RIMS Kôkyûroku Bessatsu
B9 (2008), 97–113

On the sectional geometric genus of multi-polarized manifolds and its application

By

Yoshiaki FUKUMA *

Abstract

In this paper, we will give an application of the sectional geometric genus of multi-polarized manifolds. In order to do that, first we will recall the definition of the ith sectional geometric genus of n-dimensional multi-polarized manifolds for every integer i with $0 \le i \le n$ and some fundamental results of this. We will also provide some results about the case where $i = 1$. Finally we will give an application to the dimension of global sections of adjoint bundles. In particular, for the case where $\dim X = 3$, we give an affirmative answer for a generalization of a conjecture proposed by Beltrametti and Sommese.

§ 1. Introduction

Let X be a projective variety of dimension n which is defined over the field of complex numbers and let L be an ample line bundle on X. Then the pair (X, L) is called a *polarized variety*. Moreover if X is smooth, then (X, L) is called a polarized *manifold*. Then the following three invariants of (X, L) are well-known.

(1) The degree L^n (see Definition 2.2 (1)).

(2) The sectional genus $g(L)$ (see Definition 2.2 (2)).

(3) The Δ-genus $\Delta(L)$, which is defined by

$$\Delta(L) := n + L^n - h^0(L).$$

Received September 27, 2007. Revised February 1, 2008
2000 Mathematics Subject Classification(s): 14C17, 14C20, 14D06, 14E30, 14J30, 14J35, 14J40
Key Words: Polarized varieties, ample line bundles, sectional genus, ith sectional geometric genus, adjoint bundles.
The author was partially supported by the Grant-in-Aid for Scientific Research (C) (No.17540033), Japan Society for the Promotion of Science.
*Department of Natural Science, Faculty of Science, Kochi University, Kochi 780-8520, Japan.
email:fukuma@math.kochi-u.ac.jp

By using the above invariants, various interesting results about polarized manifolds are obtained and applied to other researches. But there is a limit to study polarized manifolds by using only above three invariants.

So, in order to study polarized manifolds more deeply, the author introduced new invariants of (X, L). Let i be an integer with $0 \leq i \leq n$. In [7], [8] and [10], we proposed invariants of (X, L), the *ith sectional geometric genus* $g_i(X, L)$ (see Definition 2.3), the *ith sectional arithmetic genus* $p_a^i(X, L)$ and the *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$ (see Remark 2.2). Here we note that the ith sectional geometric genus can be regarded as a generalization of the degree and the sectional genus of polarized manifolds (see Remark 2.1 (2) and (3)).

Here we recall the reason why $g_i(X, L)$ (resp. $p_a^i(X, L)$, $\chi_i^H(X, L)$) is called the ith sectional geometric genus (resp. the ith sectional arithmetic genus, the ith sectional H-arithmetic genus) of (X, L). Let (X, L) be a polarized manifold of dimension $n \geq 2$ with $\mathrm{Bs}|L| = \emptyset$, where $\mathrm{Bs}|L|$ is the base locus of the complete linear system $|L|$. Let i be an integer with $1 \leq i \leq n$. Let X_{n-i} be the transversal intersection of general $n-i$ members of $|L|$. In this case X_{n-i} is a smooth projective variety of dimension i. Then we can prove that $g_i(X, L) = h^i(\mathcal{O}_{X_{n-i}})$ (resp. $p_a^i(X, L) = p_a(X_{n-i})$, $\chi_i^H(X, L) = \chi(\mathcal{O}_{X_{n-i}})$), that is, $g_i(X, L)$ (resp. $p_a^i(X, L)$, $\chi_i^H(X, L)$) is the geometric genus of X_{n-i} (resp. the arithmetic genus of X_{n-i}, the arithmetic genus of X_{n-i} in the sense of Hirzebruch).

Furthermore, by this consideration, the ith sectional geometric genus (resp. the ith sectional arithmetic genus, the ith sectional H-arithmetic genus) is expected to have properties similar to those of the geometric genus (resp. the arithmetic genus, the H-arithmetic genus) of i-dimensional projective manifolds (see [8, Section 3]).

On the other hand, let $L_1, \ldots, L_n$ be (ample) line bundles on X and let $\mathcal{F}$ be a coherent sheaf on X. In [17], Kleiman defined the intersection number $(L_1 \cdots L_n, \mathcal{F})$ by using the coefficient of polynomial $\chi(L_1^{\otimes t_1} \otimes \cdots \otimes L_n^{\otimes t_n} \otimes \mathcal{F})$. If $L_1 = \cdots = L_n = L$ and $\mathcal{F} = \mathcal{O}_X$, then $(L \cdots L, \mathcal{O}_X)$ is the degree of (X, L). Namely $(L_1 \cdots L_n, \mathcal{F})$ can be regarded as a generalization of the degree. So the author thought that we can also define the ith sectional invariants for (ample) line bundles $L_1, \ldots, L_{n-i}$ on X. This was a motivation of the paper [13] and [14]. In [13] and [14], we defined and investigated the ith sectional geometric genus of multi-polarized varieties. Main purpose of this paper is to give an application of this ith sectional geometric genus of multi-polarized manifolds. In order to do that, first we will recall the definition and basic properties of the ith sectional geometric genus of multi-polarized varieties. Concretely, in section 2 we will give various sectional invariants of polarized varieties, and we will also define the notion of a multi-(pre)polarized variety (see Definition 2.1 below). In section 3, we will define the ith sectional geometric genus of multi-(pre)polarized varieties of type $n-i$ for every integer i with $0 \leq i \leq n$ (see Definition 3.1). We will also give fundamental results about this invariant. In section 4, we consider the case where $i = 1$. We can regard this case as a generalization of the sectional genus of polarized manifolds, and we can get results similar to those of the sectional genus. In section 5, which is the main part of this paper, we will give an application using this invariant. Here we consider the dimension of global sections of adjoint bundles. In particular, for $\dim X = 3$, we get an affirmative answer for a generalization of a conjecture proposed by Beltrametti and Sommese (see Conjecture 5.1 and Theorem 5.2 below). Moreover for $\dim X = 3$ we will also consider a problem proposed in [12, Problem 3.2].

