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A-discriminants and Euler obstructions of toric varieties

By

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Abstract

In this short note, we introduce our recent results on the explicit description of the degrees of A-discriminant varieties introduced by Gelfand-Kapranov-Zelevinsky [10]. Our formulas can be applied also to the case where the A-discriminant varieties are higher-codimensional and their degrees are described by the geometry of the configurations A. For the detail, see [23].

§ 1. Degree formulas for A-discriminants

In this section, we first introduce the formula for the degrees of A-discriminants obtained by Gelfand-Kapranov-Zelevinsky [10] and announce our generalization in [23].

Let $M \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice (free $\mathbb{Z}$-module) of rank n and $M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M$ the real vector space associated with M. Let $A \subset M$ be a finite subset of M and denote by P its convex hull in $M_{\mathbb{R}}$. In this note, such a polytope P will be called an integral polytope in $M_{\mathbb{R}}$. If $A = \{\alpha(1), \alpha(2), \dots, \alpha(N+1)\}$, we can define a morphism $\varphi_A \colon T \longrightarrow \mathbb{P}^N$ from an algebraic torus $T := (\mathbb{C}^*)^n$ to a complex projective space $\mathbb{P}^N$ ($N := \sharp A - 1$) by

$$x = (x_1, x_2, \dots, x_n) \longmapsto [x^{\alpha(1)} \colon x^{\alpha(2)} \colon \cdots \colon x^{\alpha(N+1)}], \tag{1.1}$$

where for each $\alpha(i) \in A \subset M \simeq \mathbb{Z}^n$ we set $x^{\alpha(i)} = x_1^{\alpha(i)_1} x_2^{\alpha(i)_2} \cdots x_n^{\alpha(i)_n}$ as usual.

Definition 1.1 ([10]). Let $X_A := \overline{\mathrm{im}\ \varphi_A}$ be the closure of the image of $\varphi_A \colon T \longrightarrow \mathbb{P}^N$. Then the dual variety $X_A^* \subset (\mathbb{P}^N)^*$ of X_A is called the discriminant

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variety associated with A. If moreover X_A^* is a hypersurface in the dual projective space $(\mathbb{P}^N)^*$, then the defining homogeneous polynomial of X_A^* (which is defined up to non-zero constant multiples) is called the A-discriminant.

Note that the discriminant variety X_A^* is naturally identified with the set of Laurent polynomials $f\colon T=(\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ of the form $f(x)=\sum_{\alpha\in A} c_\alpha x^\alpha$ $(c_\alpha\in\mathbb{C})$ such that $\{x\in T \mid f(x)=0\}$ is a singular hypersurface in T. In order to introduce the degree formula for A-discriminants proved by Gelfand-Kapranov-Zelevinsky [10], we need the following.

Definition 1.2 ([10]). For a finite set $B\subset M\simeq\mathbb{Z}^n$, we define an affine $\mathbb{Z}$-sublattice $\mathrm{Aff}_{\mathbb{Z}}(B)$ of M by

$$\mathrm{Aff}_{\mathbb{Z}}(B):=\left\{\sum_{v\in B} c_v\cdot v \;\middle|\; c_v\in\mathbb{Z},\ \ \sum_{v\in B} c_v=1\right\}. \tag{1.2}$$

In this note, we sometimes denote the affine sublattice $\mathrm{Aff}_{\mathbb{Z}}(B)\subset M$ by $M(B)$. Now let Δ be a face of P and denote by $\mathbb{L}(\Delta)$ the smallest affine subspace of $M_{\mathbb{R}}$ containing Δ. Then $M(A\cap\Delta)=\mathrm{Aff}_{\mathbb{Z}}(A\cap\Delta)$ is a $\mathbb{Z}$-lattice of rank $\mathrm{dim}\Delta=\mathrm{dim}\mathbb{L}(\Delta)$ in $\mathbb{L}(\Delta)$ and we have $M(A\cap\Delta)_{\mathbb{R}}\simeq\mathbb{L}(\Delta)$. Let vol be the Lebesgue measure of $(\mathbb{L}(\Delta),M(A\cap\Delta))$ by which the volume of the fundamental domain by the action of $M(A\cap\Delta)$ on $\mathbb{L}(\Delta)$ is measured to be 1. For a subset $K\subset\mathbb{L}(\Delta)$, we set

$$\mathrm{Vol}_{\mathbb{Z}}(K):=(\mathrm{dim}\Delta)!\cdot\mathrm{vol}(K). \tag{1.3}$$

We call it the normalized volume of K with respect to the lattice $M(A\cap\Delta)$. Throughout this note, we use this normalized volume $\mathrm{Vol}_{\mathbb{Z}}$ instead of the usual one.

The following formula is obtained by Gelfand-Kapranov-Zelevinsky [10, Chapter 9, Theorem 2.8].

Theorem 1.3 ([10]). *Assume that $X_A\subset\mathbb{P}^N$ is smooth and X_A^* is a hypersurface in $(\mathbb{P}^N)^*$. Then the degree of the A-discriminant is given by the formula:*

$$\deg X_A^*=\sum_{\Delta\prec P}(-1)^{\mathrm{codim}\Delta}(1+\mathrm{dim}\Delta)\mathrm{Vol}_{\mathbb{Z}}(\Delta). \tag{1.4}$$

In order to state our generalization of Theorem 1.3 to the case where X_A^* may be higher-codimensional, recall the basic correspondence $(0\leq k\leq n=\mathrm{dim}P)$:

$$\{k\text{-dimensional faces of } P\}\overset{1:1}{\longleftrightarrow}\{k\text{-dimensional } T\text{-orbits in } X_A\} \tag{1.5}$$

proved by [10, Chapter 5, Proposition 1.9]. For a face $\Delta\prec P$ of P, we denote by T_Δ the corresponding T-orbit in X_A. We denote the value of the Euler obstruction

$\mathrm{Eu}_{X_A}\colon X_A \longrightarrow \mathbb{Z}$ of X_A on T_Δ by $\mathrm{Eu}(\Delta) \in \mathbb{Z}$. The precise definition of the Euler obstruction will be given later in Section 3. Here we simply recall that the Euler obstruction of X_A takes the value 1 on the smooth part of X_A. In particular, for $\Delta = P$ the T-orbit T_Δ is open dense in X_A and $\mathrm{Eu}(\Delta) = 1$.

Theorem 1.4 ([23]). *For $1 \leq i \leq N$, set*

$$\delta_i := \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta} \left\{ \binom{\mathrm{dim}\Delta - 1}{i} + (-1)^{i-1}(i+1) \right\} \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{1.6}$$

Then the codimension $r = \mathrm{codim} X_A^* = N - \mathrm{dim} X_A^*$ and the degree of the dual variety X_A^* are given by

$$r = \mathrm{codim} X_A^* = \min\{i \mid \delta_i \neq 0\}, \tag{1.7}$$

$$\deg X_A^* = \delta_r. \tag{1.8}$$

Remark.

1. For $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 0}$, we used the generalized binomial coefficient

$$\binom{p}{q} = \frac{p(p-1)(p-2)\cdots(p-q+1)}{q!}. \tag{1.9}$$

For example, for a vertex $\Delta = \{v\} \prec P$, we have $\binom{\mathrm{dim}\Delta - 1}{i} = \binom{-1}{i} = (-1)^i$.

