Iwahori-Hecke type algebras
associated with the Lie superalgebras
$A(m, n), B(m, n), C(n)$ and $D(m, n)$

By

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Abstract

In this paper we give Iwahori-Hecke type algebras $H_q(g)$ associated with the Lie superalgebras $g = A(m, n), B(m, n), C(n)$ and $D(m, n)$. We classify the irreducible representations of $H_q(g)$ for generic $q$.

Introduction

Recently, motivated by a question posed by V. Serganova [S] and study of the Weyl groupoids [H1][H2] associated with Nichols algebras [AS1][AS2] including generalizations of quantum groups, I. Heckenberger and the author [HY] introduced a notion of ‘Coxeter groupoids’ (in fact they can be defined as semigroups), and showed that a Matsumoto-type theorem holds for the groupoids, so they have the solvable word problem. We mention that the Coxeter groupoid associated with the affine Lie superalgebra $D^{(1)}(2, 1; x)$ was used in the study [HSTY], where Drinfeld second realizations of $U_q(D^{(1)}(2, 1; x))$ was analized by physical motivation in recent study of AdS/CFT correspondence.

It would be able to be said that one of the main purposes at present of the representation theory is to study the Kazhdan-Lusztig polynomials (cf. [Hu, 7.9]) and their versions. The polynomials are defined by using the standard and canonical bases of the Iwahori-Hecke algebras. The existence of those bases is closely related to the Matsumoto theorem of the Coxeter groups. So it would be natural to ask what to be
the Iwahori-Hecke algebras of the Coxeter groupoids. In this paper, we give a tentative answer to this question for the Coxeter groupoids \( W \) associated with the Lie superalgebras \( g = A(m, n), B(m, n), C(n) \) and \( D(m, n) \). We introduce the Iwahori-Hecke type algebra \( H_q(g) \) (in the text, it is also denoted by \( H_q(W) \)) as \( q \)-analogue of the semigroup algebra \( \mathbb{C}W/\mathbb{C}0 \), where 0 is the zero element of \( W \). We also show that if \( q \) is nonzero and not any root of unity, \( H_q(g) \) is semisimple and there exists a natural one-to-one correspondence between the equivalence classes of the irreducible representations of \( H_q(g) \) and those of the Iwahori-Hecke algebra \( H_q(W_0) \) associated with the Weyl group \( W_0 \) of the Lie algebra \( g(0) \) obtained as the even part of \( g = g(0) \oplus g(1) \).

Until now, no relation has been achieved between the groupoids treated in [SV] and this paper.

This paper is composed of the two sections. Main results and their proofs are given in Section 2. Results of [HY] used in Section 2 are introduced in Section 1.

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§ 1. Preliminary—Matsumoto-type theorem of Coxeter groupoids

This section is preliminary. Here we collect the results which have already been given in [HY] and will be used in the next section.

§ 1.1. Semigroups and Monoids

Let \( K \) be a non-empty set. Assume that \( K \) has a product map \( K \times K \to K \), \( (x, y) \mapsto xy \). We call \( K \) a semigroup if \( (xy)z = x(yz) \) for \( \forall x, y, z \in K \). We call \( K \) a monoid if \( K \) is a semigroup and there exists a unit 1 \( \in K \), that is, \( 1x = x1 = x \) for all \( x \in K \).

§ 1.2. Free semigroup \( F_{-1}(N) \) and Free monoid \( F_0(N) \)

Let \( N \) be a non-empty set. Let \( F_{-1}(N) \) be the set of all the finite sequences of elements of \( N \), that is

\[
F_{-1}(N) := \prod_{n=1}^{\infty} N^n = \{(h_1, \ldots, h_n)|n \in \mathbb{N}, h_i \in N\}.
\]

We regard \( F_{-1}(N) \) as the semigroup by

\[
(h_1, \ldots, h_m)(h_{m+1}, \ldots, h_{m+n}) = (h_1, \ldots, h_m, h_{m+1}, \ldots, h_{m+n}).
\]

Then we call \( F_{-1}(N) \) a free semigroup. Let \( F_0(N) \) be the semigroup obtained by adding the unit 1, that is, \( F_0(N) := \{1\} \cup F_{-1}(N), \ 1 \notin F_{-1}(N), \) and \( 1x = x1 = x \) for all \( x \in F_0(N) \).
§ 1.3. Semigroup generated by the generators and and defined by the relations

Let $Q = \{(x_j, y_j) | j \in J\}$ be a subset of $F_{-1}(N) \times F_{-1}(N)$, where $J$ is an index set. For $g_1, g_2 \in F_{-1}(N)$, we write $g_1 \sim g_2$ if there exist $(x, y) \in Q$ and $(f_1, f_2) \in F_0(N) \times F_0(N)$ such that either of the following (i), (ii), (iii) holds.

(i) $g_1 = f_1 x f_2 \neq g_2 = f_1 y f_2$.
(ii) $g_1 = f_1 y f_2 \neq g_2 = f_1 x f_2$.
(iii) $g_1 = g_2 = f_1 x f_2 = f_1 y f_2$.

For $g, g' \in F_{-1}(N)$, we write $g \sim g'$ if $g = g'$ or there exists $r \in \mathbb{N}$ and $g_1, \ldots, g_r \in F_{-1}(N)$ such that $g_1 = g$, $g_r = g'$, and $g_i \sim g_{i+1}$ for $1 \leq i \leq r - 1$. Then $F_{-1}(N)/\sim$ can be regarded as a semigroup by the product $[g][g'] = [gg']$, where for $g \in F_{-1}(N)$, we denote $[g] := \{g' | g' \sim g \} \in F_{-1}(N)/\sim$. We call $F_{-1}(N)/\sim$ the semigroup generated by $N$ and defined by the relations $x_j = y_j$ ($j \in J$). When there is no fear of misunderstanding, we also denote $[g]$ by its representative $g$ by abuse of notation.

§ 1.4. Free group $F_1(N)$ and Involutive free group $F_2(N)$

Let $N$ be a set. Let $N^{-1}$ be a copy of $N$ so that the bijective map $N \rightarrow N^{-1}$, $x \mapsto x^{-1}$, is given. Let $F_1(N)$ be the semigroup generated by

\[ \{e\} \cup N \cup N^{-1} \quad \text{(disjoint union)} \]

and defined by the relations

\[ ee = e, \quad ex = xe = x, \quad ex^{-1} = x^{-1}e = x^{-1}, \quad xx^{-1} = x^{-1}x = e \quad \text{for } \forall x \in N. \]

We call $F_1(N)$ the free group over $N$.

Let $F_2(N)$ be the semigroup generated by

\[ \{e\} \cup N \quad \text{(disjoint union)} \]

and defined by the relations

\[ ee = e, \quad ex = xe = x, \quad x^2 = e \quad \text{for } \forall x \in N. \]

We call $F_2(N)$ the involutive free group over $N$. Note that $F_2(N)$ can be identified with the quotient group of $F_1(N)$ in the natural sense:

\[ F_2(N) = F_1(N)/\{g_1 y_1^2 g_1^{-1} \cdots g_r y_r^2 g_r^{-1} | r \in \mathbb{N}, y_i \in N \cup N^{-1}, g_i \in F_1(N)\}. \]
§ 1.5. Action $\triangleright$ of $F_2(N)$ on $A$

Let $N$ and $A$ be non-empty sets. An action $\triangleright$ of $F_2(N)$ on $A$ is a map

$$\triangleright : F_2(N) \times A \to A$$

such that

$$e \triangleright a = a, \quad g \triangleright (h \triangleright a) = (gh) \triangleright a \quad \text{for } \forall g, \forall h \in F_2(N), \forall a \in A.$$ 

Note that $n \triangleright (n \triangleright a) = a$ for all $n \in N, a \in A$.

