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Propagation of microlocal solutions through a hyperbolic fixed point

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Abstract

The aim of this note is to review [BFRZ], where we have studied existence, uniqueness and the asymptotics of the microlocal solutions of a semiclassical pseudodifferential equation near a hyperbolic fixed point of its symbol. Here we focus on the construction of these solutions, that we write, as in [He-Sj 2], as superpositions of time-dependent WKB solutions. The large time asymptotic expansion of the phase and the symbol, which plays a crucial role, is described. We also give a detailed treatment of the one-dimensional case.

§ 1. Introduction

This paper is devoted to the description of the propagation of semiclassical singularities for pseudodifferential operators on $L^2(\mathbb{R}^d)$, $d \geq 1$, in the case where their associated Hamiltonian vector field presents a hyperbolic fixed point. We have obtained a complete description of this phenomenon in [BFRZ], and our main aim here is to review what concerns the representation of the solutions near the fixed point. For applications to scattering theory, we send the reader to [ABR1] and [ABR2].

In this first section, we recall some well-known results concerning the simplest one-dimensional operator with such a hyperbolic fixed point. We study the asymptotic expansion of the solutions to the one-dimensional Schrödinger equation

$$Pu := \left(-h^2 \frac{d^2}{dx^2} - \frac{\lambda^2}{4} x^2 \right) u = hzu, \tag{1.1}$$

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with respect to a semiclassical parameter $h \to 0$. Here λ is a positive constant (it will play an important role in the d-dimensional case), and z is a spectral parameter bounded with respect to h. In this model, the potential $-\lambda^2 x^2/4$ presents a non-degenerate barrier at $x = 0$ and the energy hz is close to the maximum value 0. In this one dimensional simple case, we describe here the solutions in terms of Weber functions. The discussion in the general multidimensional case will use a different approach.

Changing the scale $x = e^{\pi i/4}\sqrt{\frac{h}{\lambda}}\, y$, $z = -i\lambda\nu$, $u(x) = v(y)$, (1.1) becomes the Weber equation:

$$Qv := \left(-\frac{d^2}{dy^2} + \frac{y^2}{4}\right) v = \nu v. \tag{1.2}$$

About (1.2), the following facts are well known:

(Q1) $\nu = k + \frac{1}{2}$, $k \in \mathbb{N} = \{0, 1, \dots\}$, are the eigenvalues of Q, and the corresponding eigenfunctions are

$$v_k(y) = H_k(y) e^{-y^2/4}.$$

Here H_k is the Hermite polynomial of degree k.

(Q2) If $\nu \in \mathbb{C} \setminus \left(\mathbb{N} + \frac{1}{2}\right)$, the Weber function

$$D_{\nu-1/2}(y) = \frac{1}{\Gamma(\frac{1}{2} - \nu)} \int_0^{\infty} \exp\left(-\left(\frac{y^2}{4} + y\eta + \frac{\eta^2}{2}\right)\right) \eta^{-\nu-1/2} d\eta,$$

and $D_{\nu-1/2}(-y)$ are solutions to (1.2). This integral is convergent only for $\operatorname{Re}\nu < \frac{1}{2}$, but continues analytically to $\mathbb{C} \setminus (\mathbb{N} + \frac{1}{2})$ by integration by parts (it has in fact a simple pole at each point in $\mathbb{N} + \frac{1}{2}$ which is canceled by the Gamma prefactor).

Going back to the original variables, we obtain the following facts about (1.1):

(P1) If $z = -i\lambda\left(k + \frac{1}{2}\right)$, $k \in \mathbb{N}$, then zh is a resonance for P, and the corresponding resonant state is

$$u_k(x, h) = H_k\left(e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}\, x\right) e^{i\lambda x^2/(4h)}.$$

The function u_k is an "outgoing" wave for $x \to \pm\infty$ in the sense that its frequency set is included in the outgoing stable manifold of the corresponding classical Hamiltonian vector field (see (1.5)).

(P2) If $z \in \mathbb{C} \setminus -i\lambda\left(\mathbb{N} + \frac{1}{2}\right)$, then

$$\begin{aligned} u(x, h) =& D_{iz/\lambda - 1/2}\left(e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}\, x\right) \\ =& \text{const.} \int_0^{\infty} \exp\left(\frac{i\lambda}{4h}x^2 - e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}\, x\eta - \frac{1}{2}\eta^2\right) \eta^{-iz/\lambda - 1/2} d\eta, \end{aligned}$$

is a solution to (1.1). With the formal change of variable $\eta = e^{-\pi i/4}\sqrt{\frac{\lambda}{h}}\,\xi$, it becomes

$$u(x,h) = \text{const.} \int_0^{e^{\pi i/4}\infty} \exp\left(\frac{i\lambda}{h}\left(\frac{x^2}{4} + x\xi + \frac{\xi^2}{2}\right)\right)\xi^{-iz/\lambda - 1/2}\, d\xi.$$

Now we define, modifying the contour of integration to $(0,\infty)$ and inserting a cutoff function χ which is identically equal to 1 on the interval $[0,R]$ for a large $R>0$,

$$I_\mu(x,h) = \int_0^\infty \exp\left(\frac{i\lambda}{h}\left(\frac{x^2}{4} + x\xi + \frac{\xi^2}{2}\right)\right)\xi^{-\mu-1}\chi(\xi)\, d\xi. \tag{1.3}$$

Then we see that $I_{iz/\lambda - 1/2}(x,h)$ is a quasimode. More precisely, for $|x|<R$, we have

$$(P - hz)I_{iz/\lambda-1/2}(x,h) = \mathcal{O}(h^\infty).$$

In fact, the left hand side is the same integral as (1.3) with χ replaced by its derivatives, whose support does not contain any stationary point of the phase. Moreover, $u(x,h) = \text{const}\cdot I_\mu(x,h) + \mathcal{O}(h^\infty)$ on $L^2([-R,R])$.

Proposition 1.1. *Suppose μ stays in a compact subset of $\mathbb{C}\setminus\mathbb{N}$ for any h small enough. Then I_μ has an asymptotic expansion in powers of h uniformly for x in any compact subset of $\mathbb{R}\setminus\{0\}$: if $x>0$, there exists a symbol $a(x,h)\sim\sum_{k=0}^\infty a_k(x)h^k$ with $a_0=1$ such that*

$$I_\mu(x,h) = e^{-\pi i\mu/2}\Gamma(-\mu)\left(\frac{\lambda x}{h}\right)^\mu e^{i\lambda x^2/(4h)}a(x,h)$$

and if $x<0$, there exist symbols $b(x,h)\sim\sum_{k=0}^\infty b_k(x)h^k$ with $b_0=1$ and $c(x,h)\sim\sum_{k=0}^\infty c_k(x)h^k$ with $c_0=1$ such that

$$\begin{aligned} I_\mu(x,h) =& e^{\pi i\mu/2}\Gamma(-\mu)\left(\frac{\lambda|x|}{h}\right)^\mu e^{i\lambda x^2/(4h)}b(x,h) \\ &+ e^{\pi i/4}\sqrt{\frac{2\pi h}{\lambda}}|x|^{-\mu-1}e^{-i\lambda x^2/(4h)}c(x,h). \end{aligned}$$

Here $a(x,h)\sim\sum_{k=0}^\infty a_k(x)h^k$ means that for any $N\in\mathbb{N}$, $a(x,h)-\sum_{k=0}^N a_k(x)h^k = \mathcal{O}(h^{N+1})$.

Proof. We compute only the main term of the expansions. Let $f(x,\xi) = \frac{x^2}{4} + x\xi + \frac{\xi^2}{2}$ be the phase function. The values of ξ which contribute to the principal terms of the asymptotic expansion are the origin $\xi=0$ and the critical point $\xi=-x$ of $f(x,\xi)$ (only in the case $x<0$). Let $\varepsilon>0$ be a sufficiently small number and $\chi(\xi)$ a cutoff function which is 1 for $0\le\xi\le\varepsilon$ and 0 for $\xi\ge 2\varepsilon$. We set

$$I_\mu(x,h) = I_\mu^1(x,h) + I_\mu^2(x,h),$$

where

$$I_\mu^1(x,h) = \int_0^\infty e^{i\lambda f(x,\xi)/h}\xi^{-\mu-1}\chi(\xi)d\xi,$$
$$I_\mu^2(x,h) = \int_0^\infty e^{i\lambda f(x,\xi)/h}\xi^{-\mu-1}(1-\chi(\xi))d\xi.$$

For the principal term of I_μ^1, we can ignore $\dfrac{\xi^2}{2}$ in the phase and get

$$I_\mu^1(x,h) \sim e^{i\lambda x^2/(4h)}\int_0^\infty e^{i\lambda x\xi/h}\xi^{-\mu-1}\chi(\xi)d\xi.$$

It follows that

$$I_\mu^1(x,h) \sim e^{-i\pi\mu\,\mathrm{sgn}(x)/2}\Gamma(-\mu)\left(\frac{\lambda|x|}{h}\right)^\mu e^{i\lambda x^2/(4h)},$$

using the following formula for the Laplace transform

$$\int_0^\infty e^{ias/h}s^{p-1}ds = e^{i\pi p\,\mathrm{sgn}(a)/2}\Gamma(p)\left(\frac{h}{|a|}\right)^p, \quad a\in\mathbb{R}\setminus\{0\},\ 0<\mathrm{Re}\,p<1. \tag{1.4}$$

Next we calculate I_λ^2 by the stationary phase method. When $x>0$, there is no critical point and hence $I_\mu^2 = \mathcal{O}(h^\infty)$. When $x<0$, assuming $2\varepsilon < -x < R$ and using $f(x,\xi) = -\frac{x^2}{4} + \frac{(x+\xi)^2}{2}$, we get

$$\begin{aligned} I_\mu^2(x,h) =& e^{-i\lambda x^2/(4h)}\int_0^\infty \exp\left(\frac{i\lambda}{2h}(x+\xi)^2\right)\xi^{-\mu-1}(1-\chi(\xi))d\xi \\ \sim& e^{\pi i/4}\sqrt{\frac{2\pi h}{\lambda}}|x|^{-\mu-1}e^{-i\lambda x^2/(4h)}. \end{aligned}$$

□

Let us now see this result microlocally in the phase space. Let

$$p(x,\xi) = \xi^2 - \frac{\lambda^2}{4}x^2$$

be the classical Hamiltonian associated with the Schrödinger operator P. The solution of the canonical system

$$\begin{cases} \dot{x}(t) = \dfrac{\partial p}{\partial \xi}\big(x(t),\xi(t)\big) = 2\xi(t) \\ \dot{\xi}(t) = -\dfrac{\partial p}{\partial x}\big(x(t),\xi(t)\big) = \dfrac{\lambda^2}{2}x(t) \end{cases}$$

with initial data $\big(x(0),\xi(0)\big)=(x_0,\xi_0)$ is given by

$$\begin{aligned}\begin{pmatrix}x(t)\\ \xi(t)\end{pmatrix}&=\begin{pmatrix}\cosh\lambda t & \frac{2}{\lambda}\sinh\lambda t\\ \frac{\lambda}{2}\sinh\lambda t & \cosh\lambda t\end{pmatrix}\begin{pmatrix}x_0\\ \xi_0\end{pmatrix}\\ &=\begin{pmatrix}\left(\frac{x_0}{2}+\frac{\xi_0}{\lambda}\right)e^{\lambda t}+\left(\frac{x_0}{2}-\frac{\xi_0}{\lambda}\right)e^{-\lambda t}\\ \left(\frac{\lambda x_0}{4}+\frac{\xi_0}{2}\right)e^{\lambda t}+\left(-\frac{\lambda x_0}{4}+\frac{\xi_0}{2}\right)e^{-\lambda t}\end{pmatrix}.\end{aligned}$$

The origin $(x,\xi)=(0,0)$ is a fixed point, and

$$\Lambda_\pm=\Big\{(x,\xi)\in\mathbb{R}^2;\ \xi=\pm\frac{\lambda}{2}x\Big\}$$

are the outgoing and the incoming stable manifold respectively, namely, $\Lambda_\pm$ is the set of points from which the integral curve converges to the origin as $t\to\mp\infty$ respectively.

