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Birkhoff normal forms for superintegrable systems

By

Hidekazu Ito*

Abstract

We study the convergence problem of Birkhoff normalization for holomorphic Hamiltonian systems, and show that there exists a convergent Birkhoff normalization if the number of integrals is balanced with the resonance degree of the equilibrium point.

§ 1. Introduction

In this paper, we study the Birkhoff normalization for Hamiltonian systems. Let H be a holomorphic function of $z \in \mathbf{C}^{2n}$ near the origin:

$$H = H_2 + H_3 + H_4 + \cdots , \tag{1.1}$$

where H_j are homogeneous polynomials in $z = (z_1, \ldots, z_{2n})$ with $z_i = x_i, z_{n+i} = y_i$. Let X_H denote the Hamiltonian vector field

$$\dot{z} = JH_z\,, \qquad J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix},$$

where $\dot{z} = dz/dt$, $H_z = {}^t(H_{z_1}, \ldots, H_{z_{2n}})$ is the gradient vector of H and I is the identity matrix of degree n. The Poisson bracket is defined for any functions f and g as follows:

$$\{f, g\} = \langle f_z, Jg_z \rangle \quad (\langle \cdot, \cdot \rangle \text{ is the Euclidean inner product}).$$

After a linear symplectic transformation, we may assume that

$$H_2 = S + N, \qquad S = \sum_{k=1}^{n} \lambda_k x_k y_k\,, \qquad \{S, N\} = 0, \tag{1.2}$$

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*Division of Mathematics and Physics, Graduate School of Natural Science and Technology, Kanazawa University, Kanazawa 920-1192, Japan.

where N is the quadratic form with the property that the coefficient matrix of X_N is nilpotent. We allow here the degenerate situation where some of λ_k are equal to zero. Here and in what follows, a transformation is called *symplectic* if it preserves the standard symplectic structure $\sum_{k=1}^n dy_k \wedge dx_k$.

Definition 1.1. The Hamiltonian H is in *Birkhoff normal form* (or we call H itself *Birkhoff normal form*) if the identity $\{H, S\} = 0$ holds. We also say generally that a function f is in *S-normal form* if the identity $\{f, S\} = 0$ holds.

Since $\{H, S\} = X_S H = -X_H S$, the relation $\{H, S\} = 0$ implies

- H is invariant under the flow of X_S, i.e., H is averaged along orbits of X_S.
- S is invariant under the flow of X_H, i.e., S is an integral of X_H.

One can find a formal symplectic transformation $\varphi\colon z \mapsto z+O(|z|^2)$ such that $H\circ\varphi$ is in Birkhoff normal form. More precisely, we have

Theorem 1.2. *Let $H = H_2 + H_3 + \cdots$ be a Hamiltonian with $H_2 = S + N$ satisfying (1.2). Then,*

(1) *For any integer $N \geq 2$, there exists a holomorphic symplectic transformation $\varphi\colon z \mapsto z + O(|z|^2)$ such that*

$$H\circ\varphi(z) = h(z) + O(|z|^{N+1}), \qquad \{h, S\} = 0. \tag{1.3}$$

Hence there exists a formal symplectic transformation φ such that $\{H\circ\varphi, S\} = 0$.

(2) *Let $k = (k_1, \ldots, k_n) \in \mathbf{Z}^n$, $|k| = \sum_{j=1}^n |k_j|$, $\lambda = (\lambda_1, \ldots, \lambda_n)$. If the condition*

$$\langle k, \lambda\rangle \Big(= \sum_{j=1}^n k_j\lambda_j \Big) = 0 \quad (|k| \leq N) \Rightarrow k = 0 \tag{1.4}$$

holds for some integer $N \geq 4$, then the Birkhoff normal form $h(z)$ is a function (polynomial of degree $\leq [N/2]$) of n variables $\omega_k = x_k y_k$ $(k = 1, \ldots, n)$.

In the case (1.3) above, we say that $H\circ\varphi(z)$ is in *Birkhoff normal form up to order N*. The near-to-identity transformation φ is called *Birkhoff transformation.* One can prove item (1) using the generating function or Lie series technique to define a desired Birkhoff transformation. Furthermore, item (2) follows from comparison of the coefficients in both sides of the identity $\{h, S\} = 0$. See Lemma 3.1 in §3.

Under condition (1.4), which we call *non-resonance condition (up to order N)*, the vector field X_h is solved explicitly. In fact, since h is a function of $\omega = (\omega_1, \ldots, \omega_n)$,

$$\dot{x}_k = \frac{\partial h}{\partial \omega_k} x_k, \quad \dot{y}_k = -\frac{\partial h}{\partial \omega_k} y_k \qquad (k = 1, \ldots, n),$$

where

$$\frac{d}{dt}\omega_k(t) = \dot{x}_k y_k + x_k \dot{y}_k = 0$$

and therefore ω_k are integrals of X_h. Hence the solution is expressed as

$$x_k(t) = e^{t\Omega_k} x_k(0), \quad y_k(t) = e^{-t\Omega_k} y_k(0) \qquad \left(\Omega_k = \frac{\partial h}{\partial \omega_k}(\omega(0))\right).$$

This means that the Birkhoff normal form under the non-resonance condition (1.4) gives a local approximation of a given Hamiltonian by integrable one.