For $n, n' \in N$ and $a \in A$, define

$$\Theta(n, n'; a) := \{(nn')^m \triangleright a, (n'n)^m \triangleright a \mid m \in \mathbb{N} \cup \{0\}\}.$$ 

Let

$$\theta(n, n'; a) := |\Theta(n, n'; a)|.$$ 

This is the cardinality of $\Theta(n, n'; a)$, which is either in $\mathbb{N}$ or is $\infty$. One obviously has $\Theta(n, n'; a) = \Theta(n', n; a)$ and $\Theta(n, n'; n \triangleright a) = n \triangleright \Theta(n', n; a)$.

Let $a_0 := a$, $b_0 := a$, and define recursively $a_{m+1} := n \triangleright b_m$, $b_{m+1} := n' \triangleright a_m$ for all $m \in \mathbb{N} \cup \{0\}$. That is:

$$b_0 := a, \quad a_1 := n \triangleright a, \quad b_2 := n' \triangleright n \triangleright a, \quad a_3 := n \triangleright n' \triangleright n \triangleright a, \ldots$$

Then we have

$$\theta(n, n'; a) = \begin{cases} 
\infty & \text{if } a_m \neq b_m \text{ for all } m \in \mathbb{N}, \\
\min \{m \in \mathbb{N} \mid a_m = b_m\} & \text{otherwise}. 
\end{cases}$$

§ 1.6. Coxeter groupoids

Definition 1.1. [HY, Definition 1] Let $N$ and $A$ be non-empty sets. Let $\triangleright$ be a transitive action of $F_2(N)$ on $A$. For each $a \in A$ and $i, j \in N$ with $i \neq j$ let

$$m_{i,j;a} = m_{j,i;a} \in (\mathbb{N} + 1) \cup \{\infty\}$$

be such that $\theta(i, j; a) \in \mathbb{N} \Rightarrow m_{i,j;a} \in \mathbb{N} \cup \{\infty\}$ or $\theta(i, j; a) = \infty \Rightarrow m_{i,j;a} = \infty$. Set

$$m := (m_{i,j;a} \mid i, j \in N, i \neq j, a \in A).$$

Let

$$(1.1) \quad W = (W, N, A, \triangleright, m)$$
be the semigroup generated by the set
\[ \{0, e_a, s_{i,a} \mid a \in A, i \in N\} \]
and defined by the relations
\[
(1.2) \quad 00 = e_a0 = 0e_a = s_{i,a}0 = 0s_{i,a} = 0,
\]
\[
(1.3) \quad e_a^2 = e_a, \quad e_ae_b = 0 \text{ for } a \neq b,
\]
\[
(1.4) \quad s_{i}s_{j} \cdots s_{j}s_{i,a} = s_{j}s_{i} \cdots s_{i}s_{j,a} \quad (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is finite and odd},
\]
\[
(1.4) \quad s_{j}s_{i} \cdots s_{j}s_{i,a} = s_{i}s_{j} \cdots s_{i}s_{j,a} \quad (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is finite and even},
\]
where we use the convention:
\[
(1.5) \quad s_{j}s_{i,a} := s_{j,i \triangleright a}s_{i,a}, \quad s_{i}s_{j}s_{i,a} := s_{i,j \triangleright i \triangleright a}s_{j}s_{i,a}, \ldots
\]
See also (1.7) below.

§ 1.7. Sign representation

Let \( \mathbb{Z}A \) be the free \( \mathbb{Z} \)-module generated by \( A \), that is,
\[ \mathbb{Z}A = \bigoplus_{a \in A} \mathbb{Z}a. \]
Then there exists a unique semigroup homomorphism
\[ \widetilde{\mathrm{sgn}} : W \rightarrow \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}A) \]
such that
\[
(1.6) \quad \widetilde{\mathrm{sgn}}(0)(b) = 0, \quad \widetilde{\mathrm{sgn}}(e_a)(b) = \delta_{ab}b, \quad \widetilde{\mathrm{sgn}}(s_{i,a})(b) = (-1)^{\delta_{ab}i \triangleright a} b
\]
for \( a, b \in A \) and \( i \in N \), where \( \delta \) means Kronecker’s symbol. Hence for \( w \in W \) one has
\[ w \neq 0 \]
if and only if \( w = e_a \) for some \( a \in A \) or there exist \( m \in \mathbb{N} \) and \( i_j \in N \), \( b_j \in A \) with \( 1 \leq j \leq m \) such that \( b_j = i_{j+1} \triangleright b_{j+1} \) and \( w = s_{i_1,b_1} \cdots s_{i_{m-1},b_{m-1}}b_{m}s_{i_m,b_m} \). If this is the case, we use the convention
\[
(1.7) \quad s_{i_1} \cdots s_{i_{m-1}}s_{i_m,b_m} := w,
\]
and, if \( m = 0 \), \( s_{i_1} \cdots s_{i_{m-1}}s_{i_m,a} \) means \( e_a \). We note again
Lemma 1.2. (1) \( s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} \neq 0 \) for all \( a \in A \) and \( m \in \mathbb{N} \cup \{0\} \).
(2) If \( s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a} = s_{j_1} \cdots s_{j_{r-1}} s_{j_r, b} \), then \( a = b, i_1 \cdots i_m \triangleright a = j_1 \cdots j_r \triangleright b \) and \((-1)^m = (-1)^r\).

§1.8. Generalization of Root systems

Definition 1.3. [HY, Definition 2] We call a quadruple \((R, N, A, \triangleright)\) a multi-domains root system if the following conditions hold.

1. \( N \) and \( A \) are non-empty sets and \( \triangleright \) is a transitive action of \( F_2(N) \) on \( A \).
2. Let \( V_0 \) be the \(|N|\)-dimensional \( \mathbb{R} \)-linear space. Then \( R = \{(R_a, \pi_a, S_a) | a \in A\} \), where \( \pi_a = \{\alpha_{n,a} | n \in N\} \subset R_a \subset V_0 \), and \( \pi_a \) is a basis of \( V_0 \) for all \( a \in A \).
3. \( R_a = R_a^+ \cup -R_a^+ \) for all \( a \in A \), where \( R_a^+ = (\mathbb{N} \cup \{0\})\pi_a \cap R_a \).
4. For any \( i \in N \) and \( a \in A \) one has \( R \alpha_{i,a} \cap R_a = \{\alpha_{i,a}, -\alpha_{i,a}\} \).
5. \( S_a = \{\sigma_{i,a} | i \in N\} \), and for each \( a \in A \) and \( i \in N \) one has \( \sigma_{i,a} \in \text{GL}(V_0) \),
\[ \sigma_{i,a}(R_a) = R_{i \triangleright a}, \quad \sigma_{i,a}(\alpha_{i,a}) = -\alpha_{i,i \triangleright a}, \quad \sigma_{i,a}(\alpha_{j,a}) \in \alpha_{j,i \triangleright a} + (\mathbb{N} \cup \{0\})\alpha_{i,i \triangleright a} \]
for all \( j \in N \setminus \{i\} \).
6. \( \sigma_{i,i \triangleright a} \sigma_{i,a} = \text{id} \) for \( a \in A \) and \( i \in N \).
7. Let \( a \in A, i, j \in N, i \neq j, d = |(\mathbb{N} \cup \{0\})\alpha_{i,a} + (\mathbb{N} \cup \{0\})\alpha_{j,a}) \cap R_a| \). If \( d \) is finite then \( \theta(i, j; a) \) is finite and it divides \( d \).