Let us investigate the frequency set of the solutions to (1.1) (see §4.1 for the definition and some properties of the frequency set). First by Theorem 4.3, we know for any solution $u\in L^2(\mathbb{R}^d)$ to (1.1) with $\|u\|\le 1$ that

$$\mathrm{FS}\,(u)\subset\mathrm{Char}\,(P)=\Big\{(x,\xi)\in\mathbb{R}^2;\ \xi^2-\frac{\lambda^2}{4}x^2=0\Big\}=\Lambda_+\cup\Lambda_-.$$

Next we study the frequency set of the solutions we obtained in *(P1)*, *(P2)*. In case *(P1)*, by Proposition 4.2, we have

$$\mathrm{FS}\,(u_k)=\Lambda_+. \tag{1.5}$$

In case *(P2)*, we set $\Lambda_\pm=\Lambda_\pm^+\cup\Lambda_\pm^-$, where

$$\Lambda_\pm^+=\{(x,\xi)\in\Lambda_\pm;\ x>0\},\quad \Lambda_\pm^-=\{(x,\xi)\in\Lambda_\pm;\ x<0\}.$$

Then, with $\mu=i\frac{z}{\lambda}-\frac{1}{2}$, we have

$$\mathrm{FS}\,(I_\mu)=\Lambda_+\cup\Lambda_-^-.$$

More precisely, we see that

$$\mathrm{FS}\,(I_\mu^1)=\Lambda_+,\quad \mathrm{FS}\,(I_\mu^2)=\Lambda_-^-.$$

The solution $I_\mu(x,h)$ describes the wave coming from $x<0$ to the origin and scattered to the positive and the negative directions. In the same way, $I_\mu(-x,h)$ describes the wave coming from $x>0$ to the origin and scattered to the positive and the negative directions.

Thus Proposition 1.1 is interpreted as follows: The wave coming from $x<0$ along the incoming stable manifold Λ_-^- with amplitude

$$e^{\pi i/4}\sqrt{\frac{2\pi h}{\lambda}}|x|^{-\mu-1}e^{-i\lambda x^2/(4h)}$$

transmits through the barrier at the origin to Λ_+^+ with amplitude

$$e^{-\pi i\mu/2}\Gamma(-\mu)\left(\frac{\lambda x}{h}\right)^{\mu}e^{i\lambda x^2/(4h)}$$

and reflects to Λ_+^- with amplitude

$$e^{\pi i\mu/2}\Gamma(-\mu)\left(\frac{\lambda|x|}{h}\right)^{\mu}e^{i\lambda x^2/(4h)}.$$

In case (P1), on the other hand, the wave is purely outgoing. In the general case, this is related to the fact that the outgoing wave is determined by the incoming wave if and only if zh is not a resonance.

This paper is organized as follows: In §2, we state the results of [BFRZ], which deals with the multidimensional general case. In §3, we survey how to construct microlocal solutions on the outgoing stable manifold in terms of the data given of the incoming stable manifold. In §4, we recall the notion of *microlocal solution* (§4.1), sketch the theory of *expandible solution* (§4.2), perform an exact calculus for the one-dimensional example of §1 using the technique of §3 (§.4.3) and finally give some brief proofs for Propositions in §3 (§4.4).

§ 2. Microlocal Cauchy problem near a hyperbolic fixed point

§ 2.1. Classical mechanics

We suppose that the real-valued function $p(x,\xi)\in C^\infty(\mathbb{R}^{2d};\mathbb{R})$, defined in a neighborhood of the origin in $\mathbb{R}_x^d\times\mathbb{R}_\xi^d$, behaves like

$$p(x,\xi)=|\xi|^2-\sum_{j=1}^{d}\frac{\lambda_j^2}{4}x_j^2+\mathcal{O}\big((x,\xi)^3\big)\quad\text{as}\quad(x,\xi)\to(0,0),\tag{2.1}$$

where $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_d$ are constants.

Let us consider the canonical system of p:

$$\frac{d}{dt}\begin{pmatrix}x\\ \xi\end{pmatrix}=\begin{pmatrix}\nabla_\xi p\\ -\nabla_x p\end{pmatrix}.\tag{2.2}$$

The origin $(x,\xi) = (0,0)$ is a fixed point of the Hamilton vector field H_p. The linearization of H_p at the origin is

$$\frac{d}{dt}\begin{pmatrix} x \\ \xi \end{pmatrix} = F_p \begin{pmatrix} x \\ \xi \end{pmatrix}, \tag{2.3}$$

where F_p is the fundamental matrix

$$F_p := \begin{pmatrix} \frac{\partial^2 p}{\partial x \partial \xi} & \frac{\partial^2 p}{\partial \xi^2} \\ -\frac{\partial^2 p}{\partial x^2} & -\frac{\partial^2 p}{\partial \xi \partial x} \end{pmatrix}\Bigg|_{(x,\xi)=(0,0)} = \begin{pmatrix} 0 & 2\,\mathrm{Id} \\ \frac{1}{2}\mathrm{diag}\,(\lambda_j)^2 & 0 \end{pmatrix}.$$

This matrix has d positive eigenvalues $\{\lambda_j\}_{j=1}^d$ and d negative eigenvalues $\{-\lambda_j\}_{j=1}^d$. The eigenspaces $\Lambda_\pm^0$ corresponding to these positive and negative eigenvalues are respectively outgoing and incoming stable manifolds for the quadratic part p_0 of p:

$$\begin{aligned} \Lambda_\pm^0 =& \big\{(x,\xi) \in \mathbb{R}^{2d};\ \exp(tH_{p_0})(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\big\} \\ =& \big\{(x,\xi) \in \mathbb{R}^{2d};\ \xi_j = \pm\frac{\lambda_j}{2}x_j,\ j = 1,\dots,d\big\}. \end{aligned}$$

By the stable manifold theorem, we also have outgoing and incoming stable manifolds for p:

$$\Lambda_\pm = \big\{(x,\xi) \in \mathbb{R}^{2d};\ \exp(tH_p)(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\big\}.$$

The tangent space of $\Lambda_\pm$ at $(0,0)$ is $\Lambda_\pm^0$. The manifolds $\Lambda_\pm$ are Lagrangian manifolds and can be written

$$\Lambda_\pm = \Big\{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \frac{\partial \phi_\pm}{\partial x}(x)\Big\},$$

where the generating functions $\phi_\pm$ behave like

$$\phi_\pm(x) = \pm\sum_{j=1}^d \frac{\lambda_j}{4}x_j^2 + \mathcal{O}\big(|x|^3\big) \quad \text{as} \quad x \to 0. \tag{2.4}$$

Now suppose $\rho_\pm = (x_\pm,\xi_\pm) \in \Lambda_\pm \setminus \{(0,0)\}$. Then by definition $\exp(tH_p)\rho_\pm \to (0,0)$ as $t \to \mp\infty$. More precisely,

Proposition 2.1. *One has, in the sense of Definition 4.7,*

$$\exp(tH_p)(\rho_\pm) \sim \sum_{k=1}^\infty \gamma_k^\pm(t)e^{\pm\mu_k t} \quad \text{as} \quad t \to \mp\infty,$$

where

$$0 < \mu_1 < \mu_2 < \cdots$$

are the linear combinations over $\mathbb{N}$ *of* $\{\lambda_j\}_{j=1}^d$*, and in particular* $\mu_1 = \lambda_1$*. The* $\gamma_k^\pm(t)$ *are vector valued polynomials in* t*, and in particular* γ_1 *is an eigenvector of* F_p *corresponding to* $\pm\lambda_1$ *and independent of* t *(remark that* $\gamma_1 e^{-\lambda_1 t}$ *is a solution to (2.3)).*

For the proof, see the remark after Corollary 4.10. In the sequel, we will also denote the x-space projection of the vector $\gamma_1^{\pm}(\rho_\pm)$ by $X_1^{\pm}(\rho_\pm)$.

Remark. If the remainder term of p in (2.1) is independent of ξ, then p is the classical Hamitonian associated to the Schrödinger operator $P = -h^2\Delta + V(x)$:

$$p(x,\xi) = |\xi|^2 + V(x), \quad V(x) = -\sum_{j=1}^{d} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(|x|^3) \quad \text{as} \quad x \to 0. \tag{2.5}$$

The potential $V(x)$ reaches its local non-degenerate maximum 0 at the origin. In this case, by the symmetry with respect to ξ, one has

$$\phi_-(x) = -\phi_+(x), \quad \Lambda_- = \{(x,-\xi) \in \mathbb{R}^{2d}; \ \ (x,\xi) \in \Lambda_+\}.$$

In this case, if $\rho_\pm = (x, \pm\xi)$, then

$$X_1^+(\rho_+) = X_1^-(\rho_-) =: X_1(x).$$

§ 2.2. Microlocal Cauchy problem and its uniqueness

Suppose $p \in S_h^0(1)$, i.e. $p(x,\xi) \in C^\infty(\mathbb{R}^{2d};\mathbb{R})$ is uniformly bounded with respect to h with all its derivatives. Let P be the Weyl quantization of $p(x,\xi)$, namely

$$\big[Pu\big](x) := \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi/h} p\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi.$$

When p is of the form (2.5), it is the semiclassical Schrödinger operator

$$P = -h^2\Delta + V(x). \tag{2.6}$$

For a small neighborhood Ω of $(0,0)$ and $\varepsilon > 0$ small, we consider the microlocal Cauchy problem in the sense of §4.1:

$$\begin{cases} Pu = hzu & \text{in } \Omega, \\ u = u_0(x) & \text{on } \mathcal{C} := \Omega \cap \Lambda_- \cap \{|x| = \varepsilon\}. \end{cases} \tag{2.7}$$

Remark that the initial surface $\mathcal{C}$ is transversal to the Hamilton flow. The spectral parameter z may be complex but in a disc of center 0 and radius bounded with respect to h.

We start with a uniqueness result for this problem. For the proof, we send the reader to [BFRZ, Section 4]. Let r be any positive number and z complex number, which may depend on h, in a disc $D(r) := \{z \in \mathbb{C}; \ |z| < r\}$.

Theorem 2.2. *There exist a positive δ and a h-dependent finite set $\Gamma(h) \subset D(r) \cap \{z \in \mathbb{C};\ \operatorname{Im} z < -\delta\}$, whose cardinal number is bounded with respect to h, such that if* $\operatorname{dist}\big(z, \Gamma(h)\big) > h^C$ *for some $C > 0$, and if $u_0 = 0$, then the solution $u \in L^2(\mathbb{R}^d)$ of (2.7), satisfying $\|u\| \le 1$, is 0 in a neighborhood Ω' of the origin.*

Remark. In the analytic category (i.e. p is analytic near the origin and the microlocal solution is defined with the microsupport MS instead of the frequency set FS, see §4.1), we have the same theorem with more precision on the set $\Gamma(h)$. In fact, $\Gamma(h)$ is $-i\mathcal{E}_0$ modulo $\mathcal{O}(h)$, where

$$\mathcal{E}_0 = \Big\{ \sum_{j=1}^{d} \lambda_j (\alpha_j + \frac{1}{2});\ (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \Big\}$$

is the set of eigenvalues of the harmonic oscillator $-\Delta + \displaystyle\sum_{j=1}^{d} \lambda_j^2 \frac{x_j^2}{4}$, see [BFRZ].