2. Note that the number $\mathrm{codim} X_A^* - 1$ is called the dual defect of X_A.

Corollary 1.5. *Assume that X_A^* is a hypersurface in $(\mathbb{P}^N)^*$. Then the degree of the A-discriminant is given by*

$$\deg X_A^* = \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta} (1 + \mathrm{dim}\Delta) \mathrm{Vol}_{\mathbb{Z}}(\Delta) \cdot \mathrm{Eu}(\Delta). \tag{1.10}$$

The above theorem is proved by using Ernström's class formula in [7] (see [22] for another proof and its generalizations). Note that if the dual defect of X_A is zero the degree formula of X_A^* for singular X_A's was also obtained by Dickenstein-Feichtner-Sturmfels [5]. In their paper, they express the degree of X_A^* by some combinatorial invariants of A. Our formulas seem to be more directly related to the geometry of the convex polytope P.

In Section 3, we will give two combinatorial formulas for the Euler obstruction $\mathrm{Eu}_{X_A}\colon X_A \longrightarrow \mathbb{Z}$ of X_A. The first one is simpler and can be applied only to the very special but important case where the integral polytope P is sufficiently large and $A = P \cap M$ (i.e. A is saturated), whereas the second one can be applied to the general case.

To end this section, we shall briefly explain an important case to which our first formulas can be applied. Let $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = M^*$ be the dual $\mathbb{Z}$-lattice of M and set $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N$. Since $N_{\mathbb{R}}$ is the dual vector space of $M_{\mathbb{R}}$, a point $u \in N_{\mathbb{R}}$ can be considered as a linear functional on $M_{\mathbb{R}}$ and we can define the following subset of P.

Definition 1.6. In the setting above, we define the supporting face $\Delta(P, u)$ of u in P by

$$\Delta(P, u) := \left\{ v \in P \;\middle|\; \langle u, v \rangle = \min_{w \in P} \langle u, w \rangle \right\}. \tag{1.11}$$

Note that for $u = 0 \in N_{\mathbb{R}}$ we have $\Delta(P, u) = P$. Now for each face $\Delta \prec P$ of P let us set

$$\sigma_{\Delta} := \{ u \in N_{\mathbb{R}} \mid \Delta(P, u) = \Delta \}. \tag{1.12}$$

Then we obtain a decomposition of $N_{\mathbb{R}}$:

$$N_{\mathbb{R}} = \bigsqcup_{\Delta \prec P} \sigma_{\Delta}, \tag{1.13}$$

where Δ ranges through the set of faces of P. Each $\overline{\sigma_{\Delta}}$ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ and $\Sigma_P := \{ \overline{\sigma_{\Delta}} \mid \Delta \prec P \}$ defines a complete fan in $N_{\mathbb{R}}$ (see [9], [26] and so on for the definitions). We call Σ_P the normal fan of P. Then by [26, Theorem 2.13] if P is sufficiently large and $A = P \cap M$, φ_A is an embedding and X_A is naturally identified with the image of the projective embedding of the complete toric variety X_{Σ_P} associated with Σ_P and $A \subset P$. In particular, X_A is a normal variety in this case.

§ 2. Milnor fibers over singular toric varieties

In this section, we give some explicit formulas for the monodromy zeta functions of non-degenerate polynomials over possibly singular toric varieties. These formulas are considered to be natural generalizations of the previous results by Varchenko [31], Kirillov [16] and Oka [27], [28] and so on and will be used for the explicit computation of the Euler obstructions of toric varieties in the next section.

First, we recall the definitions of Milnor fibers and Milnor monodromies over singular varieties (see for example [30] for a review on this subject). Let $X \subset \mathbb{C}^N$ be a possibly singular subvariety such that $0 \in X$ and $f \colon X \longrightarrow \mathbb{C}$ be a polynomial function on X. For simplicity, we assume that $Y = f^{-1}(0)$ is a hypersurface in X containing the origin. Then the following lemma is well-known (see for example [18, Definition 1.4]).

Lemma 2.1. *For sufficiently small $\varepsilon > 0$, there exists $\eta_0 > 0$ with $0 < \eta_0 \ll \varepsilon$ such that for $0 < \forall\eta < \eta_0$ the restriction of f:*

$$X \cap B(0;\varepsilon) \cap f^{-1}(D(0;\eta) \setminus \{0\}) \longrightarrow D(0;\eta) \setminus \{0\} \tag{2.1}$$

is a topological fiber bundle over the punctured disk $D(0;\eta)\setminus\{0\} := \{z \in \mathbb{C} \mid 0 < |z| < \eta\}$, where $B(0;\varepsilon)$ is the open ball in $\mathbb{C}^N$ with radius ε centered at the origin.

Definition 2.2. A fiber of the above fibration is called the Milnor fiber of the function $f\colon X \longrightarrow \mathbb{C}$ at $0 \in X$ and we denote it by F_0.

As in the same way as the case of polynomials over $\mathbb{C}^N$ (see [25]), we can define the Milnor monodromy operators

$$\Phi_j\colon H^j(F_0;\mathbb{C}) \xrightarrow{\sim} H^j(F_0;\mathbb{C}) \quad (j = 0, 1, \ldots) \tag{2.2}$$

and the zeta-function

$$\zeta_f(t) := \prod_{j=0}^{\infty} \det(\mathrm{id} - t\Phi_j)^{(-1)^j} \tag{2.3}$$

associated with it. Since the above product is in fact finite, $\zeta_f(t)$ is a rational function of t and its degree in t is the topological Euler characteristic $\chi(F_0)$ of the Milnor fiber F_0.

Now we return to the toric case. Let $M \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice of rank n and set $M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M$. We take a finitely generated subsemigroup $\mathcal{S}$ of M such that $0 \in \mathcal{S}$ and denote by $K(\mathcal{S})$ the convex hull of $\mathcal{S}$ in $M_{\mathbb{R}}$. For simplicity, assume that $K(\mathcal{S})$ is a strongly convex polyhedral cone in $M_{\mathbb{R}}$ of the maximal dimension (i.e. $\dim K(\mathcal{S}) = n$). Then the group algebra $\mathbb{C}[\mathcal{S}]$ is finitely generated over $\mathbb{C}$ and $X(\mathcal{S}) := \mathrm{Spec}(\mathbb{C}[\mathcal{S}])$ is a (not necessarily normal) toric variety of dimension n (see [9], [10] and [26] and so on for the detail). Indeed, let $M(\mathcal{S})$ be the $\mathbb{Z}$-sublattice of rank n in M generated by $\mathcal{S}$ and consider the algebraic torus $T := \mathrm{Spec}(\mathbb{C}[M(\mathcal{S})]) \simeq (\mathbb{C}^*)^n$. Then the affine toric variety $X(\mathcal{S})$ admits a natural action of $T = \mathrm{Spec}(\mathbb{C}[M(\mathcal{S})])$ and has a unique 0-dimensional orbit. We denote this orbit point by 0 and call it the special point of $X(\mathcal{S})$. Recall that a polynomial function $f\colon X(\mathcal{S}) \longrightarrow \mathbb{C}$ on $X(\mathcal{S})$ corresponds to an element $f = \sum_{v\in\mathcal{S}} a_v \cdot v$ ($a_v \in \mathbb{C}$) of $\mathbb{C}[\mathcal{S}]$.

