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Difference algebra associated to the q-Painlevé equation of type $A_7^{(1)}$

By

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Abstract

In this article we will see the notion of decomposable extension and a property of solutions of q-Painlevé eqation of type $A_7^{(1)}$ as its example. We also show that difference fields are completely different from differential ones.

§ 1. Introduction

In his [7] the author defined the decomposable extension, a sort of difference extensions, and studied a property of solutions of q-Painlevé equation of type $A_7^{(1)}$. In this paper we introduce the decomposable extension.

The notations on difference algebra are refered to Cohn [2]. A difference field $\mathcal{K} = (K, \tau)$ is a pair of a field K and an isomorphism τ of K into K. A difference field $\mathcal{K}' = (K', \tau')$ is a difference overfield of a difference field $\mathcal{K} = (K, \tau)$ if $K' \supset K$ and $\tau'|_K = \tau$.

The following is the definition of the decomposable extension, a difference analogue of K. Nishioka's in [5].

Definition 1.1. Let $\mathcal{U}$ be a difference field, $\mathcal{L}/\mathcal{K}$ be a difference field extension in $\mathcal{U}$ of finite transcendence degree and $n = \mathrm{tr.deg}\, L/K$. We define $\mathcal{U}$-decomposable extensions inductively.

1. If $n = 0$ or 1 then $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable.

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2. When $n > 1$, $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable if there exists a difference overfield $\mathcal{E} \subset \mathcal{U}$ of $\mathcal{K}$ such that $\operatorname{tr.deg} E/K < \infty$, E is free from L over K and there exists a difference intermediate field $\mathcal{M}$ of $\mathcal{LE}/\mathcal{E}$ such that $\operatorname{tr.deg} LE/M \geq 1$, $\operatorname{tr.deg} M/E \geq 1$, $\mathcal{LE}/\mathcal{M}$ is $\mathcal{U}$-decomposable and $\mathcal{M}/\mathcal{E}$ is $\mathcal{U}$-decomposable.

The transcendence degree is related to the order of algebraic difference equations. On the one hand a solution of an algebraic difference equation of order n over a difference field $\mathcal{K}$ is an element of some difference overfield $\mathcal{L}$ of $\mathcal{K}$ satisfying $\operatorname{tr.deg} L/K \leq n$. On the other hand, for a difference field extension $\mathcal{L}/\mathcal{K}$ of $\operatorname{tr.deg} L/K = n$, every element of $\mathcal{L}$ satisfies an algebraic difference equation over $\mathcal{K}$ of order not exceeding n.

An additional requirement $\mathcal{E} = \mathcal{K}$ in the definition let us take a glance at a basic notion of the decomposable extension. In that case a $\mathcal{U}$-decomposable extension $\mathcal{L}/\mathcal{K}$ of $\operatorname{tr.deg} L/K \geq 2$ is divisible by some difference intermediate field $\mathcal{M}$ of $\mathcal{L}/\mathcal{K}$ into two $\mathcal{U}$-decomposable extensions $\mathcal{L}/\mathcal{M}$ and $\mathcal{M}/\mathcal{K}$ of positive transcendence degree. Repeating this operation we obtain a chain of difference field extensions

$$\mathcal{K} = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_n = \mathcal{L},$$

where $\operatorname{tr.deg} N_i/N_{i-1} = 1$ for all i $(1 \leq i \leq n)$. Let f be a transcendence basis of N_i/N_{i-1}. Then f satisfies some algebraic difference equation of order 1, and $\mathcal{N}_i$ is a algebraic overfield of $\mathcal{N}_{i-1}\langle f\rangle = \mathcal{N}_{i-1}(f, \tau f, \tau^2 f, \dots)$, where τ is the operator of $\mathcal{L}$. Hence the extension $\mathcal{L}/\mathcal{K}$ seems to be an extension constructed of solutions of algebraic difference equations of order 1. The extension $\mathcal{E}/\mathcal{K}$ extends the set from which we choose the coefficients of the algebraic difference equations.

§ 2. Properties and examples of decomposable extensions

In this section we introduce some properties and examples of decomposable extensions. We will mention some relations between decomposable extensions and strongly normal extensions.