In the C^∞ case, Helffer and Sjöstrand ([He-Sj 1]) have constructed the asymptotic expansion (in powers of $h^{1/2}$) of the eigenvalues at the bottom of a potential well. The set of the first terms of the expansion is $\mathcal{E}_0$. This means that $-i\mathcal{E}_0$ is included in $\Gamma(h)$ modulo $\mathcal{O}(h)$. We expect that, modulo $\mathcal{O}(h^\infty)$, $\Gamma(h)$ is the set of $-i$ times the eigenvalues obtained in [He-Sj 1].

If $u = 0$ in Ω', it is 0 also on Λ_+ by Theorem 4.3. Hence this result can be expressed as follows: The microsupport propagates from the incoming stable manifold Λ_- to the outgoing stable manifold Λ_+ under a generic assumption on the energy z.

§ 2.3. Transition operator

Theorem 2.2 says that the data u_0 given on $\Lambda_- \cap \{|x| = \varepsilon\}$ uniquely determines the solution u at any point $\rho_F = (x_F, \xi_F)$ on Λ_+ (if it exists). Our problem now is to construct u near ρ_F in terms of u_0 which, restricted to the initial surface $\mathcal{C}$, has its support in a small neighborhood of a point $\rho_I = (x_I, \xi_I) \in \mathcal{C}$.

For the sake of simplicity, we assume in the following that P is a Schrödinger operator (2.6), see [BFRZ] for the general case.

We make two generic assumptions; one is on the spectral parameter z and the other is on the initial point $\rho_I = (x_I, \xi_I) \in \mathcal{C}$ and the final point $\rho_F = (x_F, \xi_F) \in \Lambda_+$:

(A1) There exists $\nu > 0$ such that $\operatorname{dist}(z, -i\mathcal{E}_0) > \nu$.

(A2) $X_1(x_I) \cdot X_1(x_F) \neq 0$.

In particular, $X_1(x_I) \neq 0$. This means that, in case $\lambda_1 < \lambda_2$, the Hamilton flow starting from ρ_I converges to the origin tangentially to the x_1-axis. In case $\lambda_1 = \lambda_2$, also, we can assume, without loss of generality, that the x_1-axis is parallel to $X_1(x_I)$. Since p is of real principal type near ρ_I, we can modify the initial surface $\mathcal{C}$ so that it is given by $\{x_1 = \varepsilon\} \cap \Lambda_-$ near ρ_I. Hence, denoting $x_I = (\varepsilon, x_I')$, the initial data u_0 on $\mathcal{C}$ is a function of x' in a small neighborhood of x_I' and 0 elsewhere.

Before stating the existence theorem, we state two lemmas describing the behavior of classical quantities, which appear in the principal part of the representation formula of the solution u.

Lemma 2.3. *Let $x(t)$ be the x-space projection of the flow $\exp(tH_p)\rho_F$. Then the integral*

$$I_\infty(x) := \int_0^{-\infty} \Big(\Delta\phi_+(x(\tau)) - \frac{1}{2}\sum_{j=1}^{d} \lambda_j \Big)\, d\tau$$

converges.

The proof is obvious from (2.4) and Proposition 2.1.

Lemma 2.4. *For y' near x_I' and η' near ξ_I', let*

$$\rho(y', \eta') := \big(\varepsilon, y'; -\sqrt{-|\eta'|^2 - V(\varepsilon, y')}, \eta'\big)$$

be the point in $\{x_1 = \varepsilon\} \cap p^{-1}(0)$ and

$$\big(x(t, y', \eta'), \xi(t, y', \eta')\big) = \exp(tH_p)\rho(y', \eta')$$

the Hamiltonian trajectory starting from the point $\rho(y', \eta')$. Then the Jacobian

$$J(t, y', \eta') := \det \frac{\partial x(t, y', \eta')}{\partial(t, y')}$$

has the non-vanishing limit

$$J_\infty(y') := \lim_{t\to+\infty} \frac{J(t, y', \eta')}{J(0, y', \eta')}\Big|_{\eta'=\frac{\partial\phi_-}{\partial y'}(\varepsilon, y')} e^{(-\sum_{j=1}^{d}\lambda_j + 2\lambda_1)t}.$$

See (3.24) for the proof.

Theorem 2.5. *If $\operatorname{dist}(z, \Gamma(h)) \geq \nu h$, for some $\nu > 0$, then the microlocal Cauchy problem (2.7) has a solution u (unique thanks to Theorem 2.2). Microlocally near $\rho_F = (x_F, \xi_F)$, it has the following representation formula*

$$u(x, h) = \frac{h^{S/\lambda_1}}{(2\pi h)^{d/2}} \int_{\mathbb{R}^{d-1}} e^{i(\phi_+(x) - \phi_-(\varepsilon, y'))/h} d(x, y'; h) u_0(y') dy'. \tag{2.8}$$

Here

$$S = \frac{1}{2}\sum_{j=1}^{d} \lambda_j - iz, \tag{2.9}$$

and the symbol $d \in S^0_h(1)$ *has the following asymptotic expansion*

$$d(x,\eta';h) \sim \sum_{k=0}^{\infty} d_k(x,y',\ln h) h^{\hat{\mu}_k/\lambda_1}, \tag{2.10}$$

where $0 = \hat{\mu}_0 < \hat{\mu}_1 (= \mu_2 - \mu_1) < \hat{\mu}_2 < \cdots$ *is a numbering of the linear combinations of* $\{\mu_k - \mu_1\}_{k=0}^{\infty}$ *over* $\mathbb{N}$, *and* $d_k(x,y',\ln h)$ *are polynomials in* $\ln h$. *In particular,* d_0 *is independent of* $\ln h$ *and given by*

$$\begin{aligned} d_0(x,y') =& e^{-i\pi d/4} \lambda_1^{1/2 - S/\lambda_1} \exp(-\frac{S\pi}{2\lambda_1} i\,\sigma) \Gamma\left(\frac{S}{\lambda_1}\right) \\ &\times e^{I_\infty(x)} \sqrt{\frac{|\det \nabla^2_{y'} \phi_-(\varepsilon,y')|}{J_\infty(y')}} \cdot \frac{|X_1(\varepsilon,y')|}{|X_1(\varepsilon,y')\cdot X_1(x)|^{\frac{S}{\lambda_1}}} \ , \end{aligned} \tag{2.11}$$

where

$$\sigma = \operatorname{sgn}\big(X_1(x_I)\cdot X_1(x_F)\big).$$

§ 3. Construction of the microlocal solution

§ 3.1. Expression on Λ_-

On Λ_-, we first write the solution u of (2.7) by means of Fourier integral operators,

$$u(x,h) = \frac{1}{(2\pi h)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{i\psi(x,\eta')/h} b(x,\eta';h) \hat{u}_0(\eta')\, d\eta', \tag{3.1}$$

where $\hat{u}_0(\eta')$ is the h-Fourier transform of u_0:

$$\hat{u}_0(\eta') = \frac{1}{(2\pi h)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{-iy'\cdot\eta'/h} u_0(y')\, dy'.$$

Since u is a solution to the equation of (2.7), the phase function ψ should verify the eikonal equation

$$|\partial_x \psi|^2 + V(x) = 0, \tag{3.2}$$

and, if the symbol b has an expansion of the form

$$b(x,\eta';h) \sim \sum_{k=0}^{\infty} b_k(x,\eta') h^k, \tag{3.3}$$

each term b_k should satisfy the transport equation

$$2\partial_x\psi \cdot \partial_x b_k + (\Delta_x \psi - iz) b_k = i\Delta_x b_{k-1}, \quad k \geq 0. \tag{3.4}$$

Here we used the convention $b_{-1} = 0$.

On the other hand, the initial condition of (2.7) for u reads for ψ and b

$$\psi|_{x_1=\varepsilon} = x' \cdot \eta', \tag{3.5}$$

$$b_{|x_1=\varepsilon} = 1. \tag{3.6}$$

Proposition 3.1. *The local solution ψ to the Cauchy problem* (3.2), (3.5) *exists and is unique.*

In fact,

$$\Lambda_{\eta'} = \bigcup_t \exp(tH_p)(\Lambda'_{\eta'}) \subset p^{-1}(0)$$

with

$$\Lambda'_{\eta'} = \big\{ \big(\varepsilon, y', -2\sqrt{-V(\varepsilon, y') - \eta'^2}, \eta'\big);\ y' \text{ near } x'_I \big\} \subset p^{-1}(0),$$

is a Lagrangian manifold whose projection to the x-space is a diffeomorphism. Hence the generating function is the solution to (3.2), (3.5) (if the constant is suitably chosen):

$$\Lambda_{\eta'} = \Big\{ (x, \xi) \in \mathbb{R}^{2d};\ \xi = \frac{\partial \psi}{\partial x}(x) \Big\}.$$

Now let us parametrize $\Lambda'_{\eta'}$ by y':

$$\rho(y', \eta') = \big(x(y', \eta'), \xi(y', \eta')\big) := \big(\varepsilon, y'; -\sqrt{-|\eta'|^2 - V(\varepsilon, y')}, \eta'\big),$$

and let

$$\rho(t, y', \eta') = \big(x(t, y', \eta'), \xi(t, y', \eta')\big) := \exp(tH_p)\rho(y', \eta')$$

be the evolution by the Hamilton flow. $\Lambda_{\eta'}$ and Λ_- intersect transversely at a point on $x_1 = \varepsilon$. Recall that $\Lambda_- = \{(x, \xi) \in \mathbb{R}^{2n};\ \xi_j = -\frac{\lambda_j}{2} x_j + \mathcal{O}\big(|x|^2\big),\ j = 1, \dots, d\}$. We denote this point and its evolution by

$$\rho_c(\eta') = \big(x_c(\eta'), \xi_c(\eta')\big), \quad \rho_c(t, \eta') = \big(x_c(t, \eta'), \xi_c(t, \eta')\big) = \exp(tH_p)\rho_c(\eta').$$

About the Jacobian

$$J(t, y', \eta') = \det \frac{\partial x(t, y', \eta')}{\partial (t, y')}, \tag{3.7}$$

the following results are well known in the theory of WKB analysis (see [MF]).