Definition 2.3. Let $f = \sum_{v\in\mathcal{S}} a_v \cdot v \in \mathbb{C}[\mathcal{S}]$ be a polynomial function on $X(\mathcal{S})$.

(i) We define the support $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) := \{v \in \mathcal{S} \mid a_v \neq 0\} \subset \mathcal{S}. \tag{2.4}$$

(ii) We define the Newton polygon $\Gamma_+(f)$ of f to be the convex hull of $\bigcup_{v\in \mathrm{supp}(f)}(v+K(\mathcal{S}))$ in $K(\mathcal{S})$.

(iii) The union of compact faces of $\Gamma_+(f)$ is called the Newton diagram of f. We denote it by $\Gamma(f)$.

Now let us fix a function $f\in \mathbb{C}[\mathcal{S}]$ such that $0\notin \mathrm{supp}(f)$ (i.e. $f\colon X(\mathcal{S})\longrightarrow \mathbb{C}$ vanishes at the special point 0) and consider its Milnor fiber F_0 at $0\in X(\mathcal{S})$. Choose a $\mathbb{Z}$-basis of $M(\mathcal{S})$ and identify $M(\mathcal{S})$ with $\mathbb{Z}^n$. Then each element v of $\mathcal{S}\subset M(\mathcal{S})$ is identified with a $\mathbb{Z}$-vector $v=(v_1,\dots,v_n)$ and to any $g=\sum_{v\in\mathcal{S}} b_v\cdot v\in \mathbb{C}[\mathcal{S}]$ we can associate a Laurent polynomial $L(g)=\sum_{v\in\mathcal{S}} b_v\cdot x^v$ on $T=(\mathbb{C}^*)^n$. One can easily prove that the following definition does not depend on the choice of the $\mathbb{Z}$-basis of $M(\mathcal{S})$.

Definition 2.4. We say that $f=\sum_{v\in\mathcal{S}} a_v\cdot v\in\mathbb{C}[\mathcal{S}]$ is non-degenerate if for any compact face γ in $\Gamma_+(f)$ the complex hypersurface

$$\{x=(x_1,\dots,x_n)\in(\mathbb{C}^*)^n \mid L(f_\gamma)(x)=0\} \tag{2.5}$$

in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma:=\sum_{v\in\gamma\cap\mathcal{S}} a_v\cdot v$.

For each face $\Delta\prec K(\mathcal{S})$ of $K(\mathcal{S})$ such that $\Gamma_+(f)\cap\Delta\neq\emptyset$, let $\gamma_1^\Delta,\gamma_2^\Delta,\dots,\gamma_{n(\Delta)}^\Delta$ be the compact faces of $\Gamma_+(f)\cap\Delta$ of the maximal dimension (i.e. $\dim\gamma_i^\Delta=\dim\Delta-1$). In order to define the lattice distance from γ_i^Δ to the origin $0\in M_\mathbb{R}\simeq M(\mathcal{S})_\mathbb{R}$, we take the unique smallest linear subspace $\mathbb{L}(\Delta)$ of $M_\mathbb{R}$ containing Δ and consider the $\mathbb{Z}$-lattice $M(\mathcal{S}\cap\Delta)$ of rank $\dim\Delta$ such that $M(\mathcal{S}\cap\Delta)_\mathbb{R}=\mathbb{R}\otimes_\mathbb{Z} M(\mathcal{S}\cap\Delta)\simeq\mathbb{L}(\Delta)$. Then there exists a unique primitive vector u_i^Δ in the dual lattice $M(\mathcal{S}\cap\Delta)^*$ of $M(\mathcal{S}\cap\Delta)$ whose supporting face in $\Gamma_+(f)\cap\Delta$ is exactly γ_i^Δ.

Definition 2.5. We define the lattice distance $d(0,\gamma_i^\Delta)\in\mathbb{Z}_{>0}$ from γ_i^Δ to the origin $0\in M_\mathbb{R}$ to be the value of u_i^Δ on γ_i^Δ.

Then by using the normalized volume $\mathrm{Vol}_\mathbb{Z}(\gamma_i^\Delta)$ of γ_i^Δ with respect to the lattice $M(\mathcal{S}\cap\Delta)\cap\mathbb{L}(\gamma_i^\Delta)$ we have the following result.

Theorem 2.6 ([24]). *Assume that $f=\sum_{v\in\mathcal{S}} a_v\cdot v\in\mathbb{C}[\mathcal{S}]$ is non-degenerate. Then the monodromy zeta function $\zeta_f(t)$ of $f\colon X(\mathcal{S})\longrightarrow\mathbb{C}$ at $0\in X(\mathcal{S})$ is given by*

$$\zeta_f(t)=\prod_{\Gamma_+(f)\cap\Delta\neq\emptyset}\zeta_\Delta(t), \tag{2.6}$$

where for each face $\Delta\prec K(\mathcal{S})$ of $K(\mathcal{S})$ such that $\Gamma_+(f)\cap\Delta\neq\emptyset$ we set

$$\zeta_\Delta(t)=\prod_{i=1}^{n(\Delta)}\left(1-t^{d(0,\gamma_i^\Delta)}\right)^{(-1)^{\dim\Delta-1}\mathrm{Vol}_\mathbb{Z}(\gamma_i^\Delta)}. \tag{2.7}$$

Let Γ_i^Δ be the convex hull of $\gamma_i^\Delta \sqcup \{0\}$ in Δ. Then the normalized volume $\mathrm{Vol}_{\mathbb{Z}}(\Gamma_i^\Delta) \in \mathbb{Z}$ of Γ_i^Δ with respect to the lattice $M(\mathcal{S} \cap \Delta)$ is equal to $d(0, \gamma_i^\Delta) \cdot \mathrm{Vol}_{\mathbb{Z}}(\gamma_i^\Delta)$ and we obtain the following result.

Corollary 2.7 ([24]). *Assume that $f = \sum_{v \in \mathcal{S}} a_v \cdot v \in \mathbb{C}[\mathcal{S}]$ is non-degenerate. Then the Euler characteristic of the Milnor fiber F_0 of $f \colon X(\mathcal{S}) \longrightarrow \mathbb{C}$ at $0 \in X(\mathcal{S})$ is given by*

$$\chi(F_0) = \sum_{\Gamma_+(f) \cap \Delta \neq \emptyset} (-1)^{\dim \Delta - 1} \sum_{i=1}^{n(\Delta)} \mathrm{Vol}_{\mathbb{Z}}(\Gamma_i^\Delta). \tag{2.8}$$

Now recall the following correspondence ($0 \leq k \leq n$):

$$\{k\text{-dimensional faces in } K(\mathcal{S})\} \overset{1:1}{\longleftrightarrow} \{k\text{-dimensional } T\text{-orbits in } X(\mathcal{S})\}. \tag{2.9}$$

For a face Δ of $K(\mathcal{S})$, we denote by T_Δ the corresponding T-orbit in $X(\mathcal{S})$. Then we obtain the following local version of Bernstein-Khovanskii-Kushnirenko's theorem which expresses the Euler characteristic of $F_0 \cap T_\Delta$ in terms of the Newton diagram of f. Note that $F_0 \cap T_\Delta$ is a locally closed subset of T_Δ.