A solution of algebraic difference equations over a difference field $\mathcal{K}$ is an element of some difference overfield $\mathcal{L}$ of $\mathcal{K}$ satisfying the equations.

Example 2.1. Let f be a solution of an algebraic difference equation of order 1 over a difference field $\mathcal{K}$. Then $\mathcal{K}\langle f\rangle/\mathcal{K}$ is $\mathcal{K}\langle f\rangle$-decomposable because $\operatorname{tr.deg} K\langle f\rangle/K \leq 1$. Note that the difference Riccati equations over $\mathcal{K}$ are algebraic difference equations of order 1 over $\mathcal{K}$.

Lemma 2.2. *Let $\mathcal{U}$ be a difference field and $\mathcal{L}/\mathcal{K}$ a $\mathcal{U}$-decomposable extension. For any difference overfield $\mathcal{U}'$ of $\mathcal{U}$ the extension $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}'$-decomposable.*

Proof. We inductively prove this. Let $n = \mathrm{tr.deg}\, L/K$. When $n = 0$ or 1 we find the extension $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}'$-decomposable by definition. Let $n \geq 2$. There exists a difference overfield $\mathcal{E} \subset \mathcal{U} \subset \mathcal{U}'$ of $\mathcal{K}$ such that $\mathrm{tr.deg}\, E/K < \infty$, E is free from L over K and there exists a difference intermediate field $\mathcal{M}$ of $\mathcal{LE}/\mathcal{E}$ such that $\mathrm{tr.deg}\, LE/M \geq 1$, $\mathrm{tr.deg}\, M/E \geq 1$, $\mathcal{LE}/\mathcal{M}$ is $\mathcal{U}$-decomposable and $\mathcal{M}/\mathcal{E}$ is $\mathcal{U}$-decomposable. From the induction hypothesis we find that the extensions $\mathcal{LE}/\mathcal{M}$ and $\mathcal{M}/\mathcal{E}$ are both $\mathcal{U}'$-decomposable. Hence we conclude the extension $\mathcal{L}/\mathcal{K}$ to be $\mathcal{U}'$-decomposable. □

Lemma 2.3. *Let $\mathcal{U}$ be a difference field, $\mathcal{N}/\mathcal{K}$ a difference field extension in $\mathcal{U}$, and $\mathcal{L}$ an difference intermediate field of $\mathcal{N}/\mathcal{K}$. If the extensions $\mathcal{N}/\mathcal{L}$ and $\mathcal{L}/\mathcal{K}$ are $\mathcal{U}$-decomposable then $\mathcal{N}/\mathcal{K}$ is $\mathcal{U}$-decomposable.*

Proof. This is also proved by induction. □

The Galois theory of differential fields was originated and developed by Kolchin ([4]). In his [1] Bialynicki-Birula defined a strongly normal extension, by which he extended the Kolchin's Galois theory to the Galois theory for fields with operators, where a field with operators means a field together with a family of "automorphisms" and derivations of the field. We introduce the definition of strongly normal extensions with one operator.

Definition 2.4. Let $\mathcal{K} = (K, \tau_K)$ be a difference field whose operator τ_K is an automorphism of K and $\mathcal{L} = (L, \tau_L)$ a difference overfield of $\mathcal{K}$ whose operator τ_L is an automorphism of L. Then we say that $\mathcal{L}$ is a strongly normal extension of $\mathcal{K}$ if

1. The field L is a regular extension of the field K
2. The field L is finitely generated over the field K
3. $C_{\mathcal{L}} = C_{\mathcal{K}}$ and $C_{\mathcal{K}}$ is algebraically closed
4. $\langle \mathcal{L} \otimes_K \mathcal{L} \rangle = (\mathcal{L} \otimes_K 1) C_{\langle \mathcal{L} \otimes_K \mathcal{L} \rangle}$,

where $C_{\mathcal{K}} = \{a \in K | \tau_K a = a\}$ is the subfield of invariants and $\langle\ \rangle$ denotes the quotient field.

This type of strongly normal extension $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable for some difference field extension $\mathcal{U}$ of $\mathcal{L}$. We need several lemmas to prove it.

Let (K, τ) and (K', τ') be difference fields. A mapping ϕ is a difference isomorphism of (K, τ) into (onto) (K', τ') if ϕ is an isomorphism of K into (onto) K' and $\phi\tau = \tau'\phi$.