The author would like to thank the referee for giving him some valuable comments and suggestions.

§ 2. Preliminaries.

Definition 2.1. Let X be a complex projective variety of dimension n, and let $L_1, \cdots, L_k$ be ample line bundles (resp. line bundles) on X. Then $(X, L_1, \cdots, L_k)$ is called a *multi-polarized* (resp. *multi-prepolarized*) *variety of type* k. If $k = 1$, then it is called a *polarized* (resp. *prepolarized*) *variety.*
If X is smooth, then we say that $(X, L_1, \cdots, L_k)$ is a multi-polarized (resp. multi-prepolarized) *manifold* of type k.

Here we will give the definition of the degree and the sectional genus of prepolarized varieties.

Definition 2.2. Let (X, L) be a prepolarized variety of dimension n. Then the Euler-Poincaré characteristic $\chi(tL)$ is a polynomial in t of total degree at most n. We set

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(L) \binom{t+j-1}{j}.$$

(1) The *degree* L^n *of* (X, L) is defined by $L^n := \chi_n(L)$.
(2) The *sectional genus* $g(L)$ *of* (X, L) is defined by $g(L) := 1 - \chi_{n-1}(L)$.

Here we will define the ith sectional geometric genus of prepolarized varieties.

Definition 2.3. ([7]) Let (X, L) be a prepolarized variety of dimension n and let i be an integer with $0 \leq i \leq n$. Then the *ith sectional geometric genus* $g_i(X, L)$ is defined by the following:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 2.1.

(1) Since $\chi_j(L) \in \mathbb{Z}$ for every integer j with $0 \leq j \leq n$, by definition we get $g_i(X, L) \in \mathbb{Z}$ for every integer i with $0 \leq i \leq n$.

(2) If $i = 0$, then $g_0(X, L) = L^n$.

(3) If $i = 1$, then $g_1(X, L) = g(L)$. Moreover if X is smooth, then $g_1(X, L)$ can be written as follows:

$$g_1(X, L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical line bundle on X.

(4) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$.

Remark 2.2. In [7] and [8], we defined the *ith sectional arithmetic genus* $p_a^i(X, L)$ and the *ith sectional H-arithmetic genus* $\chi_i^H(X, L)$ of n-dimensional prepolarized varieties (X, L) for every integer i with $0 \leq i \leq n$ as follows:

$$\begin{aligned} p_a^i(X, L) &:= (-1)^i(\chi_{n-i}(L) - h^0(\mathcal{O}_X)), \\ \chi_i^H(X, L) &:= \chi_{n-i}(L). \end{aligned}$$

§ 3. Definition and fundamental properties.

In this section, we will give the definition of the ith sectional geometric genus of multi-prepolarized varieties and some fundamental properties.

Notation 3.1. Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X. Then $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}})$ is a polynomial in $t_1, \ldots, t_{n-i}$ of total degree at most n. So we can write $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}})$ uniquely as follows.

$$\begin{aligned} &\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}}) \\ &= \sum_{p=0}^{n} \sum_{\substack{p_1 \geq 0, \ldots, p_{n-i} \geq 0 \\ p_1 + \cdots + p_{n-i} = p}} \chi_{p_1, \ldots, p_{n-i}}(L_1, \ldots, L_{n-i}) \binom{t_1 + p_1 - 1}{p_1} \cdots \binom{t_{n-i} + p_{n-i} - 1}{p_{n-i}}. \end{aligned}$$

Definition 3.1. Let $(X, L_1, \ldots, L_{n-i})$ be an n-dimensional multi-prepolarized variety of type $n - i$ for $i \in \mathbb{Z}$ with $0 \leq i \leq n - 1$.

(1) We set

$$\chi_i^H(X, L_1, \dots, L_{n-i}) := \begin{cases} \chi_{\underbrace{1,\dots,1}_{n-i}}(L_1, \dots, L_{n-i}) & \text{if } 0 \leq i \leq n-1, \\ \chi(\mathcal{O}_X) & \text{if } i = n. \end{cases}$$

(2) The *ith sectional geometric genus* $g_i(X, L_1, \dots, L_{n-i})$ is defined by the following:

$$g_i(X, L_1, \dots, L_{n-i}) = (-1)^i(\chi_i^H(X, L_1, \dots, L_{n-i}) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 3.1.

(1) $\chi_i^H(X, L_1, \dots, L_{n-i})$ in Definition 3.1 (1) is called the *ith sectional H-arithmetic genus of* $(X, L_1, \dots, L_{n-i})$.