Proposition 3.2. *The function $J(t, y', \eta')$ verifies*

$$J(0, y', \eta') = -2\sqrt{-V(\varepsilon, y') - \eta'^2}, \tag{3.8}$$

$$\sqrt{\frac{J(t, y', \eta')}{J(0, y', \eta')}} = \exp \int_0^t \Delta\psi(x(\tau, y', \eta'), \eta') d\tau. \tag{3.9}$$

Proposition 3.3. *The solution b to the Cauchy problem (3.4), (3.6) exists locally and uniquely. In particular,*

$$b_0(x(t, y', \eta'), \eta') = e^{izt}\sqrt{\frac{J(0, y', \eta')}{J(t, y', \eta')}}. \tag{3.10}$$

§ 3.2. From Λ_- to a neighborhood of the origin

3.2.1. WKB solution to the time-dependent Schrödinger equation

The WKB solution $e^{i\psi/h}b$ in the integral (3.1) cannot be continued to a full neighborhood of the origin $(x, \xi) = (0, 0)$ because the origin is a singularity of the Hamilton vector field H_p. In order to overcome this problem, we use an idea of Helffer and Sjöstrand in [He-Sj 2], and express this WKB solution on Λ_- as h-Fourier inverse transform with respect to time of the time-dependent WKB solution:

$$e^{i\psi(x,\eta')/h}b(x, \eta'; h) = \frac{1}{(2\pi h)^{1/2}}\int_0^\infty e^{i\varphi(t,x,\eta')/h}a(t, x, \eta'; h)\, dt. \tag{3.11}$$

Note that here the factor $e^{iEt/h} = e^{izt}$ for the h-Fourier inverse transform is taken into account in the symbol a. Now u is of the form

$$u(x, h) = \frac{1}{(2\pi h)^{d/2}}\int_0^\infty \int_{\mathbb{R}^{d-1}} e^{i\varphi(t,x,\eta')/h}a(t, x, \eta'; h)\hat{u}_0(\eta')\, d\eta'\, dt. \tag{3.12}$$

The time-dependent phase $\varphi(t, x, \eta')$ and the symbol $a(t, x, \eta'; h) \sim \sum_l a_l(t, x, \eta')h^l$ satisfy respectively the eikonal equation

$$\partial_t\varphi + |\partial_x\varphi|^2 + V(x) = 0 \tag{3.13}$$

and the transport equation

$$\partial_t a_l + 2\partial_x\varphi\cdot\partial_x a_l + (\Delta_x\varphi - iz)a_l = i\Delta_x a_{l-1}, \quad l \geq 0 \tag{3.14}$$

$(a_{-1} \equiv 0)$. We should solve these equations so that (3.11) holds.

For this purpose, we first take a hypersurface

$$\Gamma_0 = \big\{(x, \xi) \in \Lambda_{\eta'};\ \psi(x, \eta') = \psi\big(x_c(\eta'), \eta'\big)\big\}$$

in $\Lambda_{\eta'}$ containing $\rho_c(\eta')$, and a Lagrangian manifold Λ_0 intersecting transversally with $\Lambda_{\eta'}$ along Γ_0. Notice that

$$\Lambda_0 \cap p^{-1}(0) = \Gamma_0.$$

Let Γ_t and Λ_t be the evolution by the Hamilton flow of Γ_0 and Λ_0 respectively:

$$\Gamma_t = \exp(tH_p)\Gamma_0 \subset \Lambda'_\eta, \quad \Lambda_t = \exp(tH_p)\Lambda_0.$$

We see that Λ_t is a Lagrangian manifold and that if ε is sufficiently small, then Λ_t (restricted suitably to a neighborhood of $\rho_c(t,\eta')$) projects nicely to the x-space for every large t and hence there exists a generating function $\varphi(t,x)$.

$$\Lambda_t = \Big\{(x,\xi)\in\mathbb{R}^{2d};\ \xi = \frac{\partial\varphi}{\partial x}(t,x)\Big\}.$$

It is determined modulo a function of t but by the eikonal equation (3.13) it is determined modulo a constant.

The x-space projection of Γ_t can be written as

$$\Pi_x\Gamma_t = \Big\{x\in\mathbb{R}^d;\ \frac{\partial\varphi}{\partial t}(t,x) = 0\Big\}.$$

In fact, if $(x,\xi)\in\Gamma_t = \Lambda_t\cap p^{-1}(0)$, then $\xi = \frac{\partial\varphi}{\partial x}$ and $\xi^2+V(x)=0$, namely, $|\frac{\partial\varphi}{\partial x}|^2 + V(x) = 0$ which means $\frac{\partial\varphi}{\partial t} = 0$ by the eikonal equation (3.13).

In other words, for each x near the curve $\{x_c(t,\eta');\ t\geq 0\}$, let $t = t(x,\eta')$ be the time at which $x\in\Pi_x\Gamma_t$. Then $t(x,\eta')$ is a critical point of φ.

Proposition 3.4. *The function $\varphi(t,x,\eta')$ defined above is a solution to* (3.13). *Moreover, there exists a neighborhood of the curve $\{x_c(t,\eta');\ t\geq 0\}$ such that for any point x there, there exists a unique $t = t(x,\eta')$ such that $x\in\Pi_x\Gamma_t$, and we have*

$$\frac{\partial\varphi}{\partial t}\big(t(x,\eta'),x,\eta'\big) = 0,\quad \frac{\partial^2\varphi}{\partial t^2}\big(t(x,\eta'),x,\eta'\big) > 0, \tag{3.15}$$

$$\psi(x,\eta') = \varphi\big(t(x,\eta'),x,\eta'\big). \tag{3.16}$$

Next, we calculate the asymptotic expansion of the right hand side of (3.11) coming from the critical point $t(x,\eta')$ using the stationary phase method. The condition that this expansion coincides with the WKB solution of the left hand side of (3.11) together with the transport equation (3.14) determines all the $a_l(t,x,\eta')$'s successively in the neighborhood of the curve. In particular,

$$b_0(x,\eta') = \frac{e^{\pi i/4}}{\sqrt{\varphi_{tt}\big(t(x,\eta'),x,\eta'\big)}}a_0\big(t(x,\eta'),x,\eta'\big). \tag{3.17}$$

3.2.2. Asymptotic behavior at large time

Both the phase function $\varphi(t,x,\eta')$ and each term of the symbol $a_k(t,x,\eta')$ is expandible in t in the sense of §4.2.

Proposition 3.5. *As $t\to+\infty$*

$$\varphi(t,x,\eta') \sim \phi_+(x) + \tilde\psi(\eta') + \phi_1(x,\eta')e^{-\mu_1 t} + \sum_{k=2}^{\infty}\phi_k(t,x,\eta')e^{-\mu_k t}, \tag{3.18}$$

where

$$\tilde{\psi}(\eta') = x_c'(\eta') \cdot \eta' - \phi_-(x_c(\eta')), \tag{3.19}$$

and $\phi_k(t, x, \eta')$ *is a polynomial in* t*. In particular* ϕ_1 *is independent of* t *and given by*

$$\phi_1(x, \eta') = -\lambda_1 X_1(x) \cdot x + \mathcal{O}\big(x^2\big) \quad \text{as} \quad x \to 0. \tag{3.20}$$

The vector $X_1(x)$ is given at the end of § 2.1.

Proposition 3.6. *Recall that we have set* $S = \frac{1}{2}\sum_{j=1}^{d} \lambda_j - iz$*. As* $t \to +\infty$*,*

$$a_l(t, x, \eta') \sim e^{-St} \sum_{k=0}^{\infty} a_{k,l}(t, x, \eta') e^{-\mu_k t}. \tag{3.21}$$

In particular, $a_{0,0}(x, \eta')$ *is independent of* t*.*

Let us calculate $a_{0,0}(0, \eta')$. To use the fact

$$a_{0,0}(0, \eta') = \lim_{t \to \infty} e^{St} a_0\big(t, x_c(t, \eta'), \eta'\big),$$

we restrict ourselves to the curve $x_c(t, \eta')$. First, we have

Proposition 3.7. *As* $t \to +\infty$*, one has*

$$\varphi_{tt}(t, x_c(t, \eta'), \eta') = |X_1(x_c(\eta'))|^2 \lambda_1^3 e^{-2\lambda_1 t}\Big(1 + \mathcal{O}\big(e^{-\hat{\mu}_1 t}\big)\Big), \tag{3.22}$$

$$\Delta\psi(x_c(t, \eta'), \eta') = \sum_{j=1}^{d} \frac{\lambda_j}{2} - \lambda_1 + \mathcal{O}\big(e^{-\hat{\mu}_1 t}\big). \tag{3.23}$$

Integrating (3.23) from 0 to t and taking its exponential, we see, by (3.9), the existence of the limit

$$\lim_{t \to +\infty} \frac{J(t, x_c'(\eta'), \eta')}{J(0, x_c'(\eta'), \eta')} \exp\Big(-\sum_{j=1}^{d} \lambda_j + 2\lambda_1\Big)t > 0. \tag{3.24}$$

This limit is a function of η', but we write $J_\infty(y')$ as function of $y' = x_c'(\eta')$.

On the other hand, since $t\big(x_c(t, \eta'), \eta'\big) = t$, the equality (3.17) can be written as

$$a_0\big(t, x_c(t, \eta'), \eta'\big) = e^{-\pi i/4} \sqrt{\varphi_{tt}\big(t, x_c(t, \eta'), \eta'\big)}\, b_0\big(x_c(t, \eta'), \eta'\big). \tag{3.25}$$

From (3.10) and (3.22), we obtain

$$\begin{aligned} a_0\big(t, x_c(t, \eta'), \eta'\big) =& e^{-\pi i/4} |X_1(x_c(\eta'))| \lambda_1^{3/2} e^{-St} \\ & \times \sqrt{\frac{J(0, x_c'(\eta'), \eta')}{J(t, x_c'(\eta'), \eta')}} e^{(\sum_{j=1}^{d} \lambda_j - 2\lambda_1)t}\Big(1 + \mathcal{O}\big(e^{-\hat{\mu}_1 t}\big)\Big). \end{aligned}$$

Taking the limit $t \to +\infty$ after multiplying by e^{St}, we get, by (3.24),

Proposition 3.8. *For* $\eta' = \frac{\partial \phi_-}{\partial y'}(\varepsilon, y')$*, one has*

$$a_{0,0}(0,\eta') = \frac{e^{-\pi i/4}\lambda_1^{3/2}\big|X_1\big(x_c(\eta')\big)\big|}{\sqrt{J_\infty(y')}}. \tag{3.26}$$

§ 3.3. From a neighborhood of the origin to Λ_+

The integral on the right hand of (3.11) is convergent when $\operatorname{Re} S > 0$. But it is well defined also for $\operatorname{Re} S \leq 0$ under the assumption **(A1)**. In fact, since $\varphi - \phi_+ - \tilde\psi(\eta') = \mathcal{O}(e^{-\lambda_1 t})$, one gets by Taylor expansion,

$$e^{-St}e^{i\left(\varphi - \phi_+ - \tilde\psi(\eta')\right)/h} = e^{-St}\sum_{m=0}^{N-1}\frac{1}{m!}\Big(\frac{i}{h}\big(\varphi - \phi_+ - \tilde\psi(\eta')\big)\Big)^m + \mathcal{O}\big(e^{-(S+N\lambda_1)t}\big).$$

The last term of the right hand side is exponentially decaying for sufficiently large N. On the other hand, the integral in t of the first term of the right hand side is a finite sum of the form

$$I_p(S+\mu_l) = \int_0^\infty e^{-(S+\mu_l)t}t^p\,dt$$

modulo convergent integrals. Here p and l are some non-negative integers, see also (3.29). We give a meaning to this integral setting $I_p(\zeta) = \dfrac{p!}{\zeta^{p+1}}$ which is the analytic extension of $I_p(\zeta)$ from $\{\zeta \in \mathbb{C};\ \operatorname{Re}\zeta > 0\}$ to $\mathbb{C}_\zeta \setminus \{0\}$. It is important to remark that the assumption **(A1)** implies $S + \mu_l \neq 0$.