Convention. We write \((R, N, A, \triangleright) \in \mathcal{R} \) if \((R, N, A, \triangleright)\) is a multi-domains root system, that is, \( \mathcal{R} = \{(R, N, A, \triangleright)\} \) denotes the family of all the multi-domains root systems.

Definition 1.4. [HY, Definition 4] Let \((R, N, A, \triangleright) \in \mathcal{R} \). Let \( m := (m_{i,j,a} | i, j \in N, i \neq j, a \in A) \) be such that \( m_{i,j,a} := |(\mathbb{N} \cup \{0\})\alpha_{i,a} + (\mathbb{N} \cup \{0\})\alpha_{j,a}) \cap R_a| \). Then we call \((W, N, A, \triangleright, m)\) the Coxeter groupoid associated with \((R, N, A, \triangleright)\).

Theorem 1.5. [HY, Theorem 1] Let \((R, N, A, \triangleright) \in \mathcal{R} \). Set \( V = \bigoplus_{a \in A} V_a \), where \( V_a = V_0 \). Let \( P_a : V \rightarrow V_a \) and \( \iota_a : V_a \rightarrow V \) be the canonical projection and the canonical inclusion map respectively. Then the assignment \( \rho : 0 \mapsto 0 \cdot \text{id}_V, e_a \mapsto \iota_a P_a, s_{i,a} \mapsto \iota_{i \triangleright a} \sigma_{i,a} P_a \), gives a faithful representation \((\rho, V)\) of the Coxeter groupoid \((W, N, A, \triangleright, m)\) associated with \((R, N, A, \triangleright)\).
§ 1.9. Matsumoto-type theorem

Define $\ell : W \to \mathbb{N} \cup \{0\} \cup \{-\infty\}$ to be the map such that $\ell(0) = -\infty$, $\ell(e_a) = 0$ for all $a \in A$, and

$$\ell(w) = \min \{m \in \mathbb{N} | w = s_{i_1} \cdots s_{i_m-1} s_{i_m, a} \text{ for some } i_1, \ldots, i_m \in N, a \in A\}$$

for all $w \in W \setminus (\{0\} \cup \{e_a | a \in A\})$; we also refer to Lemma 1.2 (1) for this definition of $\ell$. One has

(1.8) $\ell(w) = \ell(w^{-1})$

for $w \in W \setminus \{0\}$, and

(1.9) $\ell(ww') \leq \ell(w) + \ell(w')$

for $w, w' \in W$. We say that a product $w = s_{i_1} \cdots s_{i_m-1} s_{i_m, a} \in W$ is reduced if $m = \ell(w)$.

Definition 1.6. [HY, Definition 5] Let $W = (W, N, A, \triangleright, m)$ be a Coxeter groupoid. Let $\widetilde{W} = (\widetilde{W}, N, A, \triangleright, m)$ denote the semigroup generated by the set

$$\{0, \tilde{e}_a, \tilde{s}_{i, a} | a \in A, i \in N\}$$

and defined by the relations

(1.10) $00 = 0$, $0\tilde{e}_a = \tilde{e}_a 0 = 0\tilde{s}_{i, a} = \tilde{s}_{i, a} 0 = 0$,

(1.11) $\tilde{e}_a^2 = \tilde{e}_a$, $\tilde{e}_a \tilde{e}_b = 0$ for $a \neq b$, $\tilde{e}_{ia} \tilde{s}_{i, a} = \tilde{s}_{i, a} \tilde{e}_a = \tilde{s}_{i, a}$,

(1.12) $\tilde{s}_i \tilde{s}_j \cdots \tilde{s}_j \tilde{s}_{i, a} = \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_j \tilde{s}_{j, a}$ (m_{i,j;a} factors) if $m_{i,j;a}$ is finite and odd,

$\tilde{s}_j \tilde{s}_i \cdots \tilde{s}_j \tilde{s}_{i, a} = \tilde{s}_i \tilde{s}_j \cdots \tilde{s}_j \tilde{s}_{j, a}$ (m_{i,j;a} factors) if $m_{i,j;a}$ is finite and even.

Theorem 1.7. [HY, Theorem 5] (Matsumoto-type theorem of the Coxeter groupoids) Let

$$W = (W, N, A, \triangleright, m)$$

be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ (see Definition 1.4). Suppose that $m \in \mathbb{N} \cup \{0\}$, $a \in A$, and $(i_1, \ldots, i_m), (j_1, \ldots, j_m) \in N^m$ such that

$$\ell(s_{i_1} \cdots s_{i_m-1} s_{i_m, a}) = m$$

and equation

$$s_{i_1} \cdots s_{i_m-1} s_{i_m, a} = s_{j_1} \cdots s_{j_m-1} s_{j_m, a}$$

holds in $W$. Then in the semigroup $(\widetilde{W}, N, A, \triangleright, m)$ one has

$$\tilde{s}_{i_1} \cdots \tilde{s}_{i_m-1} \tilde{s}_{i_m, a} = \tilde{s}_{j_1} \cdots \tilde{s}_{j_m-1} \tilde{s}_{j_m, a}.$$
Corollary 1.8. [HY, Corollary 6] Let

\[ W = (W, N, A, \triangleright, m) \]

be the Coxeter groupoid associated with \((R, N, A, \triangleright) \in \mathcal{R}\) (see Definition 1.4). Suppose that \(m \in \mathbb{N} \cup \{0\}\), \(a \in A\), and \((i_1, \ldots, i_m) \in N^m\) such that \(\ell(s_{i_1} \cdots s_{i_{m-1}}s_{i_m,a}) < m\) holds in \((W, N, A, \triangleright, m)\). Then there exist \(j_1, \ldots, j_m \in N\) and \(t \in \{1, \ldots, m-1\}\) such that \(j_t = j_{t+1}\) and in the semigroup \((\bar{W}, N, A, \triangleright, m)\) one has the equation

\[ \bar{s}_{i_1} \cdots \bar{s}_{i_{m-1}}s_{i_m,a} = \bar{s}_{j_1} \cdots \bar{s}_{j_t} \bar{s}_{j_{t+1}} \cdots \bar{s}_{j_{m-1}} \bar{s}_{j_m,a} \].