(2) We can also define the ith sectional arithmetic genus of multi-prepolarized varieties of type $n-i$ for every integer i with $0 \leq i \leq n$, where $n = \dim X$. Namely, the *ith sectional arithmetic genus* $p_a^i(X, L_1, \dots, L_{n-i})$ is defined by the following:

$$p_a^i(X, L_1, \dots, L_{n-i}) := (-1)^i(\chi_i^H(X, L_1, \dots, L_{n-i}) - h^0(\mathcal{O}_X)).$$

(3) Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X with $1 \leq r \leq n$. Then in [6, Definition 2.1], we defined the *ith c_r-sectional geometric genus $g_i(X, \mathcal{E})$ of* $(X, \mathcal{E})$ for every integer i with $0 \leq i \leq n-r$. Let i be an integer with $0 \leq i \leq n-1$ and let $L_1, \dots, L_{n-i}$ be ample line bundles on X. By setting $\mathcal{E} := L_1 \oplus \cdots \oplus L_{n-i}$, we see that $g_i(X, \mathcal{E}) = g_i(X, L_1, \dots, L_{n-i})$.

Remark 3.2.

(1) We can prove that $\chi_{p_1,\cdots,p_{n-i}}(L_1, \dots, L_{n-i})$ is an integer for every non-negative integers $p_1, \dots, p_{n-i}$ with $0 \leq p_1 + \cdots + p_{n-i} \leq n$. So in particular we see that $g_i(X, L_1, \dots, L_{n-i})$ is an integer.

(2) If $i = 0$, then $g_0(X, L_1, \dots, L_n) = L_1 \cdots L_n$.

(3) If $i = n-1$, then $g_{n-1}(X, L_1)$ in Definition 3.1 (2) is equal to the $(n-1)$th sectional geometric genus of (X, L_1) in Definition 2.3.

(4) If $i = n$, then $g_n(X) = h^n(\mathcal{O}_X)$.

(5) In [13, Definition 2.1], we proposed more general definition of the ith sectional geometric genus of multi-polarized varieties. For details, see [13].

Here we will see a relationship between the ith sectional geometric genus of multi-polarized varieties in Definition 3.1 (2) and that of polarized varieties in Definition 2.3. By [13, Lemma 2.1] we see that $\chi_{\underbrace{1,\dots,1}_{n-i}}(L,\dots,L) = \chi_{n-i}(L)$. So we get the following:

Proposition 3.1. *Let X be a projective variety of dimension n and let L be a line bundle on X. Let i be an integer with $0 \leq i \leq n$. Then*

$$g_i(X, \underbrace{L,\dots,L}_{n-i}) = g_i(X, L).$$

Proposition 3.1 shows that the ith sectional geometric genus of multi-polarized varieties in Definition 3.1 (2) is more general than that of polarized varieties in Definition 2.3, and the author believe that the ith sectional geometric genus of multi-polarized varieties is much more useful.

The following result makes us possible to calculate the ith sectional geometric genus of several multi-polarized manifolds.

Theorem 3.1. *Let X be a smooth projective variety of dimension n, and let i be an integer with $0 \leq i \leq n-1$. Let $L_1,\dots,L_{n-i}$ be nef and big line bundles on X. Then*

$$\begin{aligned} &g_i(X, L_1,\dots,L_{n-i}) \\ &= \sum_{u=1}^{n-i}\left\{(-1)^{n-i-u}\sum_{(p_1,\cdots,p_{n-i})\in S(n-i)_u} h^0(K_X + p_1L_1 + \cdots + p_{n-i}L_{n-i})\right\} \\ &\quad + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X). \end{aligned}$$

Here

$$S(n-i)_u = \{(p_1,\cdots,p_{n-i}) \mid p_m \in \mathbb{Z},\ 0 \leq p_m \leq 1,\ \sharp\{p_m \mid p_m = 1\} = u\ \}.$$

Proof. See [13, Corollary 2.3]. □

Here we will give notation used in Theorem 3.2 below. Let X be a smooth projective variety of dimension n and let i be an integer with $1 \leq i \leq n-1$. Let $L_1,\dots,L_{n-i}$ be ample line bundles on X. Assume that $\mathrm{Bs}|L_j| = \emptyset$ for every integer j with $1 \leq j \leq n-i$. Then by Bertini's theorem, for every integer j with $1 \leq j \leq n-i$, there exists a general member $X_j \in |L_j|_{X_{j-1}}|$ such that X_j is a smooth projective variety of dimension $n-j$. (Here we set $X_0 := X$.)

Theorem 3.2. *Let n and i be integers with $n \geq 2$ and $1 \leq i \leq n-1$. Let $(X, L_1, \dots, L_{n-i})$ be an n-dimensional multi-polarized manifold of type $(n-i)$.*
(1) *Assume that* $\mathrm{Bs}|L_1| = \emptyset$. *Then*

$$g_i(X, L_1, \dots, L_{n-i}) = g_i(X_1, L_2|_{X_1}, \dots, L_{n-i}|_{X_1}).$$

(Here $X_1 \in |L_1|$ is a smooth member.)
(2) *Assume that* $\mathrm{Bs}|L_j| = \emptyset$ *for every integer j with $1 \leq j \leq n-i$. Then*

$$g_i(X, L_1, \dots, L_{n-i}) = h^i(\mathcal{O}_{X_{n-i}}).$$

Proof. See [13, Theorem 2.3]. □

Finally we will give the following formula about $g_i(X, L_1, \dots, L_{n-i})$ using intersection numbers. This result is very useful to investigate this invariant more deeply.