Theorem 2.8 ([24]). *Assume that $f = \sum_{v \in \mathcal{S}} a_v \cdot v \in \mathbb{C}[\mathcal{S}]$ is non-degenerate. Then we have*

$$\chi(F_0 \cap T_\Delta) = (-1)^{\dim \Delta - 1} \sum_{i=1}^{n(\Delta)} \mathrm{Vol}_{\mathbb{Z}}(\Gamma_i^\Delta). \tag{2.10}$$

In the rest of this section, we extend our results to non-degenerate complete intersection subvarieties in the affine toric variety $X(\mathcal{S})$. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[\mathcal{S}]$ ($1 \leq k \leq n = \dim X(\mathcal{S})$) and consider the following subvarieties of $X(\mathcal{S})$:

$$V := \{f_1 = \cdots = f_{k-1} = f_k = 0\} \subset W := \{f_1 = \cdots = f_{k-1} = 0\}. \tag{2.11}$$

Assume that $0 \in V$. Our objective here is to study the Milnor fiber G_0 of $g := f_k|_W \colon W \longrightarrow \mathbb{C}$ at $0 \in V = g^{-1}(0) \subset W$ and its monodromy zeta function $\zeta_g(t)$. We call $\zeta_g(t)$ the k-th principal monodromy zeta function of $V = \{f_1 = \cdots = f_k = 0\}$. For each face $\Delta \prec K(\mathcal{S})$ of $K(\mathcal{S})$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$, we set

$$I(\Delta) := \{j \in \{1, 2, \ldots, k-1\} \mid \Gamma_+(f_j) \cap \Delta \neq \emptyset\} \subset \{1, 2, \ldots, k-1\} \tag{2.12}$$

and $m(\Delta) := \sharp I(\Delta) + 1$. Let $\mathbb{L}(\Delta)$, $M(\mathcal{S} \cap \Delta)$, $M(\mathcal{S} \cap \Delta)^*$ be as before and $\mathbb{L}(\Delta)^*$ the dual vector space of $\mathbb{L}(\Delta)$. Then $M(\mathcal{S} \cap \Delta)^*$ is naturally identified with a subset of $\mathbb{L}(\Delta)^*$ and the polar cone

$$\Delta^\vee = \{u \in \mathbb{L}(\Delta)^* \mid \langle u, v \rangle \geq 0 \text{ for any } v \in \Delta\} \tag{2.13}$$

of Δ in $\mathbb{L}(\Delta)^*$ is a rational polyhedral convex cone with respect to $M(\mathcal{S}\cap\Delta)^* \subset \mathbb{L}(\Delta)^*$. For $j \in I(\Delta) \sqcup \{k\}$ and $u \in \mathrm{Int}\Delta^\vee \cap M(\mathcal{S}\cap\Delta)^*$, we define the u-part $f_j^u \in \mathbb{C}[\mathcal{S}\cap\Delta]$ of f_j by

$$f_j^u := \sum_{v\in\Gamma(f_j|_\Delta;u)} a_v \cdot v \in \mathbb{C}[\mathcal{S}\cap\Delta], \tag{2.14}$$

where $f_j = \sum_{v\in\Gamma_+(f_j)} a_v \cdot v$ in $\mathbb{C}[\mathcal{S}]$ and we set $f_j|_\Delta := \sum_{v\in\Gamma_+(f_j)\cap\Delta} a_v \cdot v \in \mathbb{C}[\mathcal{S}\cap\Delta]$ and

$$\Gamma(f_j|_\Delta;u) := \left\{ v \in \Gamma_+(f_j)\cap\Delta \;\middle|\; \langle u,v\rangle = \min_{w\in\Gamma_+(f_j)\cap\Delta} \langle u,w\rangle \right\}. \tag{2.15}$$

Now recall the following definition.

Definition 2.9. Let $g_1, g_2, \ldots, g_p$ be Laurent polynomials on $(\mathbb{C}^*)^n$. Then we say that the subvariety $\{x \in (\mathbb{C}^*)^n \mid g_1(x) = g_2(x) = \cdots = g_p(x) = 0\}$ of $(\mathbb{C}^*)^n$ is non-degenerate complete intersection if the p-form $dg_1 \wedge dg_2 \wedge \cdots \wedge dg_p$ does not vanish on it.

By taking a $\mathbb{Z}$-basis of $M(\mathcal{S})$ and identifying the u-parts f_j^u with Laurent polynomials $L(f_j^u)$ on $T = (\mathbb{C}^*)^n$ as before, we have the following definition which does not depend on the choice of the $\mathbb{Z}$-basis of $M(\mathcal{S})$.

Definition 2.10. We say that $(f_1, \ldots, f_k)$ is non-degenerate if for any face $\Delta \prec K(\mathcal{S})$ such that $\Gamma_+(f_k)\cap\Delta \neq \emptyset$ (including the case where $\Delta = K(\mathcal{S})$) and any $u \in \mathrm{Int}\Delta^\vee \cap M(\mathcal{S}\cap\Delta)^*$ the following two subvarieties of $(\mathbb{C}^*)^n$ are non-degenerate complete intersections.

$$\{x \in (\mathbb{C}^*)^n \mid L(f_j^u)(x) = 0 \text{ for any } j \in I(\Delta)\} \tag{2.16}$$

$$\{x \in (\mathbb{C}^*)^n \mid L(f_j^u)(x) = 0 \text{ for any } j \in I(\Delta) \sqcup \{k\}\}. \tag{2.17}$$

Remark. The above definition is slightly different from the one in [28] and so on, since our result (Theorem 2.11 below) generalizes the ones in [16], [27] and [28].

For each face $\Delta \prec K(\mathcal{S})$ of $K(\mathcal{S})$ such that $\Gamma_+(f_k)\cap\Delta \neq \emptyset$, let us set

$$f_\Delta := \left(\prod_{j\in I(\Delta)} f_j\right) \cdot f_k \in \mathbb{C}[\mathcal{S}] \tag{2.18}$$

and consider its Newton polygon $\Gamma_+(f_\Delta) \subset K(\mathcal{S})$. Let $\gamma_1^\Delta, \gamma_2^\Delta, \ldots, \gamma_{n(\Delta)}^\Delta$ be the compact faces of $\Gamma_+(f_\Delta)\cap\Delta$ ($\neq \emptyset$) such that $\dim\gamma_i^\Delta = \dim\Delta - 1$. Then for each $1 \leq i \leq n(\Delta)$

there exists a unique primitive vector $u_i^{\Delta} \in \mathrm{Int}\Delta^{\vee} \cap M(\mathcal{S} \cap \Delta)^*$ whose supporting face in $\Gamma_+(f_{\Delta}) \cap \Delta$ is exactly γ_i^{Δ}. For $j \in I(\Delta) \sqcup \{k\}$, set

$$\gamma(f_j)_i^{\Delta} := \left\{ v \in \Gamma_+(f_j) \cap \Delta \;\middle|\; \langle u_i^{\Delta}, v \rangle = \min_{w \in \Gamma_+(f_j) \cap \Delta} \langle u_i^{\Delta}, w \rangle \right\}, \tag{2.19}$$

$$d(0, \gamma(f_j)_i^{\Delta}) := \min_{w \in \Gamma_+(f_j) \cap \Delta} \langle u_i^{\Delta}, w \rangle. \tag{2.20}$$