Lemma 2.5. *Let $\mathcal{U}$ and $\mathcal{V}$ be difference fields, $\mathcal{L}/\mathcal{K}$ and $\mathcal{N}/\mathcal{J}$ difference field extensions in $\mathcal{U}$ and $\mathcal{V}$ respectively, and $\phi\colon \mathcal{U} \to \mathcal{V}$ difference isomorphism of $\mathcal{U}$ into $\mathcal{V}$ satisfying $\phi(\mathcal{L}) = \mathcal{N}$ and $\phi(\mathcal{K}) = \mathcal{J}$. If the extension $\mathcal{L}/\mathcal{K}$ is $\mathcal{U}$-decomposable then $\mathcal{N}/\mathcal{J}$ is $\mathcal{V}$-decomposable.*

Proof. This is straightforward. □

We need the following

Lemma 2.6 (Corollary 1 in [1]). *Let $\mathcal{L}/\mathcal{K}$ be a difference field extension such that the operators of $\mathcal{L}$ and $\mathcal{K}$ are surjective and $\mathcal{L} = \mathcal{K}\mathcal{C}_{\mathcal{L}}$. If L is finitely generated over K as field then $C_{\mathcal{L}}$ is finitely generated over $C_{\mathcal{K}}$.*

Proposition 2.7. *Any strongly normal extension $\mathcal{L}/\mathcal{K}$ with* $\operatorname{tr.deg} L/K \geq 2$ *is $\mathcal{U}$-decomposable for some difference overfield $\mathcal{U}$ of $\mathcal{L}$ such that $\mathcal{U}$ and $\langle \mathcal{L} \otimes_K \mathcal{L} \rangle$ are isomorphic as difference field by an extension of the naturally defined difference isomorphism of $\mathcal{L}$ onto $1 \otimes_K \mathcal{L}$.*

Proof. Put $\mathcal{L}_1 = \mathcal{L} \otimes_K 1$ and $\mathcal{L}_2 = 1 \otimes_K \mathcal{L}$. The fields L_1 and L_2 are linearly disjoint over K, so they are free over K. By definition L_2/K is finitely generated as field, which implies the extension L_1L_2/L_1 is also finitely generated as field. Since we have $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_1 C_{\mathcal{L}_1\mathcal{L}_2}$, we obtain by using Lemma 2.6 that the extension $C_{\mathcal{L}_1\mathcal{L}_2}/C_{\mathcal{L}_1}$ is finitely generated.

Let $C_{\mathcal{L}_1\mathcal{L}_2} = C_{\mathcal{L}_1}(x_1, \ldots, x_k)$ and $\mathcal{N}_i = \mathcal{L}_1(x_1, \ldots, x_i)$ for all i $(0 \leq i \leq k)$. We find that

$$\mathcal{L}_1 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_k = \mathcal{L}_1 C_{\mathcal{L}_1\mathcal{L}_2} = \mathcal{L}_1\mathcal{L}_2$$

is a finite chain of difference field extensions. Since $\operatorname{tr.deg} N_i/N_{i-1} \leq 1$ for all i $(1 \leq i \leq k)$, there exists an integer i_0 such that $\operatorname{tr.deg} L_1L_2/N_{i_0} \geq 1$ and $\operatorname{tr.deg} N_{i_0}/L_1 \geq 1$. From the definition of decomposable extensions and Lemma 2.3 we find the extensions $\mathcal{L}_1\mathcal{L}_2/\mathcal{N}_{i_0}$ and $\mathcal{N}_{i_0}/\mathcal{L}_1$ are $\mathcal{L}_1\mathcal{L}_2$-decomposable, which implies the extension $\mathcal{L}_2/\mathcal{K}$ is $\mathcal{L}_1\mathcal{L}_2$-decomposable.

