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# The geometric structure of a virtual turning point and the model of the Stokes geometry

By

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## § 1. Introduction

The notion of a virtual turning point was first introduced by Aoki-Kawai-Takei [AKT1] to describe the complete Stokes geometry associated with a higher order linear differential equation with a large parameter. It is well known that we could not describe the correct Stokes geometry without it even for a simple third order equation (see [BNR]), and hence, it is essential and indispensable for the description of the Stokes geometry.

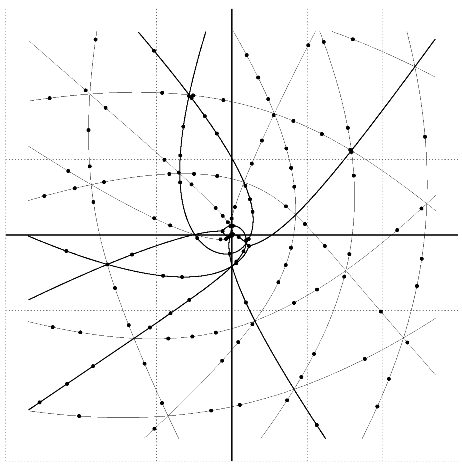

Figure 1. The Stokes geometry.

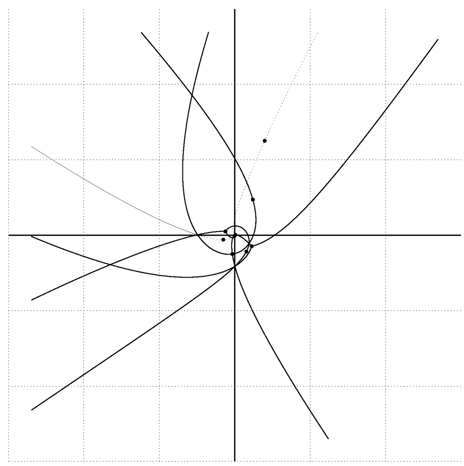

Figure 2. Our Ansatz has been applied.

One of notable facts for a virtual turning point is that the Stokes phenomenon does not necessarily occur on every portion of a new Stokes curve, i.e., a Stokes curve

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emanating from a virtual turning point. Hence an Ansatz was also given in the same paper [AKT1] to determine the inert portions of a new Stokes curve, on which the Stokes phenomena never occur.

Let us present an example which indicates the importance of the Ansatz concretely. Fig. 1 is a Stokes geometry for the Lax pair $\mathrm{NYL}_2$ of the Noumi-Yamada system $\mathrm{NY}_2$ (see [NY] and [T1] for the definitions of these systems) where a small dot denotes a virtual turning point and all the Stokes curves are drawn by solid lines. In describing this figure, as the system $\mathrm{NYL}_2$ has infinitely many virtual turning points, we have chosen finitely many virtual turning points, and the Stokes geometry is described only with them. If more and more virtual turning points were taken into account, then finally the figure would be blacked out because all the new Stokes curves form dense orbits in the complex plane.

Fig. 2 is the Stokes geometry to which the Ansatz has been applied. We draw by dotted line the portions on which the Stokes phenomena never occur, and moreover, in this figure we omit a new Stokes curve and its virtual turning point if the entire portion of the curve is inert in the above sense. We see that almost all the virtual turning points are inert in this example, and that the Ansatz is quite effective in determining these points.

Now to treat those inert portions of Stokes curves in an appropriate manner, we will introduce a boolean valued function called a state function of the Stokes geometry. The state function takes the zero value on the inert portions of Stokes curves, which are drawn by dotted lines in the figure, and otherwise it takes the value 1. Hence the Stokes geometry is, in our sense, the geometrical data (the turning points and the Stokes curves) equipped with the state function that satisfies our Ansatz. To avoid possible confusions, however, the word "**the Stokes geometry**" simply implies the geometrical data, and the data equipped with the state function will be called "**the model of the Stokes geometry**" in this paper.

Then in studying the model of the Stokes geometry, the most fundamental problem is its unique existence. This is not evident for the general Stokes geometry because we might solve a system of infinitely many equations given by the Ansatz.

The second problem is the following: Since the number of the turning points might be infinite, we usually calculate the state function for a finite subset of the turning points, and it is often said to be "the finite model" of the Stokes geometry. It is not known, however, that this finite model certainly gives an approximation of the true model.

The purpose of this paper is to answer these problems. As we see in Section 5, under some geometrical conditions, we show that the model of the Stokes geometry uniquely exists (Theorem 5.6 and Theorem 5.16), and that the finite model gives an approximation of the true model in the sense of Theorem 5.8 and Theorem 5.17.

The content of each section is as follows: Section 2 gives the definition of a virtual turning point. Then to describe the geometric structure underlying a virtual turning point, in Section 3, we introduce the Riemann manifold $\mathcal{R}_{\mathrm{sym}}$, in which our Stokes geometry (a turning point and a Stokes curve) is defined. This manifold has an important real analytic function that is deeply concerned with the configuration of the Stokes geometry. Hence we also give its several properties that are needed later on. Our Ansatz is explained with some examples in Section 4.

Section 5 is devoted to the study of our main problems. We first clarify the geometric situation where our study is undertaken. Then, in Subsection 5.2, we establish the main results for the special, but important case (Theorems 5.6 and 5.8). We construct a depth function of the Stokes geometry, which is a key in proving our results. The main results for the general case (Theorems 5.16 and 5.17) are in Subsection 5.3. The idea of the proof is to consider a Stokes path tree and its depth. We give, however, only sketches of proofs for the general case in this paper. The details are given in the paper [H4] which we are now preparing.

The author is deeply indebted to the members of the Kawai-Takei seminar. The paper is based on discussions in the seminar and their previous works. Especially Prof. T. Kawai gave me many valuable ideas and suggestions.

## § 2. Virtual Turning Points

We will consider a linear differential equation with a large parameter $\eta$ of the form:

$$Pu = \left(\eta^{-1}\frac{d}{dx} + A(x;\,\eta)\right)u = 0. \tag{2.1}$$

Here $A(x;\,\eta)$ designates an $n \times n$ matrix of formal power series of $\eta^{-1}$ in the form:

$$A(x;\,\eta) = A_0(x) + A_1(x)\eta^{-1} + A_2(x)\eta^{-2} + \dots \quad ,$$

and each matrix $A_j(x)$ $(j = 0, 1, \dots)$ consists of polynomial components of the variable $x$. Let $\Lambda(\lambda,\,x)$ denote the characteristic polynomial of the leading matrix $A_0(x)$, i.e.

$$\Lambda(\lambda,\,x) := \det(\lambda I - A_0(x)), \tag{2.2}$$

and $D(x)$ designates the discriminant of the equation $\Lambda(\lambda,\,x) = 0$ of the variable $\lambda$. Remember that a root of $D(x) = 0$ is called an **ordinary turning point**.

In what follows we always assume the following conditions (A-1) and (A-2):

- (**A-1**): All the roots of $D(x) = 0$ are simple, that is, the equation (2.1) has only simple turning points.
- (**A-2**): The analytic set $\{(\lambda, x) \in \mathbb{C}^2; \Lambda(\lambda, x) = 0\}$ is connected.

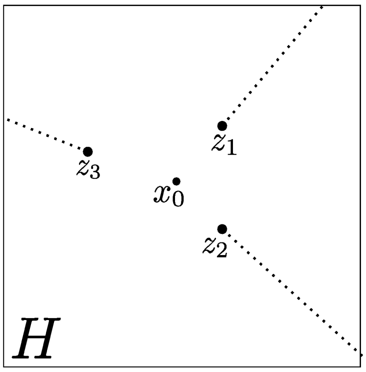


Figure 3. The cut space $H$.

Let $Z = \{z_1, z_2, \dots, z_m\}$ be the set of the ordinary turning points of (2.1), and let $H$ be the complex plane $\mathbb{C}$ equipped with cut lines as follows: We first take a point $x_0 \in \mathbb{C} \setminus Z$ such that any line passing through different ordinary turning points never passes the point $x_0$, and we fix it. Then we determine the cut lines in $H$ so that for every ordinary turning point $z_j \in Z$, there exists a cut line emanating from $z_j$ that tends to infinity, and that it is contained in the line passing through $x_0$ and $z_j$ as we see in Fig. 3.

In this cut space $H$, we denote the roots of the equation $\Lambda(\lambda, x) = 0$ of the variable $\lambda$ by:

$$\lambda_1(x),\ \lambda_2(x),\ \dots,\ \lambda_{n-1}(x),\ \lambda_n(x). \tag{2.3}$$

Then it follows from the definition of an ordinary turning point that for any point $z_l \in Z$ some roots $\lambda_i(x)$ and $\lambda_j(x)$ $(i \neq j)$ merge at $z_l$, and in this case $z_l$ is said to be of **type** $(i, j)$.

Now let us recall the definition of a virtual turning point. It was first introduced in [AKT1] as a self-intersection point of a bicharacteristic curve for the partial differential equation which is the Borel transform of the equation (2.1). The following equivalent definition is due to [AKKSST] and [T1]: Let $\mathbb{Z}_n$ (resp. $\mathbb{Z}_{n,\neq}$) denote the set of integers from 1 to $n$ (resp. $\{(i, j) \in \mathbb{Z}_n^2; i \neq j\}$) respectively.

**Definition 2.1.** A point $v \in \mathbb{C}$ is said to be a virtual turning point of type $(i, j) \in \mathbb{Z}_{n,\neq}$ if there exist a piecewise smooth path $C_v(\theta) : [0,1] \to \mathbb{C}$ with $C_v(0) = C_v(1) = v$ and a continuous function $\mu(x)$ on $C_v$ for which the following conditions are satisfied:

1. The function $\mu(x)$ is a root of the equation $\Lambda(\mu, x) = 0$ for any $x \in C_v$, and it satisfies

$$\mu(C_v(\theta)) = \lambda_i(C_v(\theta)) \text{ near } \theta = 0 \text{ and } \mu(C_v(\theta)) = \lambda_j(C_v(\theta)) \text{ near } \theta = 1. \tag{2.4}$$

2. The function $\mu(x)$ satisfies the integral relation:

$$\int_{C_v} \mu(z)dz = 0. \tag{2.5}$$

Note that an ordinary turning point is, from the logical viewpoint, a virtual turning point in the above sense. But, for the sake of convenience, we exclude ordinary turning points from the definition of virtual turning points. In what follows, a turning point means either an ordinary turning point or a virtual turning point. We can define a Stokes curve that emanates from a virtual turning point in the same way as in the case of an ordinary turning point. A Stokes curve emanating from a virtual turning point is often called a **new Stokes curve**.

*Remark.* Since every ordinary turning point considered in this paper is simple by (A-1), the path $C_v$ in Definition 2.1 can be taken away from the ordinary turning points (see Fig. 4). Hence the function $\mu(x)$ is nothing but an analytic continuation of $\lambda_i$ along $C_v$ in our case.

## § 3. The Riemann Manifold $\mathcal{R}_{\text{sym}}$

Taking Definition 2.1 into account, we may regard a virtual turning point as a zero point of some holomorphic function. To be more precise, let us define a (multi-valued) holomorphic function $\varphi(x)$ by replacing $v$ in (2.5) with the variable $x$:

$$\varphi(x) = \int_{C_x} \lambda_i(z)dz \tag{3.6}$$

where $C_x$ is some closed smooth path in $\mathbb{C} \setminus Z$ that starts from $x$, and the integration is done for an analytic continuation of $\lambda_i(z)$ along $C_x$. We assume that the consequence of the analytic continuation of $\lambda_i(z)$ is $\lambda_j(z)$ $(i \neq j)$. Then a zero point of $\varphi(x)$ certainly gives a virtual turning point of type $(i, j)$, and moreover, we will know in Sections 4 and 5 that $\varphi(x)$ is closely related to the global configuration of the Stokes geometry.

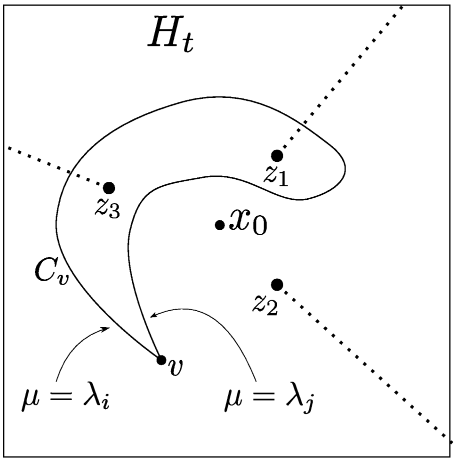


Figure 4. The path $C_v$.

The function $\varphi(x)$, however, depends on the path $C_x$, and we cannot regard it as a function defined on the base space $\mathbb{C}$. Hence, in [H3], we had introduced the Riemann manifold $\mathcal{R}_{\mathrm{sym}}$ on which $\varphi(x)$ can be lifted as a single-valued holomorphic function. In this section, we briefly review the definition of $\mathcal{R}_{\mathrm{sym}}$ and its properties that we need in the later section. See [H3] for the details and the proofs.

Let $l_x$ be the segment from $x_0$ to $x$ in $H$ (see Fig. 5). Noticing the following deformation of the path $C_x$

$$C_x = l_x + (-l_x + C_{x_0} + l_x) + (-l_x),$$

we obtain

$$\varphi(x) = \int_{x_0}^{x} (\lambda_j(z) - \lambda_i(z))dz + \int_{C_{x_0}} \lambda_i(z)dz. \tag{3.7}$$

Here the paths of integration in (3.7) satisfy the following conditions:
(**P-1**): The path of integration of the first integral is taken in $H$.
(**P-2**): The path $C_{x_0}$ in the second integral is a closed path starting from $x_0$ such that the analytic continuation of $\lambda_i(x)$ along $C_{x_0}$ is $\lambda_j(x)$ $(j \neq i)$.

Therefore $\varphi(x)$ is a sum of two integrals which have completely different structures. While the first integral in (3.7) depends on the analytic structure of the multi-valued holomorphic function $\lambda_j(z) - \lambda_i(z)$, the second one depends on the algebraic structure of the set of paths which satisfy the condition (P-2). Such a structure of paths is concretely described by **the type diagram** introduced in [H3]. For the convenience of the reader, let us briefly recall its definition. See [H3] for details. First remember

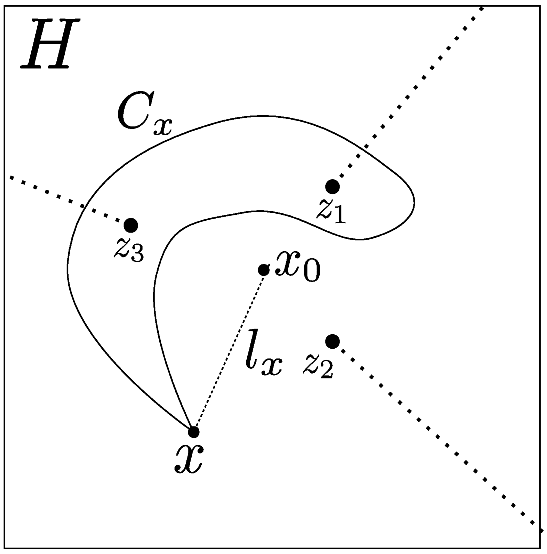


Figure 5. The deformation of the path.

that the (abstract) directional graph consists of two sets: the set of nodes and the set of edges. Here each edge is an ordered pair of nodes.

**Definition 3.1.** The type diagram of the Stokes geometry associated with the system (2.1) is a directional graph consisting of the data 1. and 2. below:

1. Each node is an integer $1, 2, \ldots, n$ where $n$ is the order of the polynomial $\Lambda(\lambda, x)$ of the variable $\lambda$.
2. Each edge is indexed by an ordinary turning point. If $v \in Z$ is of type $(i, j)$, then the edge indexed by $v$ is either an ordered pair $\{i, \{j\}\}$ or $\{j, \{i\}\}$.

*Remark.* In [AKT2] a similar graphical notion called "a bicharacteristic graph" was introduced, and it is, in a sense, a dual notion of the type diagram.

From now on, the symbol $i \xrightarrow{v} j$ (or $j \xrightarrow{v} i$) denotes the edge indexed by an ordinary turning point $v$ of type $(i, j)$. Let $L_0$ (resp. $L_1$) be the free $\mathbb{Z}$ module generated by the nodes (resp. the edges) of the type diagram respectively. We define the complex $\dot{L}$ by:

$$\dot{L}: \quad 0 \leftarrow L_0 \xleftarrow{\partial} L_1 \leftarrow 0, \tag{3.8}$$

where the morphism $\partial$ is defined by

$$\partial(i \xrightarrow{v} j) = \{j\} - \{i\} \qquad (i \xrightarrow{v} j \in L_1). \tag{3.9}$$

Then it is easy to see that the homology group $H_1(\dot{L})$ is a free $\mathbb{Z}$ module, and

$$\dim_{\mathbb{Z}}(H_1(\dot{L})) = 1 + \#Z - n. \tag{3.10}$$

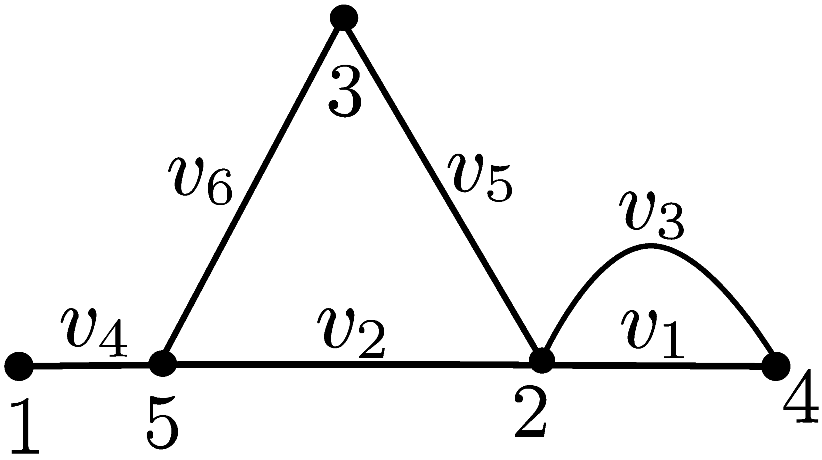


Figure 6. An example of the type diagram.

**Example 3.2.** Fig. 6 is an example of the type diagram associated with a $5 \times 5$ system that has 6 ordinary turning points. It consists of 5 nodes $\{1, 2, \ldots, 5\}$ and 6 edges indexed by the ordinary turning points $\{v_1, v_2, \ldots, v_6\}$. The edge indexed by $v_1$ (resp. $v_2$, $\ldots$, $v_6$) corresponds to an ordinary turning point $v_1$ (resp. $v_2$, $\ldots$, $v_6$) of type $(2, 4)$ (resp. $(2, 5)$, $\ldots$, $(3, 5)$) respectively. We have $\dim_{\mathbb{Z}}(H_1(\dot{L})) = 2$ by (3.10), and its basis is, for example, given by

$$D_1 : 2 \overset{v_1}{\to} 4 \overset{v_3}{\to} 2,$$

and

$$D_2 : 2 \overset{v_5}{\to} 3 \overset{v_6}{\to} 5 \overset{v_2}{\to} 2$$

For any $(i,\, j) \in \mathbb{Z}_{n,\neq}$ we set

$$L_1(i, j) = \{\sigma \in L_1;\, \partial\sigma = \{j\} - \{i\}\}, \tag{3.11}$$

that is, $L_1(i, j)$ is the set of paths from the node $i$ to the node $j$ of the type diagram.