For $j \in \{1, 2, \dots, k-1\} \setminus I(\Delta)$, we also set $\gamma(f_j)_i^{\Delta} := \emptyset$. Note that we have

$$\gamma_i^{\Delta} = \sum_{j \in I(\Delta) \sqcup \{k\}} \gamma(f_j)_i^{\Delta}, \tag{2.21}$$

$$\dim \left(\sum_{j \in I(\Delta) \sqcup \{k\}} \gamma(f_j)_i^{\Delta} \right) = \dim \gamma_i^{\Delta} = \dim \Delta - 1 \tag{2.22}$$

for any face $\Delta \prec K(\mathcal{S})$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$ and $1 \leq i \leq n(\Delta)$. For each face $\Delta \prec K(\mathcal{S})$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$, $\dim\Delta \geq m(\Delta)$ and $1 \leq i \leq n(\Delta)$, we set

$$K_i^{\Delta} := \sum_{\substack{\alpha_1 + \cdots + \alpha_{m(\Delta)} = \dim\Delta - 1 \\ \alpha_q \geq 1 \text{ for } q \leq m(\Delta)-1,\ \alpha_{m(\Delta)} \geq 0}} \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\gamma(f_{j_1})_i^{\Delta}, \dots, \gamma(f_{j_1})_i^{\Delta}}_{\alpha_1 \text{ times}}, \dots, \underbrace{\gamma(f_{j_{m(\Delta)}})_i^{\Delta}, \dots, \gamma(f_{j_{m(\Delta)}})_i^{\Delta}}_{\alpha_{m(\Delta)} \text{ times}}). \tag{2.23}$$

Here we set $I(\Delta) \sqcup \{k\} = \{j_1, j_2, \dots, k = j_{m(\Delta)}\}$ and

$$\mathrm{Vol}_{\mathbb{Z}}(\underbrace{\gamma(f_{j_1})_i^{\Delta}, \dots, \gamma(f_{j_1})_i^{\Delta}}_{\alpha_1 \text{ times}}, \dots, \underbrace{\gamma(f_{j_{m(\Delta)}})_i^{\Delta}, \dots, \gamma(f_{j_{m(\Delta)}})_i^{\Delta}}_{\alpha_{m(\Delta)} \text{ times}}) \tag{2.24}$$

is the normalized $(\dim\Delta - 1)$-dimensional mixed volume of

$$\underbrace{\gamma(f_{j_1})_i^{\Delta}, \dots, \gamma(f_{j_1})_i^{\Delta}}_{\alpha_1 \text{ times}}, \dots, \underbrace{\gamma(f_{j_{m(\Delta)}})_i^{\Delta}, \dots, \gamma(f_{j_{m(\Delta)}})_i^{\Delta}}_{\alpha_{m(\Delta)} \text{ times}} \tag{2.25}$$

(see [28, Chapter IV]) with respect to the lattice $M(\mathcal{S} \cap \Delta) \cap \mathbb{L}(\gamma_i^{\Delta})$.

Remark.

1. If $\dim\Delta - 1 = 0$, we set

$$K_i^{\Delta} = \mathrm{Vol}_{\mathbb{Z}}(\underbrace{\gamma(f_k)_i^{\Delta}, \dots, \gamma(f_k)_i^{\Delta}}_{0 \text{ times}}) := 1 \tag{2.26}$$

(in this case $\gamma(f_k)_i^{\Delta}$ is a point).

2. Let $Q_1, Q_2, \dots, Q_p$ be integral polytopes in $(\mathbb{R}^p, \mathbb{Z}^p)$. Then their normalized p-dimensional mixed volume $\mathrm{Vol}_{\mathbb{Z}}(Q_1, Q_2, \dots, Q_p) \in \mathbb{Z}$ is given by the formula

$$
\begin{aligned}
(2.27)\quad & p!\mathrm{Vol}_{\mathbb{Z}}(Q_1, Q_2, \dots, Q_p) \\
&= \mathrm{Vol}_{\mathbb{Z}}(Q_1 + Q_2 + \cdots + Q_p) \\
&\quad - \sum_{i=1}^{p} \mathrm{Vol}_{\mathbb{Z}}(Q_1 + \cdots + Q_{i-1} + Q_{i+1} + \cdots + Q_p) \\
&\quad + \sum_{1 \leq i < j \leq p} \mathrm{Vol}_{\mathbb{Z}}(Q_1 + \cdots + Q_{i-1} + Q_{i+1} + \cdots + Q_{j-1} + Q_{j+1} + \cdots + Q_p) \\
&\quad + \cdots + (-1)^{p-1} \sum_{i=1}^{p} \mathrm{Vol}_{\mathbb{Z}}(Q_i),
\end{aligned}
$$

where $\mathrm{Vol}_{\mathbb{Z}}(\,\cdot\,) \in \mathbb{Z}$ is the normalized p-dimensional volume.

Theorem 2.11 ([24]). *Assume that $(f_1, \dots, f_k)$ is non-degenerate. Then the k-th monodromy zeta function $\zeta_g(t)$ ($g = f_k|_W \colon W \longrightarrow \mathbb{C}$) is given by*

$$
(2.28)\qquad \zeta_g(t) = \prod_{\substack{\Gamma_+(f_k) \cap \Delta \neq \emptyset \\ \dim\Delta \geq m(\Delta)}} \zeta_\Delta(t),
$$

where for each face $\Delta \prec K(\mathcal{S})$ of $K(\mathcal{S})$ such that $\Gamma_+(f_k) \cap \Delta \neq \emptyset$ and $\dim\Delta \geq m(\Delta)$ we set

$$
(2.29)\qquad \zeta_\Delta(t) = \prod_{i=1}^{n(\Delta)} \left(1 - t^{d(0, \gamma(f_k)_i^\Delta)}\right)^{(-1)^{\dim\Delta - m(\Delta)} K_i^\Delta}.
$$

In particular, the Euler characteristic of the Milnor fiber G_0 of $g = f_k|_W \colon W \longrightarrow \mathbb{C}$ at $0 \in V = g^{-1}(0)$ is given by

$$
(2.30)\qquad \chi(G_0) = \sum_{\substack{\Gamma_+(f_k) \cap \Delta \neq \emptyset \\ \dim\Delta \geq m(\Delta)}} (-1)^{\dim\Delta - m(\Delta)} \sum_{i=1}^{n(\Delta)} d(0, \gamma(f_k)_i^\Delta) \cdot K_i^\Delta.
$$

§ 3. Euler obstructions of toric varieties

In this section, we give an algorithm to compute the Euler obstructions of toric varieties. A beautiful formula for the Euler obstructions of 2-dimensional toric varieties was proved by Gonzalez-Sprinberg [11]. Our result can be considered as a natural generalization of his formula.

First we recall the definition of Euler obstructions (for the detail see [13] and so on). Let X be an algebraic variety over $\mathbb{C}$. Then the Euler obstruction Eu_X of X is a $\mathbb{Z}$-valued function on X defined as follows. The value of Eu_X on the smooth part of X is defined to be 1. In order to define the value of Eu_X at a singular point $p \in X$, we take an affine open neighborhood U of p in X and a closed embedding $U \hookrightarrow \mathbb{C}^N$. Next we choose a Whitney stratification $U = \bigsqcup_{\alpha \in A} U_\alpha$ of U in $\mathbb{C}^N$. Then the values $\mathrm{Eu}_X(U_\alpha)$ of Eu_X on the strata U_α are defined by induction on codimensions of U_α as follows.