Let $\phi\colon \mathcal{L}_2 \xrightarrow{\sim} \mathcal{L}$ be the naturally defined difference isomorphism and $\{a_1, a_2, \ldots, a_l\}$ be a transcendence basis of L_1/K. Since L_1 and L_2 are free over K, a_i $(1 \leq i \leq l)$ are algebraically independent over L_2. Choose $b_1, b_2, \ldots, b_l$ to be algebraically independent over L. We extend the surjective isomorphism $\phi\colon L_2 \xrightarrow{\sim} L$ to a surjective isomorphism $\phi_1\colon L_2(a_1, \ldots, a_l) \xrightarrow{\sim} L(b_1, \ldots, b_l)$ sending a_i to b_i. Then we extend ϕ_1 to a surjective isomorphism

$$\overline{\phi_1}\colon \overline{L_1L_2} = \overline{L_2(a_1, \ldots, a_l)} \xrightarrow{\sim} \overline{L(b_1, \ldots, b_l)},$$

where overlined fields are algebraic closures. A restricted mapping $\tilde{\phi} = \overline{\phi_1}|_{L_1L_2}$ is an isomorphism of L_1L_2 into $\overline{L(b_1, \ldots, b_l)}$ and an extension of ϕ.

Let τ be the operator of the difference field $\mathcal{L}_1\mathcal{L}_2$. We define an operator τ' of $\tilde{\phi}(L_1L_2)$ as $\tau' = \tilde{\phi} \circ \tau \circ \tilde{\phi}^{-1}$. In fact τ' is an isomorphism of $\tilde{\phi}(L_1L_2)$ into $\tilde{\phi}(L_1L_2)$ because $\tilde{\phi}$, τ and $\tilde{\phi}^{-1}$ are injective homomorphisms. Then $\tilde{\phi}$ is a difference isomorphism of $\mathcal{L}_1\mathcal{L}_2 = (L_1L_2, \tau)$ onto $(\tilde{\phi}(L_1L_2), \tau')$.

We will see $\tau'|_L = \tau_L$, where τ_L is the original operator of the difference field $\mathcal{L}$. Put $\tau_2 = \tau|_{L_2}$ for convenience. Since the map ϕ is a difference isomorphism, we have $\phi\tau_2 = \tau_L\phi$. Hence for any $x \in L$ we find

$$\begin{aligned}\tau' x &= \tilde{\phi} \circ \tau \circ \tilde{\phi}^{-1}(x) = \tilde{\phi} \circ \tau(\phi^{-1}(x)) \\ &= \tilde{\phi}(\tau_2 \circ \phi^{-1}(x)) = \phi \circ \tau_2 \circ \phi^{-1}(x) \\ &= \tau_L x,\end{aligned}$$

which means $\tau'|_L = \tau_L$. Therefore $(\tilde{\phi}(L_1L_2), \tau')$ is a difference overfield of $\mathcal{L} = (L, \tau_L)$. By Lemma 2.5 we obtain that the strongly normal extension $\mathcal{L}/\mathcal{K}$ is $(\tilde{\phi}(L_1L_2), \tau')$-decomposable. □

Corollary 2.8. *Let $\mathcal{L}/\mathcal{K}$ be a strongly normal extension of* $\operatorname{tr.deg} L/K \geq 2$ *and $\mathcal{U}$ a difference overfield of $\mathcal{L}$ as in Proposition 2.7. Then the difference field extensions $\mathcal{U}/\mathcal{L}$ and $\mathcal{U}/\mathcal{K}$ are $\mathcal{U}$-decomposable.*

Proof. Put $\mathcal{L}_1 = \mathcal{L} \otimes_K 1$ and $\mathcal{L}_2 = 1 \otimes_K \mathcal{L}$. By $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_1 C_{\mathcal{L}_1\mathcal{L}_2}$ and a surjective difference isomorphism $\mathcal{L} \otimes_K \mathcal{L} \xrightarrow{\sim} \mathcal{L} \otimes_K \mathcal{L}$ sending $x \otimes y$ to $y \otimes x$ we obtain $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2 C_{\mathcal{L}_1\mathcal{L}_2}$. From Lemma 2.6 we find that the difference field extension $C_{\mathcal{L}_1\mathcal{L}_2}/C_{\mathcal{L}_2}$ is finitely generated.