**Definition 3.3.** We say that a family of paths $\{\alpha_{ij}\}_{(i,j)\in\mathbb{Z}_{n,\neq}}$ satisfies the 1-cocycle condition in the type diagram if the following conditions are satisfied:

1. (anti-symmetric) For any $(i, j) \in \mathbb{Z}_{n,\neq}$,

$$\alpha_{ij} \in L_1(i, j) \text{ and } \alpha_{ij} = -\alpha_{ji}. \tag{3.12}$$

2. (1-cocycle condition) For mutually different indices $i, j, k \in \mathbb{Z}_n$ we have

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0. \tag{3.13}$$

Note that such a family of paths always exists. Let $\{\alpha_{ij}\}$ be a family of paths that satisfies the 1-cocycle condition, and we fix it in what follows.

To link the type diagram to the function $\varphi(x)$ defined by (3.7), we will introduce the morphism $I : L_1 \to \mathbb{C}$ of $\mathbb{Z}$ modules as follows: For any edge $k \xrightarrow{w} l \in L_1$ with $w \in Z$ of type $(k, l)$, the morphism $I$ is defined by:

$$I(k \xrightarrow{w} l) = \int_{x_0}^{w} (\lambda_k(z) - \lambda_l(z))\, dz \tag{3.14}$$

where the path of integration is taken in $H$, that is, the path never crosses any cut lines in $H$. Let $F_{i,j}(x)$ denote the first integral of (3.7), i.e.,

$$F_{i,j}(x) := \int_{x_0}^{x} (\lambda_j(z) - \lambda_i(z)) dz \tag{3.15}$$

where the path of integration is again taken in $H$, and we set for any cycle $\alpha \in H_1(\dot{L})$

$$f_{i,j,\alpha}(x) := F_{i,j}(x) + I(\alpha_{ij}) + I(\alpha). \tag{3.16}$$

Note that $F_{i,j}(x)$ and $f_{i,j,\alpha}(x)$ are single-valued holomorphic functions on $H$. Then we have:

**Proposition 3.4** (Proposition 4.15 [H3]). *There exists a cycle $\alpha \in H_1(\dot{L})$ such that we have*

$$\varphi(x) = f_{i,j,\alpha}(x) \qquad (x \in H). \tag{3.17}$$

*In particular, $v$ is a turning point of type $(i, j) \in \mathbb{Z}_{n,\neq}$ if and only if $v$ is a root of the equation*

$$f_{i,j,\alpha}(x) = 0 \tag{3.18}$$

*for some $\alpha \in H_1(\dot{L})$.*

To construct the Riemann manifold $\mathcal{R}_{\mathrm{sym}}$, we need to know the structure of an analytic continuation of each $f_{i,j,\alpha}(x)$, and it is systematically described by **the shift vectors**.

**Definition 3.5.** The shift vector $r_{k \xrightarrow{w} l}$ is defined by:

$$r_{k \xrightarrow{w} l} = [k \xrightarrow{w} l + \alpha_{lk}] \in H_1(\dot{L}) \tag{3.19}$$

for any edge $k \xrightarrow{w} l$ with $w \in Z$ of type $(k, l)$. Note that $r_{k \xrightarrow{w} l} = -r_{l \xrightarrow{w} k}$ holds.

Note that the set of the shift vectors is finite. Hence, thanks to Proposition 3.6 below, we can describe every analytic continuation of $f_{i,j,\alpha}(x)$ completely once we obtain all the shift vectors using the type diagram.

**Proposition 3.6** (Proposition 4.17 [H3]). *Let $v \in Z$. Suppose that a curve $C$ crosses the cut line emanating from $v$ only once, and that $C$ never crosses any other cut line in $H$. Then the consequence of an analytic continuation of $f_{i,j,\alpha}(x)$ along $C$ has the same form $f_\xi(x)$ where the index $\xi \in \mathbb{Z}_{n,\neq} \times H_1(\dot{L})$ is given as follows:*

1. *If the type of $v$ is $(i,j)$, then $\xi = (j,\, i,\, \alpha - 2r_{i \xrightarrow{v} j})$.*
2. *If $v$ is of type $(j,k)$ for some $k \notin \{i,j\}$, then $\xi = (i,\, k,\, \alpha + r_{j \xrightarrow{v} k})$.*
3. *If $v$ is of type $(i,k)$ for some $k \notin \{i,j\}$, then $\xi = (k,\, j,\, \alpha - r_{i \xrightarrow{v} k})$.*

Taking Proposition 3.6 into account, we first construct the Riemann manifold $\pi_{\mathcal{R}} : \mathcal{R} \to \mathbb{C}$ as follows: We denote by $\Xi$ the set of indices:

$$\Xi := \left(\mathbb{Z}_{n,\neq} \times H_1(\dot{L})\right) \tag{3.20}$$

and set

$$X = \mathbb{C} \setminus \left(\underset{v \in Z}{\cup} h_v\right). \tag{3.21}$$

Here $h_v$ designates the cut line in $H$ emanating from $v \in Z$ (see Fig. 3). Note that $h_v$ is an open half line in $\mathbb{C}$, i.e., $v \notin h_v$. Let us now consider the cut space $H$ to be the set

$$H = X \sqcup \left(\underset{v \in Z}{\cup} \left(h_v^R \sqcup h_v^L\right)\right) \tag{3.22}$$

where $h_v^R$ and $h_v^L$ are copies of $h_v$, and the symbol $\sqcup$ denotes the disjoint union of sets. We make $H$ a topological space so that $h_v^R$ (resp. $h_v^L$) becomes the right (resp. left) side boundary of $X$ along $h_v$. For any point $x \in H$, let $x^*$ denote the opposite point in $\pi_H^{-1}(\pi_H(x))$ where $\pi_H : H \to \mathbb{C}$ denotes the canonical projection, that is, if $x \in h_v^R$, then $x^*$ is the point in $h_v^L$ with $\pi_H(x^*) = \pi_H(x)$. Note that $v^* = v$ holds for every $v \in Z$. We set

$$H_\Xi = \underset{\xi \in \Xi}{\sqcup} H_\xi \tag{3.23}$$

where $H_\xi$ ($\xi \in \Xi$) designates a copy of $H$, and $\pi_{H_\Xi} : H_\Xi \to H$ denotes the canonical projection. Now we are ready to construct the Riemann manifold $\mathcal{R}$ over $\mathbb{C}$ by gluing $H_\xi$'s (see Fig. 7 also).

**Definition 3.7.** We define the involution map $\mathcal{J} : H_\Xi \to H_\Xi$ as follows:

1. If $x \in h_v^R \cup h_v^L \cup \{v\}$ with $v \in Z$ of type $(i,j)$, then
   - $\mathcal{J}(x,\, i,\, j,\, \alpha) = \left(x^*,\, j,\, i,\, \alpha - 2r_{i \xrightarrow{v} j}\right)$ and $\mathcal{J}(x,\, j,\, i,\, \alpha) = \left(x^*,\, i,\, j,\, \alpha - 2r_{j \xrightarrow{v} i}\right)$ for any $\alpha \in H_1(\dot{L})$,

- $\mathcal{J}(x, k, j, \alpha) = \left(x^*, k, i, \alpha + r_{j \overset{v}{\to} i}\right)$ and
  $\mathcal{J}(x, k, i, \alpha) = \left(x^*, k, j, \alpha + r_{i \overset{v}{\to} j}\right)$ for each $\alpha \in H_1(\dot{L})$ and any $k \notin \{i, j\}$,
- $\mathcal{J}(x, i, k, \alpha) = \left(x^*, j, k, \alpha - r_{i \overset{v}{\to} j}\right)$ and
  $\mathcal{J}(x, j, k, \alpha) = \left(x^*, i, k, \alpha - r_{j \overset{v}{\to} i}\right)$ for each $\alpha \in H_1(\dot{L})$ and any $k \notin \{i, j\}$,

2. For any point $\hat{x} = (x, k, l, \alpha) \in H_\Xi$ that does not appear in 1., we define $\mathcal{J}(\hat{x}) = (x^*, k, l, \alpha)$.

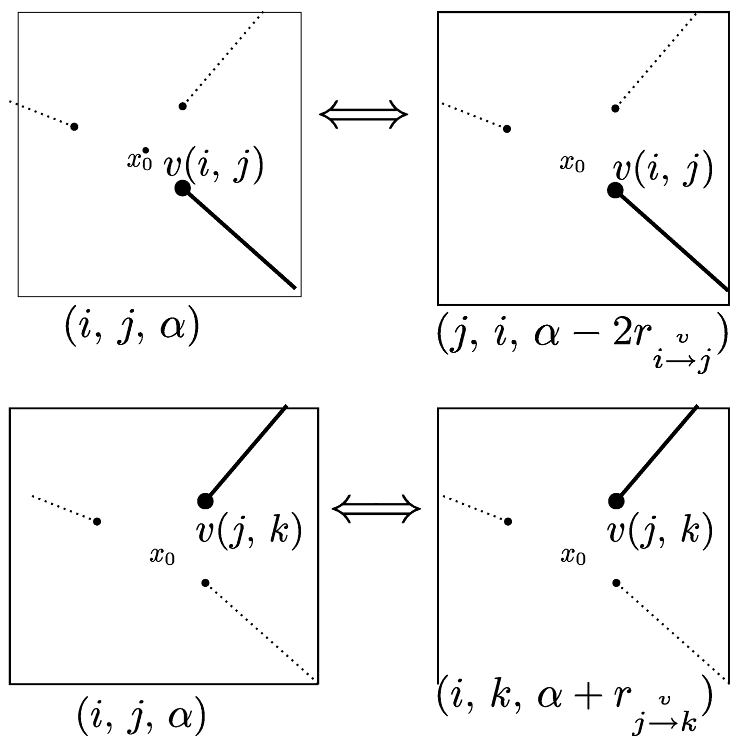


Figure 7. Gluing $H_\xi$'s.

Since $\mathcal{J}$ is an involution map on $H_\Xi$ (i.e. $\mathcal{J} \circ \mathcal{J} = \mathrm{Id}_{H_\Xi}$), it induces the equivalence relation $\overset{\mathcal{J}}{\sim}$ in $H_\Xi$ by:

$$\hat{x} \overset{\mathcal{J}}{\sim} \hat{y} \quad \text{if and only if} \quad \hat{x} = \mathcal{J}(\hat{y}) \quad \text{or} \quad \hat{x} = \hat{y}. \tag{3.24}$$

**Definition 3.8.** The Riemann manifold $\mathcal{R}$ over $\mathbb{C}$ is the set of equivalence classes $H_\Xi / \overset{\mathcal{J}}{\sim}$, and $\pi_{\mathcal{R}} : \mathcal{R} \to \mathbb{C}$ denotes the canonical projection.

Then, by the definition of $\mathcal{R}$, a family of functions

$$\{f_{i,j,\alpha}(x)\}_{(i,j,\alpha)\in\Xi} \qquad (\text{resp. } \{\lambda_j(x) - \lambda_i(x)\}_{(i,j,\alpha)\in\Xi}) \tag{3.25}$$

defines a single-valued holomorphic function on $\mathcal{R}$, and we denote it by $F(\hat{x})$ (resp. $\lambda_{\text{diff}}(\hat{x})$) respectively. We can easily prove the following integral relation:

$$F(\hat{x}_2) - F(\hat{x}_1) = \int_{\hat{x}_1}^{\hat{x}_2} \lambda_{\text{diff}}(\hat{x}) d\pi_{\mathcal{R}}(\hat{x}) \tag{3.26}$$

for any points $\hat{x}_1$ and $\hat{x}_2$ in the same connected component of $\mathcal{R}$. Indeed, if $\hat{x}_2$ is in a sufficiently small neighborhood of $\hat{x}_1$, then (3.26) follows from the from (3.16) of $f_{i,j,\alpha}(x)$, and this entails the relation for any $\hat{x}_2$ by the unique continuation property of a holomorphic function.

A turning point (resp. a Stokes curve) in $\mathcal{R}$ is now simply defined by a zero point of $F(\hat{x})$ (resp. a smooth locus of the zero set of $\operatorname{Im} F(\hat{x})$ emanating from a turning point) respectively. The space $\mathcal{R}$ is, however, a little bit large for our purpose. In fact, the holomorphic function $F(\hat{x})$ has an extra symmetry

$$F(x, i, j, \alpha) = -F(x, j, i, -\alpha), \tag{3.27}$$

and thus, the same turning point appears twice in $\mathcal{R}$. Hence we introduce the involution map $\mathcal{I}$ on $\mathcal{R}$ as:

$$\mathcal{I}(x, i, j, \alpha) = (x, j, i, -\alpha), \tag{3.28}$$

and we define the Riemann manifold $\mathcal{R}_{\text{sym}}$ as follows: Note that since $\mathcal{I}$ and $\mathcal{J}$ commute, the map $\mathcal{I}$ is well-defined on $\mathcal{R}$, and it induces the equivalence relation $\overset{\mathcal{I}}{\sim}$ in $\mathcal{R}$.

**Definition 3.9.** The Riemann manifold $\mathcal{R}_{\text{sym}}$ over $\mathbb{C}$ is the set of equivalence classes $\mathcal{R}/\overset{\mathcal{I}}{\sim}$. We denote by $\pi_{\mathcal{R}_{\text{sym}}} : \mathcal{R}_{\text{sym}} \to \mathbb{C}$ the canonical projection.

Then, by (3.27), the zero set of $F(\hat{x})$ (resp. $\operatorname{Im} F(\hat{x})$) is well-defined on $\mathcal{R}_{\text{sym}}$ respectively, and hence we can define a turning point and a Stokes curve in the same way as in $\mathcal{R}$: Set

$$\hat{Z} := \{(v, i, j, \alpha) \in \mathcal{R}_{\text{sym}};\, v \in Z,\, \{\text{the type of } v\} \cap \{i, j\} \neq \phi\} \subset \mathcal{R}_{\text{sym}}. \tag{3.29}$$

**Definition 3.10.** A point in the zero set of $F$ is said to be a **turning point**, in particular, the point $(v, i, j, r_{i \xrightarrow{v} j}) \in \mathcal{R}_{\text{sym}}$ with $v \in Z$ of type $(i, j)$ is called an **ordinary turning point** in $\mathcal{R}_{\text{sym}}$. A **Stokes curve** in $\mathcal{R}_{\text{sym}}$ emanating from a turning point $\hat{v} \in \mathcal{R}_{\text{sym}}$ is a connected component in the set

$$\{\hat{x} \in \mathcal{R}_{\text{sym}};\, \operatorname{Im} F(\hat{x}) = 0\} \setminus \left(\hat{Z} \cup \{\text{the turning points in } \mathcal{R}_{\text{sym}}\}\right) \tag{3.30}$$

whose closure contains $\hat{v}$.

Note that each Stokes curve is open in the zero set of $\operatorname{Im} F(\hat{x})$, and that each turning point has at least two Stokes curves emanating from it. We also note that, if the boundary of a Stokes curve consists of two points, one of them is, needless to say, a turning point from which the Stokes curve emanates, but that the other one is never a turning point due to the fact that $|F(\hat{x})|$ is strictly increasing along the Stokes curve (see (3.35) below).

A Stokes curve can be also defined as an integral curve as usual: We denote by $\omega$ the holomorphic 1-form on $\mathcal{R}$

$$\omega := \lambda_{\text{diff}}(\hat{x}) d\pi_{\mathcal{R}}(\hat{x}), \tag{3.31}$$

and the non-degenerate real analytic vector field $\mathcal{V}_\omega$ on the set $\mathcal{R}_{\text{sym}} \setminus \hat{Z}$ is well-defined by:

$$\mathcal{V}_\omega := \{\operatorname{Im} \omega = 0\} \qquad \text{on } \mathcal{R}_{\text{sym}} \setminus \hat{Z}. \tag{3.32}$$

Then a Stokes curve emanating from $\hat{v}$ is nothing but the maximal integral curve of $\mathcal{V}_\omega$ emanating from $\hat{v}$.

To understand the structure of the Riemann manifold $\mathcal{R}_{\text{sym}}$ concretely, we will briefly explain its local structure:

- At any point in $\mathcal{R}_{\text{sym}} \setminus \hat{Z}$, the Riemann manifold $\mathcal{R}_{\text{sym}}$ is locally isomorphic to the base space $\mathbb{C}$ with respect to $\pi_{\mathcal{R}_{\text{sym}}}$,
- At a point $\hat{x}$ in $\hat{Z}$ except for an ordinary turning point in $\mathcal{R}_{\text{sym}}$ (see Definition 3.10), it is locally a double covering space over $\mathbb{C}$ and $\hat{x}$ is its ramification point of degree 2.
- It is, however, locally isomorphic to $\mathbb{C}$ with respect to $\pi_{\mathcal{R}_{\text{sym}}}$ at an ordinary turning point in $\mathcal{R}_{\text{sym}}$.

Note that the zero set of $\operatorname{Im} F(\hat{x})$ is smooth outside $\hat{Z}$, and hence, every Stokes curve is real analytic smooth and connected.

Although the holomorphic functions $F(\hat{x})$ and $\lambda_{\text{diff}}(\hat{x})$ themselves are not well-defined on $\mathcal{R}_{\text{sym}}$, the functions

$$F^{\text{a}}(\hat{x}) := |F(\hat{x})| \quad \text{and} \quad \lambda^{\text{a}}_{\text{diff}}(\hat{x}) := |\lambda_{\text{diff}}(\hat{x})|. \tag{3.33}$$

are still well-defined on $\mathcal{R}_{\text{sym}}$, and they play an important role in the later section. Let $s$ be a Stokes curve emanating from a turning point $\hat{v}$ in $\mathcal{R}_{\text{sym}}$. We set

$$l(s; \hat{x}) := \text{"the length of the portion of } s \text{ between } \hat{v} \text{ and } \hat{x}\text{"} \qquad (\hat{x} \in s). \tag{3.34}$$

Here the length of the curve is estimated by the metric induced from the base space.

