The notion of μ-(semi) stability has played a central role in the classification of vector bundles on X. We fix an integer $r \geq 2$ and $c_i \in H^{2i}(X, \mathbb{Z})\,(1 \leq i \leq n)$. A fundamental problem concerning μ-stable bundles is the following

Problem 1. Determine r and c_i for which a vector bundle (or torsion-free sheaf) E exists on X with $\mathrm{rk}(E) = r$, $c_i(E) = c_i$ which is μ-stable with respect to H.

The most general result concerning the problem above is the following asymptotic theorem due to Maruyama.

Theorem 1.1 ([3]). *Assume that $r \geq n$. Then, for any c_1 and an integer s, there exists a μ-stable vector bundle E with $\mathrm{rk}(E) = r$, $c_1(E) = c_1$ and $c_2(E) \cdot H^{n-2} \geq s$.*

In particular, the theorem implies the existence of a sequence of μ-stable bundles $\{E_m\}_{m=1}^{\infty}$ such that their discriminants $(rc_2 - (r-1)c_1^2) \cdot H^{n-2}$ become arbitrarily

Received September 29, 2007.
2000 Mathematics Subject Classification(s): Primary 14J60. Secondary 14D20.
Key Words: *stable bundles, Calabi-Yau manifolds, Bogomolov inequality*
This research was partially supported by Grant-In-Aid for Scientific Research (C) 18540043.
*Department of Mathematical and Physical Sciences, Faculty of Science, Japan Women's University, Bunkyoku, Mejirodai 2-8-1, Japan.
email:nakashima@fc.jwu.ac.jp

large as m goes to infinity. We notice that this result is not effective in the sense that $c_2(E)\cdot H^{n-2}$ are not given explicitly as a function of s. This is because the proof of the theorem is based on the boundedness of certain family of coherent sheaves on X. So we may pose the following weaker

Problem 2. Construct a sequence of μ-stable vector bundles $\{E_m\}_{m=1}^{\infty}$ with effectively computable discriminants Δ_m with $\Delta_m \to \infty$ as $m \to \infty$.

In the case of surfaces, there are some known results on the problem above. For example, we have the following result due to Drézet-Le Potier for the projective plane $\mathbb{P}^2$.

Theorem 1.2 ([2]). *Assume that $X = \mathbb{P}^2$. There exists an explicit function δ such that if*

$$2rc_2 - (r-1)c_1^2 \geq 2r^2\delta,$$

then there exists a μ-semistable sheaf E with $\mathrm{rk}(E) = r$, $c_i(E) = c_i$.

For K3 surfaces, Yoshioka proved the following

Theorem 1.3 ([9]). *Assume that X is a K3 surface and H is general. If*

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) \geq -2,$$

then there exists a μ-semistable sheaf E with $\mathrm{rk}(E) = r$, $c_i(E) = c_i$.

In both of the results above certain strengthenings of the classical Bogomolov inequality ensure the existence of (semi-) stable sheaves and, if one excludes the case of exceptional bundles, these inequalities are *necessary* for existence. No analogous results have been obtained for varieties of dimension ≥ 3 up to now. However, inspired by superstring theory, very interesting conjectures have been recently proposed concerning the existence of stable sheaves on Calabi-Yau threefolds([1]). Motivated by these conjectures, we constructed in [7] a sequence of μ-stable bundles which violate "strong Bogomolov inequality", and thereby gave an answer to Problem 2 at the same time. In this note we review the construction of [7] and its application to the higher dimensional Brill-Noether problems proposed in [8].

The author would like to thank the organizers of the conference and Prof.E.Sato for inviting him to the conference.

§ 2. Conjectures of Douglas-Reinbacher-Yau

In their study of superstring theory, Douglas-Reinbacher-Yau proposed the following conjectures on the existence of stable sheaves on Calabi-Yau threefolds([1]).

Conjecture 1. Let X be a Calabi-Yau threefold and let $r \geq 2$ and $c_i \in H^{2i}(X, \mathbb{Z})\,(1 \leq i \leq 3)$. Assume that there exists an ample $\mathbb{R}$-divisor $\widetilde{H}$ on X such that

$$\frac{1}{2r^2}\left(2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X)\right) = \widetilde{H}^2, \tag{2.1}$$

$$\frac{1}{6r^2}(c_1^3 + 3r(rch_3 - ch_2c_1)) < \frac{2^{5/2}}{3}r \cdot \widetilde{H}^3 \tag{2.2}$$

where ch_i denotes the i-th Chern character. Then there exists a μ-stable reflexive sheaf E on X with respect to some ample divisor such that $\mathrm{rk}(E) = r$, $c_i(E) = c_i$.