Put $C_{\mathcal{L}_1\mathcal{L}_2} = C_{\mathcal{L}_2}(x_1, \ldots, x_n)$ and $\mathcal{N}_i = \mathcal{L}_2(x_1, \ldots, x_i)$ for all i $(0 \leq i \leq n)$. We find that

$$\mathcal{L}_2 = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_n = \mathcal{L}_2 C_{\mathcal{L}_1\mathcal{L}_2} = \mathcal{L}_1\mathcal{L}_2$$

is a finite chain of difference field extensions. Since $\operatorname{tr.deg} N_i/N_{i-1} \leq 1$ for all i $(1 \leq i \leq n)$ the extensions $\mathcal{N}_i/\mathcal{N}_{i-1}$ $(1 \leq i \leq n)$ are all $\mathcal{L}_1\mathcal{L}_2$-decomposable, and so $\mathcal{L}_1\mathcal{L}_2/\mathcal{L}_2$ is also $\mathcal{L}_1\mathcal{L}_2$-decomposable. By Lemma 2.5 we obtain $\mathcal{U}/\mathcal{L}$ is $\mathcal{U}$-decomposable, which implies $\mathcal{U}/\mathcal{K}$ is $\mathcal{U}$-decomposable. □

A strongly normal differential field extension is a decomposable differential field extension in the sense defined in [5], moreover a chain of strongly normal differential field extensions and algebraic ones are decomposable by grace of the universal differential field extension defined by Kolchin in [4]. However we do not have such a useful "universal" difference field extension (refer to Section 4 Appendix), instead we introduce a way of constructing somewhat similar decomposable chains.

Although the operator of a difference field is not always an automorphism, the following shows some kind of algebraically closed difference field has an automorphism operator.

Lemma 2.9. *Let $\mathcal{K} = (K, \tau_K)$ be a difference field whose operator τ_K is an automorphism of K and $\mathcal{L}$ a difference overfield of $\mathcal{K}$ such that* $\operatorname{tr.deg} L/K < \infty$. *Then algebraic closure $\overline{L}$ of L is a difference overfield of $\mathcal{L}$ with some automorphism of $\overline{L}$.*

Proof. From the theorem of Steinitz. □

Example 2.10. If a chain of difference field extensions $\mathcal{K} = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_n$ satisfies one of the following for each i $(1 \leq i \leq n)$, then $\mathcal{N}_n/\mathcal{K}$ is $\mathcal{N}_n$-decomposable.

1. $\operatorname{tr.deg} N_i/N_{i-1} \leq 1$.

2. $\mathcal{N}_i$ is the difference overfield $\mathcal{U}$ in Proposition 2.7 for some strongly normal extension $\mathcal{L}$ of $\mathcal{N}_{i-1}$ such that $\operatorname{tr.deg} L/N_{i-1} \geq 2$.

Strongly normal extensions are defined for difference fields whose operator is an automorphism. If the operator of $\mathcal{K}$ is an automorphism, for Lemma 2.9, we may take an algebraic closure of N_i to make the operator surjective.

§ 3. q-Painlevé equation of type $A_7^{(1)}$

In [7] the author studied a property of solutions of q-Painlevé equation of type $A_7^{(1)}$,

$$q\text{-}P(A_7): \quad \overline{f} f^2 \underline{f} = t(1-f),$$

where $\overline{f} = f(qt)$ and $\underline{f} = f(t/q)$, and proved that q-$P(A_7)$ has no solution in any decomposable extension of $\mathbb{C}(t)$ if $q \in \mathbb{C}^\times$ is not a root of unity. In this section we sketch the proof.

q-Painlevé equations are q-defference equations which are discrete analogs of the Painlevé equations. Grammaticos, Ramani and Papageorgiou presented in their [3] a notion called singularity confinement, by which they obtained an integrability criterion for discrete-time systems that is a discrete counter part of the Painlevé property for systems of a continuous variable. Ramani, Grammaticos and Hietarinta made several discrete Painlevé equations using the method of singularity confinement (see [9]). q-$P(A_7)$ appears in the paper of Ramani and Grammaticos ([10]).

In his [11] Sakai introduced a geometric approach to theory of the Painlevé equations, and showed both classifications of Painlevé equations and discrete Painlevé equations by rational surfaces. The notation q-$P(A_7)$ is determined by the type $A_7^{(1)}$ of the rational surface of the equation. The list of discrete Painlevé equations and their notations can be seen in the paper of H. Sakai ([12]). In addition q-$P(A_7)$ has symmetry $A_1^{(1)}$.