**Lemma 3.11.** *We have*

$$F^{\mathrm{a}}(\hat{x}_1) - F^{\mathrm{a}}(\hat{x}_2) = \int_{\hat{x}_2}^{\hat{x}_1} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s;\, \hat{x}) \tag{3.35}$$

*for any points $\hat{x}_1$ and $\hat{x}_2$ in a Stokes curve $s$ in $\mathcal{R}_{\mathrm{sym}}$.*

*Proof.* We may assume $l(s;\, \hat{x}_1) \geq l(s;\, \hat{x}_2)$. Let $\pi : \mathcal{R} \to \mathcal{R}_{\mathrm{sym}}$ be the canonical projection. Since $s \cap \hat{Z} = \phi$, we have a lift $\tilde{s}$ of $s$ in $\mathcal{R}$ with respect to $\pi$. Let $\tilde{v}$ be a turning point in $\mathcal{R}$ from which $\tilde{s}$ emanates, and we parameterize the curve $\tilde{s}$ by the length of the curve $\tilde{s}$ from $\tilde{v}$:

$$\theta = l(\tilde{s};\, \hat{x}) \qquad (\hat{x} \in \tilde{s}). \tag{3.36}$$

Then, in $\mathcal{R}$, there exists a constant $c$ being either 1 or $-1$ such that we have

$$\tilde{s}^* d\pi_{\mathcal{R}}(\hat{x}) = c \frac{\overline{\lambda_{\mathrm{diff}}(\hat{x})}}{|\lambda_{\mathrm{diff}}(\hat{x})|} d\theta \qquad (\text{for } \hat{x} = \tilde{s}(\theta)). \tag{3.37}$$

We use the same symbol $\hat{x}_i$ to denote the corresponding point in the curve $\tilde{s}$ $(i = 1, 2)$. It follows from the integral relation (3.26) that we have

$$F(\hat{x}_1) - F(\hat{x}_2) = c \int_{\hat{x}_2}^{\hat{x}_1} |\lambda_{\mathrm{diff}}(\hat{x})| dl(\tilde{s};\, \hat{x}). \tag{3.38}$$

Noticing $F(\tilde{v}) = 0$ and (3.38), the function $F(\hat{x})$ takes a constant signature at every point in $\hat{x} \in \tilde{s}$, and moreover, we have $|F(\hat{x}_1)| \geq |F(\hat{x}_2)|$. Hence we obtain

$$\begin{aligned} F^{\mathrm{a}}(\hat{x}_1) - F^{\mathrm{a}}(\hat{x}_2) &= ||F(\hat{x}_1)| - |F(\hat{x}_2)|| = |F(\hat{x}_1) - F(\hat{x}_2)| \\ &= \left| c \int_{\hat{x}_2}^{\hat{x}_1} |\lambda_{\mathrm{diff}}(\hat{x})| dl(\tilde{s};\, \hat{x}) \right| = \int_{\hat{x}_2}^{\hat{x}_1} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s;\, \hat{x}). \end{aligned} \tag{3.39}$$

The proof is now completed. □

We will introduce the several important notions of the Stokes geometry in $\mathcal{R}_{\mathrm{sym}}$.

**Definition 3.12.** If a triplet of points $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2 \in \mathcal{R}_{\mathrm{sym}} \setminus \hat{Z}$ is given by a coordinates representation of the following form:

$$\begin{aligned} \hat{x} &= (x,\, i,\, j,\, \alpha) \\ \hat{x}_1 &= (x,\, j,\, k,\, \alpha_1) \\ \hat{x}_2 &= (x,\, k,\, i,\, \alpha_2) \end{aligned} \qquad \Big(x \in H,\, \alpha, \alpha_1, \alpha_2 \in H_1(\dot{L})\Big) \tag{3.40}$$

for mutually different integers $i, j, k$ in $\mathbb{Z}_n$, then we say that these points form **a circuit index triplet**.

Note that the coordinates representation (3.40) is not unique, that is, the other coordinates representation in the form (3.40) may give the same triplet of points in $\mathcal{R}_{\mathrm{sym}}$, the following definition is, however, independent of such a choice of coordinates representations. Let $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$ be points in Stokes curves $s$, $s_1$ and $s_2$ in $\mathcal{R}_{\mathrm{sym}}$ respectively.

**Definition 3.13.** We say that $\hat{x}$ is **coherent** with respect to $\hat{x}_1$ and $\hat{x}_2$ if $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$ form a circuit index triplet with the coordinates representation (3.40) for which the following conditions are satisfied:

1. $\operatorname{Re} F(\hat{x}_1) \operatorname{Re} F(\hat{x}_2) > 0$.
2. The sum of indices of these points is zero, that is,

   $$\alpha + \alpha_1 + \alpha_2 = 0 \tag{3.41}$$

   holds.

The lemma below is easy to prove, but it will play the most fundamental role in the subsequent arguments.

**Lemma 3.14.** *If $\hat{x}$ is coherent with respect to $\hat{x}_1$ and $\hat{x}_2$, then we have*

$$F^{\mathrm{a}}(\hat{x}) = F^{\mathrm{a}}(\hat{x}_1) + F^{\mathrm{a}}(\hat{x}_2). \tag{3.42}$$

*Proof.* Suppose that the triplet has the coordinates representation (3.40). In $\mathcal{R}$ we have the following identity:

$$\begin{aligned} F(\hat{x}) + F(\hat{x}_1) + F(\hat{x}_2) &= f_{i,j,\alpha}(x) + f_{j,k,\alpha_1}(x) + f_{k,i,\alpha_2}(x) \\ &= F_{i,j}(x) + F_{j,k}(x) + F_{k,i}(x) + I(\alpha_{ij} + \alpha_{jk} + \alpha_{ki}) + I(\alpha + \alpha_1 + \alpha_2). \end{aligned} \tag{3.43}$$

Since $F_{i,j}(x)$ has the form (3.15), we get

$$F_{i,j}(x) + F_{j,k}(x) + F_{k,i}(x) = 0,$$

and we also have

$$I(\alpha_{ij} + \alpha_{jk} + \alpha_{ki}) = 0$$

because $\{\alpha_{ij}\}$ satisfies the 1-cocycle condition. Then the equality

$$F(\hat{x}) = -F(\hat{x}_1) - F(\hat{x}_2)$$

follows from (3.41). Noticing that $F(\hat{x})$, $F(\hat{x}_1)$ and $F(\hat{x}_2)$ are real numbers because the points are in Stokes curves, and that the pair $F(\hat{x}_1)$ and $F(\hat{x}_2)$ has the same signature by the condition 1. of Definition 3.13, we finally obtain the conclusion in $\mathcal{R}_{\mathrm{sym}}$:

$$F^{\mathrm{a}}(\hat{x}) = |F(\hat{x})| = |F(\hat{x}_1) + F(\hat{x}_2)| = |F(\hat{x}_1)| + |F(\hat{x}_2)| = F^{\mathrm{a}}(\hat{x}_1) + F^{\mathrm{a}}(\hat{x}_2).$$

□

## § 4. The Algorithm to Determine the State Function

Let $V$ be a subset of the set of turning points in $\mathcal{R}_{\mathrm{sym}}$, and let $S(V)$ denote the set of the Stokes curves emanating from a point in $V$. We designate by $G(V)$ the Stokes geometry consisting of the geometric data $V$ and $S(V)$.

### § 4.1. The Ansatz

Let $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$ be points in Stokes curves $s$, $s_1$ and $s_2 \in S(V)$ respectively.

**Definition 4.1.** If $\hat{x}$ is coherent with respect to $\hat{x}_1$ and $\hat{x}_2$, then we say that $\hat{x}$ is **a coherent point** of $s$ in $G(V)$, and $\hat{x}_1$ and $\hat{x}_2$ is **a co-coherent pair** of $\hat{x}$ in $G(V)$.

Note that a coherent point $\hat{x}$ may have infinitely many co-coherent pairs. Let $|S(V)| \subset \mathcal{R}_{\mathrm{sym}}$ denote the set of the points in the union of all the Stokes curves in $S(V)$.

**Definition 4.2.** A function $\mu(\hat{x})$ on $|S(V)|$ is said to be a state function of the Stokes geometry $G(V)$ if the function $\mu(\hat{x})$ satisfies the conditions below:

1. $\mu(\hat{x})$ takes boolean values, i.e., either 1 or 0.
2. The set $\{\hat{x} \in |S(V)|;\, \mu(\hat{x}) = 0\}$ is open in $|S(V)|$, and each connected component of the set $\{\hat{x} \in |S(V)|;\, \mu(\hat{x}) = 1\}$ has an interior point in $|S(V)|$.
3. For any Stokes curve $s \in S(V)$, the set of the discontinuous points of $\mu|_s(\hat{x})$ is discrete in the closure of $s$. Here $\mu|_s$ denotes the restriction of $\mu(\hat{x})$ to $s$.
4. $\mu(\hat{x})$ satisfies the Ansatz (Definition 4.3) that will be given below.

The following Ansatz was first given in Aoki-Kawai-Takei [AKT1], and it was extended by Y. Umeta [U] so that it may be applicable to some degenerate Stokes geometry.

Let $\hat{x}$ be a point in a Stokes curve $s$, then we denote by $\mathrm{st}(\hat{x})$ the unique Stokes curve passing through $\hat{x}$ (i.e., $\mathrm{st}(\hat{x}) = s$). For any coherent point $\hat{x}_0$ and any co-coherent pair $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$, we designate by

$$\mathrm{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2). \tag{4.1.1}$$

the intersection multiplicity at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$ of the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_1)_{\hat{x}_1})$ (i.e., the projection of the portion of $\mathrm{st}(\hat{x}_1)$ near $\hat{x}_1$) and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_2)_{\hat{x}_2})$ (i.e., that of $\mathrm{st}(\hat{x}_2)$ near $\hat{x}_2$), and it is said to be the intersection multiplicity at the coherent point $\hat{x}_0$ (with respect to the co-coherent pair $\hat{x}_1$ and $\hat{x}_2$).

**Definition 4.3** (**Ansatz about the state function**). A state function $\mu(\hat{x})$ of $G(V)$ satisfies the following axiom: Let $s \in S(V)$ be any Stokes curve emanating from a turning point $\hat{v} \in V$.

1. In some neighborhood of $\hat{v}$ we have

$$\mu|_s(\hat{x}) = \begin{cases} 0 & \text{if } \hat{v} \text{ is a virtual turning point,} \\ 1 & \text{if } \hat{v} \text{ is an ordinary turning point.} \end{cases} \tag{4.1.2}$$

2. If $\hat{x}_0 \in s$ is not a coherent point of $s$ in $G(V)$, then the function $\mu|_s(\hat{x})$ is continuous at $\hat{x}_0$.

3. If $\hat{x}_0 \in s$ is a coherent point of $s$ in $G(V)$, then the function $\mu|_s(\hat{x})$ is discontinuous at $\hat{x}_0$ if and only if the following relation holds:

$$\sum \mathrm{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)\mu(\hat{x}_1)\mu(\hat{x}_2) \equiv 1 \pmod 2. \tag{4.1.3}$$

   Here the sum is taken over all co-coherent pairs $(\hat{x}_1,\, \hat{x}_2)$ of $\hat{x}_0$ in $G(V)$.

Note that in considering the sum in (4.1.3) we encounter two difficulties. The first one is that the infinitely many terms may appear in the sum. This difficulty will be overcame in the next section.

The second one is the following: The intersection multiplicity $\mathrm{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)$ at a coherent point $\hat{x}_0$ may become $\infty$, that is, two local curves $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_1)_{\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_2)_{\hat{x}_2})$ in the base space identically coincide near $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$. This happens only for a quite degenerate system because the local curves $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_1)_{\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_2)_{\hat{x}_2})$ have mutually different types $(i,j)$ and $(j,k)$ at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$ respectively. Note that these local projection images are the integral curves defined by:

$$\mathrm{Im}((\lambda_i(z) - \lambda_j(z))dz) = 0 \quad \text{and} \quad \mathrm{Im}((\lambda_j(z) - \lambda_k(z))dz) = 0 \tag{4.1.4}$$

respectively. Hence, in this paper, we only consider the Stokes geometry that satisfies the following condition:

- (**A-3**): For any coherent point $\hat{x}_0$ and every co-coherent pair $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$, we have

$$|\mathrm{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)| < \infty, \tag{4.1.5}$$

  that is, the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_1)_{\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_2)_{\hat{x}_2})$ properly intersect at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$ in the base space.

We also note that the condition (4.1.3) becomes much simpler for the non-degenerate case, that is, if only one co-coherent pair $(\hat{x}_1, \hat{x}_2)$ of $\hat{x}_0$ exists and if $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_1)_{\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{st}(\hat{x}_2)_{\hat{x}_2})$ transversally intersect at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$, then (4.1.3) is nothing but

$$\mu(\hat{x}_1)\mu(\hat{x}_2) = 1, \tag{4.1.6}$$

and hence the function $\mu(\hat{x})$ is discontinuous at $\hat{x}_0$ if and only if $\mu(\hat{x}_1) = 1$ and $\mu(\hat{x}_2) = 1$ hold.

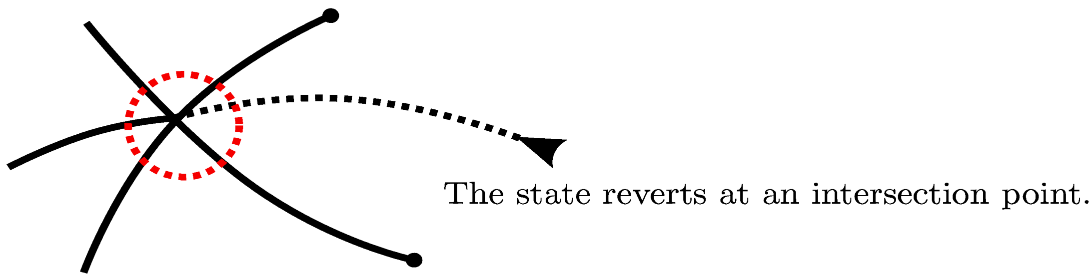


Figure 8. The Ansatz.

This is equivalent to saying, in the figure of the Stoke geometry (see Fig. 8), that the state (dotted or solid) of the Stokes curve $\mathrm{st}(\hat{x}_0)$ reverts at $\hat{x}_0$ if and only if the Stokes curves $\mathrm{st}(\hat{x}_1)$ and $\mathrm{st}(\hat{x}_2)$ are drawn by solid lines near $\hat{x}_1$ and $\hat{x}_2$ respectively.

Even if every coherent point in $G(V)$ is non-degenerate, the existence and the uniqueness of the state function of $G(V)$ is not so clear. In fact, unless we care about the analyticity that the Stokes geometry possesses, then we could easily find an example that has no model of $G(V)$, or that has many models of $G(V)$. The following example was first found by A. Shudo and K. Ikeda [SI].

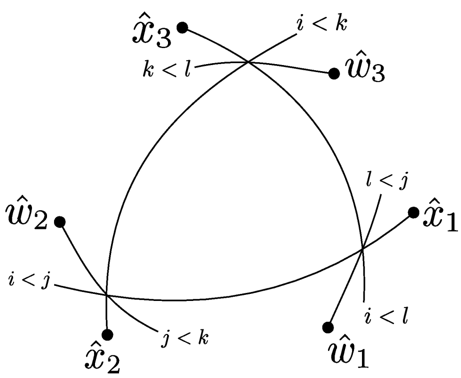


Figure 9. The example given in [SI].

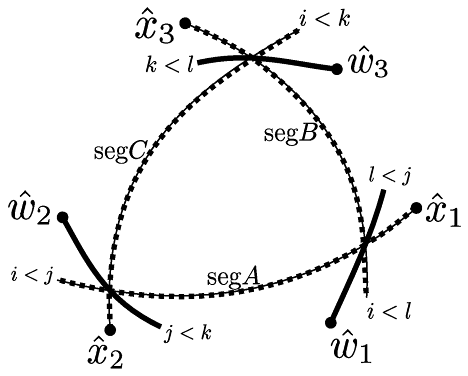


Figure 10. The model 1.

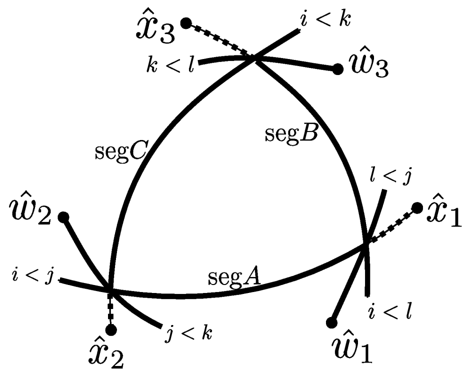


Figure 11. The model 2.

Fig. 9 is the projection image of some Stokes geometry in the base space $\mathbb{C}$. Its configuration is as follows:

- $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are virtual turning points, and $\hat{w}_1$, $\hat{w}_2$ and $\hat{w}_3$ are ordinary turning points.
- The first intersection point of the Stoke curve emanating from $\hat{x}_i$ $(i = 1, 2, 3)$ is a coherent point of this Stokes curve.
- No coherent points exist in the Stokes curve emanating from $\hat{w}_i$ $(i = 1, 2, 3)$.

Then applying the Ansatz (Definition 4.3) to Fig. 9, we can find two models Fig. 10 and Fig. 11. The example has certainly the state function, but the uniqueness of the state function does not hold. Hence the readers might think that the Ansatz would be too weak to determine the state function uniquely. However that is not true. As we will see below, this Stoke geometry is not associated with a system of differential equations because the system impose the specific local configuration upon the Stokes geometry. One of such restrictions on the Stokes geometry is given in Lemma 4.4 below.

Let $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$ be points in Stokes curves $s$, $s_1$ and $s_2$ in $\mathcal{R}_{\mathrm{sym}}$ respectively, and let $\hat{x}$ be a coherent point of $s$, and $\hat{x}_1$ and $\hat{x}_2$ its co-coherent pair. We designate by $\vec{t}$, $\vec{t_1}$ and $\vec{t_2} \in \mathbb{C}$ the tangent vectors of the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{\hat{x}})$, $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{1,\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{2,\hat{x}_2})$ at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})$ respectively. Here we equip a Stokes curve $\rho$ in $\mathcal{R}_{\mathrm{sym}}$ with the orientation so that $l(\rho;\hat{x})$ $(\hat{x} \in \rho)$ decreases, and the direction of a tangent vector is determined by the induced orientation on the local projection image of the Stokes curve (see Fig. 12).

**Lemma 4.4.** *We assume that the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{1,\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{2,\hat{x}_2})$ transversally intersect at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})$ in the base space. Then we have*

$$\vec{t} \in \{\alpha\vec{t_1} + \beta\vec{t_2};\ \alpha > 0,\ \beta > 0\}. \tag{4.1.7}$$

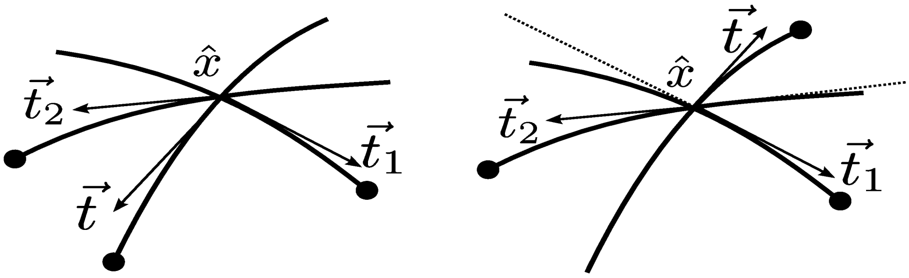


Figure 12. The local configuration.

*Proof.* Although this fact is well-known to those who are familiar with the Stokes geometry, there seems no good reference for it. For the reader's convenience I provide a complete proof here. Suppose that the triplet $\hat{x}$, $\hat{x}_1$ and $\hat{x}_2$ is represented by coordinates (3.40). We denote by $\tilde{x}$ (resp. $\tilde{x}_1$ and $\tilde{x}_2$) a point in $\mathcal{R}$ given by the coordinates (3.40) respectively. The curve $\tilde{s}$ designates a unique lift of $s$ in $\mathcal{R}$ passing through $\tilde{x}$.