If the Hamiltonian H is real analytic, one can define the real Birkhoff normal form in the same way as in Definition 1.1 with S replaced by a real quadratic normal form. Then one can choose a transformation φ in Theorem 1.2 to be real analytic. In particular, when the origin is an elliptic equilibrium point of X_H, the Birkhoff normal form h under the non-resonance condition (1.4) becomes a function of n variables $\widehat{\omega}_k = \frac{1}{2}(x_k^2 + y_k^2)$. Therefore, the flow of the vector field X_h gives rise to periodic or quasi-periodic motions on a real torus $\widehat{\omega}_k = \text{const.}$ $(k = 1, \ldots, n)$, and hence KAM theory and Nekhoroshev estimates can be applied to the Hamiltonian (1.3) with $N \geq 4$ (see [1]).

We consider complex normal form again because it is convenient to deal with also for real analytic Hamiltonians. We note that, in the non-resonance case, namely, when $\lambda_1, \ldots, \lambda_n$ are *rationally independent* (i.e, $N = \infty$ in (1.4)), there exists a formal Birkhoff transformation φ such that

$$H \circ \varphi(z) = h(\omega), \qquad \omega = (\omega_1, \ldots, \omega_n).$$

Therefore, if φ is convergent, the vector field X_H is integrable in the sense to be defined below. However, there does not exist in general a convergent Birkhoff transformation as C.L. Siegel [12] (more recently Pérez-Marco [10]) showed. Then the question arises:

Q: When does there exist a convergent Birkhoff transformation ?

One may relate this question to integrability of the original vector field. In fact, the following holds.

Theorem 1.3 ([4]). *Let $H = H_2 + H_3 + \cdots$ be a holomorphic Hamiltonian with $H_2 = S + N$ satisfying (1.2) and assume that $\lambda_1, \ldots, \lambda_n$ are rationally independent. Suppose that X_H has n integrals $G_1 (= H), G_2, \ldots, G_n$ which are holomorphic and functionally independent near the origin. Then, there exists a holomorphic Birkhoff transformation φ. Furthermore, for any integral G of X_H, $G \circ \varphi$ is a function of n variables $\omega_1, \ldots, \omega_n$.*

In the above, the functions $G_1, \ldots, G_n$ are *functionally independent* if the gradient vectors $\partial G_1/\partial z, \ldots, \partial G_n/\partial z$ are linearly independent on a dense open subset of the domain considered. Since $G_k \circ \varphi$ are functions of $\omega_1, \ldots, \omega_n$, we have $\{G_i, G_j\} = \{G_i \circ \varphi, G_j \circ \varphi\} = 0$ for any $i, j = 1, \ldots, n$. A $2n$-dimensional Hamiltonian vector field X_H is called *Liouville-integrable* or *completely integrable* if it has n integrals which are functionally independent and Poisson commuting. In the case above, X_H is called *analytically Liouville-integrable* since the integrals $G_1, \ldots, G_n$ are holomorphic. In the real case, the well-known Liouville-Arnold theorem gives the description of the phase space of smooth integrable system as the foliation of n-dimensional invariant tori on which the flow is periodic or quasi-periodic. The case above with elliptic equilibrium point corresponds to this situation.

Theorem 1.3 is a generalization of the previous results by Rüssmann [11] and Vey [13] under some nondegeneracy condition. Its proof is constructive and uses the structure of simultaneous normalization of n integrals. It is extended to simple resonance cases [5, 7]. More recently, Zung [15] generalized Theorem 1.3 to general resonance cases by developing new geometric method based on the toric characterization of Birkhoff normalization.

Theorem 1.4 ([15]). *Let $H = H_2 + H_3 + \cdots$ be a holomorphic Hamiltonian with $H_2 = S + N$ satisfying (1.2). Suppose that X_H is analytically Liouville-integrable. Then, there exists a holomorphic Birkhoff transformation φ. Furthermore, for any integral G of X_H, $G \circ \varphi$ is in S-normal form.*

Under the assumption of commuting relations among integrals, this theorem includes Theorem 1.3 as a special case. In resonance cases, however, Theorem 1.4 does not claim any further information about the Birkhoff normal $H \circ \varphi$, such as whether $X_{H \circ \varphi}$ can be solved explicitly or not.

The aim of this note is to clarify this situation in resonance cases. To proceed further, we summarize different features with Birkhoff normal forms between non-resonance and resonance cases.

In non-resonance case:

- The Birkhoff normal form $H \circ \varphi$ is uniquely determined independently of the choice of the transformation φ, while φ is not uniquely determined. It is a power series in n variables $\omega_k = x_k y_k$.
- The vector field $X_{H \circ \varphi}$ admits n Poisson commuting integrals $\omega_1, \ldots, \omega_n$ and can be solved explicitly in the new coordinates.
- The number of functionally independent integrals of X_H is at most n.

In resonance case:

- The Birkhoff normal form $H \circ \varphi$ depends on the choice of the transformation φ. It generally contains other "resonant" terms in addition to those terms consisting of $\omega_1, \dots, \omega_n$.

- The existence of a convergent Birkhoff transformation does not necessarily imply the integrability of X_H if the resonance degree q defined in §2 is greater than 1.
- The Birkhoff normal form becomes more complicated in general as the resonance degree q increases.
- It is possible that the number of functionally independent integrals exceeds n.

Our purpose is to show that the non-resonance feature holds true also in resonance case if the number of integrals and the resonance degree are balanced. It leads to the study of Birkhoff normal forms for the so-called *superintegrable (non-commutatively integrable)* systems. Superintegrability is characterized by the existence of integrals whose number is more than one half the dimension of the phase space. There are many examples of such systems in classical mechanics, such as the Kepler problem, the free rigid body motion (Euler-Poinsot system) and etc. Nevertheless, it seems that the Birkhoff normal form for superintegrable system has not been studied in detail until now.