In the next section, we also need

Proposition 1.9. [HY, Corollary 3] Let \(m \in \mathbb{N}\), \((i_1, \ldots, i_m, j) \in N^{m+1}\), and \(a \in A\), and suppose that \(\ell(s_{i_1} \cdots s_{i_m} s_{j,j \triangleright a}) = m\). Then:

1. \(m = |\sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(R_a^+) \cap -R_{i_1 \cdots i_m \triangleright a}^+|\).
2. \(\ell(s_{i_1} \cdots s_{i_m} s_{j,j \triangleright a}) = m - 1 \Leftrightarrow \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(\alpha_{j,a}) \in -R_{i_1 \cdots i_m \triangleright a}^+\).
3. \(\ell(s_{i_1} \cdots s_{i_m} s_{j,j \triangleright a}) = m + 1 \Leftrightarrow \sigma_{i_1} \cdots \sigma_{i_{m-1}} \sigma_{i_m,a}(\alpha_{j,a}) \in R_{i_1 \cdots i_m \triangleright a}^+\).

Example 1.10. Here we treat the finite dimensional simple Lie superalgebra \(D(2,1;x)\), where \(x \notin \{0, -1\}\). Note that it has 14 (positive and negative) roots. One has \(m_{i,j;a(k)} = 2 + \delta_{k0} + (1-\delta_{k0})(\delta_{ik} + \delta_{jk})\) and \(\sigma_{i,a(k)}(\alpha_{j,a(k)}) = \alpha_{j,i \triangleright a(k)} + \delta_{3,m_{i,j;a(k)}} \alpha_{i,i \triangleright a(k)}\) for \(i \neq j\). Moreover

\[ R^+_{a(k)} = \pi_{a(k)} \cup \{\alpha_{i,a(k)} + \alpha_{j,a(k)}|m_{i,j;a(k)} = 3\} \]
\[ \cup \{\alpha_{1,a(k)} + \alpha_{2,a(k)} + \alpha_{3,a(k)}\} \]
\[ \cup \{\alpha_{i,a(k)} + 2\alpha_{k,a(k)} + \alpha_{j,a(k)}|m_{i,j;a(k)} = 2\} \].

Note that \(D(2,1;1) = D(2,1) = osp(4|2)\) (see also Section 2.2).

Let \(w_{a(2)} := s_{3,a(2)} s_{2,a(0)} s_{3,a(3)} s_{1,a(3)} s_{3,a(0)} s_{2,a(2)} s_{1,a(2)}\). Then \(\rho(w_{a(2)}) = -\text{id}_{V_{a(2)}}\).

Indeed:

\[ \alpha_{1,a(2)} \mapsto -\alpha_{1,a(2)} \mapsto -\alpha_{1,a(0)} - \alpha_{2,a(0)} \mapsto -\alpha_{1,a(3)} - \alpha_{2,a(3)} - 2\alpha_{3,a(3)} \]
\[ \mapsto -\alpha_{1,a(3)} - \alpha_{2,a(3)} - 2\delta_{3,a(3)} \mapsto -\alpha_{1,a(0)} - \alpha_{2,a(0)} \mapsto -\alpha_{1,a(2)} \mapsto -\alpha_{1,a(2)}, \]
\[ \alpha_{2,a(2)} \mapsto \alpha_{1,a(2)} + \alpha_{2,a(2)} \mapsto \alpha_{1,a(0)} \mapsto \alpha_{1,a(3)} + \alpha_{3,a(3)} \mapsto \alpha_{3,a(3)} \mapsto -\alpha_{3,a(0)} \]
\[ \mapsto -\alpha_{2,a(2)} - \alpha_{3,a(2)} \mapsto -\alpha_{2,a(2)}, \]
\[ \alpha_{3,a(2)} \mapsto \alpha_{3,a(2)} \mapsto \alpha_{2,a(0)} + \alpha_{3,a(0)} \mapsto \alpha_{2,a(3)} \mapsto \alpha_{2,a(3)} \mapsto \alpha_{2,a(0)} + \alpha_{3,a(0)} \]
\[ \mapsto \alpha_{3,a(2)} \mapsto -\alpha_{3,a(2)}. \]
By Proposition 1.9(1), we have $\ell(w_{a(2)}) = |R^+_{a(2)}| = 7$, $w_{a(2)}$ is the longest word. Let $w' := (s_{3,a(2)}s_{2,a(0)})^{-1}w_{a(2)}$. Then $\ell(w') = 5$. By Theorem 1.7, $w'$ has the following four reduced expressions:

$$w' = s_{3,a(3)}s_{1,a(3)}s_{3,a(0)}s_{2,a(2)}s_{1,a(2)} = s_{1,a(1)}s_{3,a(1)}s_{1,a(0)}s_{2,a(2)}s_{1,a(2)}$$
$$= s_{1,a(1)}s_{3,a(1)}s_{2,a(1)}s_{1,a(0)}s_{2,a(2)} = s_{1,a(1)}s_{2,a(1)}s_{3,a(1)}s_{1,a(0)}s_{2,a(2)}.$$

§ 2. Main theorems—Irreducible representations of the Iwahori-Hecke type algebras $H_q(A(m, n))$, $H_q(B(m, n))$, $H_q(C(n))$ and $H_q(D(m, n))$ associated with the Lie superalgebras $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$

§ 2.1. Definition of Lie superalgebras

As for the terminology concerning Lie superalgebras, we refer to [K].

Let $v = v(0) \oplus v(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space. If $i \in \{0, 1\}$ and $j \in \mathbb{Z}$ such that $j - i \in 2\mathbb{Z}$ then let $v(j) = v(i)$. If $X \in v(0)$ (resp. $X \in v(1)$) then we write

$$\deg(X) = 0 \text{ (resp. } \deg(X) = 1)$$

and we say that $X$ is an even (resp. odd) element. If $X \in v(0) \cup v(1)$, then we say that $X$ is a homogeneous element and that $\deg(X)$ is the parity (or degree) of $X$. If $w \subset v$ is a subspace and $w = (w \cap v(0)) \oplus (w \cap v(1))$ (resp. $w \subset v(0)$, resp. $w \subset v(1)$), then we say that $w$ is a graded (resp. even, resp. odd) subspace.
Let $g = g(0) \oplus g(1)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}$-linear space equipped with a bilinear map $[ , ] : g \times g \rightarrow g$ such that $[g(i), g(j)] \subset g(i + j)$ ($i, j \in \mathbb{Z}$); we recall from the above paragraph that

$$(2.2) \quad g(i) = \{ X \in g \, | \, \deg(X) = i \}.$$  

We say that $g = (g, [ , ])$ is a ($\mathbb{C}$-)Lie superalgebra if for all homogeneous elements $X, Y, Z$ of $g$ the following equations hold.

$$[Y, X] = -(-1)^{\deg(X)\deg(Y)}[X, Y], \quad \text{(skew-symmetry)}$$

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\deg(X)\deg(Y)}[Y, [X, Z]], \quad \text{(Jacobi identity)}$$

We call the Lie algebra $g(0)$ the even part of $g$.