We may assume $F(\tilde{x}_1) < 0$. Since $\hat{x}$ is coherent with respect to $\hat{x}_1$ and $\hat{x}_2$, we have in $\mathcal{R}$

$$F(\tilde{x}) + F(\tilde{x}_1) + F(\tilde{x}_2) = 0 \tag{4.1.8}$$

and

$$F(\tilde{x}_1) < 0,\ F(\tilde{x}_2) < 0 \text{ and } F(\tilde{x}) > 0. \tag{4.1.9}$$

Then $F(\tilde{y})$ is an increasing function on the Stokes curve $\tilde{s}$ when $l(\tilde{s};\, \tilde{y})$ increases due to the fact $F(\tilde{x}) > 0$. Hence, as $\lambda_{\mathrm{diff}}(\tilde{y}) = (\lambda_j - \lambda_i)(\pi_{\mathcal{R}}(\tilde{y}))$ holds near $\tilde{x}$, the direction of $\vec{t}$ is given by

$$\vec{t} = c\overline{(\lambda_i - \lambda_j)}(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})) \tag{4.1.10}$$

for a positive constant $c > 0$. In the same way we have

$$\vec{t}_1 = -c_1\overline{(\lambda_j - \lambda_k)}(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})) \text{ and } \vec{t}_2 = -c_2\overline{(\lambda_k - \lambda_i)}(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})) \tag{4.1.11}$$

for some positive constants $c_1$, $c_2 > 0$. Finally we obtain:

$$\vec{t} = \frac{c}{c_1}\vec{t}_1 + \frac{c}{c_2}\vec{t}_2. \tag{4.1.12}$$

□

Note that if the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{1,\hat{x}_1})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{2,\hat{x}_2})$ transversally intersect at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})$, then the local projection images $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{i,\hat{x}_i})$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(s_{\hat{x}})$ also transversally intersect at the same point $(i = 1, 2)$.

The local configuration shown in the right figure of Fig. 12 is never observed in the Stokes geometry associated with a system of differential equations. Let us consider the example given by Fig. 9 again. Thus the local configuration of Fig. 9 is clearly inconsistent with Lemma 4.4, and hence the example cannot be associated with the system (2.1).

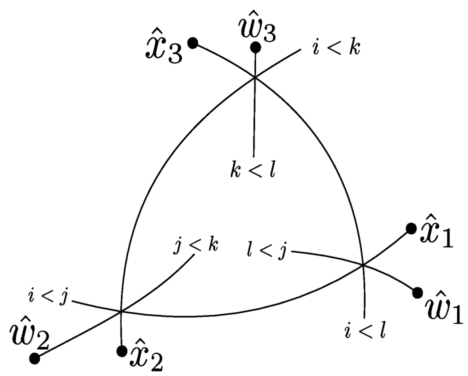


Figure 13. Another example.

Let us consider another example given in Fig. 13. Note that Fig. 13 is also the projection image of the Stokes geometry in the base space. The configuration of Fig. 13 is as follows:

- Each $\hat{w}_i$ is an ordinary turning point $(i = 1, 2, 3)$.
- The first intersection point of the Stoke curve emanating from $\hat{x}_i$ $(i = 1, 2, 3)$ is a coherent point of this Stokes curve.
- No coherent points exist in the Stokes curve emanating from $\hat{w}_i$ $(i = 1, 2, 3)$.

Note that the local configuration of Fig. 13 is consistent with Lemma 4.4. Still we observe the following facts:

1. If $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are virtual turning points, then two models exist in the same way as in the previous example.
2. If $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$ are ordinary turning points, then no model exists!

Logically speaking, the second example implies the inconsistency of our Ansatz. However, we have never encountered this kind of the Stokes geometry through the study of a huge number of concrete systems of differential equations. As a matter of fact, we can prove that no system realizes this example by using a Stokes path introduced in the next subsection.

## § 4.2. A Stokes Path

We now introduce the notion of a Stokes path in $G(V)$ that is the most fundamental tool to investigate the unique existence of the model of the Stokes geometry. Let $\hat{y}$ be a point in $|S(V)|$.

**Definition 4.5.** **A Stokes path** in $G(V)$ starting from $\hat{y}$ is generated by tracing the following walking in $|S(V)| \subset \mathcal{R}_{\mathrm{sym}}$:

1. We start from $\hat{y}$.
2. We proceed on a Stokes curve $s \in S(V)$ so that $l(s;\, \hat{x})$ decreases, where $\hat{x}$ denotes our current position in $\mathcal{R}_{\mathrm{sym}}$.
3. If we arrive at some coherent point $\hat{x}$ of a Stokes curve in $G(V)$, then we may jump to one of co-coherent points of $\hat{x}$ in $G(V)$.
4. We reach some turning point by repeating either 2. or 3. finite times. If we are at a turning point, then this walking must be terminated.

A Stokes path can be also formulated by the following schematic diagram:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & \hat{p}_k & \rightarrow & \hat{q}_k \\
 & & & & & & & & \upharpoonleft & & \\
 & & & & & & & \rightarrow & \hat{q}_{k-1} & & \\
 & & & & & & \cdots & & \downharpoonleft & & \\
 & & & & & & \cdots & & \hat{w}_{k-1} & & \\
 & & & & \hat{p}_2 & \rightarrow & \cdots & & & & \\
 & & & & \upharpoonleft & & & & & & \\
 & & \hat{p}_1 & \rightarrow & \hat{q}_1 & & & & & & \\
 & & \upharpoonleft & & \downharpoonleft & & & & & & \\
\hat{y} = \hat{p}_0 & \rightarrow & \hat{q}_0 & & \hat{w}_1 & & & & & & \\
 & & \downharpoonleft & & & & & & & & \\
 & & \hat{w}_0 & & & & & & & &
\end{array}
\tag{4.2.1}
$$

The diagram (4.2.1) implies that:

- The Stokes path starts from $\hat{y}$, and it terminates at the turning point $\hat{q}_k$. The points $\hat{p}_i$ and $\hat{q}_i$ $(i = 0, 1, 2, \ldots, k)$ belong to the same Stokes curve $s_i$ with $l(s_i;\, \hat{p}_i) \geq l(s_i;\, \hat{q}_i)$, and the horizontal arrow $\hat{p}_i \rightarrow \hat{q}_i$ indicates that we proceed on $s_i$ from $\hat{p}_i$ to $\hat{q}_i$.
- The point $\hat{q}_i$ $(i = 0, 1, \ldots, k-1)$ is coherent with respect to $\hat{w}_i$ and $\hat{p}_{i+1}$, and the vertical arrow $\hat{q}_i \rightarrow \hat{p}_{i+1}$ indicates that we jump from the coherent point $\hat{q}_i$ to its co-coherent point $\hat{p}_{i+1}$.

Note that $\hat{p}_i$ and $\hat{q}_i$ might be the same point. In this case, we jump after jumping in the same fiber of $\pi_{\mathcal{R}_{\mathrm{sym}}}$ successively.

**Definition 4.6.** Let $D$ be a Stokes path represented by the schematic diagram (4.2.1). Then the number $k$ of the diagram is said to be the depth of the Stokes path $D$, and we denote it by $\mathrm{dep}(D)$.

For any points $\hat{x}_1$ and $\hat{x}_2$ in a Stokes curve $s$, taking the integral relation (3.35) into account, we define the function $\varphi(\hat{x}_1, \hat{x}_2)$ by:

$$\varphi(\hat{x}_1, \hat{x}_2) = F^{\mathrm{a}}(\hat{x}_1) - F^{\mathrm{a}}(\hat{x}_2) = \int_{\hat{x}_2}^{\hat{x}_1} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s;\, \hat{x}). \tag{4.2.2}$$

Here $l(s;\, \hat{x})$ denotes the length of the portion in $s$ from its turning point to $\hat{x}$. Note that if $l(s;\, \hat{x}_1) > l(s;\, \hat{x}_2)$, then we have $\varphi(\hat{x}_1, \hat{x}_2) > 0$.

**Lemma 4.7.** *Let $D$ be a Stokes path starting from $\hat{y}$ represented by the schematic diagram (4.2.1). Then the function $F^{\mathrm{a}}(\hat{x})$ is strictly decreasing on the Stokes path $D$ if $\hat{x}$ moves from $\hat{y}$ to $\hat{q}_k$ along $D$. Moreover we have*

$$F^{\mathrm{a}}(\hat{y}) = \sum_{j=0}^{k} \varphi(\hat{p}_j, \hat{q}_j) + \sum_{j=0}^{k-1} F^{\mathrm{a}}(\hat{w}_j). \tag{4.2.3}$$

*Proof.* It follows from (3.35) that $F^{\mathrm{a}}(\hat{x})$ is strictly decreasing on each Stokes curve $s$ when $l(s;\, \hat{x})$ decreases. Since $\hat{q}_j$ is coherent with respect to $\hat{w}_j$ and $\hat{p}_{j+1}$ $(j = 0, 1, \ldots, k-1)$, by Lemma 3.14, we have

$$F^{\mathrm{a}}(\hat{q}_j) = F^{\mathrm{a}}(\hat{w}_j) + F^{\mathrm{a}}(\hat{p}_{j+1}) > F^{\mathrm{a}}(\hat{p}_{j+1}) \qquad (j = 0, 1, \ldots, k-1). \tag{4.2.4}$$

Hence $F^{\mathrm{a}}(\hat{x})$ is strictly decreasing on the Stokes path $D$. To obtain (4.2.3), we first note that

$$\begin{aligned} F^{\mathrm{a}}(\hat{y}) &= F^{\mathrm{a}}(\hat{p}_0) \\ &= \varphi(\hat{p}_0, \hat{q}_0) + F^{\mathrm{a}}(\hat{q}_0) && (\hat{q}_0 \text{ is coherent with respect to } \hat{w}_0 \text{ and } \hat{p}_1) \\ &= \varphi(\hat{p}_0, \hat{q}_0) + F^{\mathrm{a}}(\hat{w}_0) + F^{\mathrm{a}}(\hat{p}_1) \\ &= \varphi(\hat{p}_0, \hat{q}_0) + F^{\mathrm{a}}(\hat{w}_0) + \varphi(\hat{p}_1, \hat{q}_1) + F^{\mathrm{a}}(\hat{q}_1). \end{aligned}$$

Then by repeating the same argument as above, we have

$$\begin{aligned} F^{\mathrm{a}}(\hat{y}) &= \sum_{j=0}^{k-1} \varphi(\hat{p}_j, \hat{q}_j) + \sum_{j=0}^{k-1} F^{\mathrm{a}}(\hat{w}_j) + \varphi(\hat{p}_k, \hat{q}_k) + F^{\mathrm{a}}(\hat{q}_k) \\ &= \sum_{j=0}^{k} \varphi(\hat{p}_j, \hat{q}_j) + \sum_{j=0}^{k-1} F^{\mathrm{a}}(\hat{w}_j). && (\hat{q}_k \text{ is a turning point}). \end{aligned}$$

This completes the proof. □

Now we are ready to prove that the example given in Fig. 13 cannot be associated with a system of differential equations. Note that Fig. 14 is the same Stokes geometry as Fig. 13. We denote by $s_i$ the Stokes curve emanating from $\hat{x}_i$ $(i = 1, 2, 3)$, and we designate by $b_i$ the projection image of the coherent point of the Stokes curve $s_i$ $(i = 1, 2, 3)$.

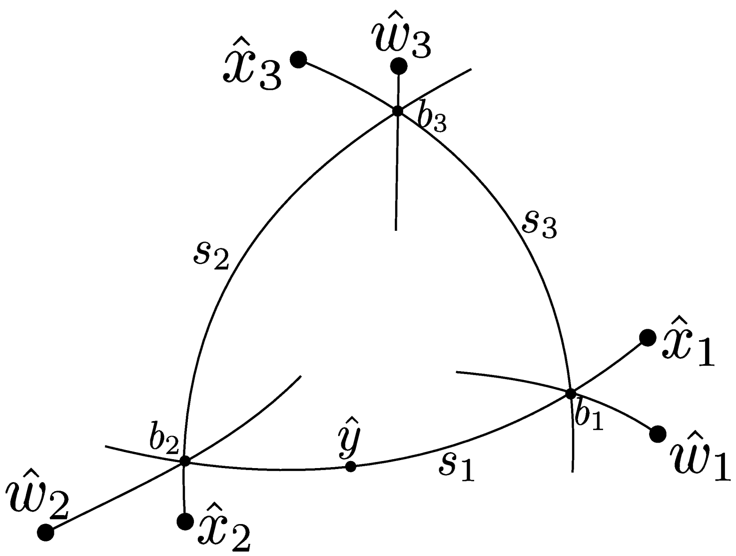


Figure 14. The same Stokes geometry as in Fig. 13

Let us consider the Stokes path $D$ by tracing the following walking: We start from the point $\hat{y}$ (see Fig. 13) in the Stokes curve $s_1$, and we proceed on $s_1$ to its turning point $\hat{x}_1$. When we reach the coherent point of $s_1$ in the fiber $\pi_{\mathcal{R}_{\rm sym}}^{-1}(b_1)$, we jump to the co-coherent point in the Stokes curve $s_3$. Then we proceed on $s_3$ to its turning point $\hat{x}_3$, and if we reach the coherent point of $s_3$ in the fiber $\pi_{\mathcal{R}_{\rm sym}}^{-1}(b_3)$, then we jump to the Stokes curve $s_2$ and proceed on $s_2$ to $\hat{x}_2$. Finally when we reach the coherent point of $s_2$ in the fiber $\pi_{\mathcal{R}_{\rm sym}}^{-1}(b_2)$, then we jump to the Stokes curve $s_1$, and this time we come to the turning point $\hat{x}_1$ along the curve $s_1$. Note that the depth of the Stokes path $D$ is 3.

The key feature of $D$ is that the starting point $\hat{y}$ appears again in the middle of the path $D$, and with the help of Lemma 4.7 we then find a contradiction. In fact, it follows form Lemma 4.7 that the function $F^{\rm a}(\hat{x})$ is strictly decreasing along the Stokes path $D$, and which implies $F^{\rm a}(\hat{y}) < F^{\rm a}(\hat{y})$. Hence we conclude that the configuration of this example is never given by the zero set of $F(\hat{x})$ and that of $\operatorname{Im} F(\hat{x})$. Otherwise stated, the configuration of Fig. 13 is not associated with the system (2.1).

We also note that we can extend the Stokes path $D$ to that with arbitrary depth; for example, if we turn around the loop in Fig. 13 twice before we reach $\hat{x}_1$, then we get the Stoke path $D$ with $\mathrm{dep}(D) = 6$, and therefore we have

$$\mathrm{dep}(\hat{y}) := \sup_{\text{D: any Stokes path from } \hat{y}} \mathrm{dep}(D) = +\infty.$$

As we will see in the next section, if the Stokes geometry satisfies some conditions, the function $\mathrm{dep}(\hat{y})$ always takes a finite value, and this fact plays a key role in proving the unique existence of the model of the Stokes geometry.

## § 5. The Unique Existence of a Model

We are considering, in this paper, the Stokes geometry associated with the system (2.1) that satisfies the conditions (A-1) and (A-2) given in Section 2, and moreover, we also assume the condition (A-3) in Section 4 so that our Ansatz is applicable to the Stokes geometry.

Let $V$ be a subset of the set of the turning points in $\mathcal{R}_{\mathrm{sym}}$. Recall that $S(V)$ denotes the totality of the Stokes curves emanating from a point in $V$, and that $G(V)$ designates the Stokes geometry consisting of the geometric data $V$ and $S(V)$.

### § 5.1. The Geometric Conditions for $\mathcal{R}_{\mathrm{sym}}$

We first recall that any point $\hat{x} \in \mathcal{R}_{\mathrm{sym}}$ has the (not necessarily unique) coordinates representation:

$$\hat{x} = (x,\, i,\, j,\, \alpha), \qquad x \in H,\ (i, j, \alpha) \in \mathbb{Z}_{n,\neq} \times H_1(\dot{L}) \tag{5.1.1}$$

where the precise definition of $H$ was given by (3.22). Since $H_1(\dot{L})$ is a free $\mathbb{Z}$ module of rank

$$\kappa = 1 + \#Z - n, \tag{5.1.2}$$

we identify, from now on, $H_1(\dot{L})$ with the set $\mathbb{Z}^\kappa$ of integer vectors, and for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa) \in H_1(\dot{L})$ we set $|\alpha| = \sum_{i=1}^{\kappa} |\alpha_i|$. Then we designates by $\mathcal{R}_{\mathrm{sym}}(k)$ ($k = 0, 1, 2, \ldots$) the set of every point in $\mathcal{R}_{\mathrm{sym}}$ that is represented by the coordinates (5.1.1) with $|\alpha| \leq k$. We say that the indices of the subset $E \subset \mathcal{R}_{\mathrm{sym}}$ are bounded if $E \subset \mathcal{R}_{\mathrm{sym}}(k)$ holds for some $k$.

We will introduce the following geometric conditions (GA-1), (GA-2) and (GA-3) for $\mathcal{R}_{\mathrm{sym}}$:

- **(GA-1) The irregular singularity at $\infty$**

There exist positive constants $C$, $R > 0$ and a constant $d > -1$ such that we have

$$\lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) \geq C|\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})|^d \qquad (|\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})| > R). \tag{5.1.3}$$

Another way of describing the situation in the base space is as follows: the Puiseux expansion of $\lambda_i(x) - \lambda_j(x)$ at $x = \infty$ can be written in the form:

$$\lambda_i(x)-\lambda_j(x) = x^d(c_0+c_1x^{-d_1}+c_2x^{-d_2}+\dots) \quad (c_0 \neq 0,\ 0 < d_1 < d_2 < \dots) \tag{5.1.4}$$

with $d > -1$ for any $(i, j) \in \mathbb{Z}_{n,\neq}$.

- **(GA-2): No geometric loops**

  Any maximal integral curve of the vector field $\mathcal{V}_\omega$ (defined by (3.32)) on $\mathcal{R}_{\mathrm{sym}} \setminus \hat{Z}$ either terminates at some point in $\hat{Z}$ or tends to infinity with respect to $\pi_{\mathcal{R}_{\mathrm{sym}}}$, that is,

$$\begin{aligned} &\lim_{\theta\to 1-0} \tau(\theta) \in \hat{Z} \quad \text{or} \quad \lim_{\theta\to 1-0} |\pi_{\mathcal{R}_{\mathrm{sym}}}(\tau(\theta))| = +\infty, \\ &\lim_{\theta\to 0+0} \tau(\theta) \in \hat{Z} \quad \text{or} \quad \lim_{\theta\to 0+0} |\pi_{\mathcal{R}_{\mathrm{sym}}}(\tau(\theta))| = +\infty \end{aligned} \tag{5.1.5}$$

  holds for any maximal integral curve $\tau(\theta) : (0, 1) \to \mathcal{R}_{\mathrm{sym}} \setminus \hat{Z}$ of $\mathcal{V}_\omega$. In the base space, this is equivalent to saying that any maximal integral curve of the (multi-valued) vector field defined by

$$\mathrm{Im}\,((\lambda_i(x) - \lambda_j(x))dx) = 0 \qquad ((i, j) \in \mathbb{Z}_{n,\neq}) \tag{5.1.6}$$

  either terminates at some ordinary turning point or tends to $x = \infty$.