In the rest of the paper, we state the results in §2 and describe in §3 the idea of the proof of the main theorem. We refer to [6] for the detailed proof.

§ 2. Statement of the results

a. The main result Let $\mathcal{R}$ be the discrete subgroup of $\mathbf{Z}^n$ defined by

$$\mathcal{R} := \{k = (k_1, \dots, k_n) \in \mathbf{Z}^n \mid \langle k, \lambda \rangle = 0\}, \quad \langle k, \lambda \rangle = \sum_{j=1}^{n} k_j \lambda_j.$$

We call this group $\mathcal{R}$ the *resonance lattice* for the quadratic form $S = \sum_{k=1}^{n} \lambda_k x_k y_k$. If $\dim_{\mathbf{Z}} \mathcal{R} = q$, we say that the quadratic form S (or the equilibrium point $z = 0$) is of *resonance degree* q and call the discrete group $\mathcal{R}$ the *resonance lattice of degree* q. Here $0 \le q \le n-1$, and the cases $q = 0$ and $q = 1$ correspond to the non-resonance and the simple resonance cases respectively.

Let $\rho^{(1)}, \dots, \rho^{(q)} \in \mathbf{Z}^n$ be the generators of the resonance lattice $\mathcal{R}$. Then, there exist $n - q$ linearly independent vectors $\rho^{(q+1)}, \dots, \rho^{(n)} \in \mathbf{Z}^n$ such that

$$\langle \rho^{(i)}, \rho^{(j)} \rangle = 0 \qquad (i = 1, \dots, q, \quad j = q+1, \dots, n). \tag{2.1}$$

We set

$$\omega_k = x_k y_k, \qquad \tau_k = \sum_{j=1}^{n} \rho_j^{(k)} \omega_j \qquad (k = 1, \dots, n), \tag{2.2}$$

where $\rho^{(k)} = (\rho_1^{(k)}, \ldots, \rho_n^{(k)})$. Furthermore, writing $\rho^{(k)} = \rho_+^{(k)} - \rho_-^{(k)}$ with $\rho_+^{(k)}$ and $\rho_-^{(k)}$ being vectors whose components are nonnegative integers, we define the monomial ω_{n+k} by

$$\omega_{n+k} = x^{\rho_+^{(k)}} y^{\rho_-^{(k)}} \qquad (k = 1, \ldots, q), \tag{2.3}$$

where we used multi-index notations. One can easily see that $\omega_1, \ldots, \omega_{n+q}$ are in S-normal form. For example, if $\lambda_k = 0$, one may take $\rho^{(k)} = \boldsymbol{e}_k$ (the unit vector) and then $\omega_{n+k} = x_k$. If $\lambda_k \neq 0$ for all $k = 1, \ldots, n$, then $\omega_{n+1}, \ldots, \omega_{n+q}$ are of degree ≥ 2.

We now state our main result.

Theorem 2.1. *Let $H = H_2 + H_3 + \cdots$ be a holomorphic function with $H_2 = S + N$ satisfying (1.2) and assume that S is of resonance degree q. Suppose that there exist $n + q$ integrals of X_H which are holomorphic and functionally independent near the origin. Then there exists a holomorphic Birkhoff transformation φ such that the Hamiltonian $H \circ \varphi$ becomes a function of $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$. Furthermore the following holds:*

(1) *The function $H \circ \varphi$ is a convergent power series in $\omega_1, \ldots, \omega_n$ that is uniquely determined independently of the choice of φ. In particular, the nilpotent part of H_2 vanishes, i.e., $N = 0$.*
(2) *The functions $\omega_1, \ldots, \omega_{n+q}$ are $n + q$ functionally independent integrals of $X_{H \circ \varphi}$.*
(3) *For any integral G of X_H, $G \circ \varphi$ is in S-normal form. It is a function of $n + q$ variables $\omega_1, \ldots, \omega_{n+q}$ and can be written as Laurent series in those variables.*

Remarks. (i) Items (1)-(3) are direct consequences of the fact that $H \circ \varphi$ is a function of the $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$. Apart from these items, we have the converse assertion to Theorem 2.1: *If there exists a holomorphic Birkhoff transformation φ such that $H \circ \varphi$ is a function of the $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$, then there exist $n + q$ integrals of X_H that are holomorphic and functionally independent near the origin.*

(ii) Theorem 2.1 is a natural generalization of Theorem 1.3, which corresponds to the case $q = 0$.

(iii) We do not assume any Poisson commuting relations among integrals. Therefore, Theorem 1.4 does not apply and the existence of a convergent Birkhoff transformation is not trivial at all. Actually, $\omega_1, \ldots, \omega_n$ are n Poisson commuting integrals of $X_{H \circ \varphi}$ and therefore X_H is Liouville-integrable near the origin. However it is not the assumption but a consequence of the theorem.

In Theorem 2.1, the vector field $X_{H \circ \varphi}$ can be solved explicitly for the new symplectic coordinates (Birkhoff coordinates), and those solutions are confined on the level set of the map $F(z) = (\omega_1, \dots, \omega_{n+q})$. The map $F(z)$ can be taken also as $F(z) = (F_1, \dots, F_{n+q})$ with

$$\begin{cases} F_i = \tau_{q+i} & (i = 1, \dots, n-q), \\ F_{n-q+i} = \tau_i & (i = 1, \dots, q), \\ F_{n+i} = \omega_{n+i} & (i = 1, \dots, q). \end{cases}$$

Then one can prove (see Lemma 3.1) that

$$\{F_i, F_j\} = 0 \qquad \text{for } \ i = 1, \dots, n-q, \quad j = 1, \dots, n+q.$$