Theorem 3.3. *Let n and i be integers with $n \geq 2$ and $0 \leq i \leq n-1$. Let $(X, L_1, \dots, L_{n-i})$ be an n-dimensional multi-prepolarized manifold. Then*

$$\begin{aligned} &g_i(X, L_1, \dots, L_{n-i}) \\ &= \begin{cases} L_1 \cdots L_n & \text{if } i = 0, \\ \sum_{k=0}^{i} \left(\sum_{(t_1, \cdots, t_{n-i}) \in S(n-i)^+_{n-k}} \frac{(-1)^k}{(t_1)! \cdots (t_{n-i})!} L_1^{t_1} \cdots L_{n-i}^{t_{n-i}} \right) T_k(X) & \\ \quad + \sum_{j=0}^{i-1} (-1)^{i-j-1} h^j(\mathcal{O}_X) & \text{if } i \neq 0. \end{cases} \end{aligned}$$

Here $S(u)^+_r := \{(t_1, \dots, t_u) \in \mathbb{Z}^{\oplus u} \mid \sum_{j=1}^{u} t_j = r \text{ and } t_k \geq 1 \text{ for any } k\}$, *and $T_k(X)$ is the Todd polynomial of weight k of the tangent bundle $\mathcal{T}_X$ of X.*

Proof. See [13, Theorem 2.4 and Corollary 2.7]. □

For further informations about the sectional geometric genus of multi-polarized varieties, see [13].

§ 4. The case where $i = 1$.

In this section, we will investigate the case where $i = 1$. (Here we always assume that X is smooth.) This case can be regarded as a generalization of the sectional genus of polarized manifolds. So it is expected that $g_1(X, L_1, \dots, L_{n-1})$ has some properties similar to those of sectional genus of polarized manifolds.

First we will provide the following results which will be needed later.

Theorem 4.1.

(1) *Let* $(X, L_1, \dots, L_{n+1})$ *be an* n*-dimensional multi-polarized manifold of type* $n+1$. *Then* $K_X + L_1 + \cdots + L_{n+1}$ *is nef.*

(2) *Let* $(X, L_1, \dots, L_n)$ *be an* n*-dimensional multi-polarized manifold of type* n. *Then* $K_X + L_1 + \cdots + L_n$ *is nef unless*

$$(X, L_1, \dots, L_n) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1)).$$

(3) *Let* $(X, L_1, \dots, L_{n-1})$ *be an* n*-dimensional multi-polarized manifold of type* $n-1$ *with* $n \geq 2$. *If* $K_X + L_1 + L_2 + \cdots + L_{n-1}$ *is not nef, then there exists* $\sigma \in \mathfrak{S}_{n-1}$ *such that* $(X, L_{\sigma(1)}, L_{\sigma(2)}, \dots, L_{\sigma(n-1)})$ *is one of the following: (Here* $\mathfrak{S}_{n-1}$ *denotes the symmetric group of degree* $n-1$*.)*

 (A) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$.

 (B) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2), \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$.

 (C) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1), \mathcal{O}_{\mathbb{Q}^n}(1), \dots, \mathcal{O}_{\mathbb{Q}^n}(1))$.

 (D) X *is a* $\mathbb{P}^{n-1}$*-bundle over a smooth projective curve* B *and* $L_j|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ *for any fiber* F *and every integer* j *with* $1 \leq j \leq n-1$.

Proof. See [14, Theorem 5.1.1 and Theorem 5.2.1]. Here we note that this is a direct consequence of [19, Theorems 1, 2 and 3]. □

By Theorem 3.3 we can get the following formula using intersection numbers.

Proposition 4.1. *Let* X *be a smooth projective variety of dimension* n, *and let* $L_1, \dots, L_{n-1}$ *be line bunldes on* X. *Then*

$$g_1(X, L_1, \dots, L_{n-1}) = 1 + \frac{1}{2}\left(K_X + \sum_{j=1}^{n-1} L_j\right) L_1 \cdots L_{n-1}.$$

By using Proposition 4.1 we can prove the non-negativity of $g_1(X, L_1, \dots, L_{n-1})$ and get a classification of $(X, L_1, \dots, L_{n-1})$ with $g_1(X, L_1, \dots, L_{n-1}) = 0$.

Theorem 4.2. *Let* X *be a smooth projective variety of dimension* $n \geq 2$. *Let* $L_1, \dots, L_{n-1}$ *be ample line bundles on* X. *Then*

(1) $g_1(X, L_1, \dots, L_{n-1}) \geq 0$ *holds.*

(2) *If* $g_1(X, L_1, \dots, L_{n-1}) = 0$, *then* $(X, L_{\sigma(1)}, L_{\sigma(2)}, \dots, L_{\sigma(n-1)})$ *is one of the following: (Here* $\sigma \in \mathfrak{S}_{n-1}$*.)*

 (A) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$.

(B) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2), \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(1))$.

(C) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1), \mathcal{O}_{\mathbb{Q}^n}(1), \dots, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(D) *X is a $\mathbb{P}^{n-1}$-bundle over a projective line $\mathbb{P}^1$ and $L_j|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for any fiber F and j with $1 \leq j \leq n-1$.*

Proof. (See [14, Theorem 6.1.1].) Here we will give a sketch of proof of this theorem.

If $K_X + L_1 + \cdots + L_{n-1}$ is nef, then by Proposition 4.1 we have

$$g_1(X, L_1, \dots, L_{n-1}) \geq 1.$$

So we may assume that $K_X + L_1 + \cdots + L_{n-1}$ is not nef. Then $(X, L_1, \dots, L_{n-1})$ is one of the types in Theorem 4.1 (3). In each case, we can calculate $g_1(X, L_1, \dots, L_{n-1})$ and we get its non-negativity.