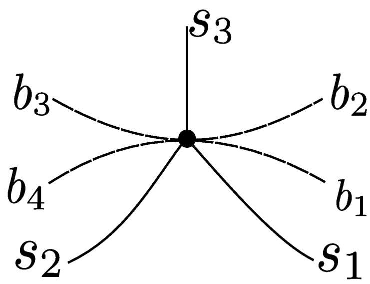


Figure 15. The transversality at an ordinary turning point.

- **(GA-3): The transversality at an ordinary turning point**

  For any ordinary turning point $x_0 \in Z$ of type $(i, j)$ and any $k \notin \{i, j\}$, we have

$$\frac{\displaystyle\lim_{x\to x_0} \frac{\lambda_i(x) - \lambda_j(x)}{\sqrt{x - x_0}}}{(\lambda_k(x_0) - \lambda_j(x_0))^{\frac{3}{2}}} \notin \{\mathbb{R},\ \sqrt{-1}\mathbb{R}\}. \tag{5.1.7}$$

A geometric implication of this condition is that for any Stokes curve $s$ emanating from $x_0$ and for any bifurcated curve $b$ emanating from $x_0$, the curves $s$ and $b$ are not tangent each other at $x_0$.

To exemplify this assumption we show Fig. 15. Here $b_1$,$b_2$,$b_3$ and $b_4$ are bifurcated curves and $s_1$, $s_2$ and $s_3$ are ordinary Stokes curves. Clearly the tangent line of $b_p$ $(p = 1, 2, 3, 4)$ and that of $s_q$ $(q = 1, 2, 3)$ at $x_0$ are mutually different. Hence the condition (GA-3) is satisfied in this figure.

We need the following geometric finiteness lemma. Let $\hat{x}_1$ and $\hat{x}_2$ be points in a Stokes curve $s$ in $\mathcal{R}_{\mathrm{sym}}$, and we denote by $[\hat{x}_1,\, \hat{x}_2]$ the closed portion in $s$ between $\hat{x}_1$ and $\hat{x}_2$. We also denote by $l(s;\, \hat{x}_1, \hat{x}_2)$ the length of the portion $[\hat{x}_1,\, \hat{x}_2]$ of $s$.

**Lemma 5.1 (The geometric finiteness lemma).** *We assume the first two conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$. Then we have:*

1. *The number of turning points belonging to $\mathcal{R}_{\mathrm{sym}}(k)$ is finite for any $k$.*
2. *There exists a positive integer $m$ such that if some point of a Stokes curve $s$ belongs to $\mathcal{R}_{\mathrm{sym}}(k)$, then we have $s \subset \mathcal{R}_{\mathrm{sym}}(k+m)$.*
3. *For any compact set $K \subset \mathbb{C}$ there exists a compact set $\tilde{K} \subset \mathbb{C}$ such that we have*
$$[\hat{x}_1,\, \hat{x}_2] \subset \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(\tilde{K})$$
*for any Stokes curve $s$ and any points $\hat{x}_1$ and $\hat{x}_2$ in $s \cap \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(K)$.*
4. *For any compact set $K \subset \mathbb{C}$ there exists a constant $C_K > 0$ satisfying that we have*
$$l(s;\, \hat{x}_1,\, \hat{x}_2) \le C_K \tag{5.1.8}$$
*for any Stokes curve $s$ and any points $\hat{x}_1$ and $\hat{x}_2$ in $s \cap \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(K)$,*

*Proof.* For the assertion 1.: Since turning points are the images of the roots of the holomorphic $F(\hat{x})$ in $\mathcal{R}$, they are discrete in $\mathcal{R}_{\mathrm{sym}}$, and thus, it is sufficient to show that $f_{i,\,j,\,\alpha}(x)$ defined by (3.16) has no roots near $x = \infty$ in $H$ for any $(i, j) \in \mathbb{Z}_{n,\neq}$ and for any $|\alpha| \le k$. Then it follows from the condition (GA-1) that there exists a positive constant $R$ such that we have

$$|F_{i,j}(x)| > \max_{(i,j)\in\mathbb{Z}_{n,\neq},\,|\alpha|\le k} |I(\alpha_{ij}) + I(\alpha)| \qquad (x \in H,\, |x| > R). \tag{5.1.9}$$

Hence we obtain assertion 1.

For the assertions 2., 3. and 4.: It follows from the conditions (GA-1) and (GA-2) that, in the base space, each Stokes curve tending to infinity is tangent to some line at $\infty$

and the other Stokes curve (i.e., those contained in some compact set in the base space) terminates at some ordinary turning point. Hence for any Stokes curve $s$ in $\mathcal{R}_{\mathrm{sym}}$, the length of the portion $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K) \cap s$ is uniformly bounded. By these observations, the assertions are intuitively clear. We need, however, some preparations for the geometry to prove these facts precisely, and we refer the reader to [H4] for the details. □

## § 5.2. The Unique Existence of the Model for the Special Case

Let $V$ be a subset of the set of the turning points in $\mathcal{R}_{\mathrm{sym}}$. In this subsection, we will consider the unique existence of the model of the Stokes geometry for the special case, that is, the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ (see Subsection 5.1) are satisfied, and moreover, the condition (†) for $V$ that will be given below is also assumed. The important example of this case is that the number of the turning points in $V$ is finite.

**Definition 5.2 (The condition (†) for $V$).** We say that $V$ satisfies the condition (†) if the projection image $\pi_{\mathcal{R}_{\mathrm{sym}}}(V)$ of $V$ is discrete in the base space.

Note that the condition (†) is satisfied if one of the following conditions holds:

1. The set $V$ is finite, or equivalently, the indices of $V$ are bounded.
2. $\mathrm{Rank}_{\mathbb{Z}}\, H_1(\dot{L}) \leq 2$.
3. We have $I(\sigma) \in \mathbb{Q}[\sqrt{-1}] = \mathbb{Q} + \mathbb{Q}\sqrt{-1}$ for every cycle $\sigma \in H_1(\dot{L})$. Here $I$ was defined by (3.14).

We have already introduced the depth of a Stokes path (Definition 4.6). Now we will define the depth for every point in $|S(V)|$ by a depth function that will be given in the following proposition: Let $K$ be a compact subset in $\mathbb{C}$, and we set

$$|S(K; V)| = |S(V \cap \pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K))| \cap \pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K). \tag{5.2.1}$$

**Proposition 5.3.** *We assume the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ and the condition (†) for $V$. Then there exists a function* $\mathrm{dep} : |S(V)| \to \mathbb{Z}_{\geq 0}$ *that satisfies the following conditions 1., 2., 3. and 4., and we call it a depth function of the Stokes geometry $G(V)$.*

1. *The set $|S(V)|_{\leq k} := \{\hat{x} \in |S(V)|;\ \mathrm{dep}(\hat{x}) \leq k\}$ is open in $|S(V)|$ for any $k \in \mathbb{Z}_{\geq 0}$, and for any Stokes curve $s$ emanating from $\hat{v} \in V$ we have $\mathrm{dep}|_s(\hat{x}) = 0$ in some neighborhood of $\hat{v}$. Here $\mathrm{dep}|_s(\hat{x})$ designates the restriction of $\mathrm{dep}(\hat{x})$ to $s$.*
2. *We have $\mathrm{dep}(\hat{x}_1) \geq \mathrm{dep}(\hat{x}_2)$ for any points $\hat{x}_1$ and $\hat{x}_2$ with $l(s; \hat{x}_1) \geq l(s; \hat{x}_2)$ in a Stokes curve $s \in S(V)$.*

3. *If $\hat{x}_0$ is a coherent point of a Stokes curve in $G(V)$, then we have*

$$\operatorname{dep}(\hat{x}_0) > \max\{\operatorname{dep}(\hat{x}_1), \operatorname{dep}(\hat{x}_2)\} \tag{5.2.2}$$

*for every co-coherent pair $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$ in $G(V)$.*

4. *For any compact set $K \subset \mathbb{C}$, the function* $\operatorname{dep}(\hat{x})$ *is uniformly bounded on $|S(K; V)|$.*

*Proof.* Let $\hat{y} \in |S(V)|$, and we set

$$D(\hat{y}) := \{\text{Any Stokes path starting from } \hat{y} \text{ in } G(V)\}. \tag{5.2.3}$$

Noticing $D(\hat{y}) \neq \phi$, we define the function $\operatorname{dep}(\hat{y})$ by:

$$\operatorname{dep}(\hat{y}) = \sup_{D \in D(\hat{y})} \operatorname{dep}(D) \qquad (\hat{y} \in |S(V)|). \tag{5.2.4}$$

First we will show that $\operatorname{dep}(\hat{y})$ takes a finite value for every $\hat{y} \in |S(V)|$. We have two lemmas:

**Lemma 5.4.** *Let $s \in S(V)$. Then the set of coherent points of $s$ in $G(V)$ is discrete in the closure of $s$.*

**Lemma 5.5.** *For any compact set $K \subset \mathbb{C}$ the function* $\operatorname{dep}(\hat{y})$ *defined by (5.2.4) is uniformly bounded on the set $|S(K; V)|$.*

For any point $\hat{x}$ of a Stokes curve $s$, let us denote by $\operatorname{tp}(\hat{x})$ (resp. $\operatorname{st}(\hat{x})$) the turning point from which $s$ emanates (resp. the Stokes curve $s$) respectively, and recall that $[\hat{x}, \operatorname{tp}(\hat{x})]$ designates the portion of $s$ between $\hat{x}$ and $\operatorname{tp}(\hat{x})$.

*Proof.* We will prove Lemma 5.4 and Lemma 5.5 in four steps:
**(Step 1)**: We first prove that the function $F^{\mathrm{a}}(\hat{y})$ is uniformly bounded on $|S(K; V)|$, that is, there exists a constant $M > 0$ such that we have

$$F^{\mathrm{a}}(\hat{y}) \leq M \qquad (\hat{y} \in |S(K; V)|). \tag{5.2.5}$$

Let us take a compact set $\tilde{K} \subset \mathbb{C}$ given by 3. of Lemma 5.1. It follows from the definition (3.25) of $\lambda_{\mathrm{diff}}(\hat{x})$ that the function $\lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x})$ does not depend on indices $\alpha \in H_1(\dot{L})$, and thus, there exists a constant $C_1$ satisfying

$$\lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) \leq C_1 \qquad (\hat{x} \in \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(\tilde{K})). \tag{5.2.6}$$

Let $s$ be a Stokes curve in $\mathcal{R}_{\mathrm{sym}}$. Then, by the integral relation (3.35), we have

$$F^{\mathrm{a}}(\hat{y}) = \int_{\operatorname{tp}(\hat{y})}^{\hat{y}} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s; \hat{x}) \qquad (\hat{y} \in s). \tag{5.2.7}$$

Thanks to 4. of Lemma 5.1, there exists a positive constant $C_{\tilde{K}}$ that satisfies

$$l(s;\,\hat{y}) \leq C_{\tilde{K}} \tag{5.2.8}$$

for any Stokes curve $s \in S(V \cap \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(\tilde{K}))$ and any point $\hat{y} \in s \cap \pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(\tilde{K})$, and thus, the estimate (5.2.5) follows from (5.2.6), (5.2.7) and (5.2.8).

**(Step 2)**: We will prove the assertion that there exists a compact set $K_1 \subset \mathbb{C}$ such that every relevant portion of any Stokes path starting from $|S(K;\,V)|$ is contained in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(K_1)$; to be more precise, if such a Stokes path $D$ is represented by the diagram (4.2.1), then $[p_i,\,\mathrm{tp}(\hat{p}_i)]$, $[q_i,\,\mathrm{tp}(\hat{q}_i)]$, $[w_i,\,\mathrm{tp}(\hat{w}_i)]$ and $D$ itself are contained in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(K_1)$.

By the condition (GA-1), there exist positive constants $C_2$, $R_1 > 0$ and a constant $d > -1$ such that

$$\lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) \geq C_2 |\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})|^d \qquad (|\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})| \geq R_1). \tag{5.2.9}$$

Then we take a constant $R > R_1$ sufficiently large so that it satisfies the conditions

$$K \subset B_R := \{x \in \mathbb{C};\, |x| \leq R\} \tag{5.2.10}$$

and

$$C_2 \int_R^{2R} r^d dr > M \tag{5.2.11}$$

where $M$ is the constant given in (Step 1). Then it follows from Lemma 4.7 that for a Stokes path $D$ starting from a point $\hat{y} \in |S(K;\,V)|$ which is represented by (4.2.1), we obtain

$$\begin{aligned} M \geq F^{\mathrm{a}}(\hat{y}) &= \sum_{j=0}^{k} \varphi(\hat{p}_j, \hat{q}_j) + \sum_{j=0}^{k-1} F^{\mathrm{a}}(\hat{w}_j) \\ &\geq \sum_{j=0}^{k} \varphi(\hat{p}_j, \hat{q}_j) = \sum_{j=0}^{k} \int_{\hat{q}_j}^{\hat{p}_j} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(\mathrm{st}(\hat{p}_j)\,;\hat{x}) \\ &\geq C_2 \int_R^{\psi(\hat{q}_k)} r^d dr \end{aligned} \tag{5.2.12}$$

where $\psi(\hat{q}_k) = \max\{|\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{q}_k)|,\, R\}$. Hence, taking (5.2.11) into account, $D$ itself is contained in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(B_{2R})$. Since $F^{\mathrm{a}}(\hat{x})$ is a decreasing function along a Stokes path, we have

$$M \geq F^{\mathrm{a}}(\hat{y}) \geq \max\{F^{\mathrm{a}}(\hat{p}_i),\, F^{\mathrm{a}}(\hat{q}_i),\, F^{\mathrm{a}}(\hat{w}_i)\}. \tag{5.2.13}$$

Hence, by the same argument as above, we conclude that the relevant portions are also contained in $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(B_{3R})$.

**(Step 3)**: We will prove Lemma 5.4.

We set

$$E_{tp}(K_1) := K_1 \cap \pi_{\mathcal{R}_{\mathrm{sym}}}(V) \tag{5.2.14}$$

where $K_1$ was determined in (Step 2). Then the fact that $E_{tp}(K_1)$ is a finite set follows from the condition (†). Let us denote by $E_{st}(K_1)$ the Stokes curves in the base space emanating from every point in $E_{tp}(K_1)$. Here a Stokes curve in the base space implies the projection image of that in $\mathcal{R}_{\mathrm{sym}}$, and it coincides with that of the usual definition in the base space. Note that a Stokes curve in the base space might have self-intersection points. The set $E_{int}(K_1)$ designates all the points in $K_1$ at which some Stokes curves in $E_{st}(K_1)$ properly intersect each other, or at which a Stokes curve in $E_{st}(K_1)$ intersects itself (a self-intersection point). We can easily see that $E_{st}(K_1)$ and $E_{int}(K_1)$ are finite sets by 4. of Lemma 5.1, and thus, we get

$$\epsilon_1 := \inf_{x,y\in E(K_1),\, x\neq y} |x-y| > 0 \tag{5.2.15}$$

where $E(K_1) = E_{tp}(K_1) \cup E_{int}(K_1)$.

Let $D$ be a Stokes path in $G(V)$ starting from some point in $|S(K;V)|$, and let $\hat{x}_0$ be a coherent or co-coherent point that appears in $D$. Then we can easily observe $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0) \in E_{int}(K_1)$. Indeed, the relevant portion of $D$ are contained in $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K_1)$ by (Step 2), in particular, all the relevant turning points $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{p}_i))$, $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{q}_i))$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{w}_i))$ belong to $E_{tp}(K_1)$. We have, thus, obtained $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0) \in E_{int}(K_1)$ by the condition (A-3) (this condition implies, in particular, that $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$ is either a proper intersection point of different Stokes curves or a self-intersection point of a Stokes curve).

Then there exists a constant $\epsilon > 0$ such that we have

$$l(\mathrm{st}(\hat{x}_0);\, \hat{x}_0) \geq \epsilon > 0 \tag{5.2.16}$$

for every coherent or co-coherent point $\hat{x}_0$ in any Stokes path in $G(V)$ starting from $|S(K;V)|$. This is clear by (5.2.15) if $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{x}_0)) \neq \pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$ because $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{x}_0))$ $\in E_{tp}(K_1)$ and $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0) \in E_{int}(K_1)$ hold by the above observations. If $\pi_{\mathcal{R}_{\mathrm{sym}}}(\mathrm{tp}(\hat{x}_0)) = \pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_0)$, then (5.2.16) follows from the trivial fact: Let $s \in E_{st}(K_1)$ and $v \in E_{tp}(K_1)$ be its turning point. Then the point $v$ appears in the curve $s$ at most finitely many times by 4. of Lemma 5.1.

Now we will prove Lemma 5.4. Let $s$ be a Stokes curve emanating from a turning point in $V \cap \pi_{\mathcal{R}_{\text{sym}}}^{-1}(K)$, and let $\hat{x}_0 \in \pi_{\mathcal{R}_{\text{sym}}}^{-1}(K)$ be a coherent point of $s$ in $G(V)$. It is enough to show $\pi_{\mathcal{R}_{\text{sym}}}(\hat{x}_0) \in E_{int}(K_1)$ because $E_{int}(K_1)$ is a finite set. Let $\hat{x}_1$ and $\hat{x}_2$ be a co-coherent pair of $\hat{x}_0$ in $G(V)$. Then the diagram:

$$\begin{array}{ccc} & \hat{x}_2 \to \mathrm{tp}(\hat{x}_2) & \\ & \uparrow & \\ \hat{x}_0 \to & \hat{x}_0 & \\ & \downarrow & \\ & \hat{x}_1 & \end{array}, \tag{5.2.17}$$

is a Stokes path in $G(V)$ starting from $|S(K;\,V)|$. This entails $\pi_{\mathcal{R}_{\text{sym}}}(\hat{x}_0) \in E_{int}(K_1)$.