In this case, we have $n-q$ commuting vector fields $X_{F_1}, \dots, X_{F_{n-q}}$ as well as their integrals $F_1, \dots, F_{n+q}$. Since H is a function of $\tau_{q+1}, \dots, \tau_n$, one may replace one of $\tau_{q+1}, \dots, \tau_n$ by H. This corresponds to the situation that were first studied as superintegrable systems by Nekhoroshev [9], and more recently is reformulated as the extended integrability by Bogoyavlenski [2]. Furthermore, the case with $F(z) = (\omega_1, \dots, \omega_{n+q})$ can be considered as a complex analytic version of superintegrable system with singularities in the sense of Michenko-Fomenko [8] (see also [3]). See [6] for details.

b. Relations between the number of integrals and the resonance degree We can derive from Theorem 2.1 the following consequence for generally non-integrable systems. Here and in what follows, the number of integrals, denoted by $\sharp$(integrals), means the number of integrals of X_H which are holomorphic and functionally independent near the origin.

Corollary 2.2. *Let $H = H_2 + H_3 + \cdots$ be a holomorphic function with $H_2 = S + N$ satisfying (1.2) and let q be a nonnegative integer. Then the following holds:*

(1) *If S is of resonance degree q, then $\sharp$(integrals) $\leq n+q$.*

(2) *If $\sharp$(integrals) $= n+q$, then S is of resonance degree $\geq q$.*

(3) *If $\sharp$(integrals) $= 2n-1$, then S is of resonance degree $n-1$ and the Birkhoff normal form $H \circ \varphi$ in Theorem 2.1 is a convergent power series in one variable $S = \sum_{k=1}^{n} \lambda_k x_k y_k$.*

The proof of this corollary is straightforward. In fact, items (1) and (2) follow from the item (3) of Theorem 2.1. Item (3) follows from (2) since the resonance degree q is at most $n-1$ and τ_n is an integer multiple of $H_2 = S$ in the case $q = n-1$.

It seems very special that an n degrees of freedom system possesses $2n-1$ integrals, but there are some well-known examples such as the Kepler problem and the Calogero model. The latter is a model describing the motions of interacting particles on the line and is defined by the Hamiltonian

$$H(x,y) = \sum_{k=1}^{n} (y_k^2 + \alpha^2 x_k^2) + \sum_{1 \le k < l \le n} \frac{1}{(x_k - x_l)^2} \qquad (\alpha \in \mathbf{R} \setminus \{0\}),$$

to which item (3) above can be applied near equilibria.

c. Real analytic case When the Hamiltonian is real analytic, Theorem 2.1 can be stated in real analytic category with S replaced by real quadratic normal form. We state it here only in the case of elliptic equilibrium point, where the real quadratic normal form is given by

$$\widehat{S} = \sum_{k=1}^{n} \frac{\alpha_k}{2}(x_k^2 + y_k^2) \qquad (\alpha_k \in \mathbf{R}).$$

We say that $\widehat{S}$ is of resonance degree q if the resonance lattice

$$\mathcal{R} := \{k \in \mathbf{Z}^n \mid \langle k, \alpha \rangle = 0\}, \qquad \alpha = (\alpha_1, \dots, \alpha_n),$$

is of dimension q over $\mathbf{Z}$. Correspondingly, we replace ω_k and τ_k by

$$\begin{cases} \widehat{\omega}_k = \dfrac{1}{2}(x_k^2 + y_k^2), \qquad \widehat{\tau}_k = \displaystyle\sum_{j=1}^{n} \rho_j^{(k)} \widehat{\omega}_j \qquad (k = 1, \dots, n), \\ \widehat{\omega}_{n+k} = \operatorname{Im} f_k(x,y) \qquad (k = 1, \dots, q) \end{cases}$$

with

$$f_k(x,y) = \omega_{n+k}\Big(\frac{x+iy}{\sqrt{2}}, \frac{y+ix}{\sqrt{2}}\Big) = \prod_{j=1}^{n} \left(\frac{x_j + iy_j}{\sqrt{2}}\right)^{\rho_{+j}^{(k)}} \left(\frac{y_j + ix_j}{\sqrt{2}}\right)^{\rho_{-j}^{(k)}},$$

where $i = \sqrt{-1}$ and $\operatorname{Im} f_k(x,y)$ denotes the imaginary part of the complex-valued function $f_k(x,y)$ in the real variable $(x,y) \in \mathbf{R}^{2n}$. Then we have

Theorem 2.3. *In the assumptions of Theorem 2.1, suppose that H is real analytic near the origin and that S is replaced by the real polynomial $\widehat{S}$. Then there exists a real analytic symplectic transformation $\varphi\colon z \mapsto z + O(|z|^2)$ such that $\{H \circ \varphi, \widehat{S}\} = 0$ and items (1), (2) and (3) of Theorem 2.1, except the second assertion of (3), hold with ω_k, τ_j replaced by $\widehat{\omega}_k$ and $\widehat{\tau}_j$. Moreover, each connected component of the regular level set of the real map $F(z) = (\widehat{\omega}_1, \dots, \widehat{\omega}_{n+q})$ is a torus of dimension $n - q$.*

Remark. This theorem holds also when $\widehat{\omega}_{n+k}$ are defined as the real parts of $f_k(x,y)$.