Moreover, by the above proof, we see that if $g_1(X, L_1, \dots, L_{n-1}) = 0$, then $K_X + L_1 + \cdots + L_{n-1}$ is not nef. By calculating $g_1(X, L_1, \dots, L_{n-1})$, we can get a classification of $(X, L_1, \dots, L_{n-1})$ with $g_1(X, L_1, \dots, L_{n-1}) = 0$. □

Furthermore we can also prove the following:

Theorem 4.3. *Let X be a smooth projective variety of dimension $n \geq 2$ and let $L_1, \dots, L_{n-1}$ be ample line bundles on X. Assume that $g_1(X, L_1, \dots, L_{n-1}) = 1$. Then $(X, L_1, \dots, L_{n-1})$ is one of the following:*

(1) *$(X, L_1, \dots, L_{n-1})$ satisfies $K_X + L_1 + \cdots + L_{n-1} = \mathcal{O}_X$.*

(2) *X is a $\mathbb{P}^{n-1}$-bundle over an elliptic curve C and $L_j|_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for any integer j with $1 \leq j \leq n-1$, where F is a fiber of the bundle.*

Proof. See [14, Theorem 6.1.2]. □

See the paper [14] for further informations about the case where $i = 1$.

§ 5. Application

In this section we are going to provide one application to the dimension of global sections of adjoint bundles. First of all, we will give the following theorem, which shows the relation between the dimension of global sections of adjoint bundles and the ith sectional geometric genus, and which is a generalization of [11, Theorem 2.1].

Theorem 5.1. *Let X be a smooth projective variety with $\dim X = n \geq 2$, let $L_1, \cdots, L_m$ be nef and big line bundles on X and let L be a nef line bundle, where $m \geq 1$. Then*

$$\begin{aligned}
&h^0(K_X + L_1 + \cdots + L_m + L) - h^0(K_X + L_1 + \cdots + L_m) \\
&= \sum_{s=0}^{n-1} \sum_{(k_1, \cdots, k_{n-s-1}) \in A^m_{n-s-1}} g_s(X, L_{k_1}, \cdots, L_{k_{n-s-1}}, L) \\
&\quad - \sum_{s=0}^{n-2} \binom{m-1}{n-s-2} h^s(\mathcal{O}_X).
\end{aligned}$$

Here $A^p_t := \{(k_1, \cdots, k_t) \mid k_l \in \{1, \cdots, p\}, k_i < k_j \text{ if } i < j\}$, *and we set*

$$\sum_{(k_1, \cdots, k_{n-s-1}) \in A^m_{n-s-1}} g_s(X, L_{k_1}, \cdots, L_{k_{n-s-1}}, L) = \begin{cases} 0 & \text{if } n-s-1 > m, \\ g_{n-1}(X, L) & \text{if } s = n-1. \end{cases}$$

Remark 5.1. In the proof of Theorem 5.1 and Proposition 5.1 we will use the following for convenience: Let X be a projective variety of dimension n and let $H_1, \ldots, H_{n+1}$ be line bundles on X. Then we set $g_{-1}(X, H_1, \ldots, H_{n+1}) = 0$ and $h^{-1}(\mathcal{O}_X) = 0$.

Proof. First we note that we need the following proposition in this proof.

Proposition 5.1. *Let X be a projective variety of dimension n and let i be an integer with $0 \leq i \leq n-1$. Let $A, B, L_1, \cdots, L_{n-i-1}$ be line bundles on X. Then*

$$\begin{aligned}
&g_i(X, A+B, L_1, \cdots, L_{n-i-1}) \\
&= g_i(X, A, L_1, \cdots, L_{n-i-1}) + g_i(X, B, L_1, \cdots, L_{n-i-1}) \\
&\quad + g_{i-1}(X, A, B, L_1, \cdots, L_{n-i-1}) - h^{i-1}(\mathcal{O}_X).
\end{aligned}$$

Proof. See [13, Corollary 2.4 and Remark 2.6]. □

We are going to prove the formula in Theorem 5.1 by induction on m.
Assume that $m = 1$. Here we note that $L_1 + L$ is nef and big. Then by [7, Theorem

2.3] and Proposition 5.1, we see that

$$\begin{aligned}
&h^0(K_X+L_1+L)-h^0(K_X+L_1)\\
&=g_{n-1}(X,L_1+L)+h^n(\mathcal{O}_X)-h^{n-1}(\mathcal{O}_X)\\
&\quad -g_{n-1}(X,L_1)-h^n(\mathcal{O}_X)+h^{n-1}(\mathcal{O}_X)\\
&=g_{n-1}(X,L)+g_{n-2}(X,L_1,L)-h^{n-2}(\mathcal{O}_X)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^1_{n-s-1}} g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L)\\
&\quad -\sum_{s=0}^{n-2}\binom{0}{n-s-2}h^s(\mathcal{O}_X).
\end{aligned}$$

Next we assume that the formula in Theorem 5.1 is true for $m\leq r$. We consider the case where $m=r+1$. Here we set $A:=L_{r+1}+L$. Then A is nef and big. By assumption

$$\begin{aligned}
&h^0(K_X+L_1+\cdots+L_r+L_{r+1}+L)-h^0(K_X+L_1+\cdots+L_r)\\
&=h^0(K_X+L_1+\cdots+L_r+A)-h^0(K_X+L_1+\cdots+L_r)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^r_{n-s-1}} g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},A)\\
&\quad -\sum_{s=0}^{n-2}\binom{r-1}{n-s-2}h^s(\mathcal{O}_X).
\end{aligned}\tag{5.1}$$