Conjecture 2. Let X be Calabi-Yau threefold and let D be a smooth ample divisor on X. Let $r \geq 2$, $c_i \in H^{2i}(D, \mathbb{Z})\,(i = 1, 2)$. Assume that c_1 lies in the image of $H^{1,1}(X) \to H^{1,1}(D)$ and

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(D) > 0.$$

Then there exists a μ-stable bundle E on D with $\mathrm{rk}(E) = r$, $c_i(E) = c_i$.

We briefly comment on the physical background of the above conjectures. First, μ-stable holomorphic vector bundles on a Calabi-Yau threefold correspond to BPS particles in type IIA superstring theory and their rank and Chern classes are identified with the charges of these BPS particles. According to the attractor mechanism, the assumptions on Chern classes in the DRY conjectures imply the existence of extremal black hole solutions of supergravity. These solutions are indistinguishable from BPS particles, hence we conclude that the existence of μ-stable bundles should follow.

It is natural to ask whether the claimed sufficient conditions for the existence of stable sheaves are in fact necessary conditions. If this is the case, from the equality (2.1) the following inequality of Chern classes should hold for any stable bundle on Calabi-Yau threefolds:

$$(2rc_2 - (r-1)c_1^2) \cdot H > \frac{r^2}{12}c_2(X) \cdot H \tag{2.3}$$

since, by the Hodge index theorem, for any ample divisor H, we have

$$(\widetilde{H}^2 \cdot H)^3 \geq (\widetilde{H}^3)^2 \cdot H^3 > 0.$$

This may be considered as a strengthening of the well-known Bogomolov inequality

$$(2rc_2 - (r-1)c_1^2) \cdot H \geq 0$$

since we have $c_2(X) \cdot H \geq 0$ for Calabi-Yau threefolds. Further, combining (2.1) with (2.2), we would obtain an upper bound for c_3 of μ-stable bundles.

Unfortunately it turns out that the inequality (2.3) cannot hold in general. In fact, some counter-examples are given by Jardim([1]) and the author([6]). However, one expects that a strong form of Bogomolov inequality might hold, if we replace $\frac{1}{12}c_2(X)\cdot H$ in the RHS of (2.3) by some other positive constant. So we introduce the following general definition for varieties not necessarily Calabi-Yau.

Definition 2.1. Let X be a smooth projective variety of dimension $n \geq 2$ and let H be an ample line bundle on X. Let $\alpha = \alpha(X, H)$ be a positive real number depending on X and H. We say that the *strong Bogomolov inequality of type* α holds if, for any H-stable bundles of rank r and Chern classes c_i, we have

$$(2rc_2 - (r-1)c_1^2) \cdot H^{n-2} \geq r^2\alpha.$$

Later we shall show that such inequality fails for large class of complete intersections.

§ 3. Construction of stable bundles

In this section we review a method of constructing stable sheaves via extensions given in [5],[6],[7]. We assume that in the rest of this note all varieties are defined over $\mathbb{C}$.

Definition 3.1. Let X be a smooth projective variety of dimension $n \geq 2$ and let H be an ample line bundle on X. The *minimal H-degree* $d_{\min}(H)$ is defined as follows.

$$d_{\min}(H) := \min\{L \cdot H^{n-1} \mid L \in \mathrm{Pic}(X),\ L \cdot H^{n-1} > 0\}.$$

A line bundle L on X is said to be *H-minimal* if $L \cdot H^{n-1} = d_{\min}(H)$.

Remark. Let X is a variety such that the Picard group $\mathrm{Pic}(X)$ is generated by an ample line bundle $\mathcal{O}_X(1)$, then $\mathcal{O}_X(1)$ itself is $\mathcal{O}_X(1)$-minimal.