§ 3. Sketch of the proof of Theorem 2.1

We give a sketch of the proof of Theorem 2.1 in two steps. For details, we refer to [6].

a. First we describe the idea of the proof of the existence of a convergent Birkhoff transformation. The proof is purely constructive and uses the rapidly convergent iteration technique. First of all, we note the following

Lemma 3.1. *Let $S = \sum_{j=1}^{n} \lambda_j x_j y_j$ and $\mathcal{R}$ the resonance lattice of degree q for S. Let f be a power series written as $f = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} x^\alpha y^\beta$. Then the following holds:*

(1) *f is in S-normal form if and only if*

$$c_{\alpha\beta} = 0 \qquad \text{if } \ \alpha - \beta \notin \mathcal{R}.$$

(2) *Let $\tau_1, \ldots, \tau_n$ be the functions given by (2.2). Then f is in S-normal form if and only if the following $n - q$ identities hold:*

$$\{f, \tau_k\} = 0 \qquad (k = q + 1, \ldots n).$$

(3) *The monomials $\omega_1, \ldots, \omega_{n+q}$ given by (2.2) and (2.3) are in S-normal form and functionally independent.*

In fact, direct calculations yield the following formulae:

$$\{f, S\} = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} \langle \alpha - \beta, \lambda \rangle x^\alpha y^\beta, \qquad \{f, \tau_k\} = \sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} \langle \alpha - \beta, \rho^{(k)} \rangle x^\alpha y^\beta.$$

Then, it is straightforward to prove items (1)-(3) except the functional independence of $\omega_1, \ldots, \omega_{n+q}$. The last assertion can be shown by carrying out elementary transformations for the Jacobian matrix $\partial(\omega_1, \ldots, \omega_{n+q})/\partial(z_1, \ldots, z_{2n})$.

We have the following crucial fact about the algebra of all power series in S-normal form.

Lemma 3.2. *Let $S = \sum_{j=1}^{n} \lambda_j x_j y_j$ be a quadratic form of resonance degree q and let $\mathcal{B}$ be the set of all power series of $z \in \mathbf{C}^{2n}$ in S-normal form. Then $\mathcal{B}$ is the Lie algebra generated by a finite number of monomials $v_1, \ldots, v_N$ $(N \geq n + q)$ such that*

(1) *$v_i = \omega_i$ for $i = 1, \ldots, n + q$,*

(2) *$v_{n+q+1}, \ldots, v_N$ can be written as the quotients of two monomials in $\omega_1, \ldots, \omega_{n+q}$.*

Proof. By Lemma 3.1 (2), the S-normal forms are invariant under the $\mathbf{T}^{n-q}$-action generated by the vector fields $X_{i\tau_{q+1}}, \ldots, X_{i\tau_n}$ with $i = \sqrt{-1}$ (This fact played a key role in proving Zung's theorem [15]). This implies that $\mathcal{B}$ is finitely generated. See [6] for its elementary proof without using the knowledge of invariant theory.

Let us choose the generators $v_1, \ldots, v_N$ satisfying (1). To see item (2), let $x^\alpha y^\beta$ be a monomial in S-normal form. Then, since

$$\alpha - \beta = \sum_{j=1}^{q} c_j \rho^{(j)} \in \mathcal{R} \qquad (c_j \in \mathbf{Z}),$$

we have

$$x^\alpha y^\beta = x^{\alpha-\beta}(xy)^\beta, \qquad x^{\alpha-\beta} = \prod_{j=1}^{q} \left(x^{\rho^{(j)}}\right)^{c_j},$$

where $(xy)^\beta = \prod_{j=1}^{n}(x_j y_j)^{\beta_j}$. Here $x^{\rho^{(j)}}$ can be expressed as

$$x^{\rho^{(j)}} = x^{\rho_+^{(j)} - \rho_-^{(j)}} = \frac{x^{\rho_+^{(j)}} y^{\rho_-^{(j)}}}{(xy)^{\rho_-^{(j)}}} = \frac{\omega_{n+j}}{(xy)^{\rho_-^{(j)}}}.$$

This proves item (2). □

The desired Birkhoff transformation is obtained as the composition of infinite number of symplectic transformations. Each step is described in the following lemma. Below, $P_N W$ denotes the sum of all terms in S-normal form contained in W.

Lemma 3.3. *Let $H = H_2 + H_3 + \cdots$ be a holomorphic function with $H_2 = S + N$ satisfying (1.2). Assume that it is in Birkhoff normal form up to order $s_1 + d - 1$ ($s_1 = 2,\ d \geq 1$). Then there exists a unique polynomial W of the form*

$$W = W^{d+2} + \cdots + W^{2d+1} \quad \textit{with} \quad P_N W = 0\,, \tag{3.1}$$

W^l being homogeneous polynomials of degree l, such that the time-1 map $\varphi = \exp X_W$ takes H into Birkhoff normal form up to order $s_1 + 2d - 1$.

This proves the existence of a formal Birkhoff transformation as the limit of iteration procedure. We will give a proof of this lemma below in describing the simultaneous normalization of integrals.

In the following, for any power series $f = f(z)$ with $f(0) = 0$, we use the notation

$$f = f^0 + f^1 + \cdots; \quad f^0 \not\equiv 0 \quad (\deg f^0 = s \geq 1),$$

where f^d ($d = 0, 1, \ldots$) denotes the homogeneous polynomial of degree $s + d$. We say that f is in *S-normal form up to order $s + d$* if the polynomial $f^0 + f^1 + \cdots + f^d$ is in S-normal form.

The structure of the simultaneous normalization is described as follows:

Lemma 3.4. *Let $H = H_2 + H_3 + \cdots$ be a holomorphic function with $H^0 = H_2 = S + N$ satisfying (1.2). Let G be an integral of X_H. Assume that H is in S-normal form up to order $s_1 + d - 1$ ($s_1 = \deg H^0 = 2$). Then G is in S-normal form up to order $s + d - 1$ ($s = \deg G^0$).*

This lemma can be proved by comparing homogeneous parts of the identity $\{G, H\} = 0$ ([6, 5]).