§2.2. Lie superalgebras $\mathfrak{gl}(m+1|n+1)$ and $\mathfrak{osp}(m|n)$

Let $m, n \in \mathbb{N} \cup \{0\}$. Let:

$$\mathcal{D}_{m+1|n+1} := \{ (p_1, \ldots, p_{m+n+2}) \in \mathbb{Z}^{m+n+2} \mid p_i \in \{0, 1\}, \sum_{i=1}^{m+n+2} p_i = n+1 \}.$$  

For $i, j \in \{1, \ldots, m+n+2\}$, let $E_{i,j}$ denote the $(m+n+2) \times (m+n+2)$ matrix having 1 in $(i, j)$ position and 0 otherwise, that is, the $(i, j)$-matrix unit. Let $E_{m+n+2}$ denote the $(m+n+2) \times (m+n+2)$ unit matrix, that is,

$$E_{m+n+2} = \sum_{i=1}^{m+n+2} E_{i,i}.$$  

Denote by $M_{m+n+2}(\mathbb{C})$ the $\mathbb{C}$-linear space of the $(m+n+2) \times (m+n+2)$-matrices, i.e.,

$$M_{m+n+2}(\mathbb{C}) = \oplus_{i,j=1}^{m+n+2} \mathbb{C} E_{i,j}.$$  

Let $d = (p_1, \ldots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$. The Lie superalgebra $\mathfrak{gl}(m+1|n+1) = gl(d)$ is defined by

$$\mathfrak{gl}(d)(0) = \oplus_{1 \leq i \leq m+n+2} \mathbb{C} E_{i,i}, \quad \mathfrak{gl}(d)(1) = \oplus_{1 \leq i \leq m+n+2} \mathbb{C} E_{i,j},$$

and $[X, Y] = XY - (-1)^{r_1 r_2} YX$ for $X \in \mathfrak{gl}(d)(r_1)$ and $Y \in \mathfrak{gl}(d)(r_2)$,

where $XY$ and $YX$ mean the matrix product, that is, $E_{i,j} E_{k,l} = \delta_{j,k} E_{i,l}$. Define the $\mathbb{C}$-linear map $\operatorname{str} : \mathfrak{gl}(d) \rightarrow \mathbb{C}$ by $\operatorname{str}(E_{i,j}) = \delta_{i,j} (-1)^{p_i}$. The Lie subsuperalgebra \{ $X \in \mathfrak{gl}(d) \mid \operatorname{str}(X) = 0$ \} of $\mathfrak{gl}(d)$ is denoted as $\mathfrak{s}(m+1|n+1) = \mathfrak{s}(d)$. The finite dimensional simple Lie superalgebra $A(m, n)$ is defined as follows, where $m + n \geq 1$. Let $\mathfrak{z}$ be the one dimensional ideal $\mathbb{C} E_{m+n+2}$ of $\mathfrak{gl}(d)$. If $m \neq n$, then $A(m, n)$ means $\mathfrak{sl}(d)$. On the other hand, $A(n, n)$ means $\mathfrak{sl}(d)/\mathfrak{z}$, and is also denoted as $\mathfrak{psl}(n+1|n+1)$.

Let $d = (p_1, \ldots, p_{m+2n}) \in \mathcal{D}_{m|2n}$. Define the map $\theta : \{1, \ldots, m+2n\} \rightarrow \{1, \ldots, m+2n\}$ by $\theta(i) = m + 2n + 1 - i$. Assume that $p_{\theta(i)} = p_i$. Let $g_i \in \{1, -1\}$ be such that $g_i = -1$ if $p_i = 1$ and $i < \theta(i)$ and $g_i = 1$ otherwise. We have an automorphism $\Omega$ of
\[ \text{gl}(d) \text{ defined by } \Omega(E_{i,j}) = -(-1)^{p_{i}p_{j}+p_{j}}g_{i}g_{j}E_{\theta(j),\theta(i)} \]. The Lie superalgebra \( \mathfrak{osp}(m|2n) \) means \( \{ X \in \mathfrak{gl}(d) | \Omega(X) = X \} \). We also denote \( \mathfrak{osp}(m|2n) \) as follows:

\begin{align*}
B(m - 1, n) &= \mathfrak{osp}(2m - 1|2n) & \text{if } m, n \in \mathbb{N}, \\
D(m + 1, n) &= \mathfrak{osp}(2m + 2|2n) & \text{if } m, n \in \mathbb{N}, \\
C(n + 1) &= \mathfrak{osp}(2|2n) & \text{if } n \in \mathbb{N}.
\end{align*}

We also note that \( \mathfrak{osp}(2m + 1|0) \), \( \mathfrak{osp}(0|2n) \), and \( \mathfrak{osp}(2m|0) \) are isomorphic to the simple Lie algebras of type \( B_{m} \) (if \( m \geq 2 \)), \( C_{n} \) (if \( n \geq 3 \)) and \( D_{m} \) (if \( m \geq 4 \)) respectively, so \( \mathfrak{osp}(2m + 1|0) \cong \mathfrak{o}_{2m+1}, \mathfrak{osp}(0|2n) \cong \mathfrak{sp}_{2n} \) and \( \mathfrak{osp}(2m|0) \cong \mathfrak{o}_{2m} \).

\section*{2.3. Definition of Iwahori-Hecke type algebras}

\textbf{Definition 2.1.} Let \( W = (W, N, A, \triangleright, m) \) be the groupoid introduced in (1.1). Assume that \( A \) is finite. Let \( q \in \mathbb{C} \). Let \( H_{q}(W) \) be the \( \mathbb{C} \)-algebra (with 1) generated by

\[ \{ E_{a}, T_{i,a} | a \in A, i \in N \} \]

and defined by the relations

\begin{align*}
E_{a}^{2} &= E_{a}, \\
E_{i\triangleright a}T_{i,a}E_{a} &= T_{i,a}, \\
\sum_{a \in A} E_{a} &= 1 \\
E_{a}E_{b} &= 0 & \text{if } a \neq b, \\
(T_{i,a} - qE_{a})(T_{i,a} + E_{a}) &= 0 & \text{if } i \triangleright a = a, \\
T_{i,i\triangleright a}T_{i,a} &= E_{a} & \text{if } i \triangleright a \neq a, \\
T_{i}T_{j} \cdots T_{j}T_{i,a} &= T_{j}T_{i} \cdots T_{j}T_{j,a} & \text{if } m_{i,j;a} \text{ is finite and odd}, \\
T_{j}T_{i} \cdots T_{j}T_{i,a} &= T_{i}T_{j} \cdots T_{i}T_{j,a} & \text{if } m_{i,j;a} \text{ is finite and even},
\end{align*}

where, in (2.12)-(2.13), we use the same convention as that of (1.5) with \( s_{i,a} \) in place of \( T_{i,a} \).

\textbf{Lemma 2.2.} Let \( W = (W, N, A, \triangleright, m) \) be the Coxeter groupoid associated with an element \( (R, N, A, \triangleright) \) of \( \mathcal{R} \) (see Definition 1.4). Assume that \( A \) is finite. Then there exists a map \( f : W \to H_{q}(W) \) such that

\begin{align*}
f(0) &= 0, \quad f(e_{a}) = E_{a}, \\
f(s_{i,a}w) &= T_{i,a}f(w) & \text{if } w \in W \setminus \{0\} \text{ and } \ell(s_{i,a}w) = 1 + \ell(w).
\end{align*}
Further, as a $\mathbb{C}$-linear space, $H_q(W)$ is spanned by $f(W \setminus \{0\})$. In particular, if $W$ is finite, then

\[(2.16) \quad \dim H_q(W) \leq |W| - 1.\]

**Proof.** Let $\tilde{W}$ be the semigroup introduced in Definition 1.6 for $W$. It is easy to show that there exists a unique semigroup homomorphism $\tilde{f}: \tilde{W} \rightarrow H_q(W)$ such that $\tilde{f}(0) = 0$, $\tilde{f}(e_a) = E_a$ and $\tilde{f}(s_{i,a}) = T_{i,a}$. By Theorem 1.7, there exists a unique map $f: W \rightarrow H_q(W)$ such that $f(0) = 0$ and $f(w) = \tilde{f}(\tilde{s}_{i_1} \cdots \tilde{s}_{i_{m-1}} \tilde{s}_{i_m, a})$ if $w \in W \setminus \{0\}$, $\ell(w) = m$ and $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_m, a}$. Then $f$ satisfies (2.14)-(2.15), as desired.