Moreover by assumption we have

$$\begin{aligned}
&h^0(K_X+L_1+\cdots+L_r+L_{r+1})-h^0(K_X+L_1+\cdots+L_r)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^r_{n-s-1}} g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L_{r+1})\\
&\quad -\sum_{s=0}^{n-2}\binom{r-1}{n-s-2}h^s(\mathcal{O}_X).
\end{aligned}\tag{5.2}$$

By (5.1), (5.2) and Proposition 5.1, we have

$$
\begin{aligned}
&h^0(K_X+L_1+\cdots+L_r+L_{r+1}+L)-h^0(K_X+L_1+\cdots+L_r+L_{r+1})\\
&=h^0(K_X+L_1+\cdots+L_r+A)-h^0(K_X+L_1+\cdots+L_r)\\
&\quad-\bigl(h^0(K_X+L_1+\cdots+L_r+L_{r+1})-h^0(K_X+L_1+\cdots+L_r)\bigr)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^r_{n-s-1}}\bigl(g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},A)\\
&\qquad\qquad-g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L_{r+1})\bigr)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^r_{n-s-1}}\bigl(g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L)\\
&\qquad\qquad+g_{s-1}(X,L_{k_1},\cdots,L_{k_{n-s-1}},L_{r+1},L)-h^{s-1}(\mathcal{O}_X)\bigr)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^{r+1}_{n-s-1}}g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L)-\sum_{s=1}^{n-1}\binom{r}{n-s-1}h^{s-1}(\mathcal{O}_X)\\
&=\sum_{s=0}^{n-1}\sum_{(k_1,\cdots,k_{n-s-1})\in A^{r+1}_{n-s-1}}g_s(X,L_{k_1},\cdots,L_{k_{n-s-1}},L)-\sum_{s=0}^{n-2}\binom{r}{n-s-2}h^{s}(\mathcal{O}_X).
\end{aligned}
$$

Therefore we get the assertion. □

Here we propose the following conjecture which is a generalization of a conjecture by Beltrametti and Sommese (see [2, Conjecture 7.2.7]).

Conjecture 5.1. *Let $(X, L_1, \ldots, L_{n-1})$ be a multi-polarized manifold of type $n-1$ with $\dim X = n$. Assume that $K_X + L_1 + \cdots + L_{n-1}$ is nef. Then $h^0(K_X + L_1 + \cdots + L_{n-1}) > 0$.*

If $n = 2$, then this conjecture is true (see [2, Theorem 7.2.6]). By using Theorem 5.1, we can prove this conjecture for $\dim X = 3$.

Theorem 5.2. *Let X be a smooth projective 3-fold. Let L_1 and L_2 be ample line bundles on X. Assume that $K_X + L_1 + L_2$ is nef. Then $h^0(K_X + L_1 + L_2) > 0$.*

Proof. By Theorem 5.1 we have

$$h^0(K_X + L_1 + L_2) - h^0(K_X + L_2) = g_2(X, L_1) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X).$$

Here we use the following theorem:

Theorem 5.3. *Let (X, L) be a polarized manifold of dimension 3. Assume that $\kappa(K_X + L) \geq 0$. Then $g_2(X, L) \geq h^1(\mathcal{O}_X)$.*

Proof. See [10, Theorem 3.2.1 and Theorem 3.3.1 (2)]. □

Assume that $\kappa(K_X + L_1) \geq 0$. Then by Theorem 5.3, we see that $g_2(X, L_1) \geq h^1(\mathcal{O}_X)$. Hence $h^0(K_X + L_1 + L_2) \geq g_1(X, L_1, L_2)$.

On the other hand since $\kappa(K_X + L_1) \geq 0$ and $g_1(X, L_1, L_2) \in \mathbb{Z}$, we have

$$g_1(X, L_1, L_2) = 1 + \frac{1}{2}(K_X + L_1 + L_2)L_1L_2 \geq 2.$$

Therefore $h^0(K_X + L_1 + L_2) \geq 2$.

Assume that $\kappa(K_X + L_1) = -\infty$. Then in particular $\kappa(X) = -\infty$. Here we note that $g_1(X, L_1, L_2) \geq 1$ because $K_X + L_1 + L_2$ is nef. Furthermore $g_2(X, L_1) \geq h^2(\mathcal{O}_X) \geq 0$ by [9, Corollary 2.4].

If $h^1(\mathcal{O}_X) = 0$, then

$$\begin{aligned} & h^0(K_X + L_1 + L_2) - h^0(K_X + L_2) \\ & = g_2(X, L_1) + g_1(X, L_1, L_2) - h^1(\mathcal{O}_X) \\ & \geq 1. \end{aligned}$$

Therefore we may assume that $h^1(\mathcal{O}_X) > 0$. Then we take the Albanese map of X $\alpha : X \to \mathrm{Alb}(X)$. By taking the Stein factorization, if necessary, we may assume that there exists a surjective morphism $\beta : X \to W$ with connected fibers, where W is a normal projective variety. Let F be a general fiber of β. Then $\dim F = 1$ or 2 and $\kappa(F) = -\infty$.