Let $G_1 = H, G_2, \ldots, G_{n+q}$ be integrals of the vector field X_H that are holomorphic and functionally independent near the origin. By Ziglin's lemma ([14, 4]), we may assume that the lowest order parts $G_1^0, \ldots, G_{n+q}^0$ are functionally independent.

Let $\deg G_i^0 = s_i \geq 1$ ($s_1 = 2$). In view of Lemma 3.4, assume that G_i ($i = 1, \ldots, n+q$) are in S-normal form up to order $s_i + d - 1$. Then they can be written in the form

$$G_i(z) = g_i(z) + \widehat{G}_i(z)\,; \quad g_i = P_N g_i\,, \quad \widehat{G}_i = O(|z|^{s_i+d}). \tag{3.2}$$

We call g_i and $\widehat{G}_i$ the *normal form part* of G_i and the *remainder part* of G_i respectively.

Let $\varphi = \exp X_W$ be a symplectic transformation with W of the form (3.1). Then it can be written in the form

$$\varphi(z) = z + JW_z(z) + O(|z|^{2d+1})$$

and hence we have

$$G_i \circ \varphi(z) = g_i(z) + \{g_i(z), W(z)\} + \widehat{G}_i(z) + O(|z|^{s_i+2d}).$$

It turns out that $G_i \circ \varphi$ are in S-normal form up to order $s_i + 2d - 1$ if and only if W satisfies equations

$$\{g_i(z), W(z)\} = -P_R \widehat{G}_i(z) + O(|z|^{s_i+2d}) \qquad (i = 1, \ldots, n+q),$$

where $P_R = I - P_N$, namely $P_R \widehat{G}_i = \widehat{G}_i - P_N \widehat{G}_i$. By comparing homogeneous parts of degree $s_i + l$, we obtain the recursive relations

$$\{g_i^0, W^{l+2}\} = -P_R \widehat{G}_i^l - \sum_{\nu=1}^{l-d} \{g_i^\nu, W^{l+2-\nu}\} \quad (i = 1, \ldots, n+q) \tag{3.3}$$

for $l = d, d+1, \ldots, 2d-1$. We note that $g_1^0 = H_2 = S + N$. Assuming W^ν ($\nu = d+2, \ldots, l+1$) to be determined in such a way that $P_N W^\nu = 0$, one can determine a unique polynomial W^{l+2} satisfying (3.3) with $i = 1$ and the condition $P_N W^{l+2} = 0$. See [6] ([5]) for its proof. In the case when H_2 is semi-simple, i.e., $H_2 = S$, this can be easily checked. In fact, setting $W^{l+2} = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta$, the equation above reads

$$\sum_{\alpha,\beta \in \mathbf{Z}_+^n} c_{\alpha\beta} \langle \beta - \alpha, \lambda \rangle x^\alpha y^\beta = \text{known terms}$$

Then, the coefficients $c_{\alpha\beta}$ are uniquely determined if $\beta - \alpha \notin \mathcal{R}$. This proves the claim.

We now estimate the polynomial W determined above by using the fact that W satisfies the system of $n+q$ equations (3.3). Since g_i^ν are functions of $n+q$ variables $\omega_1, \ldots, \omega_{n+q}$, we have

$$\{g_i^\nu, W^{l+2-\nu}\} = \sum_{j=1}^{n+q} \frac{\partial g_i^\nu}{\partial \omega_j} \{\omega_j, W^{l+2-\nu}\} \qquad (\nu = 0, 1, \ldots, l-d).$$

Here $\partial g_i^\nu / \partial \omega_j$ is not generally holomorphic at the origin. However it turns out that

$$a_{ij}^\nu(z) := M(z) \frac{\partial g_i^\nu}{\partial \omega_j}, \qquad M(z) = \prod_{j=1}^{n+q} \omega_j,$$

are polynomials in z for any i, j, ν, and hence holomorphic at the origin. Multiplying the both sides of (3.3) by $M(z)$, we have the system of $n+q$ equations

$$\sum_{j=1}^{n+q} a_{ij}^0(z) \{\omega_j, W^{l+2}\} = F_i^l(z) \qquad (i = 1, \ldots, n+q),$$

where

$$F_i^l(z) = -M(z) P_R \widehat{G}_i^l - \sum_{\nu=1}^{l-d} \sum_{j=1}^{n+q} a_{ij}^\nu(z) \{\omega_j, W^{l+2-\nu}\}.$$

The functional independence of the lowest order parts $g_1^0, \ldots, g_{n+q}^0$ is equivalent to the condition

$$\det \left(a_{ij}^0(z) \right)_{i,j=1,\ldots,n+q} \neq 0 \qquad \text{on a dense open subset } \Omega' \text{ of } \mathbf{C}^{2n}.$$

One can prove the following

Lemma 3.5. *Let G_i $(i = 1, \ldots, n+q)$ be holomorphic functions of the form (3.2), and assume that their lowest order parts are functionally independent. Let W be the polynomial given in Lemma 3.3. Then $D_k W^{l+2} := \{\omega_k, W^{l+2}\}$ can be expressed as follows:*

$$D_k W^{l+2} = \frac{q_k^l(z)}{p(z)} \qquad (k = 1, \ldots, n+q\,;\ l = d, \ldots, 2d-1) \tag{3.4}$$

with

$$\begin{cases} p(z) = \det \left(a_{ij}^0(z) \right)_{i,j=1,\ldots,Q}, \\[2ex] q_k^l(z) = \det \begin{array}{c} \begin{array}{ccccc} \phantom{a_{11}^0} & & (k) & & \phantom{a_{1Q}^0} \end{array} \\ \begin{pmatrix} a_{11}^0 & \cdots & F_1^l & \cdots & a_{1Q}^0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{Q1}^0 & \cdots & F_Q^l & \cdots & a_{QQ}^0 \end{pmatrix} \end{array}, \end{cases} \qquad (Q = n+q).$$

Here the polynomials $q_k^l(z)$ are divisible by $p(z)$.