(i) If U_α is contained in the smooth part of U, we set $\mathrm{Eu}_X(U_\alpha) = 1$.

(ii) Assume that for $k \geq 0$ the values of Eu_X on the strata U_α such that $\mathrm{codim} U_\alpha \leq k$ are already determined. Then for a stratum U_β such that $\mathrm{codim} U_\beta = k+1$ the value $\mathrm{Eu}_X(U_\beta)$ is defined by

$$\mathrm{Eu}_X(q) = \sum_{U_\beta \subsetneqq \overline{U_\alpha}} \chi(U_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon)) \cdot \mathrm{Eu}_X(U_\alpha) \tag{3.1}$$

for sufficiently small $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, where $q \in U_\beta$ and f is a holomorphic function defined in an open neighborhood of q in $\mathbb{C}^N$ such that $U_\beta \subset f^{-1}(0)$ and $(q; \mathrm{grad} f(q)) \in T^*_{U_\beta}\mathbb{C}^N \setminus \left(\bigcup_{U_\beta \subsetneqq \overline{U_\alpha}} \overline{T^*_{U_\alpha}\mathbb{C}^N} \right)$.

Now let us return to the toric case. Let $N \simeq \mathbb{Z}^n$ be a $\mathbb{Z}$-lattice of rank n and σ a strongly convex rational polyhedral cone in $N_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} N$. Taking the dual $\mathbb{Z}$-lattice M of N and the polar cone $\sigma^\vee$ of σ in $M_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} M$, we obtain a semigroup $\mathcal{S}_\sigma := \sigma^\vee \cap M$ and an affine toric variety $X := U_\sigma = \mathrm{Spec}(\mathbb{C}[\mathcal{S}_\sigma])$. If we denote by T_α the T-orbit which corresponds to a face Δ_α of $\sigma^\vee$, we obtain a decomposition $X = \bigsqcup_{\Delta_\alpha \prec \sigma^\vee} T_\alpha$ of $X = U_\alpha$ into T-orbits. By the above recursive definition (ii) of Eu_X, in order to compute the Euler obstruction $\mathrm{Eu}_X \colon X \longrightarrow \mathbb{Z}$ it suffices to determine the following numbers.

Definition 3.1. For two faces Δ_α, Δ_β of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$ (i.e. $T_\beta \subsetneqq \overline{T_\alpha}$), we define the linking number $l_{\alpha,\beta} \in \mathbb{Z}$ of T_α along T_β as follows. First we choose a reference point $q \in T_\beta$ and a closed embedding $\iota \colon X = U_\sigma \hookrightarrow \mathbb{C}^N$. Then we set

$$l_{\alpha,\beta} := \chi(T_\alpha \cap f^{-1}(\eta) \cap B(q;\varepsilon)) \tag{3.2}$$

for sufficiently small $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, where f is a holomorphic function defined in an open neighborhood of q in $\mathbb{C}^N$ such that $T_\beta \subset f^{-1}(0)$ and $(q; \mathrm{grad} f(q)) \in T^*_{T_\beta}\mathbb{C}^N \setminus \left(\bigcup_{\Delta_\beta \precneqq \Delta_\kappa} \overline{T^*_{T_\kappa}\mathbb{C}^N} \right)$.

Note that the above definition of the linking number $l_{\alpha,\beta}$ does not depend on the choice of $q \in T_\beta$, ι, ε, η and f and so on. Since this linking number $l_{\alpha,\beta}$ can be defined also by taking a normal slice of T_β at a point $q \in T_\beta$, we can apply our Theorem 2.8 to a generic linear form on the normal slice and express $l_{\alpha,\beta}$ in terms of the geometry of the cones Δ_α and Δ_β as follows. First take the smallest linear subspace $\mathbb{L}(\Delta_\beta)$ of $M_\mathbb{R}$ containing Δ_β and consider the $\mathbb{Z}$-lattice $M_\beta := M \cap \mathbb{L}(\Delta_\beta)$ of rank $\dim\Delta_\beta$. Next set $\mathbb{L}(\Delta_\beta)' := M_\mathbb{R}/\mathbb{L}(\Delta_\beta)$ and let $p_\beta \colon M_\mathbb{R} \longrightarrow \mathbb{L}(\Delta_\beta)'$ be the natural projection. Then $M'_\beta := p_\beta(M) \subset \mathbb{L}(\Delta_\beta)'$ is a $\mathbb{Z}$-lattice of rank $n - \dim\Delta_\beta$ in $\mathbb{L}(\Delta_\beta)'$. By the condition $\Delta_\beta \precneqq \Delta_\alpha$, the set $K_{\alpha,\beta} := p_\beta(\Delta_\alpha) \subset \mathbb{L}(\Delta_\beta)'$ is a proper convex cone with apex $0 \in \mathbb{L}(\Delta_\beta)'$ in $\mathbb{L}(\Delta_\beta)'$.

Definition 3.2. We define the normalized relative subdiagram volume $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)$ of Δ_α along Δ_β by

$$\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta) := \mathrm{Vol}_\mathbb{Z}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}), \tag{3.3}$$

where $\Theta_{\alpha,\beta}$ is the convex hull of $K_{\alpha,\beta} \cap (M'_\beta \setminus \{0\})$ in the closed convex cone $K_{\alpha,\beta} \subset \mathbb{L}(\Delta_\beta)' \simeq \mathbb{R}^{\mathrm{codim}\Delta_\beta}$ and $\mathrm{Vol}_\mathbb{Z}(K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta})$ is the $(\dim\Delta_\alpha - \dim\Delta_\beta)$-dimensional normalized volume of $K_{\alpha,\beta} \setminus \Theta_{\alpha,\beta}$ with respect to the lattice $M'_\beta \cap \mathbb{L}(K_{\alpha,\beta})$.

Theorem 3.3 ([23]). *For any pair $(\Delta_\alpha, \Delta_\beta)$ of faces of $\sigma^\vee$ such that $\Delta_\beta \precneqq \Delta_\alpha$, the linking number $l_{\alpha,\beta}$ of T_α along T_β is given*

$$l_{\alpha,\beta} = (-1)^{\dim\Delta_\alpha - \dim\Delta_\beta - 1}\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta). \tag{3.4}$$

Now let us consider the case of projective toric varieties associated with lattice points. We inherit the situation and the notations in Section 1. First assume that the polytope P is sufficiently large and $A = P \cap M$ so that the line bundle on X_{Σ_P} associated with P is very ample and $X_A \simeq X_{\Sigma_P}$ (see the end of Section 1 and [26, Theorem 2.13]). In this case, X_A is normal. If we denote by T_α the T-orbit in X_A which corresponds to a face Δ_α of P, we obtain a decomposition $X_A = \bigsqcup_{\Delta_\alpha \prec P} T_\alpha$ of X_A into T-orbits. Now let Δ_α, Δ_β be two faces of P such that $\Delta_\beta \precneqq \Delta_\alpha$. Since there exists a T-invariant affine open subset of X_A containing both T_α and T_β, we can define the linking number $l_{\alpha,\beta}$ of T_α along T_β by the previous arguments. For example, if we choose a vertex $v \in \Delta_\beta$ of the smaller face Δ_β, for the maximal cone $\sigma \in \Sigma_P$ in the normal fan Σ_P which corresponds to the 0-dimensional face $\{v\} \prec P$ of P, we have T_α, $T_\beta \subset U_\sigma$. In order to give a formula for $l_{\alpha,\beta}$, let us fix such $v \in P$ and $\sigma \in \Sigma_P$. Then, by the dilation action of the multiplicative group $\mathbb{R}_{>0}$ on $M_\mathbb{R}$, we have the equality $\mathbb{R}_+(P - v) = \sigma^\vee$ in $M_\mathbb{R}$ which gives rise to the natural correspondence:

$$\{\text{faces of } P \text{ containing } v\} \overset{1:1}{\longleftrightarrow} \{\text{faces of } \sigma^\vee\}. \tag{3.5}$$