A coherent sheaf Q is said to be *of pure codimension one*, if Q has the form $\iota_*\mathcal{L}$ for an integral divisor $\iota : D \hookrightarrow X$ and a line bundle $\mathcal{L}$ on D. The following result is proved for Calabi-Yau manifolds in [6], and for general projective varieties in [5], [7].

Proposition 3.2. *Let X be a smooth projective variety of dimension $n \geq 2$ such that $H^1(\mathcal{O}_X) = 0$. Let Q be a torsion-free sheaf or a sheaf of pure codimension one such* *that $c_1(Q)$ is H-minimal. Let U be a non-zero vector space and let E be a coherent sheaf with* $\mathrm{Hom}(E, \mathcal{O}_X) = 0$ *which fits in a non-split extension*

$$0 \to U \otimes \mathcal{O}_X \to E \to Q \to 0.$$

Then E is a μ-stable torsion-free sheaf.

Example 3.3. Let $U \subset \mathrm{Ext}^1(Q, \mathcal{O}_X)$ be a non-zero subspace. Under the isomorphism

$$\mathrm{Hom}(U, \mathrm{Ext}^1(Q, \mathcal{O}_X)) \cong \mathrm{Ext}^1(Q, U^\vee \otimes \mathcal{O}_X),$$

the inclusion $U \hookrightarrow \mathrm{Ext}^1(Q, \mathcal{O}_X)$ corresponds to the following extension, which is called the *universal extension.*

$$0 \to U^\vee \otimes \mathcal{O}_X \to E \to Q \to 0.$$

By Proposition 3.2, E is μ-stable if $c_1(Q)$ is H-minimal.

Example 3.4. Assume that there exists a divisor $D \in |L|$ which is smooth and irreducible and let $\iota : D \hookrightarrow X$ denote the inclusion. Let $\mathcal{L}$ be a line bundle on D which is generated by global sections. We extend the evaluation map $H^0(D, \mathcal{L}) \otimes \mathcal{O}_D \to \mathcal{L}$ to the map $\varphi : H^0(D, \mathcal{L}) \otimes \mathcal{O}_X \to \iota_*\mathcal{L}$. It is well-known that the kernel of φ is locally free and is called the *elementary transformation* of $H^0(D, \mathcal{L}) \otimes \mathcal{O}_X$ along $\mathcal{L}$. We denote its dual by E. Thus E fits in the exact sequence

$$0 \to E^\vee \to H^0(D, \mathcal{L}) \otimes \mathcal{O}_X \to \iota_*\mathcal{L} \to 0.$$

By taking $\mathcal{H}om_{\mathcal{O}_X}(\quad, \mathcal{O}_X)$, we obtain the exact sequence

$$0 \to H^0(D, \mathcal{L})^\vee \otimes \mathcal{O}_X \to E \to \iota_*(L_{|D} \otimes \mathcal{L}^\vee) \to 0$$

since $\mathcal{E}xt^1_{\mathcal{O}_X}(\iota_*\mathcal{L}, \mathcal{O}_X) \cong \iota_*(L_{|D} \otimes \mathcal{L}^\vee)$. Since we have $H^0(E^\vee) = 0$, it follows from Proposition 3.2 that E is a μ-stable bundle.

§ 4. Counter-examples to strong Bogomolov inequality

In this section we use the construction in the previous section to give some examples of stable bundles which violate the strong Bogomolov inequality([7]).

Proposition 4.1. *Let X be a smooth projective variety of dimension $n \geq 2$ such that $H^1(\mathcal{O}_X) = 0$ and* $\mathrm{Pic}(X)$ *is generated by a very ample line bundle $\mathcal{O}_X(1)$. Let $D \in |\mathcal{O}_X(1)|$ be a smooth irreducible divisor. Let E_m be the dual of the elementary* *transformation of* $H^0(D, \mathcal{O}_D(m)) \otimes \mathcal{O}_X$ *along* $\mathcal{O}_D(m)$ *for sufficiently large* m*. Then* E_m *is an* $\mathcal{O}_X(1)$*-stable vector bundle of rank*

$$r_m = h^0(D, \mathcal{O}_D(m))$$

and

$$c_1(E_m) = \mathcal{O}_X(1), \quad c_2(E_m) = m\mathcal{O}_X(1)^2.$$

Furthermore, E_m *has non-trivial moduli space, that is,* $\mathrm{Ext}^1(E_m, E_m) \neq 0$.