We show

\[(2.17) \quad \forall w \in W, \forall i \in N, \forall a \in A, T_{i,a}f(w) \in \mathbb{C}f(s_{i,a}w) + \mathbb{C}f(w).\]

If $s_{i,a}w = 0$, then clearly $T_{i,a}f(w) = 0$ holds. If $w \neq 0$, $s_{i,a}w \neq 0$ and $\ell(s_{i,a}w) = 1 + \ell(w)$, then (2.17) follows from (2.15). Assume that $w \neq 0$, $s_{i,a}w \neq 0$ and $\ell(s_{i,a}w) \neq 1 + \ell(w)$. Then by (1.8) and Proposition 1.9, we have $\ell(s_{i,a}w) = \ell(w) - 1$, so $f(w) = T_{i,i\triangleright a}f(s_{i,a}w)$. Since $T_{i,a}f(w) = T_{i,a}T_{i,i\triangleright a}f(s_{i,a}w)$, we have $T_{i,a}f(w) = f(s_{i,a}w)$ if $i \triangleright a \neq a$, and $T_{i,a}f(w) = (q - 1)f(w) + qf(s_{i,a}w)$ otherwise. Hence we have (2.17), as desired.

It is clear from (2.17) that the rest of the statement follows. \hfill $\square$

**Notation.** Let $r \in \mathbb{N}$. Let $V_0^{(r)}$ be the $r$-dimensional $\mathbb{R}$-linear space with a basis $\{\varepsilon_i | 1 \leq i \leq r\}$. Let $V_0^{(r),'}$ be the subspace of $V_0^{(r)}$ formed by the elements $\sum_{i=1}^r x_i \varepsilon_i$ with $x_i \in \mathbb{R}$ and $\sum_{i=1}^r x_i = 0$, so $\dim V_0^{(r),'} = r - 1$. For a non-zero element $x = \sum_{i=1}^{|N|} x_i \varepsilon_i$ of $V_0^{(r)}$ with $x_i \in \mathbb{R}$, define $\overline{x} \in \text{GL}(V_0^{(r)})$ by $\overline{x}(\varepsilon_j) = \varepsilon_j - 2x_j(\sum_{i=1}^{|N|} x_i^2)^{-1}x$, that is, $\overline{x}$ is the reflection of $V_0^{(r)}$ with respect to the hyperplane of $V_0^{(r)}$ orthogonal to $x$. Note that if $x \in V_0^{(r),'}$, then $\overline{x}(V_0^{(r),'}) = V_0^{(r),'}$.

**§ 2.4. Basic of Iwahori-Hecke algebras**

For the basic facts about the Iwahori-Hecke algebras, we refer to [GU]. Let $W = (W, N, A, \triangleright, \mathbf{m})$ be the groupoid introduced in (1.1). In this subsection we always assume that

\[(2.18) \quad |A| = 1 \text{ and } N \text{ and } W \text{ are finite.}\]

Let $a \in A$, so $A = \{a\}$. Then $W \setminus \{0\}$ is nothing but the Coxeter group associated with the Coxeter system $(W \setminus \{0\}, \{s_{i,a}, i \in N\})$. In this case, we also denote $H_q(W)$ and $T_{i,a}$ by $H_q(W \setminus \{0\})$ and $T_i$ respectively. That is, $H_q(W \setminus \{0\})$ is the $\mathbb{C}$-algebra (with 1) generated by $T_i$ ($i \in N$) and defined by the relations

\[(2.19) \quad (T_i - q)(T_i + 1) = 0, \]

\[(2.20) \quad T_iT_j \cdots T_jT_i = T_jT_i \cdots T_jT_i \quad (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is odd}, \]

\[(2.20) \quad T_jT_i \cdots T_jT_i = T_iT_j \cdots T_iT_j \quad (m_{i,j;a} \text{ factors}) \quad \text{if } m_{i,j;a} \text{ is even}. \]
It is well-known that \( \dim H_q(W \setminus \{0\}) = |W \setminus \{0\}|. \) In this paper we fix a complete set of non-equivalent irreducible representations of \( H_q(W \setminus \{0\}) \) by

\[
\{ \rho_{H_q(W \setminus \{0\}), \lambda} : H_q(W \setminus \{0\}) \to \text{End}_\mathbb{C}(V_{H_q(W \setminus \{0\}), \lambda}) | \lambda \in \Lambda_{H_q(W \setminus \{0\})} \},
\]

where \( \Lambda_{H_q(W \setminus \{0\})} \) is an index set. Define the polynomial \( P_{W \setminus \{0\}}(q) \) by

\[
P_{W \setminus \{0\}}(q) := \sum_{w \in W \setminus \{0\}} q^{\ell(w)}.
\]

This is called the Poincaré polynomial of \( W \setminus \{0\}. \)

It is well-known [GU] (see also [CR, (25.22) and (27.4)]) that for \( q \in \mathbb{C} \setminus \{0\}, \) the following three conditions are equivalent.

(i) \( P_{W \setminus \{0\}}(q) \neq 0 \) holds.
(ii) \( H_q(W \setminus \{0\}) \) is a semisimle algebra.
(iii) The map

\[
\bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \rho_{H_q(W \setminus \{0\}), \lambda} : H_q(W \setminus \{0\}) \to \bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \text{End}_\mathbb{C}(V_{H_q(W \setminus \{0\}), \lambda})
\]

defined by \( X \mapsto \bigoplus_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} \rho_{H_q(W \setminus \{0\}), \lambda}(X) \) is a \( \mathbb{C} \)-algebra isomorphism.

In particular,

\[
q \cdot P_{W \setminus \{0\}}(q) \neq 0 \implies \dim H_q(W \setminus \{0\}) = \sum_{\lambda \in \Lambda_{H_q(W \setminus \{0\})}} (\dim V_{H_q(W \setminus \{0\}), \lambda})^2.
\]

Assume that \( N = \{1, 2, \ldots, n\} \) and \( m_{i,i+1;a} = 3 \) and \( m_{i,j;a} = 2 \ (|j - i| \geq 2) \). Then \( W \) is the Coxeter groupoid associated with \( (R, N, A, \triangleright) \in \mathcal{R} \) such that \( V_0 = V_0^{(n+1)}, \)
\[
R^+_a = \{ \varepsilon_i - \varepsilon_j | 1 \leq i < j \leq n + 1 \}, \quad \alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1} \quad \text{and} \quad \sigma_{i,a} = \overline{\sigma}_{\alpha_{i,a}}.
\]

We also denote \( W \setminus \{0\} \) by \( W(B_n) \) and \( W(C_n) \).

Note that \( \dim H_q(W(B_n)) = 2^n n! \).