Note that this correspondence is compatible with the ones introduced before. Therefore, by taking the two faces of $\sigma^\vee$ which correspond to Δ_α and Δ_β through this correspondence, we can define the normalized relative subdiagram volume $\mathrm{RSV}_\mathbb{Z}(\Delta_\alpha, \Delta_\beta)$ of Δ_α

along Δ_β. Hence by Theorem 3.3 we can calculate the linking number $l_{\alpha,\beta}$ by the combinatorial formula

$$l_{\alpha,\beta} = (-1)^{\dim\Delta_\alpha - \dim\Delta_\beta - 1}\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta). \tag{3.6}$$

Consequently, the Euler obstruction $\mathrm{Eu}_{X_A} : X_A \longrightarrow \mathbb{Z}$ of X_A is calculated as follows. Since Eu_{X_A} is constant on each stratum T_α, we denote by $\mathrm{Eu}(\Delta_\alpha)$ the value of Eu_{X_A} on T_α. Then all the values $\mathrm{Eu}(\Delta_\alpha)$ are determined by induction on codimensions of faces of P by the rule:

(i) $\mathrm{Eu}(P) := \mathrm{Eu}_{X_A}(T) = 1$,

(ii) $\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \precneqq \Delta_\alpha} (-1)^{\dim\Delta_\alpha - \dim\Delta_\beta - 1}\mathrm{RSV}_{\mathbb{Z}}(\Delta_\alpha, \Delta_\beta) \cdot \mathrm{Eu}(\Delta_\alpha)$.

Combining these results with the combinatorial description of the intersection cohomology complexes of toric varieties obtained by Denef-Loeser [4], Fieseler [8], Stanley [29] and so on, we can easily compute the characteristic cycle of the intersection cohomology complex of arbitrary normal toric variety (see [2] for another approach to this problem). For the definition of intersection cohomology complexes, see [12] and so on. We can also compute the Chern-Mather classes of complete toric varieties by using the results of Ehlers (unpublished) and Barthel-Brasselet-Fieseler [1]. See [23] for the detail.

From now on, we give a combinatorial geometric description of Eu_{X_A} for general finite subsets $A \subset M \simeq \mathbb{Z}^n$. Without loss of generality, we may assume that the rank of the $\mathbb{Z}$-lattice $M(A)$ generated by A is n. For each face Δ_α of P, consider the smallest affine subspace $\mathbb{L}(\Delta_\alpha)$ of $M_{\mathbb{R}}$ containing Δ_α and the $\mathbb{Z}$-lattice $M_\alpha := M(A \cap \Delta_\alpha)$ generated by $A \cap \Delta_\alpha$ in $\mathbb{L}(\Delta_\alpha)$. Now let us fix two faces Δ_α, Δ_β of P such that $\Delta_\beta \precneqq \Delta_\alpha$. By taking a suitable affine transformation of the lattice $M(A)$, we may assume that the origin of $M(A)$ is a vertex of the smaller face Δ_β. By using this choice of the origin $0 \in \Delta_\beta \subset M(A)$, let us consider the subsemigroup $\mathcal{S}_\alpha$ of M_α generated by $A \cap \Delta_\alpha$. Although $\mathcal{S}_\alpha$ depends also on Δ_β and so on, we denote it by $\mathcal{S}_\alpha$ to simplify the notation. Denote by M_α/Δ_β the quotient lattice $M_\alpha/(M_\alpha \cap \mathbb{L}(\Delta_\beta))$ of rank $\dim\Delta_\alpha - \dim\Delta_\beta$. Then the following definitions are essentially due to [10].

Definition 3.4 ([10]).

(i) We denote by $\mathcal{S}_\alpha/\Delta_\beta$ the image of $\mathcal{S}_\alpha \subset M_\alpha$ in the quotient $\mathbb{Z}$-lattice M_α/Δ_β.

(ii) We denote by $K(\mathcal{S}_\alpha/\Delta_\beta)$ (resp. $K_+(\mathcal{S}_\alpha/\Delta_\beta)$) the convex hull of $\mathcal{S}_\alpha/\Delta_\beta$ (resp. $(\mathcal{S}_\alpha/\Delta_\beta) \setminus \{0\}$) in $(M_\alpha/\Delta_\beta)_{\mathbb{R}}$ and set $K_-(\mathcal{S}_\alpha/\Delta_\beta) := \overline{K(\mathcal{S}_\alpha/\Delta_\beta) \setminus K_+(\mathcal{S}_\alpha/\Delta_\beta)}$. We call $K_-(\mathcal{S}_\alpha/\Delta_\beta)$ the subdiagram part of the semigroup $\mathcal{S}_\alpha/\Delta_\beta$ and denote by $u(\mathcal{S}_\alpha/\Delta_\beta)$ its normalized volume with respect to the $\mathbb{Z}$-lattice $M_\alpha/\Delta_\beta \subset (M_\alpha/\Delta_\beta)_{\mathbb{R}}$.

Finally let us recall the definition of the index $i(\Delta_\alpha, \Delta_\beta) \in \mathbb{Z}_{>0}$ given by [10, Chapter 5, (3.1)].

Definition 3.5 ([10]). For two faces Δ_α, Δ_β of P such that $\Delta_\beta \precneqq \Delta_\alpha$, we define $i(\Delta_\alpha, \Delta_\beta)$ as the index

$$i(\Delta_\alpha, \Delta_\beta) := [M_\alpha \cap \mathbb{L}(\Delta_\beta) \colon M_\beta]. \tag{3.7}$$

Now recall that by [10, Chapter 5, Proposition 1.9] we have the basic correspondence:

$$\{\text{faces of } P\} \overset{1:1}{\longleftrightarrow} \{T\text{-orbits in } X_A\}. \tag{3.8}$$

For a face $\Delta_\alpha \prec P$ of P, we denote by T_α the corresponding T-orbit in X_A. We also denote by $\mathrm{Eu}(\Delta_\alpha)$ the value of the Euler obstruction $\mathrm{Eu}_{X_A} \colon X_A \longrightarrow \mathbb{Z}$ on T_α as before. Then by Theorem 2.8 and [10, Chapter 5, Theorem 3.1] all the values $\mathrm{Eu}(\Delta_\alpha)$ are determined by:

(i) $\mathrm{Eu}(P) = 1$,

(ii) $\mathrm{Eu}(\Delta_\beta) = \sum_{\Delta_\beta \precneqq \Delta_\alpha} (-1)^{\dim\Delta_\alpha - \dim\Delta_\beta - 1} i(\Delta_\alpha, \Delta_\beta) \cdot u(\mathcal{S}_\alpha/\Delta_\beta) \cdot \mathrm{Eu}(\Delta_\alpha)$.