The formula (3.4) plays a key role in making estimate of W. To explain it, let $r > 0$ be a small parameter and define a polydisk

$$\Omega_r := \{z \in \mathbf{C}^{2n} \,|\, |z_i| < \delta_i r \quad (i = 1, \dots, 2n)\ \}$$

with some constants $0 < \delta_i < 1$. Here we can choose the constants δ_i in such a way that

$$|p(z)| \geq c_1 r^s \quad \text{on} \quad \Delta_r := \{\, z \in \mathbf{C}^{2n} \,|\, |z_i| = \delta_i r \quad (i = 1, \dots, 2n)\,\},$$

where $z = (z_1, \dots, z_{2n})$, $s = \deg p(z)$ and $c_1 > 0$ is a constant which is independent of r.

Let $A(\Omega_r)$ be the space of power series in z which are absolutely convergent on $\overline{\Omega}_r$, where $\overline{\Omega}_r$ is the closure of Ω_r. For a function $f = f^0 + f^1 + \cdots \in A(\Omega_r)$, we introduce the notations

$$|f|_r := \max_{z \in \overline{\Omega}_r} |f(z)|\,, \quad \|f\|_r := \sum_{d=0}^{\infty} |f^d|_r\,, \qquad \|f\|_{r,m} := \frac{\|f\|_r}{r^m}\,,$$

where m is an arbitrary integer. We note that the holomorphic function $f \in A(\Omega_r)$ attains the maximum $|f|_r$ at a point belonging to Δ_r. Then, using the formula (3.4), we have

$$\|D_k W^{l+2}\|_r \leq \frac{\|q_k^l(z)\|_r}{\min\limits_{z \in \Delta_r} |p(z)|} \leq \frac{1}{c_1} \|q_k^l(z)\|_{r,s}\,.$$

This is the fundamental step leading to the estimate of W in the form

$$\|W\|_r \leq c_2 |\!|\!| \widehat{G} |\!|\!|_r,$$

where $c_2 > 0$ is a constant independent of r, and

$$|\!|\!| \widehat{G} |\!|\!|_r := \sum_{i=1}^{n+q} \|\widetilde{\widehat{G}}_i\|_{r, s_i - 2}.$$

Here the notation $\widetilde{\widehat{G}_i}$ denotes the majorant series of $\widehat{G}_i$. See [6] for the proof as well as proofs of the later part.

Let us introduce the notation

$$|\!|\!| g |\!|\!|_r := \sum_{i=1}^{n+q} \|\widetilde{g}_i\|_{r, s_i - 2}.$$

Then, the final estimate of one iteration step is given as follows:

Lemma 3.6. *In addition to the assumption of Lemma 3.5, assume that* $G_i \in A(\Omega_r)$ $(i = 1, \ldots, n+q)$. *Then, there exists a sufficiently small* $r_0 > 0$ *such that, for any* $0 < r' < r < r_0$, $\sigma = r - \dfrac{2}{5}(r - r')$ *and* $\rho = r - \dfrac{1}{5}(r - r')$, *the symplectic transformation* $\varphi = \exp X_W$ *described in Lemma 3.3 is holomorphic on the domain* Ω_σ *and takes* Ω_σ *into* Ω_ρ. *Moreover, the normal form part* g_i' *and the remainder part* $\widehat{G}_i'$ *of* $G_i' = G_i \circ \varphi$ *satisfy the following estimates:*

$$\|g'\|_{r'} \le \left(\frac{r'}{r}\right)^2 \left\{ \|g\|_r + \|\widehat{G}\|_r \left(\frac{r'}{r}\right)^d \right\},$$

$$\|\widehat{G}'\|_{r'} \le c_3 \|\widehat{G}\|_r \left\{ \frac{\|\widehat{G}\|_r}{r^2 \left(1 - \dfrac{r'}{r}\right)^{2n+5}} + \frac{\left(\dfrac{5r'}{4r' + r}\right)^{2d+2}}{\left(1 - \dfrac{r'}{r}\right)^{2n+2}} \right\}.$$

Here c_3 *is a positive constant independent of* r, r'.

This estimate is good enough to prove uniform convergence of the iteration procedure. For the proof of this lemma and the uniform convergence, we refer to [6].

b. We give a brief sketch of the proof of the fact that the Birkhoff normal form $H \circ \varphi$ depends only on $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$ and is uniquely determined.