Proposition 3.5 and Proposition 3.6 together with the formula (1.4) lead to the following proposition:

Proposition 3.9. *On* Λ_+*, one has*

$$\int_0^\infty e^{i\varphi(t,x,\eta')/h}a(t,x,\eta';h)\,dt = h^{S/\lambda_1}e^{i(\phi_+(x)+\tilde\psi(\eta'))/h}c(x,\eta';h), \tag{3.27}$$

where the symbol c *has an asymptotic expansion of the form*

$$c(x,\eta';h) \sim \sum_{k=0}^\infty c_k(x,\eta',\ln h)h^{\hat\mu_k/\lambda_1}.$$

Here $0 = \hat\mu_0 < \hat\mu_1 < \hat\mu_2 < \cdots$ *are the linear combinations over* $\mathbb{N}$ *of the set* $\{\mu_k - \mu_1\}_{k=1}^\infty$*, and in particular,* c_0 *is independent of* $\ln h$ *and given by*

$$c_0(x,\eta') = \Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp(\frac{S\pi}{2\lambda_1}i\operatorname{sgn}\phi_1)}{\lambda_1|\phi_1(x)|^{S/\lambda_1}}a_{0,0}(x,\eta'). \tag{3.28}$$

Remark. The set $\{\mu_k\}_{k\in\mathbb{N}}$ is a subset of $\{\hat\mu_k\}_{k\in\mathbb{N}}$ and they satisfy the additive property:

$$\{\mu_k\}_{k\in\mathbb{N}} + \{\mu_k\}_{k\in\mathbb{N}} = \{\mu_k\}_{k\in\mathbb{N}}, \quad \{\hat\mu_k\}_{k\in\mathbb{N}} + \{\hat\mu_k\}_{k\in\mathbb{N}} = \{\hat\mu_k\}_{k\in\mathbb{N}}. \tag{3.29}$$

The principal symbol $c_0(x,\eta')$ satisfies the transport equation

$$2\partial_x\phi_+\cdot\partial_x c_0 + (\Delta\phi_+ - iz)c_0 = 0. \tag{3.30}$$

This is an ordinary differential equation along Hamilton flows on Λ_+. Let $\rho=(x,\xi)$ be a point on Λ_+ and

$$\big(x(t),\xi(t)\big) = \exp(tH_p)\rho.$$

Then on $x=x(t)$, (3.30) becomes

$$\frac{d}{dt}\Big[c_0\big(x(t),\eta'\big)\Big] + \big(\Delta\phi_+(x(t)) - iz\big)c_0 = 0.$$

The solution is given by

$$c_0\big(x(t),\eta'\big) = e^{izt-\int_0^t\Delta\phi_+(x(\tau))d\tau}c_0(x,\eta').$$

This and (3.28) lead us to

$$\begin{aligned} c_0(x,\eta') =& e^{-izt+\int_0^t\Delta\phi_+(x(\tau))d\tau}c_0\big(x(t),\eta'\big) \\ =& e^{-izt+\int_0^t\Delta\phi_+(x(\tau))d\tau}\Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp\big(\frac{S\pi}{2\lambda_1}i\,\mathrm{sgn}\,\phi_1\big)}{\lambda_1|\phi_1(x(t))|^{S/\lambda_1}}a_{0,0}\big(x(t),\eta'\big). \end{aligned} \tag{3.31}$$

On the other hand, using Proposition 2.1

$$x(t) = X_1(x)e^{\lambda_1 t} + \mathcal{O}\big(e^{(\mu_2-\delta)t}\big) \quad \text{as} \quad t\to-\infty,$$

and (3.20), we have

$$\phi_1(x(t)) = -\lambda_1X_1(x_c(\eta'))\cdot X_1(x)e^{\lambda_1 t} + \mathcal{O}\big(e^{(\mu_2-\delta)t}\big) \quad \text{as} \quad t\to-\infty, \tag{3.32}$$

for any $\delta>0$. Inserting this into (3.31) and taking the limit $t\to-\infty$, we obtain

Proposition 3.10. *For x near x_F, one has*

$$c_0(x,\eta') = \Gamma\left(\frac{S}{\lambda_1}\right)\frac{\exp(-\frac{S\pi}{2\lambda_1}i\,\sigma)}{\lambda_1|\lambda_1X_1(x_c(\eta'))\cdot X_1(x)|^{S/\lambda_1}}e^{I_\infty(x)}a_{0,0}(0,\eta'), \tag{3.33}$$

where

$$\sigma = \mathrm{sgn}\,\big(X^-(x_c(\eta'))\cdot X^+(x)\big).$$

Microlocally on Λ_+, we can write u, using (3.12) and Proposition 3.9, as

$$\begin{aligned} u(x,h) =& \frac{h^{S/\lambda_1}}{(2\pi h)^{d/2}}\int_{\mathbb{R}^{d-1}} e^{i(\phi_+(x)+\tilde\psi(\eta'))/h}c(x,\eta';h)\hat u_0(\eta')\,d\eta' \\ =& \frac{h^{S/\lambda_1}}{(2\pi h)^{d-1/2}}\int_{\mathbb{R}^{2d-2}} e^{i(\phi_+(x)+\tilde\psi(\eta')-y'\cdot\eta')/h}c(x,\eta';h)u_0(y')\,d\eta'\,dy'. \end{aligned} \tag{3.34}$$

§ 3.4. Stationary phase method with respect to η'

We apply the stationary phase method for the integral (3.34) with respect to η'. The phase function is

$$\tilde{\psi}(\eta') - y' \cdot \eta' = x_c'(\eta') \cdot (\eta' - y') - \phi_-(x_c(\eta')).$$

Since $\eta' = \frac{\partial \phi_-}{\partial x'}(x_c(\eta'))$, $y' = x_c'(\eta')$ at the critical point $\eta' = \eta'(y')$, and the critical value is $-\phi_-(\varepsilon, y')$. Moreover, this critical point is non-degenerate. In fact, since

$$\frac{\partial^2 \phi_-}{\partial x'^2}(x_c(\eta')) \cdot \frac{\partial x_c'(\eta')}{\partial \eta'} = \frac{\partial}{\partial \eta'}\Big(\frac{\partial \phi_-}{\partial x'}(x_c(\eta'))\Big) = \frac{\partial}{\partial \eta'}\xi_c'(\eta') = \mathrm{Id},$$

one has

$$\frac{\partial^2 \tilde{\psi}}{\partial \eta'^2} = \frac{\partial x_c'(\eta')}{\partial \eta'} = \Big(\frac{\partial^2 \phi_-}{\partial x'^2}(x_c(\eta'))\Big)^{-1} \tag{3.35}$$

and by (2.4)

$$\frac{\partial^2 \phi_-}{\partial x'^2} = -\frac{1}{2}\mathrm{diag}\,(\lambda_2, \dots, \lambda_d) + \mathcal{O}(|x'|).$$

Proposition 3.11. *We have*

$$\frac{1}{(2\pi h)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{i(\tilde{\psi}(\eta') - y' \cdot \eta')/h} c(x, \eta'; h)\, d\eta' = e^{-i\phi_-(\varepsilon, y')/h} d(x, y'; h), \tag{3.36}$$

where $d(x, \eta'; h)$ is as in Theorem 2.5.

The asymptotic form (2.10) of the symbol d follows from (3.29) and the formula (2.11) of the principal term d_0 of d follows from (3.26), (3.33) and (3.35). Thus we get Theorem 2.5.

§ 4. Appendix

§ 4.1. Microlocal solution and frequency set

We say that a distribution $u(x; h) \in L^2(\mathbb{R}^d)$ depending on h with $||u|| \leq 1$ is equal to 0 at a point (x_0, ξ_0) in the phase space $T^*\mathbb{R}^d$ if there exists an open neighborhood ω of (x_0, ξ_0) such that for all N,

$$Tu(x, \xi; h) \equiv \int e^{i(x-y)\cdot\xi/h - (x-y)^2/(2h)} u(y; h)\, dy = \mathcal{O}_N(h^N) \quad \text{as} \quad h \to 0$$

uniformly on ω. The complement of the set of such points is called *frequency set* and denoted by $\mathrm{FS}(u)$. $\mathrm{FS}(u)$ is a closed set, see [Ma].

For a pseudo-differential operator $P = Op_h^W(p)$, u is said to be a *microlocal solution* at (x_0, ξ_0) of the equation $Pu = 0$ if $(x_0, \xi_0) \notin \mathrm{FS}\,(Pu)$.

Here are some fundamental properties of frequency set, for more details see [Ma].

Proposition 4.1. *If u is independent of h, then*

$$\mathrm{FS}(u) = \mathrm{WF}(u) \cup (\operatorname{Supp} u \times \{0\})\,,$$

where $\mathrm{WF}(u)$ *is the wave front set of* u.

Proposition 4.2. *Let $u(x,h) = a(x,h)e^{i\phi(x)/h}$, where $\phi(x)$ is a real-valued C^∞ function in a domain Ω in $\mathbb{R}^d$ and $a(x,h)$ is a C^∞ symbol on Ω, i.e. $a(x,h)$ is bounded in Ω uniformly with respect to h with all its derivatives. Then*

$$\mathrm{FS}(u) \subset \Big\{(x,\xi) \in \mathbb{R}^{2d};\ \xi = \frac{\partial\phi}{\partial x}(x)\Big\}.$$

Let now $u \in L^2(\mathbb{R}^d)$, $\|u\|_{L^2} \leq 1$, satisfy $Pu = 0$ with a real-valued symbol $p \in S^0_h(1)$.

Theorem 4.3 (Propagation of singularities)**.** *The frequency set of u is included in the characteristic set*

$$\mathrm{FS}(u) \subset \mathrm{Char}\,(P) := \big\{(x,\xi) \in \mathbb{R}^{2d};\ p(x,\xi) = 0\big\}.$$

Moreover, if $\exp(tH_p)(x_0,\xi_0)$ exists for $t \in (T_0,T_1)$, $(T_0 < 0 < T_1)$ for $(x_0,\xi_0) \in \mathbb{R}^{2d}$,

$$(x_0,\xi_0) \in \mathrm{FS}(u) \quad \Longleftrightarrow \quad \forall t \in (T_0,T_1), \quad \exp(tH_p)(x_0,\xi_0) \in \mathrm{FS}(u).$$

§ 4.2. Expandible solution

Let $\nu(x,\nabla_x)$ be a vector field of the form

$$\nu(x,\nabla_x) = A(x)x\cdot\nabla_x, \quad A(0) = \mathrm{diag}(\lambda_1,\dots,\lambda_d), \tag{4.1}$$

where $0 < \lambda_1 \leq \dots \leq \lambda_d$ are positive constants, and consider the Cauchy problem

$$\begin{cases} \partial_t u + \nu(x,\nabla_x)u = v(t,x) \\ u_{|t=0} = w(x). \end{cases} \tag{4.2}$$

We denote by $\exp(t\nu)(x_0)$ the solution to the system of ordinary differential equations

$$\begin{cases} \dot{x}(t) = A(x(t))x(t) \\ x_{|t=0} = x_0. \end{cases} \tag{4.3}$$

Then

$$\begin{aligned} \frac{d}{dt}\Big[u\big(t,\exp(t\nu)(x_0)\big)\Big] &= \big(\partial_t + \nu(x,\nabla_x)\big)u\big(t,\exp(t\nu)(x_0)\big) \\ &= v\big(t,\exp(t\nu)(x_0)\big). \end{aligned}$$

Hence

$$u\big(t,\exp(t\nu)(x_0)\big) = w(x_0) + \int_0^t v\big(s,\exp(s\nu)(x_0)\big)\,ds.$$

Put now $x = \exp(t\nu)(x_0)$. Since $x_0 = \exp(-t\nu)(x)$, $\exp(s\nu)(x_0) = \exp(-(t-s)\nu)(x)$, we get

$$u(t,x) = w\big(\exp(-t\nu)(x)\big) + \int_0^t v\big(t-s,\exp(-s\nu)(x)\big)\,ds. \tag{4.4}$$

When $\nu = \nu_0 = \displaystyle\sum_{j=1}^d \lambda_j x_j \frac{\partial}{\partial x_j}$, in particular,

$$\exp(-t\nu)(x) = \big(e^{-\lambda_1 t}x_1, \dots, e^{-\lambda_d t}x_d\big).$$

Let Ω be a suitable neighborhood of 0 in $\mathbb{R}^d$.