§ 4. An example

In this section, we give an example of integral convex polytopes for which the degree of the A-discriminant is easily computed by our method.

Consider the 3-dimensional case. For a $\mathbb{Z}$-basis $\{m_1, m_2, m_3\}$ of $M \simeq \mathbb{Z}^3$, let P be the 3-dimensional simplex with vertices $v_1 = m_1$, $v_2 = m_2$, $v_3 = 2m_3$, $v_4 = 0$ and set $A := P \cap M = \{0, m_1, m_2, m_3, 2m_3\}$. Then we can easily check that the condition in [26, Theorem 2.13] is satisfied. Namely the line bundle on X_{Σ_P} associated with P is very ample and $X_A \simeq X_{\Sigma_P}$ in $\mathbb{P}^4$ in this case.

Let us compute the values of the Euler obstruction Eu_{X_A} of X_A by our algorithm. For $\alpha \subset \{1, 2, 3, 4\}$, we denote by Δ_α the face of P whose vertices are $\{v_i \mid i \in \alpha\}$. We can easily determine the values of Eu_{X_A} on the 2 and 3-dimensional T-orbits:

$$\mathrm{Eu}(P) = \mathrm{Eu}(\Delta_{123}) = \mathrm{Eu}(\Delta_{124}) = \mathrm{Eu}(\Delta_{134}) = \mathrm{Eu}(\Delta_{234}) = 1. \tag{4.1}$$

Starting from the values (4.1), we can determine the values of the Euler obstruction Eu_{X_A} on 1-dimensional T-orbits:

$$\mathrm{Eu}(\Delta_{12}) = 0, \quad \mathrm{Eu}(\Delta_{13}) = \mathrm{Eu}(\Delta_{14}) = \mathrm{Eu}(\Delta_{23}) = \mathrm{Eu}(\Delta_{24}) = \mathrm{Eu}(\Delta_{34}) = 1. \tag{4.2}$$

For example, $\mathrm{Eu}(\Delta_{12})$ is computed as follows. Since $M_{12} = \mathbb{Z}(m_1 - m_2)$, $M'_{12} = \mathbb{Z}m_2 + \mathbb{Z}m_3$. Therefore we have

$$\begin{aligned}
\mathrm{Eu}(\Delta_{12}) &= -\mathrm{RSV}_{\mathbb{Z}}(P, \Delta_{12})\mathrm{Eu}(P) + \mathrm{RSV}_{\mathbb{Z}}(\Delta_{123}, \Delta_{12})\mathrm{Eu}(\Delta_{123}) \\
&\quad + \mathrm{RSV}_{\mathbb{Z}}(\Delta_{124}, \Delta_{12})\mathrm{Eu}(\Delta_{124}) \qquad (4.3)\\
&= -2 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 0. \qquad (4.4)
\end{aligned}$$

Moreover, the values of the Euler obstruction Eu_{X_A} on 0-dimensional T-orbits are determined from the values (4.1) and (4.2):

$$\mathrm{Eu}(\Delta_1) = \mathrm{Eu}(\Delta_2) = 0, \quad \mathrm{Eu}(\Delta_3) = \mathrm{Eu}(\Delta_4) = 1. \tag{4.5}$$

For example, $\mathrm{Eu}(\Delta_1)$ is computed as follows.

$$\begin{aligned}
\mathrm{Eu}(\Delta_1) &= \mathrm{RSV}_{\mathbb{Z}}(P, \Delta_1)\mathrm{Eu}(P) - \mathrm{RSV}_{\mathbb{Z}}(\Delta_{123}, \Delta_1)\mathrm{Eu}(\Delta_{123}) \qquad (4.6)\\
&\quad - \mathrm{RSV}_{\mathbb{Z}}(\Delta_{124}, \Delta_1)\mathrm{Eu}(\Delta_{124}) - \mathrm{RSV}_{\mathbb{Z}}(\Delta_{134}, \Delta_1)\mathrm{Eu}(\Delta_{134}) \\
&\quad + \mathrm{RSV}_{\mathbb{Z}}(\Delta_{12}, \Delta_1)\mathrm{Eu}(\Delta_{12}) + \mathrm{RSV}_{\mathbb{Z}}(\Delta_{13}, \Delta_1)\mathrm{Eu}(\Delta_{13}) \\
&\quad + \mathrm{RSV}_{\mathbb{Z}}(\Delta_{14}, \Delta_1)\mathrm{Eu}(\Delta_{14}) \\
&= 2 \cdot 1 - 1 \cdot 1 - 1 \cdot 1 - 2 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0. \qquad (4.7)
\end{aligned}$$

Now let us compute the codimension and degree of the dual variety X_A^* of X_A. By (1.6), δ_1, δ_2 and δ_3 are computed as follows.

$$\begin{aligned}
\delta_1 &= \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta}(1 + \dim\Delta)\mathrm{Vol}_{\mathbb{Z}}(\Delta)\mathrm{Eu}(\Delta) \qquad (4.8)\\
&= (1+3) \cdot 2 \cdot 1 - (1+2) \cdot (1+1+2+2) \cdot 1 \qquad (4.9)\\
&\quad + (1+1) \cdot (1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1) \\
&\quad - (1+0) \cdot 1 \cdot (0+0+1+1) = 0, \\
\delta_2 &= \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta}\left\{\binom{\dim\Delta - 1}{2} - 3\right\}\mathrm{Vol}_{\mathbb{Z}}(\Delta)\mathrm{Eu}(\Delta) \qquad (4.10)\\
&= (1-3) \cdot 2 \cdot 1 - (0-3) \cdot (1+1+2+2) \cdot 1 \qquad (4.11)\\
&\quad + (0-3) \cdot (1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1) \\
&\quad - (1-3) \cdot 1 \cdot (0+0+1+1) = 0, \\
\delta_3 &= \sum_{\Delta \prec P} (-1)^{\mathrm{codim}\Delta}\left\{\binom{\dim\Delta - 1}{3} + 4\right\}\mathrm{Vol}_{\mathbb{Z}}(\Delta)\mathrm{Eu}(\Delta) \qquad (4.12)\\
&= (0+4) \cdot 2 \cdot 1 - (0+4) \cdot (1+1+2+2) \cdot 1 \qquad (4.13)\\
&\quad + (0+4) \cdot (1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 2 \cdot 1) \\
&\quad - (-1+4) \cdot 1 \cdot (0+0+1+1) = 2.
\end{aligned}$$

By Theorem 1.4, we obtain

$$\mathrm{codim} X_A^* = 3, \quad \deg X_A^* = 2. \tag{4.14}$$

In this case, we can easily check these results by direct computation. Indeed, note that

$$X_A = U_1 \cup U_2 \cup U_3 \cup U_4, \tag{4.15}$$

$$U_1 \simeq U_2 \simeq \{(x,y,z,w) \in \mathbb{C}^4 \mid xy = z^2\}, \quad U_3 \simeq U_4 \simeq \mathbb{C}^3. \tag{4.16}$$

Here U_i denotes the affine toric variety which corresponds to a vertex v_i of P ($i = 1,2,3,4$).

For the list of X_A with large dual defect, see the recent results in [3] and [6].

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