Assume that $\dim F = 2$. Then by Proposition 5.1 we obtain

$$\begin{aligned} h^0(K_F + L_1|_F + L_2|_F) &= g(F, L_1|_F + L_2|_F) - h^1(\mathcal{O}_F) \\ &= g(F, L_1|_F) + g(F, L_2|_F) + (L_1|_F)(L_2|_F) - 1 - h^1(\mathcal{O}_F). \end{aligned}$$

Since $\kappa(F) = -\infty$, we have $g(F, L_j|_F) \geq h^1(\mathcal{O}_F) \geq 0$ for $j = 1, 2$ by [5, Theorem 2.1]. Hence we see that if $h^0(K_F + L_1|_F + L_2|_F) = 0$, then $g(F, L_1|_F) = 0$, $g(F, L_2|_F) = 0$ and $(L_1|_F)(L_2|_F) = 1$. So we get $(L_j|_F)^2 = 1$, $h^1(\mathcal{O}_F) = 0$ and $(F, L_j|_F) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ for $j = 1, 2$ by [18, 2.3 Corollary]. Hence $K_F + L_1|_F + L_2|_F$ is not nef and this is a contradiction because $K_X + L_1 + L_2$ is nef. Therefore $h^0(K_F + L_1|_F + L_2|_F) > 0$ and by [3, Lemma 4.1] we get $h^0(K_X + L_1 + L_2) > 0$.

Assume that $\dim F = 1$. Then $F \cong \mathbb{P}^1$ and $g(F) = 0$. Hence by the Riemann-Roch theorem

$$\begin{aligned} h^0(K_F + L_1|_F + L_2|_F) &= \chi(\mathcal{O}_F) + \deg(K_F + L_1|_F + L_2|_F) \\ &= \deg(K_F + L_1|_F + L_2|_F) + 1 \\ &\geq 1 \end{aligned}$$

because $K_F+L_1|_F+L_2|_F$ is nef. Therefore by [3, Lemma 4.1] we get $h^0(K_X+L_1+L_2)>0$. This completes the proof of Theorem 5.2. □

By Theorem 4.1 (3) and Theorem 5.2 we can prove the following.

Corollary 5.1. *Let (X, L_1, L_2) be a 3-dimensional multi-polarized manifold of type 2. Then the following are equivalent one another.*

(1) $h^0(K_X+L_1+L_2)=0$.

(2) $\kappa(K_X+L_1+L_2)=-\infty$.

(3) $(X, L_{\sigma(1)}, L_{\sigma(2)})$ *is one of the following: (Here σ is an element of the symmetric group of degree 2.)*

- (3.1) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(1))$.
- (3.2) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2), \mathcal{O}_{\mathbb{P}^3}(1))$.
- (3.3) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1), \mathcal{O}_{\mathbb{Q}^3}(1))$.
- (3.4) *X is a $\mathbb{P}^2$-bundle over a smooth projective curve B and $L_j|_F=\mathcal{O}_{\mathbb{P}^2}(1)$ for any fiber F and every integer j with $1\leq j\leq 2$.*

Remark 5.2. Here we give some comments about Conjecture 5.1. Let X be a smooth projective variety of dimension $n\geq 2$ and let $\mathcal{E}$ be an ample vector bundle of rank $n-1$ on X. Then we propose the following problem related to Conjecture 5.1:

Problem 5.1.
Assume that $K_X+\det\mathcal{E}$ is nef. Then is $h^0(K_X+\det\mathcal{E})$ positive ?

If $n=2$, then the answer is positive. If $n=3$ and $\mathcal{E}=L_1\oplus L_2$ for ample line bundles L_1 and L_2, then the answer is positive by Theorem 5.2. But at present we don't have the answer in general. On the other hand, in [1] and [16], Ambro and Kawamata proposed the following conjecture which is a simple case of the original one ([16, Conjecture 2.1]):

Conjecture 5.2. *Let (X, L) be a polarized manifold. Assume that K_X+L is nef. Then $h^0(K_X+L)>0$ holds.*

Of course if Conjecture 5.2 is true, then the answer of Problem 5.1 is positive. But we don't know whether Conjecture 5.2 is true or not for $n\geq 3$.

Finally we consider the following problem related to Conjecture 5.2 (see [12, Problem 3.2]).

Problem 5.2. *For any fixed positive integer n, we set*

$$\mathcal{P}_n := \{\ (X, L) : \text{polarized manifold} \mid \dim X = n \text{ and } \kappa(K_X + L) \geq 0\},$$
$$\mathcal{M}_n := \{\ r \in \mathbb{N} \mid h^0(r(K_X + L)) > 0 \text{ for any } (X, L) \in \mathcal{P}_n\},$$
$$m(n) := \begin{cases} \min \mathcal{M}_n & \text{if } \mathcal{M}_n \neq \emptyset, \\ \infty & \text{if } \mathcal{M}_n = \emptyset. \end{cases}$$

Then determine $m(n)$.

Remark 5.3. By [12, Theorem 2.8], we have $m(1) = 1$ and $m(2) = 1$.

For the case where $\dim X = 3$, we obtain the following result.

Theorem 5.4. *Let (X, L) be a polarized manifold of dimension* 3.