We choose sufficiently large m so that $\mathcal{O}_D(m)$ is globally generated and $h^i(\mathcal{O}_D(m)) = 0$ for $i > 0$. Let E_m denote the dual of the elementary transformation of $H^0(D, \mathcal{O}_D(m)) \otimes \mathcal{O}_X$. Then the stability of E_m follows from Example 3.3.

By the Riemann-Roch formula, we have the following asymptotic formula for r_m:

$$\begin{aligned} r_m &= \chi(\mathcal{O}_D(m)) \\ &= \frac{\mathcal{O}_X(1)^n}{(n-1)!} m^{n-1} + O(m^{n-2}). \end{aligned}$$

Thus

$$\begin{aligned} &(2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2) \cdot \mathcal{O}_X(1)^{n-2} \\ &= \frac{2(\mathcal{O}_X(1)^n)^2}{(n-1)!} m^n + O(m^{n-1}). \end{aligned}$$

This implies that the discriminants of E_m become arbitrarily large as m goes to infinity. Notice that these values are effectively computable once X, D and m are given. Further, for any $\alpha > 0$, we have

$$r_m^2 \alpha = \left(\frac{\mathcal{O}_X(1)^n}{(n-1)!} \right)^2 \alpha m^{2n-2} + O(m^{2n-3}).$$

Thus we see that the sequence $\{E_m\}_{m=1}^{\infty}$ yields counter-examples to the strong Bogomolov inequality for sufficiently large m. We obtain the following result in the case of surfaces.

Theorem 4.2. *Let* X *be a smooth projective surface with* $H^1(\mathcal{O}_X) = 0$ *and assume that* $\mathrm{Pic}(X)$ *is generated by a very ample line bundle* $\mathcal{O}_X(1)$*. Then the strong Bogomolov inequality of type* α *fails for any* $\alpha > 2$.

Hence, by Noether-Lefschetz theorem, we obtain

Corollary 4.3. *Let* $X \subset \mathbb{P}^n$ *be a general smooth complete intersection surface of type* $(d_1, d_2, \ldots, d_{n-2})$*. Assume that*

$$\left(\sum_{i=1}^{n-2} d_i - (n+1) \right) \prod_{i=1}^{n-2} d_i > 72$$

and that $X \neq (2)$, (3), (4), $(2,2)$, $(3,2)$, $(2,2,2)$. *Then, for* $m >> 0$, E_m *satisfies*

$$2r_m c_2(E_m) - (r_m - 1)c_1(E_m)^2 < \frac{r_m^2}{12} c_2(X).$$

The assumptions of the corollary are satisfied for a general complete intersection surface of type $(5, d)$ in $\mathbb{P}^4$ with $d >> 0$, which is a divisor with ample canonical bundle of a quintic Calabi-Yau threefold. In particular, we obtain a negative answer to the following problem posed by Douglas et al.([1]).

Problem. Let S be a simply connected surface with ample or trivial canonical bundle. Then does the inequality

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(D) \geq 0$$

hold for any μ-stable bundles on S with non-trivial moduli space?

For varieties of dimension ≥ 3, we obtain the following

Theorem 4.4. *Let* X *be a smooth projective variety of dimension* $n \geq 3$. *Assume that* $H^1(\mathcal{O}_X) = 0$ *and* $\mathrm{Pic}(X)$ *is generated by a very ample line bundle* $\mathcal{O}_X(1)$. *Then the strong Bogomolov inequality of type* α *fails for any* $\alpha > 0$.

The next result shows that the inequality of type (2.3) does not hold in any dimension ≥ 3.

Corollary 4.5. *Let* $X \subset \mathbb{P}^N$ *be a general complete intersection Calabi-Yau manifold of dimension* $n \geq 3$ *with sufficiently large multidegree. Then the strong Bogomolov inequality*

$$(2rc_2 - (r-1)c_1^2) \cdot \mathcal{O}_X(1)^{n-2} \geq \frac{r^2}{12} c_2(X) \cdot \mathcal{O}_X(1)^{n-2}$$

fails.

§ 5. Higher dimensional Brill-Noether problems

In this section we review an approach to the "higher dimensional Brill-Noether problem" posed in [8] by means of the construction in section 3.