Assume that \( N = \{1, 2, \ldots, n\} \) and \( m_{i,i+1;a} = 3 \) (\( 1 \leq i \leq n - 3 \)), \( m_{n-1,n;a} = 4 \) and \( m_{i,j;a} = 2 \) (\(|j - i| \geq 2\)). Then \( W \) is the Coxeter groupoid associated with \( (R, N, A, \triangleright) \in \mathcal{R} \) such that \( V_0 = V_0^{(n)}, \)
\[
R^+_a = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n \} \cup \{ \varepsilon_i | 1 \leq i \leq n \},
\]
\[
\alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n - 1),
\]
\[
\alpha_{n,a} = \varepsilon_n \quad \text{and} \quad \sigma_{i,a} = \overline{\sigma}_{\alpha_{i,a}}.
\]

We also denote \( W \setminus \{0\} \) by \( W(D_n) \).

Note that \( \dim H_q(W(D_n)) = 2^{n-1} n! \). Then \( W \) is the Coxeter groupoid associated with \( (R, N, A, \triangleright) \in \mathcal{R} \) such that \( V_0 = V_0^{(n)}, \)
\[
R^+_a = \{ \varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j | 1 \leq i < j \leq n \},
\]
\[
\alpha_{i,a} = \varepsilon_i - \varepsilon_{i+1} \quad (1 \leq i \leq n - 1),
\]
\[
\alpha_{n,a} = \varepsilon_n - \varepsilon_{n-1} \quad \text{and} \quad \sigma_{i,a} = \overline{\sigma}_{\alpha_{i,a}}.
\]
It is well-known (cf. [C, Theorem 10.2.3 and Proposition 10.2.5]) that

\begin{align}
P_{S_{n+1}}(q) & = \prod_{r=1}^{n} \frac{q^{r+1} - 1}{q - 1}, \\
P_{W(B_n)}(q) & = \prod_{r=1}^{n} \frac{q^{2r} - 1}{q - 1}, \\
P_{W(D_n)}(q) & = \frac{q^n - 1}{q - 1} \prod_{r=1}^{n-1} \frac{q^{2r} - 1}{q - 1}.
\end{align}

§ 2.5. Iwahori-Hecke type algebra $H_q(A(m, n))$ associated with the Lie superalgebra $A(m, n)$

Let

\[ \triangleright: S_{m+n+2} \times \mathcal{D}_{m+1|n+1} \to \mathcal{D}_{m+1|n+1} \]

denote the usual (left) action of the symmetric group $S_{m+n+2}$ on $\mathcal{D}_{m+1|n+1}$ by permutations, that is, for $\sigma \in S_{m+n+2},$

\[ \sigma \triangleright (p_1, \ldots, p_{m+n+2}) = (p_{\sigma^{-1}(1)}, \ldots, p_{\sigma^{-1}(m+n+2)}). \]

Let $\sigma_i := (i, i+1) \in S_{m+n+2}$. Let $W$ be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \ldots, m + n + 1\}$, $A = \mathcal{D}_{m+1|n+1}$, $i \triangleright d = \sigma_i \triangleright d$, $V_0 = V_0^{(m+n+2)}$, $R_d^+ = \{\varepsilon_i - \varepsilon_j | 1 < j \leq m + n + 1\}$, $\alpha_{i,d} = \varepsilon_i - \varepsilon_{i+1}$ and $\sigma_{i,d} = \overline{\sigma}_{\alpha_{i,d}}$. Denote $H_q(W)$ by $H_q(A(m, n))$. Then $H_q(A(m, n))$ is the $\mathbb{C}$-algebra (with 1) generated by

\[ \{E_d | d \in \mathcal{D}_{m+1|n+1}\} \cup \{T_i,d | 1 \leq i \leq m + n + 1, d \in \mathcal{D}_{m+1|n+1}\} \]

and defined by the relations (2.6)-(2.11) and the relations

\begin{align}
T_{i,j} & = T_{i,j} T_{j,i} T_{i,j} & \text{if } |i-j| = 1, \\
T_{i,j} & = T_{i,j} T_{i,j} & \text{if } |i-j| \geq 2.
\end{align}

Define $d_e, d_o \in \mathcal{D}_{m+1|n+1}$ by

\[ d_e := (0, \ldots, 0, 1, \ldots, 1), \quad d_o := (1, \ldots, 1, 0, \ldots, 0). \]

For $d = (p_1, \ldots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1}$, define the two elements

\[ \tau_+, \tau_- \in S_{m+n+2} \]
\[ p_{\tau \pm \cdot d}(i) = \frac{1 \mp 1}{2} \quad \text{and} \quad \tau_{\pm \cdot d}(i) \leq \tau_{\pm \cdot d}(j) \quad \text{if} \ 1 \leq i \leq j \leq m + 1, \]
\[ p_{\tau \pm \cdot d}(i) = \frac{1 \pm 1}{2} \quad \text{and} \quad \tau_{\pm \cdot d}(i) \leq \tau_{\pm \cdot d}(j) \quad \text{if} \ m + 2 \leq i \leq j \leq m + n + 2. \]

Then \( \tau_{+, d} \) (resp. \( \tau_{-, d} \)) is the minimal length element among the elements \( \sigma \in S_{m+n+2} \) satisfying the condition that for any \( i \), \( i \)-th component of \( d_e \) (resp. \( d_o \)) is the same as \( \sigma(i) \)-th component \( p_{\sigma(i)} \) of \( d \).

**Example 2.3.** Assume that \( m = n = 1 \). Then \( \mathcal{D}_{2|2} = \{ d_e = (0, 0, 1, 1), d_1 = (0, 1, 0, 1), d_2 = (1, 0, 0, 1), d_4 = (1, 0, 1, 0), d_o = (1, 1, 0, 0) \} \). Then \( \tau_{+, d} \) (resp. \( \tau_{-, d} \)) is the minimal length element among the elements \( \sigma \in S_{m+n+2} \) satisfying the condition that for any \( i \), \( i \)-th component of \( d_e \) (resp. \( d_o \)) is the same as \( \sigma(i) \)-th component \( p_{\sigma(i)} \) of \( d \).

Now we consider \( |W| \). Recall \( \rho \) and \( d_e \) from Theorem 1.5 and (2.32) respectively. It is easy to see that \( P_{d_e} \rho(e_{d_e} W e_{d_e}) \iota_{d_e} \subset (\sum_{i=1}^{m+n+2} E_{\sigma(i)i})_{|V_{0}^{(m+n+2), \prime}} \sigma \in S_{m+n+2}, \sigma(\{1, \ldots, m+1\}) = \{1, \ldots, m+1\} \). Hence \( |e_{d_e} W e_{d_e}| \leq (m+1)!(n+1)! \) by Theorem 1.5, so \( |W \setminus \{0\}| = |\mathcal{D}_{m+1|n+1}|^{2} |e_{d_e} W e_{d_e}| \leq \frac{((m+n+2)!)^{2}}{(m+1)!(n+1)!} \). Hence by (2.16), we conclude

\[ \dim H_q(A(m, n)) \leq \frac{((m+n+2)!)^{2}}{(m+1)!(n+1)!}. \]