(1) *If $0 \leq \kappa(K_X + L) \leq 2$, then $h^0(K_X + L) > 0$.*

(2) *If $\kappa(K_X + L) = 3$, then $h^0(2(K_X + L)) \geq 3$.*

Proof. For a proof of (1), see [15]. Here we give a proof of (2). Assume that $\kappa(K_X + L) = 3$. By taking a reduction of (X, L), we may assume that $K_X + L$ is nef. (For the definition of a reduction of (X, L), see e.g. [7, Definition 1.1 (3)].) By Theorem 5.1 we see that

$$\begin{aligned} h^0(K_X + (K_X + L) + L) - h^0(K_X + (K_X + L)) \\ = g_2(X, L) + g_1(X, K_X + L, L) - h^1(\mathcal{O}_X). \end{aligned} \tag{5.3}$$

By Proposition 4.1 we see that

$$g_1(X, K_X + L, L) = 1 + (K_X + L)^2 L. \tag{5.4}$$

Since $K_X + L$ is nef and big, we have $(K_X + L)^3 > 0$, $(K_X + L)^2 L > 0$ and $(K_X + L)L^2 > 0$. We also note that

$$g_2(X, L) \geq h^1(\mathcal{O}_X) \tag{5.5}$$

by Theorem 5.3.

Assume that $(K_X + L)^2 L = 1$. Then by [2, Proposition 2.5.1] we have $(K_X + L)L^2 = 1$ and $L^3 = 1$. Therefore $g(L) = 2$. By a classification of polarized manifolds whose sectional genus is equal to two ([4]), we see that (X, L) satisfies the following: $\mathcal{O}_X(K_X) = \mathcal{O}_X$, $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ and $h^0(L) > 0$. Hence by (5.3), (5.4) and (5.5) we get

$$\begin{aligned}
&h^0(2K_X+2L)\\
&= h^0(2K_X+L)+g_2(X,L)-h^1(\mathcal{O}_X)+g_1(X,K_X+L,L)\\
&\geq h^0(L)+g_1(X,K_X+L,L)\\
&\geq 3.
\end{aligned}$$

Assume that $(K_X+L)^2L \geq 2$. Then by (5.3), (5.4) and (5.5) we have $h^0(2K_X+2L) \geq 3$. So we get the assertion. □

By Theorem 5.4 we obtain the following.

Corollary 5.2. $m(3) \leq 2$.

Remark 5.4. Let (X, L) be a polarized 3-fold.

(1) Assume that $\kappa(K_X+L)=3$. Then we can get the following: (For details, see the forthcoming paper [15].)

 (1.1) A lower bound for $h^0(m(K_X+L))$ for every integer $m \geq 2$.

 (1.2) A type of (X, L) with $h^0(2(K_X+L)) = 3$ or 4.

(2) Assume that $0 \leq \kappa(K_X+L) \leq 2$. Then we can also get a lower bound for $h^0(m(K_X+L))$ for every integer $m \geq 1$. (For details, see [15].)

References

[1] F. Ambro, Ladders on Fano varieties, *Algebraic geometry*, **9**, *J. Math. Sci.*, **94** (1999), 1126–1135.

[2] M. C. Beltrametti and A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties,* de Gruyter Expositions in Math. **16**, Walter de Gruyter, Berlin, NewYork, 1995.

[3] J. A. Chen and C. D. Hacon, Linear series of irregular varieties, *Algebraic Geometry in East Asia* (Kyoto 2001), 143–153, World Sci. Publishing, River Edge, NJ, 2002.

[4] T. Fujita, Classification of polarized manifolds of sectional genus two, *the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata* (1987), 73–98.

[5] Y. Fukuma, A lower bound for the sectional genus of quasi-polarized surfaces, *Geom. Dedicata* **64** (1997), 229–251.

[6] Y. Fukuma, On the c_r-sectional geometric genus of generalized polarized manifolds, *Japan. J. Math.* **29** (2003), 335–355.

[7] Y. Fukuma, On the sectional geometric genus of quasi-polarized varieties, I, *Comm. Algebra* **32** (2004), 1069–1100.

[8] Y. Fukuma, Problems on the second sectional invariants of polarized manifolds, *Mem. Fac. Sci. Kochi Univ. Ser.A Math.* **25** (2004), 55–64.
[9] Y. Fukuma, A lower bound for the second sectional geometric genus of polarized manifolds, *Adv. Geom.* **5** (2005), 431–454.
[10] Y. Fukuma, On the second sectional H-arithmetic genus of polarized manifolds, *Math. Z.* **250** (2005), 573–597.
[11] Y. Fukuma, On a conjecture of Beltrametti-Sommese for polarized 3-folds, *Internat. J. Math.* **17** (2006), 761-789.
[12] Y. Fukuma, On the dimension of global sections of adjoint bundles for polarized 3-folds and 4-folds, *J. Pure Appl. Algebra* **211** (2007), 609–621.
[13] Y. Fukuma, Invariants of ample line bundles on projective varieties and their applications, I, *Kodai Math. J.* **31** (2008), 219–256.
[14] Y. Fukuma, Invariants of ample line bundles on projective varieties and their applications, II, preprint.
[15] Y. Fukuma, Effective non-vanishing of global sections of multiple adjoint bundles for polarized 3-folds, in preparation.
[16] Y. Kawamata, On effective non-vanishing and base-point-freeness, *Asian J. Math.* **4** (2000), 173–182.
[17] S. L. Kleiman, Toward a numerical theory of ampleness, *Ann. of Math.* **84** (1966), 293–344.
[18] A. Lanteri and M. Palleschi, About the adjunction process for polarized algebraic surfaces, *J. Reine Angew. Math.* **352** (1984), 15–23.
[19] Y. G. Ye and Q. Zhang, On ample vector bundles whose adjoint bundles are not numerically effective, *Duke Math. J.* **60**, (1990) 671–687.