For the beginning of the proof we prove the following Lemma independent of equations.

Lemma 3.1. *Let $\mathcal{K}$ be a difference field, $\mathcal{U}$ a difference overfield of $\mathcal{K}$, $\mathcal{D}/\mathcal{K}$ a $\mathcal{U}$-decomposable extension and $f \in D$. Suppose f satisfies the following; for any difference overfield $\mathcal{L} \subset \mathcal{U}$ of $\mathcal{K}$ such that* $\operatorname{tr.deg} L/K < \infty$ *and* $\operatorname{tr.deg} L\langle f\rangle/L \leq 1$, *the element f is algebraic over L. Then f is algebraic over K.*

Proof. Assume that f is transcendental over K. Choose $(\mathcal{L}, \mathcal{N})$ be an element of

$$\{(\mathcal{L}, \mathcal{N}) \mid \mathcal{K} \subset \mathcal{L} \subset \mathcal{N},\ \operatorname{tr.deg} L/K < \infty,\ \mathcal{N}/\mathcal{L} \text{ is } \mathcal{U}\text{-decomposable}, \\ f \in N,\ f \text{ is transcendental over } L\}$$

which has the minimal transcendence degree. The choice is guaranteed because $(\mathcal{K}, \mathcal{D})$ satisfies the conditions. Since f is transcendental over L, by the hypothesis we find $\operatorname{tr.deg} N/L \geq 2$. By the definition of decomposable extensions there exists a difference overfield $\mathcal{E} \subset \mathcal{U}$ of $\mathcal{L}$ such that $\operatorname{tr.deg} E/L < \infty$, E is free from N over L and there exists a difference intermediate field $\mathcal{M}$ of $\mathcal{N}\mathcal{E}/\mathcal{E}$ such that $\mathcal{N}\mathcal{E}/\mathcal{M}$ and $\mathcal{M}/\mathcal{E}$ are both $\mathcal{U}$-decomposable extensions of positive transcendence degree.

Then we have $f \in NE$ and $\operatorname{tr.deg} NE/M < \operatorname{tr.deg} N/L$, which imply f is algebraic over M. Thus we obtain $\mathcal{M}\langle f\rangle/\mathcal{E}$ is $\mathcal{U}$-decomposable by Lemma 2.3. Hence we find f is algebraic over E from $\operatorname{tr.deg} M\langle f\rangle/E < \operatorname{tr.deg} N/L$, which contradicts the fact that N and E are free over L. Therefore f is algebraic over K. □

From here C denotes an algebraically closed field of characteristic 0, t an element transcendental over C and q an element of $C^\times$ which is not a root of unity. Furthermore let $\mathcal{K}$ be a difference overfield of $(C(t), t \mapsto qt)$ whose operator is surjective, and $\mathcal{U}$ a difference overfield of $\mathcal{K}$. We may take the field of Puiseux series or $\mathcal{N}_n$ in Example 2.10 as $\mathcal{U}$ for example.

The author proved in [6] that solutions of q-$P(A_7)$ are all transcendental over $C(t)$ in the case q is not a root of unity. Hence the following theorem shows that if q is not a root of unity then q-$P(A_7)$ has no solution in any $\mathcal{U}$-decomposable extension of $(C(t), t \mapsto qt)$, where $\mathcal{U}$ is an arbitrary difference overfield of $(C(t), t \mapsto qt)$.

Theorem 3.2. *Let $\mathcal{D}/\mathcal{K}$ be a $\mathcal{U}$-decomposable extension and $f \in D$ a solution of q-$P(A_7)$. Then f is algebraic over K.*

This is proved from Lemma 3.1 and the following proposition.

Proposition 3.3. *Let $f \in U$ be a solution of q-$P(A_7)$ and $\mathcal{L} \subset \mathcal{U}$ a difference overfield of $\mathcal{K}$ with finite transcendence degree. If* $\operatorname{tr.deg} L\langle f\rangle/L \leq 1$ *then f is algebraic over L.*

Proof. It is enough to prove this for algebraically closed L. Then we find the operator of $\mathcal{L}$ is surjective by Lemma 2.9. Let τ be the operater of $\mathcal{U}$. For a polynomial $F = \sum a_{ij} Y^i Y_1^j \in L[Y, Y_1]$, we define $F^* = \sum (\tau a_{ij}) Y^i Y_1^j$.