Definition 4.4. We write $u(t,x) \in \mathcal{O}^\infty(e^{-\mu t}|x|^M)$ if for every $\varepsilon > 0$, $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$,

$$D_t^k D_x^\alpha u(t,x) = \mathcal{O}(e^{-(\mu-\varepsilon)t}|x|^{(M-|\alpha|)_+}),$$

in $[0,\infty)\times\Omega$.

The map $\exp(-t\nu) : \Omega \to \Omega$ is well defined and

$$|\exp(-t\nu)(x)| = \mathcal{O}(e^{-\lambda_1 t}|x|), \quad |D_t^k D_x^\alpha \exp(-t\nu)(x)| = \mathcal{O}(e^{-\lambda_1 t}),$$

for $x \in \Omega, t \geq 0$ and for all $k \in \mathbb{N}, \alpha \in \mathbb{N}^d$. It is easy to check the following lemmas.

Lemma 4.5. *Suppose $w = 0$. If $v \in \mathcal{O}^\infty(e^{-\lambda t}|x|^N)$ with $N\lambda_1 \geq \lambda$, then $u \in \mathcal{O}^\infty(e^{-\lambda t}|x|^N)$.*

Lemma 4.6. *If $w \in \mathcal{O}(|x|^N)$ and $v = 0$, then $u \in \mathcal{O}^\infty(e^{-N\lambda_1 t}|x|^N)$.*

We will see that the solution u to the Cauchy problem (4.2) is expandible in the following sense:

Definition 4.7. Let $\mu_1 < \mu_2 < \cdots$ be the series of linear combinations over $\mathbb{N}$ of $\lambda_1, \dots, \lambda_d$. A function $u(t,x) \in C^\infty\big([0,\infty)\times\Omega\big)$ is said to be expandible if there exist u_k $(k = 1, 2, \dots)$ polynomials in t with smooth coefficients in $x \in \Omega$ such that, for any $N \in \mathbb{N}$,

$$u(t,x) - \sum_{k=1}^N e^{-\mu_k t}u_k(t,x) = \mathcal{O}^\infty\big(e^{-\mu_{N+1}t}\big).$$

First let us look for the homogeneous solution of the Cauchy problem

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u}{\partial x_j} = e^{-\mu t} \sum_{|\alpha|=N} c_\alpha(t) x^\alpha \\ u_{|t=0} = 0, \end{cases} \tag{4.5}$$

where $c_\alpha(t)$ are polynomials in t.

The function $u_1 = e^{-\mu t} \sum_{|\alpha|=N} a_\alpha(t) x^\alpha$ satisfies

$$\partial_t u_1 + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u_1}{\partial x_j} = e^{-\mu t} \sum_{|\alpha|=N} \Big(a'_\alpha(t) + \big(\sum_{j=1}^{d} \lambda_j \alpha_j - \mu \big) a_\alpha(t) \Big) x^\alpha.$$

Hence if u_1 satisfies the first equation of (4.5), $a_\alpha(t)$ should satisfy

$$a'_\alpha(t) + \delta_\alpha a_\alpha(t) = c_\alpha(t), \quad \delta_\alpha = \sum_{j=1}^{d} \lambda_j \alpha_j - \mu. \tag{4.6}$$

The equation (4.6) has a polynomial solution $a_\alpha(t)$ with

$$\deg a_\alpha = \begin{cases} \deg c_\alpha & \text{if } \delta_\alpha \neq 0 \\ \deg c_\alpha + 1 & \text{if } \delta_\alpha = 0. \end{cases}$$

Set $u_2 := u - u_1$, the function u_2 satisfies

$$\begin{cases} \partial_t u_2 + \sum_{j=1}^{d} \lambda_j x_j \frac{\partial u_2}{\partial x_j} = 0 \\ u_{2|t=0} = - \sum_{|\alpha|=N} a_\alpha(0) x^\alpha, \end{cases}$$

which leads to

$$u_2 = - \sum_{|\alpha|=N} a_\alpha(0) x^\alpha e^{-(\sum_{j=1}^{d} \lambda_j \alpha_j) t}.$$

Thus the solution to (4.5) is given by

$$u(t,x) = \sum_{|\alpha|=N} \Big(e^{-\mu t} a_\alpha(t) - e^{-(\sum_1^d \lambda_j \alpha_j) t} a_\alpha(0) \Big) x^\alpha.$$

Proposition 4.8. *Suppose $v(t,x)$ is expandible and $v = \mathcal{O}(|x|^N)$ with $N \geq 1$. Then the solution $u(t,x)$ of the Cauchy problem*

$$\begin{cases} \partial_t u + \nu(x, \nabla_x) u = v(t,x) \\ u_{|t=0} = 0 \end{cases} \tag{4.7}$$

is also expandible.

Proof. We set $v \sim v^{(N)} + v^{(N+1)} + \cdots$, where $v^{(M)}$ is homogeneous of order M:

$$v^{(M)} = \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} c_\alpha^k(t) x^\alpha.$$

In the case where $\nu = \nu_0$, Proposition 4.8 holds by the preceeding argument.

In the general case, let $\nu \sim \nu^{(0)} + \nu^{(1)} + \cdots$, where $\nu^{(k)}$ is homogeneous of order $k+1$. Expanding also $u \sim u^{(N)} + u^{(N+1)} + \cdots$, the equation becomes

$$\Big[\partial_t + (\nu^{(0)} + \nu^{(1)} + \cdots)\Big]\Big(\sum_{M=N}^{\infty} u^{(M)}\Big) = \sum_{M=N}^{\infty} v^{(M)},$$

which leads to

$$\begin{aligned} \partial_t u^{(N)} + \nu^{(0)} u^{(N)} \qquad &= v^{(N)}, \\ \partial_t u^{(N+1)} + \nu^{(0)} u^{(N+1)} &= v^{(N+1)} - \nu^{(1)} u^{(N)}, \end{aligned}$$

and in general for $M \geq N$,

$$\partial_t u^{(M)} + \nu^{(0)} u^{(M)} = v^{(M)} - \nu^{(1)} u^{(M-1)} - \cdots - \nu^{(M-N)} u^{(N)}.$$

Hence we can check inductively that

$$u^{(M)} = \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} a_\alpha^k(t) x^\alpha,$$

with

$$\deg a_\alpha^k(t) \leq \max\Big(\deg c_\alpha^k(t), \max_{|\beta|<M} \deg a_\beta^k(t)(+1)\Big),$$

where $(+1)$ occurs only for a finite number of α for each k. Therefore, for each k, $\deg a_\alpha^k$ is uniformly bounded with respect to M, since it is so for $c_\alpha^k(t)$.

For each M, we have

$$u^{(M)} \sim \sum_{k=1}^{\infty} e^{-\mu_k t} \sum_{|\alpha|=M} a_\alpha^k(t) x^\alpha.$$

There exists d_k independent of M such that $\deg a_\alpha^k \leq d_k$ for all α.

We can construct a realization $\bar{u}$ such that

$$\bar{u} \sim u^{(N)} + u^{(N+1)} + \cdots, \quad \bar{u}_{|t=0} = 0.$$

Letting $\tilde{u} := u - \bar{u}$, it remains to show the existence of an expandible solution $\tilde{u} = \mathcal{O}(|x|^\infty)$ such that

$$\begin{cases} \partial_t \tilde{u} + \nu(x, \nabla_x)\tilde{u} = \tilde{v} \\ \qquad\qquad \tilde{u}_{|t=0} = 0. \end{cases}$$

This is done by proving the following proposition by induction in N:

$$\tilde{u} = \tilde{u}_N + \tilde{v}_N$$

with expandible and $\mathcal{O}(|x|^\infty)$ function $\tilde{u}_N$ and $\mathcal{O}^\infty(e^{-\mu_N t}|x|^\infty)$ function $\tilde{v}_N$. We leave this to the reader. $\square$

Theorem 4.9. *Proposition 4.8 holds for time-dependent vector field*

$$\tilde{\nu}(t,x,\nabla_x) = A(t,x)x\cdot\nabla_x, \quad A(t,x) = A(x) + \tilde{A}(t,x), \tag{4.8}$$

where $A(x)$ is as in (4.1) *and $\tilde{A}(t,x)$ is expandible.*

Remark. If we add $\mu_0 = 0$ in the definition of expandibility, Theorem 4.9 holds without the assumption $N \geq 1$.

Corollary 4.10. *Suppose that a function $s(t,x)$ is expandible*

$$s(t,x) \sim \sum_{k=0}^{\infty} e^{-\mu_k t} s_k(t,x)$$

and that $s_0(x)$ is independent of t. If $v(t,x)$ is expandible in the form

$$v(t,x) \sim \sum_{k=0}^{\infty} e^{-(\mu_k + s_0(0))t} v_k(t,x),$$

then the solution of the Cauchy problem

$$\begin{cases} \partial_t u + \big(\tilde{\nu}(t,x,\nabla_x) + s(t,x)\big)u = v \\ \hfill u_{|t=0} = 0, \end{cases}$$

is also expandible in the same form

$$u(t,x) \sim \sum_{k=0}^{\infty} e^{-(\mu_k + s_0(0))t} u_k(t,x).$$

Remark. The solution to the homogeneous equation

$$\begin{cases} \partial_t u + \tilde{\nu}(t,x,\nabla_x)u = 0 \\ \hfill u_{|t=0} = w, \end{cases}$$

is also expandible since $u - \chi(t)w(x) =: \bar{u}$, where $\chi(t)$ is a cutoff function near $t=0$, satisfies

$$\begin{cases} \partial_t \bar{u} + \tilde{\nu}(t,x,\nabla_x)\bar{u} = -\chi(t)\tilde{\nu}w \\ \hfill \bar{u}_{|t=0} = 0, \end{cases}$$

which means by Theorem 4.9 that $\bar{u}$ is expandible.

Recall that $u = w\big(\exp t\nu(x)\big)$ when the vector field is independent of t (i.e. $\tilde{\nu} = \nu$). Taking x_j as the initial data w, we see that $\exp t\nu(x)$ is expandible. This fact also implies that the Hamilton flow $\big(x(t), \xi(t)\big) = \exp(tH_p)(x^0, \xi^0)$ on the incoming stable manifold Λ_- is expandible. In fact, $x(t)$ satisfies

$$\begin{cases} \dot{x}(t) = \nabla_\xi p(x, \nabla_x \phi_-(x)) \\ x(0) = x^0, \end{cases}$$

where $\nabla_\xi p(x, \nabla_x \phi_-) = -\operatorname{diag}(\lambda_1, \dots, \lambda_d)x + \mathcal{O}(|x|^2)$.

§ 4.3. The one-dimensional model

Eventually we come back to the one-dimensional model

$$P = -h^2 \frac{d^2}{dx^2} - \frac{\lambda^2}{4} x^2,$$

and recover the results of §1 by using the constructions of §3. Here $\lambda > 0$.