(**The Final Step**): We are ready to prove Lemma 5.5. First we will prove that there exists a positive constant $\delta_{K_1} > 0$ such that

$$\int_{\hat{x}_2}^{\hat{x}_1} \lambda_{\text{diff}}^{\text{a}}(\hat{x})dl(s;\,\hat{x}) \geq \delta_{K_1} l(s;\,\hat{x}_1,\,\hat{x}_2)^{\frac{3}{2}} \tag{5.2.18}$$

for any Stokes curve $s$ and for any points $\hat{x}_1$ and $\hat{x}_2$ in $s \cap \pi_{\mathcal{R}_{\text{sym}}}^{-1}(K_1)$ with $l(s;\,\hat{x}_1) \geq l(s;\,\hat{x}_2)$. This estimate can be proved in the following way: For any $\rho > 0$, we set

$$Z_\rho := \{x \in \mathbb{C};\, \text{dist}(x,\,Z) < \rho\} \tag{5.2.19}$$

where $\text{dist}(\cdot,\,Z)$ denotes the distance in $\mathbb{C}$ from the set $Z$. Noticing that the zero set of $\lambda_{\text{diff}}^{\text{a}}(\hat{x})$ in $\mathcal{R}_{\text{sym}}$ is given by

$$\{(v,i,j,\alpha) \in \mathcal{R}_{\text{sym}};\, v \in Z,\, \{i,j\} = \text{"the type of } v\text{"}\}, \tag{5.2.20}$$

we let $\hat{U}_\rho$ denote the open set given by the union of the connected components of $\pi_{\mathcal{R}_{\text{sym}}}^{-1}(Z_\rho)$ that contains a zero point of $\lambda_{\text{diff}}^{\text{a}}(\hat{x})$. Since the estimate (5.2.18) clearly holds outside of $\hat{U}_\rho$, taking $\rho > 0$ sufficiently small, we may consider the estimate in $\hat{U}_\rho$. Then, since every ordinary turning point is simple by the condition (A-1), there exist positive constants $C_3$ and $C_4$ such that we have

$$\lambda_{\text{diff}}^{\text{a}}(\hat{x}) \geq C_3\, \text{dist}\left(\pi_{\mathcal{R}_{\text{sym}}}(\hat{x}),\,Z\right)^{\frac{1}{2}} \qquad (\hat{x} \in \hat{U}_\rho), \tag{5.2.21}$$

and for any Stokes curve $s$ and for any points $\hat{x}_1$ and $\hat{x}_2$ in the same connected component of $s \cap \hat{U}_\rho$, we also have

$$\max\left\{\text{dist}\left(\pi_{\mathcal{R}_{\text{sym}}}(\hat{x}_1),\,Z\right),\, \text{dist}\left(\pi_{\mathcal{R}_{\text{sym}}}(\hat{x}_2),\,Z\right)\right\} \geq C_4 l(s;\,\hat{x}_1,\,\hat{x}_2). \tag{5.2.22}$$

Let $s$ be a Stokes curve, and let $\hat{x}_1$ and $\hat{x}_2$ be points in the same connected component of $s \cap \hat{U}_\rho$ with $l(s;\, \hat{x}_1) \geq l(s;\, \hat{x}_2)$, and set

$$\epsilon := \max\left\{\operatorname{dist}\left(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_1),\, Z\right),\, \operatorname{dist}\left(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_2),\, Z\right)\right\}. \tag{5.2.23}$$

Note that, by (5.2.22), we have $\epsilon \geq C_4 l(s;\, \hat{x}_1,\, \hat{x}_2)$. If there exists a point $\hat{x}_3 \in [\hat{x}_1, \hat{x}_2]$ satisfying $\operatorname{dist}(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_3),\, Z) < \dfrac{\epsilon}{2}$, then we have

$$\int_{\hat{\mathrm{x}}_1}^{\hat{x}_2} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s;\, \hat{x}) \geq C_3 \int_{\frac{\epsilon}{2}}^{\epsilon} r^{\frac{1}{2}} dr \geq \frac{2C_3}{3}\left(\frac{\epsilon}{2}\right)^{\frac{3}{2}} \geq C_5 l(s;\, \hat{x}_1,\, \hat{x}_2)^{\frac{3}{2}}$$

for some positive constant $C_5$. If not, we have

$$\int_{\hat{\mathrm{x}}_1}^{\hat{x}_2} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(s;\, \hat{x}) \geq C_3 \left(\frac{\epsilon}{2}\right)^{\frac{1}{2}} l(s;\, \hat{x}_1,\, \hat{x}_2) \geq C_6 l(s;\, \hat{x}_1,\, \hat{x}_2)^{\frac{3}{2}}$$

for some positive constant $C_6$. This entails (5.2.18).

Then it follows from (5.2.16) and (5.2.18) that there exists a positive constant $\delta > 0$ that satisfies

$$F^{\mathrm{a}}(\hat{w}) = \int_{\mathrm{tp}(\hat{w})}^{\hat{w}} \lambda^{\mathrm{a}}_{\mathrm{diff}}(\hat{x}) dl(\mathrm{st}(\hat{w});\, \hat{x}) \geq \delta_{K_1} \min\left\{1,\, l(\mathrm{st}(\hat{w});\, \hat{w})^{\frac{3}{2}}\right\} \geq \delta. \tag{5.2.24}$$

for any coherent or co-coherent point $\hat{w}$ in a Stokes path $D$ starting from $|S(K;\, V)|$. Suppose that $D$ is represented by the diagram (4.2.1), then we have

$$\begin{aligned} M \geq F^{\mathrm{a}}(\hat{y}) &= \sum_{i=0}^{k} \varphi(\hat{p}_i, \hat{q}_i) + \sum_{i=0}^{k-1} F^{\mathrm{a}}(\hat{w}_i) \\ &\geq \sum_{i=0}^{k-1} F^{\mathrm{a}}(\hat{w}_i) \geq k\delta, \end{aligned} \tag{5.2.25}$$

and thus, we have obtained

$$\operatorname{dep}(D) = k \leq \frac{M}{\delta}. \tag{5.2.26}$$

The proof has been completed. □

*Remark.* Lemmas 5.4 and 5.5 do not hold without the condition (†), and therefore, we need to take another strategy to establish the similar lemmas for the general case. This will be the main subject in the next subsection.

Let us come back to the proof of Proposition 5.3. We will make sure that the function $\operatorname{dep}(\hat{y})$ defined by (5.2.4) satisfies the conditions 1., 2. and 3. of this proposition:

The condition 1. follows from Lemma 5.4. The condition 2. is clear by the definition (5.2.4). We will show the condition 3. Let $\hat{x}_0$ be a coherent point, and $\hat{x}_1$ and $\hat{x}_2$ a co-coherent pair of $\hat{x}$. If $D$ is the schematic diagram of a Stokes path starting from $\hat{x}_1$ with $\mathrm{dep}(D) = k$, then the diagram $D_1$:

$$
\begin{array}{ccc}
 & & D \\
 & & \uparrow \\
\hat{x}_0 & \to & \hat{x}_0 \\
 & & \downarrow \\
 & & \hat{x}_2
\end{array} \tag{5.2.27}
$$

represents a Stokes path starting from $\hat{x}_0$ with $\mathrm{dep}(D_1) = k + 1$, and from which the condition 3. follows. $\square$

*Remark.* A depth function of the Stokes geometry $G(V)$ is not necessarily unique. We usually adopt $\mathrm{dep}(\hat{x})$ defined by (5.2.4) as a depth function of $G(V)$. However only properties of $\mathrm{dep}(\hat{x})$ described by conditions 1., 2., 3. and 4. in the proposition will be used in the subsequent arguments.

The main theorem for the special case is:

**Theorem 5.6.** *We assume the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ which were given in Subsection 5.1, and we also assume the condition (†) for $V$ given by Definition 5.2. Then the unique model $M(V)$ of the Stokes geometry $G(V)$ exists.*

*Proof.* We fix a depth function $\mathrm{dep}(\hat{x})$ of the Stokes geometry $G(V)$, and we set

$$
\begin{aligned}
|S(V)|_k &:= \{\hat{y} \in |S(V)|;\ \mathrm{dep}(\hat{y}) = k\}, \\
|S(V)|_{\le k} &:= \{\hat{y} \in |S(V)|;\ \mathrm{dep}(\hat{y}) \le k\} \qquad (k = 0, 1, 2, \dots).
\end{aligned} \tag{5.2.28}
$$

We first prove the existence of the model: Let us construct the state function $\mu(\hat{x})$ on $|S(V)|$. By the induction with respect to $k \in \mathbb{Z}_{\ge 0}$, we will construct the function $\mu(x)$ on each set $|S(V)|_k$ successively, and we will simultaneously prove that the set $V^E_{\le k}$ is finite for each $k$. Here the set $V^E_{\le k}$ is defined as follows: Suppose that $\mu(\hat{x})$ has been constructed on the set $|S(V)|_{\le k}$, then we define the set $|S(V)|^E_{\le k}$ by

$$
|S(V)|^E_{\le k} := \{\hat{y} \in |S(V)|_{\le k};\ \mu(\hat{y}) = 1\}, \tag{5.2.29}
$$

and

$$
V^E_{\le k} := \{\mathrm{tp}(\hat{y}) \in V;\ \hat{y} \in |S(V)|^E_{\le k}\} \tag{5.2.30}
$$

where $\mathrm{tp}(\hat{y})$ denotes the turning point from which the Stokes curve passing through $\hat{y}$ emanates.

First let us construct $\mu(\hat{x})$ on $|S(V)|_0$. Let $s$ be a Stokes curve emanating from $\hat{v} \in V$, and set

$$s_0 := s \cap |S(V)|_0. \tag{5.2.31}$$

Then by the condition 2. of Proposition 5.3 the set $s_0$ is connected, and it follows from (5.2.2) that in $s_0$ no coherent point of $s$ exists. Hence $\mu(x)$ is a constant function in $s_0$ due to the condition 2. of the Ansatz (Definition 4.3), and thus, its constant value is determined by (4.1.2), i.e:

$$\mu(\hat{x}) = \begin{cases} 0 & \text{if } \hat{v} \text{ is a virtual turning point,} \\ 1 & \text{if } \hat{v} \text{ is an ordinary turning point.} \end{cases} \qquad (\hat{x} \in s_0). \tag{5.2.32}$$

Then $V_0^E$ is a subset of the set of ordinary turning points in $\mathcal{R}_{\mathrm{sym}}$, and thus, it is finite.

Now suppose that the function $\mu(\hat{x})$ has been constructed on $|S(V)|_{k-1}$ and that the set $V^E_{\leq k-1}$ is finite ($k \geq 1$). Let $s(\theta) : (0,1) \to \mathcal{R}_{\mathrm{sym}}$ be a Stokes curve emanating from $\hat{v} \in V$, and set

$$s_k := s \cap |S(V)|_k,\ s_{<k} := s \cap |S(V)|_{\leq k-1} \text{ and } s_{\leq k} := s \cap |S(V)|_{\leq k}. \tag{5.2.33}$$

We will first construct $\mu(\hat{x})$ on the portion $s_k$. We set

$$\theta_k = \inf\{\theta \in (0,1);\ \mathrm{dep}(s(\theta)) = k\} \text{ and } \hat{x}_k = s(\theta_k). \tag{5.2.34}$$

Note that since $s_{<k}$ is non-empty, we have $\theta_k > 0$ and $\hat{x}_k \in s_k$ by the condition 1. of Proposition 5.3. Since the coherent points of $s$ in $G(V)$ is discrete by Lemma 5.4, and since a discontinuous point of $\mu|_s(\hat{x})$ is a coherent point of $s$ in $G(V)$, the set of discontinuous points of $\mu|_s(\hat{x})$ is discrete in the closure of $s$.

If we want to know whether $\mu|_s(\hat{x})$ is discontinuous or continuous at the point $\hat{x}_k$, then it suffices to construct $\mu|_s(\hat{x})$ near $\hat{x}_k$ because $s_{<k}$ is non-empty and $\mu(\hat{x})$ is already defined on $s_{<k}$ by the induction hypothesis. If we want to know, moreover, all the discontinuous points of $\mu|_s(\hat{x})$ in $s_k$, then we determine $\mu|_s(\hat{x})$ on $s_k$ successively because $s_k$ is connected. Hence it suffices to show that these discontinuous points are determined by $\mu(\hat{x})$ in $|S(V)|_{<k}$, which has already constructed there.

Let $\hat{x}_0 \in s_k$ be a coherent point of $s$ in $G(V)$, and let us consider the sum in (4.1.3). If $\hat{x}_1$ and $\hat{x}_2$ form a co-coherent pair of $\hat{x}_0$ in $G(V)$, then by (5.2.2) we have

$$\mathrm{dep}(\hat{x}_1) < k \quad \text{and} \quad \mathrm{dep}(\hat{x}_2) < k, \tag{5.2.35}$$

and we can calculate each term

$$\mathrm{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)\mu(\hat{x}_1)\mu(\hat{x}_2) \tag{5.2.36}$$

in (4.1.3), where $\mu(\hat{x}_1)$ and $\mu(\hat{x}_2)$ are determined by (5.2.35). Note that the number of terms in the sum may be infinite, and hence, we need to make sure that the number of those pairs satisfying the condition

$$\mu(\hat{x}_1) = 1 \quad \text{and} \quad \mu(\hat{x}_2) = 1 \tag{5.2.37}$$

is finite.

Let $\hat{x}_1$ and $\hat{x}_2$ be a co-coherent pair of $\hat{x}_0$ in $G(V)$ that satisfies (5.2.37). By (5.2.35) and (5.2.37), the turning points $\mathrm{tp}(\hat{x}_1)$ and $\mathrm{tp}(\hat{x}_2)$ belong to $V^E_{\leq k-1}$. It follows from the condition (A-3) and 4. of Lemma 5.1 that different Stokes curves give an only finite number of co-coherent pairs of $\hat{x}_0$. Note that different points in the same Stokes curve may form a co-coherent pair of $\hat{x}_0$, and the number of those pairs is also finite. Since $V^E_{\leq k-1}$ is a finite set by the induction hypothesis, we conclude that the number of non-zero terms in the sum is finite; hence the sum makes sense. This entails that we have constructed $\mu(\hat{x})$ on $s_k$, and thus, on $|S(V)|_{\leq k}$.

Now we will show that $V^E_{\leq k}$ is a finite set. Since the set $V^E_{\leq k-1}$ is finite due to the induction hypothesis, the indices of $V^E_{\leq k-1}$ are bounded. It follows from 2. of Lemma 5.1 that there exists a positive integer $m$ such that we have

$$|S(V)|^E_{\leq k-1} \subset \mathcal{R}_{\mathrm{sym}}(m). \tag{5.2.38}$$

Let $\hat{v}$ be a turning point in $V^E_{\leq k}$ that is not an ordinary turning point in $\mathcal{R}_{\mathrm{sym}}$. Then we can find a Stokes curve $s$ emanating from $\hat{v} \in V$ with $\mu(\hat{y}) = 1$ for some point $\hat{y} \in s_{\leq k}$. Since $\hat{v}$ is a virtual turning point, the function $\mu|_s(\hat{x})$ is discontinuous at some point $\hat{x}_0 \in s_{\leq k}$ due to the fact $\mu(\hat{y}) = 1$. Hence there exists a co-coherent pair $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$ in $G(V)$ satisfying (5.2.37). Since $\hat{x}_1$ and $\hat{x}_2$ are points in $|S(V)|^E_{\leq k-1}$, they belong to $\mathcal{R}_{\mathrm{sym}}(m)$ by (5.2.38). Since $\hat{x}_0$ is coherent with respect to $\hat{x}_1$ and $\hat{x}_2$, the sum of indices of these points is zero (see Definition 3.13 and (3.41)). Therefore we have

$$\hat{x}_0 \in \mathcal{R}_{\mathrm{sym}}(2m + 2\tau) \tag{5.2.39}$$

where $\tau$ is given by the maximum of $|r|$ for all the shift vectors $r \in H_1(\dot{L})$ (Definition 3.5). Hence it follows from 2. of Lemma 5.1 that the indices of $V^E_{\leq k}$ are bounded, and thus, $V^E_{\leq k}$ is a finite set by 1. of the same lemma. The construction of the state function has been completed.

Since a depth function itself is not unique, we need to prove the uniqueness of the state function: Let $\mu_1(\hat{x})$ and $\mu_2(\hat{x})$ be arbitrary state functions on $|S(V)|$, and let us show

$$\mu_1(\hat{x}) = \mu_2(\hat{x}) \qquad (\hat{x} \in |S(V)|). \tag{5.2.40}$$

Suppose $\mu_1(\hat{x}) \neq \mu_2(\hat{x})$ at some point. We set

$$Y := \{\hat{y} \in |S(V)|;\, \mu_1(\hat{y}) \neq \mu_2(\hat{y})\} \neq \phi,$$

and

$$k := \min_{\hat{y} \in Y} \operatorname{dep}(\hat{y}). \tag{5.2.41}$$

Let $\hat{y} \in Y$ be a point with $\operatorname{dep}(\hat{y}) = k$ in a Stokes curve $s(\theta) : (0,1) \to \mathcal{R}_{\mathrm{sym}}$. By the definition of $k$, we have

$$\mu_1(\hat{x}) = \mu_2(\hat{x}) \qquad (\hat{x} \in |S(V)|_{\leq k-1}) \tag{5.2.42}$$

where $|S(V)|_{-1} = \phi$. Set

$$\theta_0 = \inf\{\theta \in (0,1);\, \mu_1(s(\theta)) \neq \mu_2(s(\theta))\} \text{ and } \hat{x}_0 = s(\theta_0). \tag{5.2.43}$$

Since $\mu_1(s(\theta)) = \mu_2(s(\theta))$ holds near $\theta = 0$, we have $\theta_0 > 0$. Then either $\mu_1|_s(\hat{x})$ or $\mu_2|_s(\hat{x})$ is discontinuous at $\hat{x}_0$, and hence, $\hat{x}_0$ is a coherent point of $s$ in $G(V)$. It follows from (5.2.35) and (5.2.42) that we have

$$\sum \operatorname{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)\mu_1(\hat{x}_1)\mu_1(\hat{x}_2) = \sum \operatorname{mul}_{\hat{x}_0}(\hat{x}_1,\, \hat{x}_2)\mu_2(\hat{x}_1)\mu_2(\hat{x}_2)$$

where the both sums are taken over all co-coherent pairs $(\hat{x}_1,\, \hat{x}_2)$ of $\hat{x}_0$ in $G(V)$. This implies that the both $\mu_1(\hat{x})$ and $\mu_2(\hat{x})$ coincide near $\hat{x}_0$, and that contradicts the definition of $\hat{x}_0$. Hence (5.2.40) holds. □

We have the following corollary.

**Corollary 5.7.** *We assume the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ and the condition (†) for $V$. Then the unique model $M(V)$ of the Stokes geometry $G(V)$ satisfies the following compactification condition: For any compact set $K \subset \mathbb{C}$, there exists a positive integer $k$ such that the state function is identically zero in the set*

$$|S(K;\, V)| \cap (\mathcal{R}_{\mathrm{sym}} \setminus \mathcal{R}_{\mathrm{sym}}(k))\,. \tag{5.2.44}$$

*Proof.* It follows from the condition 4. of Proposition 5.3 that the depth of every point in $|S(K;\, V)|$ is uniformly bounded, and thus, we have

$$\{\hat{x} \in |S(K;\, V)|;\; \mu(\hat{x}) = 1\} \subset |S(V)|^{E}_{\leq k}$$

for some $k$ where $|S(V)|^E_{\leq k}$ was given by (5.2.29). Then the set $V^E_{\leq k}$ is finite as it was shown in the proof of Theorem 5.6, and the indices of $|S(V)|^E_{\leq k}$ are bounded. Therefore we obtain the result. □

We assume that $V$ is the set of all the turning points in $\mathcal{R}_{\mathrm{sym}}$. Note that $V$ is a countable set. Let

$$V_1 \subset V_2 \subset V_3 \subset \dots \tag{5.2.45}$$

be an increasing sequence of subsets of $V$ such that each $V_i$ is a finite set. We suppose that the sequence is exhaustive, that is, for any finite subset $W$ of $V$ there exists a positive integer $k$ such that $W \subset V_k$. Let $\mu(\hat{x})$ (resp. $\mu_k(\hat{x})$) be the state function of the Stokes geometry $G(V)$ (resp. $G(V_k)$) respectively. We regard the function $\mu_k(\hat{x})$ as a function on $|S(V)|$ by the zero extension.