Assume that $f \notin L$. Then the transformations $f_i = \tau^i f$ $(i = 0, 1, 2, \dots)$ are all transcendental over L. From $\operatorname{tr.deg} L\langle f\rangle/L \le 1$ we find that f and f_1 are algeraically dependent over L. Take an irreducible polynomial $F \in L[Y, Y_1]$ such that $F(f, f_1) = 0$.

Put

$$F_0 = (Y_1Y^2)^{n_0} F\left(\frac{qt(1-Y)}{Y_1Y^2}, Y\right)$$

and

$$F_1 = (Y_1^2Y)^{n_1} F^*\left(Y_1, \frac{qt(1-Y_1)}{Y_1^2Y}\right),$$

where $n_0 = \deg_Y F$ and $n_1 = \deg_{Y_1} F$. We easily find that $F_0, F_1 \in L[Y, Y_1] \setminus \{0\}$. From

$$F_0(f_1, f_2) = (f_2f_1^2)^{n_0} F\left(\frac{qt(1-f_1)}{f_2f_1^2}, f_1\right) = (f_2f_1^2)^{n_0} F(f, f_1) = 0$$

and

$$F_1(f, f_1) = (f_1^2f)^{n_1} F^*\left(f_1, \frac{qt(1-f_1)}{f_1^2f}\right) = (f_1^2f)^{n_1} F^*(f_1, f_2) = 0,$$

we obtain $F^* \mid F_0$ and $F \mid F_1$, where we note that all the f_i are transcendental over L.

However we find the nonexistence of such a polynomial F from the subsequent lemma, a statement analogous to Theorem 1 in the paper of Noumi and Okamoto ([8]), where they defined an invariant divisor by a polynomial like F. Hence f is an element of L. □

Lemma 3.4. *Let $q \in C^\times$ be not a root of unity, (L, τ) be a difference overfield of $(C(t), t \mapsto qt)$ whose operator τ is surjective, Y and Y_1 algebraically independent over L, ϕ_0 an isomorphism such that*

$$\begin{aligned} \phi_0\colon\ L(Y, Y_1) &\longrightarrow L(Y, Y_1) \\ Y &\longmapsto \frac{qt(1-Y)}{Y_1Y^2} \\ Y_1 &\longmapsto Y \\ L \ni x &\longmapsto x \in L \end{aligned}$$

and ϕ_1 an isomorphism such that

$$\begin{aligned} \phi_1\colon\ L(Y, Y_1) &\longrightarrow L(Y, Y_1). \\ Y &\longmapsto Y_1 \\ Y_1 &\longmapsto \frac{qt(1-Y_1)}{Y_1^2Y} \\ L \ni x &\longmapsto \tau x \in L \end{aligned}$$

For a polynomial $F \in L[Y, Y_1]$ we define F^* as in the proof of Proposition 3.3. Then there is no irreducible polynomial $F \in L[Y, Y_1] \setminus (L[Y] \cup L[Y_1])$ such that $F^* \mid (Y_1Y^2)^{n_0}\phi_0 F$ and $F \mid (Y_1^2Y)^{n_1}\phi_1 F$, where $n_0 = \deg_Y F$ and $n_1 = \deg_{Y_1} F$.

Proof. Assume there exists such F. Put $F_0 = (Y_1Y^2)^{n_0}\phi_0 F$ and $F_1 = (Y_1^2Y)^{n_1}\phi_1 F$. Then it follows that

$$n_1 = \deg_{Y_1} F^* \leq \deg_{Y_1} F_0 \leq n_0 = \deg_Y F \leq \deg_Y F_1 \leq n_1,$$

which implies $n_0 = n_1$. Put $n = n_0 = n_1 \geq 1$.