Recall that the Hamilton flow, the stable manifold and the generating function of $p(x, \xi) = \xi^2 - \frac{\lambda^2}{4} x^2$ are given by

$$\exp(tH_p)(x, \xi) = \begin{pmatrix} \cosh \lambda t & \frac{2}{\lambda} \sinh \lambda t \\ \frac{\lambda}{2} \sinh \lambda t & \cosh \lambda t \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix},$$

$$\Lambda_\pm = \Big\{ (x, \xi) \in \mathbb{R}^2;\ \xi = \pm \frac{\lambda}{2} x \Big\}, \quad \phi_\pm(x) = \pm \frac{\lambda}{4} x^2.$$

We construct a solution of the form

$$u(x, h) = \frac{1}{\sqrt{2\pi h}} \int_0^\infty e^{i\varphi(t,x)/h} a(t, x; h) dt,$$

where $v(t, x; h) = e^{i\varphi/h} a(t, x; h)$, $a(t, x; h) = \displaystyle\sum_{j=0}^\infty a_j(t, x) h^j$, satisfies the time-dependent Schrödinger equation

$$ih\partial_t v - h^2 \partial_x^2 v + (V(x) + hz)v = 0. \tag{4.9}$$

From (4.9) follow the eikonal equation

$$\varphi_t + \varphi_x^2 - \frac{\lambda^2}{4} x^2 = 0 \tag{4.10}$$

and the transport equation

$$\partial_t a_j + 2\varphi_x \partial_x a_j + (\varphi_{xx} - iz) a_j = i\partial_x^2 a_{j-1}, \quad j \geq 0, \quad \text{and} \quad a_{-1} \equiv 0. \tag{4.11}$$

Let us first solve (4.10). Take a point $(\varepsilon, -\frac{\lambda}{2}\varepsilon)$, $\varepsilon > 0$, on Λ_- and a Lagrangian manifold

$$\Lambda_0 = \Big\{(x,\xi) \in \mathbb{R}^2;\ \xi = \frac{\lambda}{2}(x - 2\varepsilon)\Big\}$$

passing by the point transversely to Λ_-. The evolution of Λ_0 by the Hamilton flow is

$$\Lambda_t = \exp(tH_p)\Lambda_0 = \Big\{(x,\xi) \in \mathbb{R}^2;\ \xi = \frac{\lambda}{2}(x - 2\varepsilon e^{-\lambda t})\Big\}.$$

This is a Lagrangian manifold and the generating function is

$$\varphi(t,x) = \frac{\lambda}{4}x^2 - \lambda\varepsilon x e^{-\lambda t} + C(t), \tag{4.12}$$

where $C(t)$ is arbitrary but independent of x. Substituting (4.12) to the eikonal equation (4.10), we get

$$\varphi(t,x) = \frac{\lambda}{4}x^2 - \lambda\varepsilon x e^{-\lambda t} + \frac{\lambda}{2}\varepsilon^2 e^{-2\lambda t}. \tag{4.13}$$

Next we solve the transport equations. With φ given by (4.13), (4.11) becomes

$$\partial_t a_j + \lambda(x - 2\varepsilon e^{-\lambda t})\partial_x a_j + (\frac{\lambda}{2} - iz)a_j = i\partial_x^2 a_{j-1}. \tag{4.14}$$

This is an ordinary differential equation along the curve $x(t) = \varepsilon e^{-\lambda t} + \delta e^{\lambda t}$ for any δ. Thus, $\tilde{a}_j(t) := a_j\big(t, x(t)\big)$ satisfies

$$\partial_t \tilde{a}_j + (\frac{\lambda}{2} - iz)\tilde{a}_j = i\partial_x^2 a_{j-1}\big(t, x(t)\big).$$

Let $j = 0$. The initial condition $a_0(0,x) = C(x)$ gives

$$a_0(t, x(t)) = C(\varepsilon + \delta)e^{-(\lambda/2 - iz)t},$$

namely

$$a_0(t,x) = C(\varepsilon + xe^{-\lambda t} - \varepsilon e^{-2\lambda t})e^{-(\lambda/2 - iz)t}.$$

Thus we obtain the following formula about the principal term:

$$\begin{aligned}\sqrt{2\pi h}\, u(x,h) \sim \int_0^\infty & e^{i(\lambda x^2/4 - \lambda\varepsilon x e^{-\lambda t} + \lambda\varepsilon^2 e^{-2\lambda t}/2)/h} \\ & \times C(\varepsilon + xe^{-\lambda t} - \varepsilon e^{-2\lambda t})e^{-(\lambda/2 - iz)t} dt.\end{aligned} \tag{4.15}$$

4.3.1. Asymptotic expansion on Λ_- When $x < 0$, there is no critical point. This implies that

$$\Lambda_-^- \cap \mathrm{FS}(u) = \emptyset. \tag{4.16}$$

When $x > 0$, the critical point $t = t(x)$ of φ is

$$x = \varepsilon e^{-\lambda t(x)}, \quad t(x) = -\frac{1}{\lambda} \log \frac{x}{\varepsilon},$$

and the critical value is

$$\varphi\big(t(x), x\big) = -\frac{\lambda}{4} x^2.$$

Moreover

$$\varphi_{tt}(t(x), x) = \lambda^3 x^2 > 0, \quad a_0(t(x), x) = C(\varepsilon) \left(\frac{x}{\varepsilon}\right)^{(\frac{\lambda}{2} - iz)/\lambda}.$$

Hence by the stationary phase method, one obtains

$$\begin{aligned} u(x,h) \sim & \frac{e^{\pi i/4}}{\sqrt{\varphi_{tt}(t(x), x)}} a_0(t(x), x) e^{-i\lambda x^2/(4h)} \\ = & C(\varepsilon) \frac{e^{\pi i/4}}{\lambda^{3/2} x} \left(\frac{x}{\varepsilon}\right)^{\frac{S}{\lambda}} e^{-i\lambda x^2/(4h)}, \end{aligned} \tag{4.17}$$

where $S = \frac{\lambda}{2} - iz$.

4.3.2. Asymptotic expansion on Λ_+ Here we calculate the contribution from $t = +\infty$ of the integral (4.15). It gives the asymptotic expansion of u on Λ_+.

Since

$$\varphi(t, x) \sim \frac{\lambda}{4} x^2 - \lambda \varepsilon x e^{-\lambda t}, \quad C(\varepsilon + x e^{-\lambda t} - \varepsilon e^{-2\lambda t}) \sim C(\varepsilon),$$

as $t \to +\infty$, we have

$$u(t, x) \sim \frac{C(\varepsilon)}{\sqrt{2\pi h}} e^{i\lambda x^2/(4h)} \int_0^\infty e^{-i\lambda \varepsilon x e^{-\lambda t}/h} e^{-St} dt.$$

By the change of variables $\lambda i \varepsilon x e^{-\lambda t}/h = s$,

$$dt = -\frac{ds}{\lambda s}, \quad e^{-St} = \left(\frac{hs}{i\lambda \varepsilon x}\right)^{S/\lambda},$$

we get

$$u(x,h) \sim e^{-\pi S i \operatorname{sgn}(x)/(2\lambda)} \frac{C(\varepsilon)}{\lambda \sqrt{2\pi h}} \Gamma\left(\frac{S}{\lambda}\right) \left(\frac{h}{\lambda \varepsilon |x|}\right)^{S/\lambda} e^{i\lambda x^2/(4h)}. \tag{4.18}$$

4.3.3. Transition operator Finally we compute u on Λ_+ when the initial data u_0 is given on $\mathcal{C} \subset \Lambda_-$. In this one-dimensional setting, $\mathcal{C}$ is the set with two points $\{(\varepsilon, -\lambda\varepsilon), (-\varepsilon, \lambda\varepsilon)\}$. We give an abritrary number $u_0 \in \mathbb{C}$ at $(\varepsilon, -\lambda\varepsilon)$, and we fix 0 at $(-\varepsilon, \lambda\varepsilon)$.

Comparing the initial condition and the results (4.16), (4.17) in §4.3.1, the arbitrary function $C(\varepsilon)$ is determined:

$$u_0 = C(\varepsilon)\frac{e^{\pi i/4}}{\lambda^{3/2}x}\left(\frac{x}{\varepsilon}\right)^{S/\lambda} e^{-i\lambda x^2/(4h)}\Big|_{x=\varepsilon} = C(\varepsilon)\frac{e^{\pi i/4}}{\lambda^{3/2}\varepsilon}e^{-i\lambda\varepsilon^2/(4h)},$$

that is,

$$C(\varepsilon) = \lambda^{3/2}e^{-\pi i/4}\varepsilon e^{i\lambda\varepsilon^2/(4h)}u_0.$$

Substituting this into (4.18), we obtain

$$u(x,h) \sim \frac{e^{-\pi i/4}}{\sqrt{2\pi h}}e^{-i\pi S\,\mathrm{sgn}(x)/(2\lambda)}\sqrt{\lambda}\varepsilon\Gamma\left(\frac{S}{\lambda}\right)\left(\frac{h}{\lambda\varepsilon|x|}\right)^{\frac{S}{\lambda}} e^{i\lambda(x^2+\varepsilon^2)/(4h)}u_0. \tag{4.19}$$

In other words, the transition operator $\mathcal{J}$ is a multiplication by a function of x, ε, z, h, and the principal term of its asymptotic expansion in h is given by

$$\mathcal{J} \sim \frac{e^{-\pi i/4}}{\sqrt{2\pi h}}e^{-i\pi S\,\mathrm{sgn}(x)/(2\lambda)}\sqrt{\lambda}\varepsilon\Gamma\left(\frac{S}{\lambda}\right)\left(\frac{h}{\lambda\varepsilon|x|}\right)^{\frac{S}{\lambda}} e^{i\lambda(x^2+\varepsilon^2)/(4h)}.$$

§ 4.4. Brief proofs of the propositions in §3

Proof of Proposition 3.2 : Differentiating the canonical equation

$$\frac{d}{dt}x(t,y',\eta') = 2\frac{\partial\psi}{\partial x}$$

with respect to (t,y'), one gets $\displaystyle\frac{d}{dt}\frac{\partial x(t,y',\eta')}{\partial(t,y')} = 2\frac{\partial^2\psi}{\partial x^2}\frac{\partial x(t,y',\eta')}{\partial(t,y')}$. Hence taking the determinant,

$$\frac{d}{dt}\det\frac{\partial x(t,y',\eta')}{\partial(t,y')} = 2\mathrm{Tr}\frac{\partial^2\psi}{\partial x^2}\det\frac{\partial x(t,y',\eta')}{\partial(t,y')},$$

i.e.

$$J(t,y',\eta') = J(0,y',\eta')\cdot\exp\Big(2\int_0^t \Delta\psi\big(x(\tau,y',\eta'),\eta'\big)\,d\tau\Big). \tag{4.20}$$

□

Proof of Proposition 3.3 : The transport equation (3.4) is an ordinary differential equation along the Hamilton flow and $b_0\big(x(t,y',\eta'),\eta'\big) = \beta(t)$ satisfies

$$\frac{d}{dt}\beta(t) + (\Delta\psi - iz)\beta(t) = 0.$$

The initial condition is $\beta(0) = b_0\big(x(0,y',\eta'),\eta'\big) = b_0(y',\eta') = 1$. Then

$$b_0\big(x(t,y',\eta'),\eta'\big) = \exp\Big(izt - \int_0^t \Delta\psi\big(x(\tau,y',\eta'),\eta'\big)\,d\tau\Big), \tag{4.21}$$

and (3.10) follows from (4.21) and (4.20). □

Proof of Proposition 3.5 : We first show that φ is expandible, see [He-Sj 2]. Let us introduce new symplectic local coordinates (x,ξ) centered at $(0,0)$ such that Λ_- is given by $x=0$ and Λ_+ is given by $\xi=0$. Then

$$p(x,\xi)=A(x,\xi)x\cdot\xi,$$

where the matrix $A(0,0)$ has the eigenvalues $\lambda_1,\dots,\lambda_d$ and we may assume that

$$A(0,0)=\operatorname{diag}(\lambda_1,\dots,\lambda_d).$$

The curve γ now becomes $\big(0,\xi_0(t)\big)$, where $\xi_0(t)=\mathcal{O}\big(e^{-\mu_1 t}\big)$.