Definition 5.1. Let X be a smooth projective variety of dimension $n \geq 2$ with $H^1(\mathcal{O}_X) = 0$. For a coherent sheaf E on X, its *Mukai vector* $v(E)$ is the following element of the rational cohomology ring $H^*(X, \mathbb{Q}) = \bigoplus_{i=1}^n H^{2i}(X, \mathbb{Q})$.

$$v(E) := \mathrm{ch}(E) \cdot \sqrt{\mathrm{Td}(X)}.$$

For given v, let $\mathcal{M}(v)$ denote the moduli space of μ-stable torsion-free sheaves E on X with $v(E) = v$ with respect to H.

When X is a K3 surface, these moduli spaces have been extensively studied by Mukai([4]). We introduce the following generalization of classical Brill-Noether locus.

Definition 5.2. For non-negative integers i, j, we define *the Brill-Noether locus* $\mathcal{M}(v)_{i,j}$ *of type* (i,j) to be the following locally closed subset of $\mathcal{M}(v)$:

$$\mathcal{M}(v)_{i,j} := \{E \in \mathcal{M}(v) \mid \dim H^0(E) = i, \dim \operatorname{Ext}^1(E, \mathcal{O}_X) = j\}.$$

We put a reduced induced scheme structure on $\mathcal{M}(v)_{i,j}$. We pose the following *higher dimensional Brill-Noether problem*, which may be regarded as a refinement of Problem 1 in section 2.

Problem. Determine the integers i, j for which $\mathcal{M}(v)_{i,j}$ is non-empty. If it is non-empty, describe its geometric structure.

Definition 5.3. A coherent sheaf Q is said to be *regular* if

$$H^1(Q) = \operatorname{Ext}^2(Q, \mathcal{O}_X) = 0.$$

We denote by $\mathcal{M}(v)_{i,j}^{reg}$ the open subset of $\mathcal{M}(v)_{i,j}$ consisting of regular sheaves. The next proposition shows how μ-stable sheaves contained in different Brill-Noether loci are related under the universal extension.

Proposition 5.4. *Let Q be as in Proposition 3.2 with $Q \in \mathcal{M}(v)_{i,j}^{reg}$. For a subspace $U \subset \operatorname{Ext}^1(Q, \mathcal{O}_X)$ of dimension $s(0 < s \leq j)$, the universal extension E corresponding to U belongs to $\mathcal{M}(v^s)_{i+s,j-s}^{reg}$ where $v^s := v + sv(\mathcal{O}_X)$. Furthermore, we have*

$$\dim \operatorname{Ext}^1(E, E) = \dim \operatorname{Ext}^1(Q, Q) + s(j - i - s).$$

Remark. The proposition above can be used to show that $\mathcal{M}(v^s)_{s,j-s}$ is birational to a Grassmann fibration over $\mathcal{M}(v)_{0,j}$ in many cases(cf.[8]).

References

[1] Douglas, M.R., Reinbacher, R. and Yau, S.-T., Branes, bundles and attractors: Bogomolov and beyond, arXive:math.AG/0604597

[2] Drézet J.-M. and Le Potier J., Fibrés stables et fibrés exceptionnels sur $\mathbb{P}_2$, *Ann. Sci. École Norm. Sup.* **18** (1985), 193–243

[3] Maruyama, M., Moduli of stable sheaves II, *J. Math. Kyoto Univ.* **18** (1978), 557–614

[4] Mukai, S., On the moduli spaces of bundles on K3 surfaces I, Vector bundles on algebraic varieties. Oxford University Press, Oxford 1987

[5] Nakashima, T., Reflection of sheaves on a Calabi-Yau variety, *Asian J. Math.* **6** (2002), 567–577

[6] Nakashima, T., A construction of stable vector bundles on Calabi-Yau manifolds, *J. Geom. Phys.* **49** (2004), 224–230

[7] Nakashima, T., Strong Bogomolov inequality for stable vector bundles, *J. Geom. Phys.* **57** (2007), 1977–1983

[8] Nakashima, T., Brill-Noether problems in higher dimensions, *Forum Math.* **20** (2008), 145–161

[9] Yoshioka, K., Moduli spaces of stable sheaves on abelian surfaces, *Math. Ann.* **321** (2001), 817–884