**Theorem 5.8.** *We assume the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ and the condition (†) for $V$. Then we have*

$$\lim_{i\to\infty} M(V_i) = M(V), \tag{5.2.46}$$

*that is, for any $\hat{x} \in |S(V)|$ we have*

$$\mu_i(\hat{x}) \to \mu(\hat{x}) \qquad (i \to \infty). \tag{5.2.47}$$

*This convergence is uniform in $|S(K; V)|$ for any compact set $K \subset \mathbb{C}$.*

*Proof.* Let $\mathrm{dep}(\hat{x})$ be a depth function of $G(V)$. Since the restriction of $\mathrm{dep}(\hat{x})$ to $|S(V_i)|$ is also a depth function of $G(V_i)$, we take, in what follows, $\mathrm{dep}(\hat{x})$ as a depth function for all the Stokes geometries $G(V_i)$ and $G(V)$.

By (Step-2) in the proof of Lemma 5.5, there exists a compact set $K_1 \subset \mathbb{C}$ such that every relevant portion of any Stokes path starting from $|S(K; V)|$ is contained in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(K_1)$. We denote by $D^E_K$ the set of every Stokes path $D$ starting from $|S(K; V)|$ with $\mu(\hat{x}_0) = 1$ for some point $\hat{x}_0 \in D$. Set

$$V_K := \{\mathrm{tp}(\hat{x}_0);\, \hat{x}_0 \in |D^E_K|,\, \mu(\hat{x}_0) = 1\}. \tag{5.2.48}$$

Since we have $|D^E_K| \subset |S(K_1; V)|$, it follows from Corollary 5.7 that there exists a positive constant $m$ such that

$$\{\hat{x} \in |D^E_K|;\, \mu(\hat{x}) = 1\} \subset \mathcal{R}_{\mathrm{sym}}(m). \tag{5.2.49}$$

Hence the indices of $V_K$ are bounded, and thus, $V_K$ is a finite set.

We take an integer $i_K$ sufficiently large so that $V_K \subset V_{i_K}$ is satisfied. Now we want to claim

$$\mu(\hat{x}) = \mu_{i_K}(\hat{x}) \qquad (\hat{x} \in |D^E_K|). \tag{5.2.50}$$

The claim can be proved by an argument similar to the proof of Theorem 5.6, but we need much more precise argument: Set

$$Y := \{\hat{x} \in |D^E_K|;\, \mu(\hat{x}) \neq \mu_{i_K}(\hat{x})\},$$

and suppose that $Y$ were non-empty. Let us take a point $\hat{y} \in Y$ such that $\mathrm{dep}(\hat{y})$ achieve a minimal value $k \geq 0$. Let $s(\theta) : (0, 1) \to \mathcal{R}_{\mathrm{sym}}$ be a Stokes curve passing through $\hat{y}$, and we set

$$\theta_0 = \inf\{\theta \in (0, 1);\, \mu(s(\theta)) \neq \mu_{i_K}(s(\theta))\} \text{ and } \hat{x}_0 = s(\theta_0). \tag{5.2.51}$$

We can easily see that the turning point $\mathrm{tp}(\hat{y})$ also belongs to $V_{i_K}$. Indeed, if $\mu_{i_K}(\hat{y}) = 1$, then we have $\mathrm{tp}(\hat{y}) \in V_{i_K}$. If not, then we have $\mu(\hat{y}) = 1$, and this entails $\mathrm{tp}(\hat{y}) \in V_K \subset V_{i_K}$. Therefore we obtain $\mu|_s(\hat{x}) = \mu_{i_K}|_s(\hat{x})$ near $\mathrm{tp}(\hat{y})$, and hence we get $\theta_0 > 0$; in particular, $\hat{x}_0$ is a coherent point of $s$ in $G(V)$. Let us consider the sums

$$\begin{aligned} &\sum_{(\hat{x}_1, \hat{x}_2) \in \mathrm{CC}} \mathrm{mul}_{\hat{x}_0}(\hat{x}_1, \hat{x}_2)\mu(\hat{x}_1)\mu(\hat{x}_2) \qquad \text{and} \\ &\sum_{(\hat{x}_1, \hat{x}_2) \in \mathrm{CC}_{i_K}} \mathrm{mul}_{\hat{x}_0}(\hat{x}_1, \hat{x}_2)\mu_{i_K}(\hat{x}_1)\mu_{i_K}(\hat{x}_2) \end{aligned} \tag{5.2.52}$$

where CC (resp. $\mathrm{CC}_{i_K}$) is the set of all the co-coherent pairs of $\hat{x}_0$ in $G(V)$ (resp. $G(V_{i_K})$). Apparently we have

$$\mathrm{CC}_{i_K} \subset \mathrm{CC}\,. \tag{5.2.53}$$

Let $(\hat{x}_1, \hat{x}_2) \in \mathrm{CC}$. Suppose that $\hat{x}_1$ and $\hat{x}_2$ belong to $|D^E_K|$. Then it follows from (5.2.2) and the definition of $k$ that we have

$$\mu(\hat{x}_1)\mu(\hat{x}_2) = \mu_{i_K}(\hat{x}_1)\mu_{i_K}(\hat{x}_2). \tag{5.2.54}$$

Note that if $\mu(\hat{x}_1)\mu(\hat{x}_2) = 1$, then we have $(\hat{x}_1, \hat{x}_2) \in \mathrm{CC}_{i_K}$ by (5.2.54).

Now suppose that $\hat{x}_1 \notin |D^E_K|$. Then, by the definition of $D^E_K$, we have $\mu(\hat{x}_1) = 0$. We will prove $\mu_{i_K}(\hat{x}_1) = 0$. If $\mu_{i_K}(\hat{x}_1) = 1$, then there exists a Stokes path $D$ in $G(V_{i_K})$ starting from $\hat{x}_1$ that reaches an ordinary turning point thanks to Lemma 5.9 below. Since $D$ is also a Stokes path in $G(V)$ starting from $\hat{x}_1$, by noticing that $\hat{x}_1$ is a co-coherent point of the coherent point $\hat{x}_0$ in $G(V)$, we obtain a Stokes path in $G(V)$ starting from some point in $|S(K;\, V)|$ that reach an ordinary turning point, and that passes the point $\hat{x}_1$ also. Since $\mu(\hat{x}) = 1$ holds in a neighborhood of an ordinary turning point, we have $\hat{x}_1 \in |D_K^E|$, which is a contradiction. Hence we have obtained $\mu_{i_K}(\hat{x}_1) = 0$. This implies that (5.2.54) still holds in this case.

By these observations, the sums in (5.2.52) coincide, and $\mu|_s(\hat{x})$ is equal to $\mu_{i_K}|_s(\hat{x})$ near $\hat{x}_0$. That contradicts the definition of $\hat{x}_0$, and we have obtained (5.2.50).

By employing the same argument as above, we also have

$$\mu(\hat{x}) = \mu_{i_K}(\hat{x}) = 0 \qquad (\hat{x} \in |S(K;V)| \setminus |D_K^E|), \tag{5.2.55}$$

and hence, we finally obtain

$$\mu(\hat{x}) = \mu_{i_K}(\hat{x}) \qquad (\hat{x} \in |S(K;V)|). \tag{5.2.56}$$

This completes the proof of Theorem 5.8. □

**Lemma 5.9.** *Let $V$ be a subset of the turning points in $\mathcal{R}_{\mathrm{sym}}$, and we assume the geometric conditions (GA-1) and (GA-2) for $\mathcal{R}_{\mathrm{sym}}$ and the condition (†) for $V$. Let $\hat{y}$ be a point in $|S(V)|$ such that $\mu(\hat{y}) = 1$ holds for the state function $\mu(\hat{x})$ of $G(V)$. Then there exists a Stokes path $D$ starting from $\hat{y}$ such that $\mu(\hat{x}) = 1$ holds for every point $\hat{x} \in D$; in particular, it reaches some ordinary turning point in $\mathcal{R}_{\mathrm{sym}}$.*

*Proof.* Let $s(\theta) : (0,1) \to \mathcal{R}_{\mathrm{sym}}$ be a Stokes curve that passes thorough $\hat{y}$. We may assume that $s(0) = \mathrm{tp}(\hat{y})$ and $s(\theta_1) = \hat{y}$ for some $\theta_1 > 0$. If $\mu(s(\theta)) = 1$ holds for any $\theta \in (0, \theta_1]$, then it suffices to take the portion of $s$ from $\hat{y}$ to $\mathrm{tp}(\hat{y})$ as a desired Stokes path. Now suppose $\mu(\theta_0) = 0$ for some $0 < \theta_0 < \theta_1$. Set

$$\theta_0 := \sup\{\theta \in (0, \theta_1]; \mu(\hat{x}) = 0\} > 0 \tag{5.2.57}$$

Then $\hat{x}_0 = s(\theta_0)$ is a discontinuous point of $\mu|_s(\hat{x})$, and thus, $\hat{x}_0$ is a coherent point in $s$. Moreover there exists at least one pair of co-coherent points $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$ with $\mu(\hat{x}_1) = 1$ and $\mu(\hat{x}_2) = 1$ because the sum in (5.2.52) takes an odd number. Then, taking the co-coherent point $\hat{x}_1$ as $\hat{y}$, we will do the same procedure again.

Since the depths of all the Stokes path starting from $\hat{y}$ are uniformly bounded due to Lemma 5.5, by repeating the above procedure finitely many times, we can reach some ordinary turning point. This completes the proof. □

## § 5.3. The Unique Existence of the Model for the General Case

In this subsection, we will consider the unique existence of the model of the Stokes geometry for the general case; that is, we assume the geometric conditions (GA-1), (GA-2) and (GA-3) for $\mathcal{R}_{\mathrm{sym}}$ but we do not assume the condition (†) for $V$ (Definition 5.2) which was supposed in the previous subsection.

For the general case, the set of coherent points in a Stokes curve may be dense in that curve, and this causes many difficulties; for example, the depths of Stoke paths are not uniformly bounded. One of noticeable facts is that almost all the coherent points are isolated from the ordinary turning points in the sense that any Stokes path starting from its co-coherent point cannot reach an ordinary turning point. To exclude the isolated coherent points in this sense, we will introduce the notion of a Stokes path tree as follows: Let $\hat{y}$ be a point in $|S(V)|$.

**Definition 5.10.** **A Stokes path tree** starting from $\hat{y}$ in $G(V)$ is a binary tree represented by the following schematic diagram:

$$
\begin{array}{llllllll}
 & & & & & \hat{p}_k & \to\ \hat{q}_k & \\
 & & & & & \upharpoonleft & & \\
 & & & & \to & \hat{q}_{k-1} & & \\
 & & & & \cdots & \downharpoonleft & & \\
 & & & & \cdots & \hat{p}_{k+1} & \to \hat{q}_{k+1} & \\
 & & & \hat{p}_3 & \to \cdots & & & \\
 & & & \upharpoonleft & & & & \\
 & \hat{p}_1 & \to & \hat{q}_1 & & & & \\
 & \upharpoonleft & & \downharpoonleft & & & & \\
 & & & \hat{p}_4 & \to \cdots & & & \\
 \hat{y} = \hat{p}_0 \to & \hat{q}_0 & & & & & & \qquad\qquad \text{(the bottom side)} \\
 & & & \hat{p}_5 & \to \cdots & & & \\
 & \downharpoonleft & & \upharpoonleft & & & & \\
 & \hat{p}_2 & \to & \hat{q}_2 & & & & \\
 & & & \downharpoonleft & & & & \\
 & & & \hat{p}_6 & \to \cdots & & &
\end{array}
\tag{5.3.1}
$$

for which the conditions below are satisfied:

1. The tree is a finite graph.

2. Every bottom node is an ordinary turning point in $V$ (i.e. $\hat{q}_k$, $\hat{q}_{k+1}$, etc. are ordinary turning points).

3. Any path from $\hat{y}$ to a bottom node in the diagram (5.3.1) gives a Stokes path starting from $\hat{y}$ in $G(V)$, and it is called "a Stokes path in the Stokes path tree".

Then we define the depth of a Stokes path tree $T$ by:

$$\operatorname{dep}(T) = \max_{D\text{: a Stokes path in } T} \operatorname{dep}(D). \tag{5.3.2}$$

Now we have the following lemma whose proof goes in the same way as that for Lemma 4.7. Remember that the function $\varphi(\hat{x}_1, \hat{x}_2)$ was defined by (4.2.2).

**Lemma 5.11.** *For a Stokes path tree starting from $\hat{y}$ that is represented by the schematic diagram (5.3.1), we have*

$$F^{\mathrm{a}}(\hat{y}) = \sum \varphi(\hat{p}_i, \hat{q}_i) \tag{5.3.3}$$

*where the sum is taken over all the horizontal arrows $\hat{p}_i \to \hat{q}_i$ in (5.3.1).*

Note that not every point $\hat{x}$ in $|S(V)|$ has a Stokes path tree starting from $\hat{x}$ itself. Hence the following notions make sense:

**Definition 5.12.** Let $\hat{x}_0$ be a coherent point of a Stokes curve $s$ in $G(V)$, and let $\hat{x}_1$ and $\hat{x}_2$ be a co-coherent pair of $\hat{x}_0$ in $G(V)$. If there exist Stokes path trees starting from $\hat{x}_1$ and $\hat{x}_2$ in $G(V)$ respectively, then we say that $\hat{x}_0$ is a **strong coherent point** of $s$ in $G(V)$, and that $\hat{x}_1$ and $\hat{x}_2$ is a **strong co-coherent pair** of $\hat{x}_0$ in $G(V)$.

For the general case, we cannot expect the existence of a depth function. Hence we introduce another notion: a weak depth function given by the following.

**Proposition 5.13.** *We assume the geometric conditions (GA-1), (GA-2) and (GA-3) for $\mathcal{R}_{\mathrm{sym}}$. Then there exists a* **weak depth function**

$$\operatorname{dep} : |S(V)| \to \mathbb{Z}_{\geq 0} \cup \{-1\}$$

*of the Stokes geometry $G(V)$ for which the following conditions are satisfied:*

1. *The set $|S(V)|_{\leq k} := \{\hat{x} \in |S(V)|;\ \operatorname{dep}(\hat{x}) \leq k\}$ is open in $|S(V)|$ for any $k \in \mathbb{Z}$, and for any Stokes curve $s$ emanating from $\hat{v} \in V$ we have*

$$\operatorname{dep}|_s(\hat{x}) = \begin{cases} -1 & \text{if } \hat{v} \text{ is a virtual turning point,} \\ 0 & \text{if } \hat{v} \text{ is an ordinary turning point,} \end{cases} \tag{5.3.4}$$

   *in some neighborhood of $\hat{v}$. Here $\operatorname{dep}|_s(\hat{x})$ designates the restriction of $\operatorname{dep}(\hat{x})$ to $s$.*

2. *We have $\operatorname{dep}(\hat{x}_1) \geq \operatorname{dep}(\hat{x}_2)$ for any points $\hat{x}_1$ and $\hat{x}_2$ with $l(s; \hat{x}_1) \geq l(s; \hat{x}_2)$ in a Stokes curve $s \in S(V)$.*

3. *If $\hat{x}_0$ is a* **strong** *coherent point of a Stokes curve in $G(V)$, then we have*

$$\operatorname{dep}(\hat{x}_0) > \max\{\operatorname{dep}(\hat{x}_1), \operatorname{dep}(\hat{x}_2)\} \tag{5.3.5}$$

   *for every* **strong** *co-coherent pair $\hat{x}_1$ and $\hat{x}_2$ of $\hat{x}_0$ in $G(V)$.*

*4. For any compact set $K \subset \mathbb{C}$, the function* $\operatorname{dep}(\hat{x})$ *is uniformly bounded on* $|S(K;\, V)|$.

*Proof.* For any $\hat{y} \in |S(V)|$, we set

$$T(\hat{y}) = \{\text{Stokes path trees starting from } \hat{y} \text{ in } G(V)\}, \tag{5.3.6}$$

and

$$\operatorname{dep}(\hat{y}) = \begin{cases} -1 & \text{if } T(\hat{y}) = \phi \\ \\ \sup\limits_{T \in T(\hat{y})} \operatorname{dep}(T) & \text{otherwise} \end{cases}. \tag{5.3.7}$$

Now we have the following two lemmas which are similar to Lemmas 5.4 and 5.5. Note that Lemma 5.14 explains why we have introduced a strong coherent point.

**Lemma 5.14.** *Let $s \in S(V)$. Then the set of strong coherent points of $s$ in $G(V)$ is discrete in the closure of $s$.*

**Lemma 5.15.** *For any compact set $K \subset \mathbb{C}$ the function* $\operatorname{dep}(\hat{y})$ *defined by (5.3.7) is uniformly bounded on the set* $|S(K;\, V)|$.

We give a sketch of the proofs for Lemmas 5.14 and 5.15. We refer the readers to [H4] for the details.

*Proof.* We first prove Lemma 5.15. Let $K$ be a compact set in the base space. By (Step-2) of the proof of Lemma 5.5, there exists a compact set $\tilde{K} \subset \mathbb{C}$ such that relevant portions of a Stokes path tree starting from a point in $|S(K;\, V)|$ are contained in $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(\tilde{K})$. Here the set of relevant portions of a Stokes path tree $T$ is, by definition, the union of relevant portions of each Stokes path in $T$ (for the definition of those of a Stokes path, see (Step-2) of the proof of Lemma 5.5).

In what follows, we consider a Stokes path tree $T$ starting only from $|S(K;\, V)|$. Hence relevant portions of $T$ are always contained in $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(\tilde{K})$.

Let $D$ be a Stokes path in a Stokes path tree $T$, and let us consider a subset $B$ of $D$ represented by the following diagram:

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & \hat{p}_{j+3} \\
 & \hat{p}_{j+2} \arrow[r] & \hat{q}_{j+2} \arrow[u] \arrow[d] \\
\hat{p}_{j+1} \arrow[r] & \hat{q}_{j+1} \arrow[u] \arrow[d] & \hat{w}_{j+2} \\
\hat{q}_{j} \arrow[u] \arrow[d] & \hat{w}_{j+1} & \\
\hat{w}_{j} & &
\end{tikzcd} \tag{5.3.8}$$

Here $B$ consists of 3 consecutive strong coherent points of the Stokes path $D$ ($\hat{q}_j$, $\hat{q}_{j+1}$ and $\hat{q}_{j+2}$) and their strong co-coherent pairs $((\hat{w}_j, \hat{p}_{j+1})$, $(\hat{w}_{j+1}, \hat{p}_{j+2})$ and $(\hat{w}_{j+2}, \hat{p}_{j+3}))$. We call such a subset $B$ of $D$ a 2-length block of the Stokes path $D$. Set

$$\varphi(B) := \varphi(\hat{p}_{j+1},\, \hat{q}_{j+1}) + \varphi(\hat{p}_{j+2},\, \hat{q}_{j+2}) + F^{\mathrm{a}}(\hat{w}_j) + F^{\mathrm{a}}(\hat{w}_{j+1}) + F^{\mathrm{a}}(\hat{w}_{j+2}). \tag{5.3.9}$$

Suppose that we can prove the following claim: There exists a positive constant $\delta > 0$ such that we have

$$\varphi(B) \geq \delta > 0 \tag{5.3.10}$$

for any block $B$ of a Stokes path in a Stokes path tree starting from $|S(K;\, V)|$. Then we have

$$F^{\mathrm{a}}(y) \geq \delta \left[\frac{\mathrm{dep}(T)}{4}\right] \tag{5.3.11}$$

for a Stokes path tree $T$ starting from $\hat{y} \in |S(K;\, V)|$. We come to the conclusion of Lemma 5.15 since $F^{\mathrm{a}}(\hat{y})$ is uniformly bounded on $|S(K;\, V)|$. Hence it suffices to show the claim.