We check the following proposition by induction:
(H)$_N$ The function φ verifies $\varphi=\psi_N+r_N$ where $\psi_N=\mathcal{O}^\infty(e^{-\lambda_1 t}|x|)$ is expandible,

$$r_N=\mathcal{O}^\infty(e^{-N\lambda_1 t}|x|^{N+1})\quad\text{and}\quad r_{N|t=0}=0.$$

By Taylor expansion with respect to r_N, we get

$$\partial_t r_N+\tilde\nu_N r_N=f_N+\mathcal{O}\big(e^{-2N\lambda_1 t}|x|^{2N+1}\big),$$

where

$$\begin{aligned}\tilde\nu_N:=&\nabla_\xi p(x,\nabla_x\psi_N)\cdot\nabla_x\\ \nabla_\xi p(x,\nabla_x\psi_N)=&A(x,0)x+\mathcal{O}^\infty\big(|x|^2e^{-\lambda_1 t}\big)\end{aligned}$$

is expandible, and

$$f_N=-\big(\partial_t\psi_N+p(x,\nabla_x\psi_N)\big)$$

is $\mathcal{O}^\infty\big(e^{-N\lambda_1 t}|x|^{N+1}\big)$ and expandible. Let ρ_N be the solution to

$$\begin{cases}\partial_t\rho_N+\tilde\nu_N\rho_N=f_N,\\ \rho_{N|t=0}=0.\end{cases}$$

then, by Theorem 4.9 and Lemma 4.5, which holds also for t-dependent $\tilde\nu$, $\rho_N=\mathcal{O}\big(e^{-N\lambda_1 t}|x|^{N+1}\big)$ is expandible. Now we put

$$\varphi=(\psi_N+\rho_N)+(r_N-\rho_N)=:\psi_{2N}+r_{2N}.$$

We see that $r_{2N}=\mathcal{O}^\infty(e^{-2N\lambda_1 t}|x|^{2N+1})$ since it satisfies

$$\begin{cases}\partial_t r_{2N}+\tilde\nu_N r_{2N}=\mathcal{O}^\infty\big(e^{-2N\lambda_1 t}|x|^{2N+1}\big)\\ r_{2N|t=0}=0.\end{cases}$$

Hence *(H)*$_N$ implies *(H)*$_{2N}$.

It remains to prove *(H)*$_1$. We first see that

$$\varphi(t,x)=\mathcal{O}^\infty(e^{-\lambda_1 t})$$

uniformly in a neighborhood of $x=0$ which means, in the original coordinates, $\varphi(t,x)=\phi_+(x)+\mathcal{O}^\infty\big(e^{-\lambda_1 t}\big)$, see also [He-Sj 2]. This estimate implies

$$\varphi(t,x)=\varphi(t,0)+x\cdot\nabla_x\varphi(t,0)+\mathcal{O}^\infty\big(e^{-\lambda_1 t}|x|^2\big).$$

Differentiating the eikonal equation

$$\partial_t\varphi+A(x,\nabla_x\varphi)x\cdot\nabla_x\varphi=0$$

with respect to x, and substituting $x=0$, $\xi(t):=\nabla_x\varphi(t,0)$ satisfies

$$\dot\xi(t)+{}^tA(0,\xi(t))\xi(t)=0.$$

Then $\xi(t)$ is expandible by Remark 4.2 since ${}^tA(0,0)=\mathrm{diag}(\lambda_1,\dots,\lambda_d)$. Hence *(H)*$_1$ holds.

It follows that φ is expandible also in the original coordinates.

Next we show (3.18). First, the limit of (3.18) on $x=x_c(t,\eta')$ is

$$\tilde\psi(\eta')=\lim_{t\to+\infty}\varphi\big(t,x_c(t,\eta'),\eta'\big).$$

On the other hand, since $\big(x_c(t,\eta'),\xi_c(t,\eta')\big)\in\Lambda_t\cap\Lambda_-$, one has

$$\begin{aligned}\frac{d}{dt}\left[\varphi(t,x_c(t,\eta'),\eta')\right]=&(\partial_x\varphi)\big(t,x_c(t,\eta'),\eta'\big)\cdot\dot x_c(t,\eta')=\xi_c(t,\eta')\cdot\dot x_c(t,\eta')\\=&\partial_x\phi_-\big(x_c(t,\eta')\big)\cdot\dot x_c(t,\eta')=\frac{d}{dt}\Big[\phi_-\big(x_c(t,\eta')\big)\Big].\end{aligned}$$

This means that $\varphi\big(t,x_c(t,\eta'),\eta'\big)-\phi_-\big(x_c(t)\big)$ is independent of t. Recalling that $\phi_-(0)=0$, we have

$$\begin{aligned}\tilde\psi(\eta')=&\lim_{t\to+\infty}\Big(\varphi\big(t,x_c(t,\eta'),\eta'\big)-\phi_-\big(x_c(t,\eta')\big)\Big)\\=&\varphi\big(0,x_c(0,\eta'),\eta'\big)-\phi_-\big(x_c(0,\eta')\big)\\=&x_c'(\eta')\cdot\eta'-\phi_-\big(x_c(\eta')\big).\end{aligned}$$

Finally we compute the asymptotic behavior (3.20) of $\phi_1(x,\eta')$ as $x\to0$. Without loss of generality, we can change the canonical coordinates such that $\phi_\pm(x)=\pm\sum_{j=1}^d\frac{\lambda_j^2}{4}x_j^2$. We develop $\varphi(t,x,\eta')$ in Taylor series at $x=x_c(t)$. Recalling that $\varphi\big(t,x_c(t)\big)=\phi_-(x(t))$ and $\nabla\varphi\big(t,x_c(t)\big)=\nabla\phi_-(x(t))$, we obtain

$$\begin{aligned}\varphi(t,x)=&\phi_-(x(t))+\nabla\phi_-(x(t))\cdot\big(x-x_c(t)\big)\\&+\frac12\big\langle\nabla^2\phi_+(x_c(t))(x-x_c(t)),x-x_c(t)\big\rangle+\mathcal{O}\big(e^{-\lambda_1 t}|x-x_c(t)|^2\big).\end{aligned}$$

Here we used (3.5) for the quadratic terms. The third term of the right hand side is equal to $\phi_+\big(x - x_c(t)\big)$. Thus by Proposition 2.1, we have

$$\begin{aligned}\varphi(t,x) &= \phi_+(x) - \sum_{j=1}^{d} \lambda_j x_j(t) x_j + \mathcal{O}\big(e^{-\lambda_1 t}|x|^2\big) + \mathcal{O}\big(e^{-\mu_2 t}\big) \\ &= \phi_+(x) + \Big(- \lambda_1 X_1(x)\cdot x + \mathcal{O}\big(|x|^2\big)\Big) e^{-\lambda_1 t} + \mathcal{O}\big(e^{-\mu_2 t}\big).\end{aligned}$$

□

Proof of Proposition 3.6 : The transport equation (3.14) is of the form

$$\partial_t u + Ax\cdot\partial_x u + s(t,x)u = v(t,x) \tag{4.22}$$

with $a_l = u$. Here $s(t,x) = \Delta_x\varphi - iz$ is expandible by Proposition 3.5:

$$s(t,x) \sim \sum_{k=0}^{\infty} e^{-\mu_k t} s_k(t,x), \quad s_0(x) = \Delta\phi_+(x) - iz.$$

Let $r(t,x)$ be the solution to the Cauchy problem

$$\begin{cases} \partial_t r + Ax\cdot\partial_x r = -s \\ r_{|t=0} = 0, \end{cases}$$

and set

$$u = e^r\tilde{u}, \quad v = e^r\tilde{v}.$$

Then the 0th order term of (4.22) vanishes:

$$\partial_t\tilde{u} + Ax\cdot\partial_x\tilde{u} = \tilde{v}(t,x).$$

By Theorem 4.9, $r(t,x)$ is expandible:

$$r(t,x) \sim \sum_k e^{-\mu_k t} r_k(t,x),$$

and in particular we see that

$$r_0(t,x) = r_0(t,x) - s_0(0)t = r_0(t,x) - St.$$

Then again by Theorem 4.9, $e^{St}u(t,x)$ is expandible, since so is $e^{St}v(t,x)$. □

Proof of Proposition 3.9 : By the change of variables $e^{-t} = s$, $\varphi - \phi_+ - \tilde{\psi} = \phi_1\sigma^{\mu_1}$, (3.18) implies

$$\begin{aligned}\sigma \sim & s\Big(1 + \sum_{k=2}^{\infty} \frac{\phi_k(-\log s, x)}{\phi_1(x)} s^{\mu_k - \mu_1}\Big)^{1/\mu_1} \\ \sim & s\Big(1 + \sum_{k=1}^{\infty} \rho_k(-\log s, x) s^{\hat{\mu}_k}\Big).\end{aligned}$$

This form of expansion is invariant for the inverse, i.e. the variable s is solved in terms of σ in the same form

$$s \sim \sigma\Big(1 + \sum_{k=1}^{\infty} f_k(-\log\sigma, x)\sigma^{\hat{\mu}_k}\Big).$$

By (3.21),

$$\begin{aligned}\int_0^\infty e^{\varphi/h} a_0 dt =& e^{i(\phi_+ + \tilde{\psi})/h} \int_0^1 e^{i\phi_1\sigma^{\mu_1}/h} s^S \sum a_{0,k}(-\log s, x) s^{\mu_k - 1} ds \\ =& e^{i(\phi_+ + \tilde{\psi})/h} \int_0^{\alpha^{1/\mu_1}} e^{i\phi_1\sigma^{\mu_1}/h} \sum b_k(-\log\sigma, x)\sigma^{S+\hat{\mu}_k - 1} d\sigma,\end{aligned}$$

where $\alpha = \frac{\varphi(0,x) - \phi_+(x) - \tilde{\psi}(\eta)}{\phi_1(x)}$, and putting furthermore $\sigma^{\mu_1} = \tau$,

$$= e^{i(\phi_+ + \tilde{\psi})/h} \int_0^\alpha e^{i\phi_1\tau/h} \sum \beta_k(-\log\tau, x)\tau^{(S+\hat{\mu}_k)/\mu_1 - 1} d\tau,$$

where $\beta_0(x)$, $b_0(x)$ are functions only of x and $\beta_0(x) = \frac{1}{\mu_1} b_0(x) = \frac{1}{\mu_1} a_{0,0}(x)$.

The last integral is not well-defined for $\frac{S+\hat{\mu}_k}{\mu_1} \in -\mathbb{N} = \{0, -1, -2, \ldots\}$, that is,

$$z = z_{\alpha,N} = -i\Big(\sum_{j=1}^{d} (\alpha_j + \frac{1}{2})\lambda_j - N\lambda_1\Big),$$

for some $\alpha_1, \ldots, \alpha_d, N \in \mathbb{N}$. If $\alpha_1 \geq N$, this z is excluded by (**A1**). On the contrary, the other cases corresponding to $\alpha_1 < N$ never occur because we already know, by the argument at the beginning of § 3.3, that our solution u is holomorphic outside $-ih\mathcal{E}_0$.

Finally, the proposition follows from formula (1.4) and the fact that

$$\int_0^1 e^{i\phi_1\tau/h} \tau^{\mu-1} (\log\tau)^m d\tau = \Big(\frac{\partial}{\partial\mu}\Big)^m \int_0^1 e^{i\phi_1\tau/h} \tau^{\mu-1} d\tau.$$

□

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