From $F \mid F_1$ there exists a polynomial $P \in L[Y, Y_1] \setminus \{0\}$ such that $F_1 = PF$. Since $\deg_Y P = \deg_Y F_1 - \deg_Y F = 0$, we find $P \in L[Y_1]$. Hence we express F as

$$F = \sum_{i,j} a_{ij} Y^i Y_1^j, \quad a_{ij} \in L,$$

and we obtain the following equations by comparing the coefficients of powers of Y in $F_1 = PF$,

$$(qt)^{n-j}(1-Y_1)^{n-j}Y_1^{2j}\left(\sum_{i=0}^{n} \tau a_{i,n-j} Y_1^i\right) = P(Y_1)\sum_{i=0}^{n} a_{ji}Y_1^i, \quad (0 \leq j \leq n).$$

Calculation shows $P = pY_1^n(1-Y_1)^{\frac{n}{2}}$, where $n/2$ is a positive integer. Then comparing the coefficients in the above equations, we obtain $q^{\frac{n}{2}} = 1$, which is a contradiction. □

§ 4. Appendix

The universal differential field extension is defined as follows, and its existence is proved for any differential field by Kolchin in [4].

Definition 4.1. A necessary and sufficient condition for a differential field extension $\mathcal{U}/\mathcal{K}$ to be universal is that for every finitely generated differential field extension $\mathcal{K}_1$ of $\mathcal{K}$ with $\mathcal{K}_1 \subset \mathcal{U}$ and every finitely generated differential field extension $\mathcal{L}$ of $\mathcal{K}_1$ there exists an differential isomorphism of $\mathcal{L}$ over $\mathcal{K}_1$ into $\mathcal{U}$.

On the contrary the following theorem is proved.

Theorem 4.2. *Let $\mathcal{K}$ be a difference field of characteristic 0. Then there does not exist such a difference overfield $\mathcal{U}$ of $\mathcal{K}$ that for any finitely generated difference overfield $\mathcal{K}_1 \subset \mathcal{U}$ of $\mathcal{K}$ and any finitely generated difference overfield $\mathcal{L}$ of $\mathcal{K}_1$ there exists a difference isomorphism of $\mathcal{L}$ over $\mathcal{K}_1$ into $\mathcal{U}$.*

Proof. Assume there exists such $\mathcal{U} = (U, \tau)$. Choose x to be transcendental over K. The field $K(x)$ equipped with an extension τ' of $\tau|_K$ sending x to x is a finitely generated difference overfield of $\mathcal{K}$. By the hypothesis there exists an difference isomorphism ϕ of $(K(x), \tau')$ into $\mathcal{U}$ over $\mathcal{K}$. Put $y = \phi x$. Then we obtain $\phi(K(x)) = K(y)$ and $\tau y = \tau \circ \phi(x) = \phi(x) = y$.

Difference fields $K(y^{\frac{1}{2}})$ equipped with extensions of $\tau|_K$,

$$\begin{aligned}\tau_i\colon\ K(y^{\frac{1}{2}}) &\longrightarrow\ \ K(y^{\frac{1}{2}}), \qquad (i=1,2)\\ y^{\frac{1}{2}}\ \ &\longmapsto (-1)^{i-1}y^{\frac{1}{2}}\end{aligned}$$

respectively are finitely generated difference overfields of $\mathcal{K}(y)$. By our assumption there exists a difference isomorphism ϕ_i $(i=1,2)$ of $(K(y^{\frac{1}{2}}),\tau_i)$ into $\mathcal{U}$ over $\mathcal{K}(y)$. Since $(\phi_i(y^{\frac{1}{2}}))^2=\phi_i(y)=y$, we have expressions,

$$\phi_i(y^{\frac{1}{2}})=(-1)^{k_i}z,\ k_i\in\mathbb{Z}\quad\text{for } i=1,2,$$

where $z\in U$ denotes a square root of y.

By the definition of ϕ_i we have $\phi_i\circ\tau_i=\tau\circ\phi_i$. Hence we obtain

$$\begin{aligned}\tau(z)&=(-1)^{k_i}\tau(\phi_i(y^{\frac{1}{2}}))=(-1)^{k_i}\phi_i\circ\tau_i(y^{\frac{1}{2}})\\ &=(-1)^{k_i}\phi_i((-1)^{i-1}y^{\frac{1}{2}})=(-1)^{i-1}z\quad\text{for } i=1,2.\end{aligned}$$

However this contradicts that the characteristic of K is 0. □

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