To prove the claim we need some preparations. First let $T$ be a Stokes path tree with $\mathrm{dep}(T) = 1$. Then $T$ is represented by the schematic diagram:

$$\begin{array}{rcl} & & \hat{p}_1 \to \hat{q}_1 \\ & & \uparrow \\ y = \hat{p}_0 \to \hat{q}_0 & & \\ & & \downarrow \\ & & \hat{p}_2 \to \hat{q}_2 \end{array} \quad . \tag{5.3.12}$$

Here $\hat{q}_1$ and $\hat{q}_2$ (resp. $[\hat{p}_1,\, \hat{q}_1]$, $[\hat{p}_2,\, \hat{q}_2]$) are ordinary turning points (resp. portions of ordinary Stokes curves) respectively. Since we have

$$F^{\mathrm{a}}(y) = F^{\mathrm{a}}(\hat{p}_0) \geq \varphi(\hat{p}_1,\, \hat{q}_1) + \varphi(\hat{p}_2,\, \hat{q}_2), \tag{5.3.13}$$

and since the number of ordinary Stokes curves is finite, there exists a positive constant $\delta_1$ that satisfies

$$F^{\mathrm{a}}(y) \geq \delta_1 > 0 \tag{5.3.14}$$

for any Stokes path tree represented by the diagram (5.3.12). This implies that for an arbitrary Stokes path tree $T$ starting from $y \in |S(K;\, V)|$ we have

$$\mathrm{dep}(T) \geq 1 \Longrightarrow F^{\mathrm{a}}(y) \geq \delta_1 > 0 \tag{5.3.15}$$

because $T$ contains at least one Stokes path subtree $T'$ of $T$ with $\mathrm{dep}(T') = 1$. Note that there exists a Stokes path tree of any depth that contains only one such a subtree. Therefore the estimate (5.3.11) does not follow from (5.3.14) directly.

Next let $B$ be a 2-length block represented by the diagram (5.3.8), and set

$$\tilde{B} := B \cup [\hat{w}_j, \mathrm{tp}(\hat{w}_j)] \cup [\hat{w}_{j+1}, \mathrm{tp}(\hat{w}_{j+1})] \cup [\hat{w}_{j+2}, \mathrm{tp}(\hat{w}_{j+2})]. \tag{5.3.16}$$

Note that $\varphi(B)$ is equal to the sum of integrals (4.2.2) along all portions of Stokes curves contained in $\tilde{B}$, i.e.,

$$\begin{aligned} \varphi(B) = {} & \varphi(\hat{p}_{j+1}, \hat{q}_{j+1}) + \varphi(\hat{p}_{j+2}, \hat{q}_{j+2}) \\ & + \varphi(\hat{w}_j, \mathrm{tp}(\hat{w}_j)) + \varphi(\hat{w}_{j+1}, \mathrm{tp}(\hat{w}_{j+1})) + \varphi(\hat{w}_{j+2}, \mathrm{tp}(\hat{w}_{j+2})). \end{aligned} \tag{5.3.17}$$

Then it follows from (5.2.18) that for any points $\hat{x}_1$ and $\hat{x}_2$ in $|\tilde{B}|$ we have

$$\varphi(B) \geq \delta_{\tilde{K}} \,\mathrm{dist}\left(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_1),\, \pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x}_2)\right)^{\frac{3}{2}} \tag{5.3.18}$$

where a positive constant $\delta_{\tilde{K}}$ depends only on $\tilde{K}$ (note that the estimate (5.2.18) has been obtained independently of the condition (†)).

Now we are ready to prove the claim. Let $B$ be a 2-length block represented by the diagram (5.3.8). We will consider the following 3 cases:

**Case 1:** One of turning points $\mathrm{tp}(\hat{w}_j)$, $\mathrm{tp}(\hat{w}_{j+1})$ and $\mathrm{tp}(\hat{w}_{j+2})$ is virtual.

We may suppose that $\mathrm{tp}(\hat{w}_j)$ is a virtual turning point. Since $B$ is a subset of Stokes path tree, there exists a Stokes path tree $T'$ starting from $\hat{w}_j$. Moreover, as $\mathrm{tp}(\hat{w}_j)$ is a virtual turning point, we have $\mathrm{dep}(T') \geq 1$. Hence, by (5.3.15), we obtain

$$\varphi(B) \geq F^{\mathrm{a}}(\hat{w}_j) \geq \delta_1. \tag{5.3.19}$$

**Case 2:** Turning points $\mathrm{tp}(\hat{w}_j)$, $\mathrm{tp}(\hat{w}_{j+1})$ and $\mathrm{tp}(\hat{w}_{j+2})$ are ordinary, and at least two of them are different points.

Applying (5.3.18) to this case, we have

$$\varphi(B) \geq \delta_{\tilde{K}} \min_{x,y \in Z,\, x \neq y} \mathrm{dist}(x,y)^{\frac{3}{2}} > 0 \tag{5.3.20}$$

where $Z$ is the set of the ordinary turning points in the base space.

**Case 3:** Turning points $\mathrm{tp}(\hat{w}_j)$, $\mathrm{tp}(\hat{w}_{j+1})$ and $\mathrm{tp}(\hat{w}_{j+2})$ are ordinary, and they coincide.

Set

$$\hat{w} := \mathrm{tp}(\hat{w}_j) = \mathrm{tp}(\hat{w}_{j+1}) = \mathrm{tp}(\hat{w}_{j+2}),$$

and the ordinary turning point $\hat{w}$ is assumed to be of type $(l, m)$. Then the Stokes curve $\mathrm{st}(\hat{p}_{j+1})$ (resp. $\mathrm{st}(\hat{p}_{j+2})$) is of type $(l, k)$ (resp. $(m, k)$) for some $k$ respectively. Hence, if the projection image of a coherent point $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{q}_i)$ $(i = j, j+1, j+2)$ is sufficiently close to $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{w})$, then the local projection images of 3 Stokes curves $\mathrm{st}(\hat{q}_i)$, $\mathrm{st}(\hat{w}_i)$ and $\mathrm{st}(\hat{p}_{i+1})$ intersect transversally at $\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{q}_i)$ by the assumption (GA-3). Since each intersection in $\pi_{\mathcal{R}_{\mathrm{sym}}}(\tilde{B})$ is transversal if $\pi_{\mathcal{R}_{\mathrm{sym}}}(\tilde{B})$ is sufficiently small, we can obtain the following fact by applying Lemma 4.4 to the configuration of $\pi_{\mathcal{R}_{\mathrm{sym}}}(\tilde{B})$: There exists a sufficiently small constant $\epsilon > 0$ such that

$$\pi_{\mathcal{R}_{\mathrm{sym}}}(\tilde{B}) \not\subseteq U_\epsilon(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{w})) := \{z \in \mathbb{C}; \mathrm{dist}(z, \pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{w})) < \epsilon\} \tag{5.3.21}$$

holds for any 2-length block $B$ of Case 3 (see [H4] for the details; these kinds of configurations of the Stokes geometry were also studied in [U]). Therefore, as $\tilde{B}$ contains a point $\hat{x}$ with

$$\mathrm{dist}(\pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{w}), \pi_{\mathcal{R}_{\mathrm{sym}}}(\hat{x})) \geq \epsilon, \tag{5.3.22}$$

the estimate

$$\varphi(B) \geq \epsilon^{\frac{3}{2}} > 0 \tag{5.3.23}$$

follows from (5.3.18).

We have obtained the estimate (5.3.10) for all cases, and this completes the proof. □

Lemma 5.14 is deduced from Lemma 5.15 as follows:

*Proof.* Let $K$ and $\tilde{K}$ be compact sets given in the proof of Lemma 5.15, and let $W$ be a subset of $V$. We say that a binary tree $T$ is a $W$-Stokes path tree if $T$ satisfies the conditions 1. and 3. of Definition 5.10 and the following condition 2'.:

2'. Every bottom node of $T$ belongs to $W$.

Let $O$ be the set of ordinary turning points in $\mathcal{R}_{\mathrm{sym}}$. Then an $O$-Stokes path tree is a usual Stokes path tree.

We define the subset $V_k(\tilde{K}; W)$ of $V$ in the following way; A turning point $\hat{v} \in V$ belongs to $V_k(\tilde{K}; W)$ if and only if there exist a Stokes curve $s$ emanating from $\hat{v}$ and a point $\hat{y} \in s$ such that there exists a $W$-Stokes path tree $T$ starting from $\hat{y}$ which satisfies the conditions 1. and 2. below.

1. $\mathrm{dep}(T) \leq k$.

2. Relevant portions of $T$ are contained in $\pi^{-1}_{\mathcal{R}_{\mathrm{sym}}}(\tilde{K})$.

Then, by the definition of a Stokes path tree, we have

$$V_{k+1}(\tilde{K};\, W) \subset V_1(\tilde{K};\, V_k(\tilde{K};\, W)) \tag{5.3.24}$$

for any $k \geq 0$. If $W'$ is a finite set, then $V_1(\tilde{K};\, W')$ is also finite by Lemma 5.1. Hence $V_k(\tilde{K};\, O)$ is a finite set for any $k \geq 0$. Note that an element in $V_k(\tilde{K};\, O)$ is a constructible turning point of level at most $k$ in the sense of [H2].

Let $T$ be a Stokes path tree starting from $|S(K;\, V)|$, and set

$$k := \sup_{\hat{y} \in |S(K;\, V)|} \mathrm{dep}(\hat{y}) < \infty. \tag{5.3.25}$$

We can obtain

$$\{\text{Turning points in relevant portions of } T\} \subset V_{\mathrm{dep}(T)}(\tilde{K};\, O) \subset V_k(\tilde{K};\, O), \tag{5.3.26}$$

and from which Lemma 5.14 follows. □

It is easy to see that $\mathrm{dep}(\hat{y})$ satisfies all properties of a week depth function by these lemmas. The proof of Proposition 5.13 has been completed. □

The main theorem for the general case is:

**Theorem 5.16.** *We assume the geometric conditions (GA-1), (GA-2) and (GA-3) for $\mathcal{R}_{\mathrm{sym}}$ which were given in Subsection 5.1. Then there exists a model $M_c(V)$ of the Stokes geometry $G(V)$, and we call it the compact model of $G(V)$. The compact model $M_c(V)$ satisfies the following compactification condition: For any compact set $K \subset \mathbb{C}$, there exists a positive integer $k$ such that the state function is identically zero in the set*

$$|S(K;\, V)| \cap (\mathcal{R}_{\mathrm{sym}} \setminus \mathcal{R}_{\mathrm{sym}}(k))\,. \tag{5.3.27}$$

*Conversely any model of the Stoke geometry $G(V)$ that satisfies the compactification condition coincides with the compact model $M_c(V)$.*

*Proof.* One of the differences between a depth function in the special case and a weak one in this subsection is that the weak one takes the negative value $-1$. We cannot determine uniquely the state function $\mu(\hat{x})$ on $|S(V)|_{-1}$ by our Ansatz. For example, let us again consider the example given in Fig. 13 with $\hat{x}_i$ being a virtual turning point $(i = 1, 2, 3)$, which has two models. This example has no depth function, but certainly it has a weak one. In fact, if we set $\mathrm{dep}(\hat{x}) = -1$ on the Stokes curve emanating from $\hat{x}_i$ ($i = 1, 2, 3$) and $\mathrm{dep}(\hat{x}) = 0$ on that emanating from $\hat{w}_i$ ($i = 1, 2, 3$), then $\mathrm{dep}(\hat{x})$ satisfies every condition of a weak depth function. As Fig. 10 and Fig. 11 suggest us, the state function is allowed to take different values on $|S(V)|_{-1}$.

Note that the both models given by Fig. 10 and Fig. 11 satisfy the compactification condition in this theorem, and hence, our theorem does not hold for the uniqueness. However that is not a contradiction because the example is never associated with a system of differential equations (see also the remark after this proof).

Taking these observations into account, we specify an initial value of the state function at each point in $|S(V)|_{-1}$, and those for the compact model are given by:

$$\mu(\hat{x}) = 0 \qquad (\hat{x} \in |S(V)|_{-1}). \tag{5.3.28}$$

Now it is easy to see that $\mu(x)$ satisfies the Ansatz at every point in $|S(V)|_{-1}$ by the following facts:

**(F-1)**: For any point in $\hat{x} \in |S(V)|_{-1}$, $\mathrm{tp}(\hat{x})$ is a virtual turning point due to the conditions 1. and 2. of Proposition 5.13.

**(F-2)**: Let us assume that the state function $\mu(\hat{x})$ satisfies the initial condition (5.3.28). Then for any coherent point $\hat{x}_0$ of a Stokes curve $s \in S(V)$ (note that we do **not** assume $\hat{x}_0 \in |S(V)|_{-1}$), if $\hat{x}_0$ is not strong, then the sum in (4.1.3) of the Ansatz takes the zero value, that is, the state function is continuous at $\hat{x}_0$. This claim can be proved in the following way: Let $\hat{x}_1$ and $\hat{x}_2$ be co-coherent pair of $\hat{x}_0$ in $G(V)$. Then, since $\hat{x}_0$ is not strong, either $\hat{x}_1$ or $\hat{x}_2$ is a point from which no Stokes path tree starts. We suppose $\hat{x}_1$ to be such a point. Then we have $\mathrm{dep}(\hat{x}_1) = -1$. In fact, this is clear for the weak depth function defined by (5.3.7), and this also holds for arbitrary weak depth functions (we omit its details). Now we get $\mu(\hat{x}_1) = 0$ by (5.3.28). Hence we can conclude that for every co-coherent pair $(\hat{x}_1, \hat{x}_2)$ of $\hat{x}_0$ in $G(V)$, either $\mu(\hat{x}_1)$ or $\mu(\hat{x}_2)$ is equal to zero, and the sum is also zero.

Then we can construct the state function on each $|S(V)|_k$ ($k \geq 0$) by the induction with respect to $k$. In fact the above facts imply that, by replacing coherent points with strong ones in the argument, we can construct the state function for the compact model as we have constructed one in the proof for Theorem 5.6. We refer the readers to [H4] for the details of the construction.

Now we will prove the uniqueness with the compactification condition. Let $\mu(\hat{x})$ be the state function of the compact model, and let $\mu_1(\hat{x})$ be a state function satisfying the compactification condition.

Suppose that the initial conditions of $\mu(\hat{x})$ and $\mu_1(\hat{x})$ coincide, that is, both state functions satisfy (5.3.28). Then, since the fact (F-2) holds for $\mu(\hat{x})$ and $\mu_1(\hat{x})$ because of the zero initial condition, we obtain $\mu(\hat{x}) \equiv \mu_1(\hat{x})$ by the same argument as in Theorem 5.6.

Let us suppose $\mu(\hat{x}) \neq \mu_1(\hat{x})$ for some point in $|S(V)|_{-1}$. Since $\mu(\hat{x}) = 0$ on $|S(V)|_{-1}$, we get a point $\hat{y} \in |S(V)|_{-1}$ with $\mu_1(\hat{y}) = 1$. Then we apply the procedure described in the proof of Lemma 5.9 to the point $\hat{y}$. Since a path generated by the procedure can not reach an ordinary turning point due to the fact (F-1), we obtain an infinite length of a Stokes path $D$ with

$$\mu_1(\hat{x}) = 1 \qquad (\hat{x} \in D). \tag{5.3.29}$$

Here $D$ consists of the infinite coherent points $\{\hat{q}_i\}_{i=0}^{\infty}$ and the co-coherent points $\{\hat{p}_i\}_{i=0}^{\infty}$ and $\{\hat{w}_i\}_{i=0}^{\infty}$ which generate the infinite version of the schematic diagram ($\hat{p}_0 = \hat{y}$) defined by (4.2.1) with letting $k \to \infty$.

Then there exists a compact set $K_1 \subset \mathbb{C}$ such that

$$\{\text{Relevant portions of } D\} \subset \pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K_1). \tag{5.3.30}$$

Here the precise definition of the relevant portions of $D$ was given in (Step-2) of the proof of Lemma 5.5. Note that the fact (5.3.30) also follows from (Step-2), for which we do not need the condition (†).

It follows from (5.3.29) and the compactification condition that there exists an integer $k_1$ that satisfies $D \subset \mathcal{R}_{\mathrm{sym}}(k_1)$, and thus, we have

$$\{\text{Relevant portions of } D\} \subset \mathcal{R}_{\mathrm{sym}}(k) \tag{5.3.31}$$

for some integer $k$. Hence, by 1. of Lemma 5.1, the number of the relevant turning points

$$E_{tp} := \{\mathrm{tp}(\hat{p}_i)\}_{i=0}^{\infty} \cup \{\mathrm{tp}(\hat{q}_i)\}_{i=0}^{\infty} \cup \{\mathrm{tp}(\hat{w}_i)\}_{i=0}^{\infty} \tag{5.3.32}$$

is finite. This implies, in particular, that the same turning point appears in the relevant portions of $D$ infinitely many times.

Since the number of coherent points in $\pi_{\mathcal{R}_{\mathrm{sym}}}^{-1}(K_1) \cap \mathcal{R}_{\mathrm{sym}}(k)$ which are generated by the Stokes curves emanating from the finite set $E_{tp}$ is again finite, the same coherent point appears in $\{\hat{q}_i\}_{i=0}^{\infty}$ infinitely many times. Since the function $F^{\mathrm{a}}(\hat{x})$ is strictly decreasing along $D$, the same point never appears twice in $D$. This is a contradiction because $\{\hat{q}_i\}_{i=0}^{\infty} \subset D$. Thus the uniqueness with the compactification condition has been proved. □

If the compactification condition described in Theorem 5.16 would be added in our Ansatz, then we could obtain the complete answer, that is, the unique model of $G(V)$ always exists for the Stokes geometry associated with a real system. For such a Stokes geometry, however, the author does not know whether the compactification condition is independent of the Ansatz.

The compact model also appears as a limit model of finite approximations for the Stokes geometry $G(V)$. Let $V$ be a set of all the turning points in $\mathcal{R}_{\mathrm{sym}}$ (note that $V$ is a countable set), and let $\{V_i\}_{i=1}^{\infty}$ be an increasing sequence of subsets of $V$ such that each $V_i$ is finite and $\{V_i\}$ is exhaustive. Note that since $V_i$ is a finite set, the model $M(V_i)$ of the Stokes geometry $G(V_i)$ uniquely exists by Theorem 5.6.

**Theorem 5.17.** *Assume the geometric conditions (GA-1), (GA-2) and (GA-3) for $\mathcal{R}_{\mathrm{sym}}$. Then we have*

$$\lim_{i\to\infty} M(V_i) = M_c(V). \tag{5.3.33}$$

*The above convergence is uniform in $|S(K;V)|$ for any compact set $K \subset \mathbb{C}$.*

*Proof.* The proof is the same as that for Theorem 5.8. □

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