Let us denote H and G_k in place of $H \circ \varphi$, $G_k \circ \varphi$. By Lemma 3.2, H and G_k are functions of $n + q$ variables $\omega_1, \ldots, \omega_{n+q}$. Since $G_1, \ldots, G_{n+q}$ are integrals of X_H, we have the identities $\{H, G_k\} = 0$ $(k = 1, \ldots, n + q)$. Then we have

$$\{H, G_k\} = \sum_{i=1}^{n+q} \left(\sum_{j=1}^{n+q} \frac{\partial G_k}{\partial \omega_j} \{\omega_i, \omega_j\} \right) \frac{\partial H}{\partial \omega_i} = 0 \qquad (k = 1, \ldots, n + q),$$

which can be written in vector form

$$DG(\omega) A H_\omega = 0.$$

Here

$$DG(\omega) = \left(\frac{\partial G_i}{\partial \omega_j}\right)_{i,j=1,\ldots,n+q}, \; A = \Big(\{\omega_i, \omega_j\}\Big)_{i,j=1,\ldots,n+q}, \; H_\omega = \begin{pmatrix} H_{\omega_1} \\ \vdots \\ H_{\omega_{n+q}} \end{pmatrix}.$$

The functional independence of $G_1, \ldots, G_{n+q}$ implies that $DG(\omega)$ is nonsingular for a generic set of variables, and then we have the identity

$$AH_\omega = 0. \tag{3.5}$$

By simple computations, we see that

$$\begin{cases} \{\omega_i, \omega_j\} = 0 & (i, j = 1, \ldots, n), \\ \{\omega_i, \omega_{n+j}\} = -\rho_i^{(j)} \omega_{n+j} & (i = 1, \ldots, n, \quad j = 1, \ldots, q). \end{cases}$$

Using these relations and the linear independence of $\rho^{(1)}, \ldots, \rho^{(q)}$, one can deduce from comparison of the first n rows of (3.5) that

$$\frac{\partial H}{\partial \omega_{n+j}} = 0 \qquad (j = 1, \ldots, q).$$

Furthermore, comparing $(n+j)$-th rows of (3.5) for $j = 1, \ldots, q$, one can prove (see [6] for details) that

$$\frac{\partial H}{\partial \tau_j} = 0 \qquad (j = 1, \ldots, q).$$

Hence $H \circ \varphi$ depends only on the variables $\tau_{q+1}, \ldots, \tau_n$ and in particular, the quadratic form H_2 does not have the nilpotent part.

Finally, the proof of the uniqueness of the Birkhoff normal form goes as follows. Let φ_1 be a convergent Birkhoff transformation such that $H \circ \varphi_1$ can be written as a convergent power series in $n - q$ variables $\tau_{q+1}, \ldots, \tau_n$. Suppose that H is taken into Birkhoff normal form by another Birkhoff transformation φ_2. Then the transformation $\varphi = \varphi_1^{-1} \circ \varphi_2$ takes the Birkhoff normal form $K_1 = H \circ \varphi_1$ into another Birkhoff normal form $K_2 = H \circ \varphi_2$. Our aim is to prove $K_1 = K_2$.

Lemma 3.7. *Let $h(z)$ be a power series in z which depends only on $\tau_{q+1}, \ldots, \tau_n$, and let W be a power series with $W(0) = 0$ in S-normal form. Then h is invariant under the map* $\exp X_W$.

Proof. First we note that $\{h, W\} = 0$. In fact, by Lemma 3.1 (2), we have

$$\{h, W\} = \sum_{k=q+1}^{n} \frac{\partial h}{\partial \tau_k} \{\tau_k, W\} = 0.$$

We recall the Baker-Cambell-Hausdorff formula

$$h \circ \exp X_W = \sum_{m=0}^{\infty} \frac{1}{m!} ad_W^m h\,; \quad ad_W^0 h = h, \quad ad_W^m h = \{ad_W^{m-1} h, W\} \\ (m = 1, 2, \ldots).$$

Then we see that the identity $\{h, W\} = 0$ implies the identity $h \circ \exp X_W = h$. □

We note that the transformation $\varphi \colon z \mapsto z + O(|z|^2)$ can be written in the form

$$\begin{cases} \varphi = \varphi^{(\nu)} \circ \psi\,; \quad \varphi^{(\nu)} = \varphi_1 \circ \cdots \circ \varphi_\nu, \quad \varphi_\nu = \exp X_{W_\nu} \quad (\nu = 1, 2, \ldots) \\ \psi(z) = z + O(|z|^{2^\nu + 1}), \end{cases}$$

where ν is an arbitrary positive integer and W_ν is a polynomial of the form $W_\nu = W^{d+2} + \cdots + W^{2d+1}$ with $d = 2^{\nu-1}$ $(\nu = 1, 2, \ldots)$. Then, one can prove that each polynomial W_ν is in S-normal form. In fact, suppose that $W_1, \ldots, W_\nu$ are in S-normal form. Then, using Lemma 3.7, we have $K_1 \circ \varphi^{(\nu)} = K_1$ because K_1 depends only on $\tau_{q+1}, \ldots, \tau_n$. Let $h = K_1 \circ \varphi^{(\nu)} (= K_1)$ and $\varphi_{\nu+1} = \exp X_{W_{\nu+1}}$ with $W_{\nu+1} = W^{d+2} + \cdots + W^{2d+1}$ $(d = 2^\nu)$. Then we have

$$h \circ \varphi_{\nu+1} = h(z) + \{h, W_{\nu+1}\} + O(|z|^{s_0+2d})$$

Since this function is in S-normal form at least up to order $s_0 + 2d - 1$, $\{h, W_{\nu+1}\}$ has to be in S-normal form up to order $s_0 + 2d - 1$. In particular, its lowest order part $\{h^0, W^{d+2}\}$ is in S-normal form. Since $h^0 = S$, this implies that $\{S, W^{d+2}\} = 0$ and hence W^{d+2} is in S-normal form. One can also prove inductively that $W^{d+3}, \ldots, W^{2d+1}$ (and hence $W_{\nu+1}$) are in S-normal form. Then it follows from Lemma 3.7 that $h \circ \varphi_{\nu+1} = h$. By induction, it leads to the proof of $K_1 = K_2$ and completes the proof of (1) of Theorem 2.1.

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