**Proposition 2.4.** Let \( V \) and \( W \) be finite dimensional \( \mathbb{C} \)-linear spaces, and let \( l : H_q(S_{m+1}) \rightarrow \text{End}_{\mathbb{C}}(V) \) and \( r : H_q(S_{n+1}) \rightarrow \text{End}_{\mathbb{C}}(W) \) be \( \mathbb{C} \)-algebra homomorphisms, i.e., representations. Let \( l \otimes r : H_q(S_{m+1}) \otimes H_q(S_{n+1}) \rightarrow \text{End}_{\mathbb{C}}(V \otimes W) \) denote the tensor representation of \( l \) and \( r \) in the ordinary sense. Let \( C_{V \otimes W,d} \) be copies of the \( \mathbb{C} \)-linear space \( V \otimes W \), indexed by \( d \in \mathcal{D}_{m+1|n+1} \). Let \( C_{V \otimes W} := \oplus_{d \in D_{m+1|n+1}} C_{V \otimes W} \). Let \( P_d : C_{V \otimes W} \rightarrow C_{V \otimes W,d} \) and \( \iota_d : C_{V \otimes W,d} \rightarrow C_{V \otimes W} \) denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique \( \mathbb{C} \)-algebra homomorphism \( l \otimes^A m n \, r : H_q(A(m, n)) \rightarrow \text{End}_{\mathbb{C}}(C_{V \otimes W}) \) satisfying the following conditions:

(i) For each \( d \in \mathcal{D}_{m+1|n+1} \), one has \( (l \otimes^A m n \, r)(E_d) = \iota_d \circ P_d \),
(ii) For each \( i \in \{1, \ldots, m+n+1\} \) and each \( d = (p_1, \ldots, p_{m+n+2}) \in \mathcal{D}_{m+1|n+1} \), one has

\[ (l \otimes^A m n \, r)(T_{i,d}) = \begin{cases} P_{\sigma_{i,d}} \circ \iota_d & \text{if } p_i \neq p_{i+1}, \\ \iota_d \circ (l(T_{\tau_{+, d}(i)}) \otimes \text{id}_W) \circ P_d & \text{if } p_i = p_{i+1} = 0, \\ \iota_d \circ (\text{id}_V \otimes r(T_{\tau_{-, d}(i)})) \circ P_d & \text{if } p_i = p_{i+1} = 1. \end{cases} \]
Figure 2. Dynkin diagrams of the Lie superalgebra $A(1,1)$

Figure 3. Braid relation
Proof. This can be checked directly. Refer to Figure 3. We explain by using an example. Denote \((\boxtimes^{A(m,n)} r)(T'_{i',d'})\) by \(S'_{i',d'}\) for any \(d'\) and \(i'\). Let \(d = (p_1, \ldots, p_{m+n+2}) \in D_{m+1|n+1}\) and assume \(p_i = p_{i+1} = 0\) and \(p_{i+2} = 1\). Let \(d_1 := i \triangleright d\), \(d_2 := (i + 1) \triangleright d_1\), \(d_3 := i \triangleright d_2\), \(d_4 := (i + 1) \triangleright d\), \(d_5 := i \triangleright d_4\) and \(d_6 := (i + 1) \triangleright d_4\). Then

\[
\begin{align*}
d &= d_1 = (p_1, \ldots, p_{i-1}, 0, 0, 1, p_{i+2}, \ldots, p_{m+n+2}), \\
d_2 &= d_4 = (p_1, \ldots, p_{i-1}, 0, 1, 0, p_{i+2}, \ldots, p_{m+n+2}), \\
d_3 &= d_5 = d_6 = (p_1, \ldots, p_{i-1}, 1, 0, 0, p_{i+2}, \ldots, p_{m+n+2}).
\end{align*}
\]

Note that \(\tau_{+.d_5} = \sigma_i \sigma_{i+1} \tau_{+.d}\). Hence \(\tau_{+.d_5}^{-1}(i + 1) = \tau_{+.d}^{-1}(i)\). Then we have \(S_{i,d} = \iota_d \circ (1(T_{\tau_{+.d}^{-1}(i)}(i)) \otimes \text{id}_W) \circ P_d\), \(S_{i+1,d_1} = \iota_{d_2} \circ P_d\), \(S_{i,d_2} = \iota_{d_3} \circ P_{d_2}\), \(S_{i+1,d} = \iota_{d_2} \circ P_{d}\) and \(S_{i,d_4} = \iota_{d_3} \circ P_{d_2}\). Hence we have \(S_{i,d_2} S_{i+1,d_1} S_{i,d} = \iota_{d_3} \circ (1(T_{\tau_{+.d}^{-1}(i)}(i)) \otimes \text{id}_W) \circ P_d\), as desired. \(\square\)

For \(\lambda \in \Lambda_{H_q(S_{m+1})}\) and \(\mu \in \Lambda_{H_q(S_{n+1})}\), we denote \(\rho_{\lambda} \otimes^{A(m,n)} \rho_{\mu}\) by \(\rho_{q; \lambda, \mu}^{A(m,n)}\) and we denote \(C_{V \otimes W}, P_{d}, \iota_{d}\) for \(V = V_{H_q(S_{m+1}), \lambda}\) and \(W = V_{H_q(S_{n+1}), \mu}\) by \(C_{q; \lambda, \mu}^{A(m,n)}, P_{d}^{\lambda, \mu}, \iota_{d}^{\lambda, \mu}\) respectively.

**Theorem 2.5.** Let \(q \in \mathbb{C}\) and assume that

\[
q P_{S_{m+1}}(q) P_{S_{n+1}}(q) \neq 0.
\]

Then the \(\mathbb{C}\)-algebra homomorphism

\[
\bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} \rho_{q; \lambda, \mu}^{A(m,n)} : H_q(A(m, n)) \rightarrow \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}} \text{End}_{\mathbb{C}}(C_{q; \lambda, \mu}^{A(m,n)})
\]

is an isomorphism. Further we have

\[
\dim H_q(A(m, n)) = \frac{(m + n + 2)!}{(m + 1)!(n + 1)!}.
\]

Moreover \(H_q(A(m, n))\) is a semisimple \(\mathbb{C}\)-algebra and a complete set of non-equivalent irreducible representations of \(H_q(A(m, n))\) is given by \(\{\rho_{q; \lambda, \mu}^{A(m,n)}|(\lambda, \mu) \in \Lambda_{H_q(S_{m+1})} \times \Lambda_{H_q(S_{n+1})}\}\).

**Proof.** Define the \(\mathbb{C}\)-algebra homomorphism

\[
f_1 : H_q(S_{m+1}) \otimes H_q(S_{n+1}) \rightarrow H_q(A(m, n))
\]
by $f_1(T_i \otimes 1) = T_{i,e}$ and $f_1(1 \otimes T_j) = T_{m+1+j,e}$. Let

$$R_{\lambda,\mu} := (t_{d_{\lambda}}^{\lambda,\mu} \circ P_{d_{\lambda}}^{\lambda,\mu}) \text{End}_C(C_{q;\lambda,\mu}^{A(m,n)}))(t_{d_{\mu}}^{\lambda,\mu} \circ P_{d_{\mu}}^{\lambda,\mu}).$$

Let $f_2$ denote the homomorphism of (2.41). It follows from (2.40) that $H_q(S_{m+1}) \otimes H_q(S_{n+1})$ is a semisimple $\mathbb{C}$-algebra. This implies

$$\text{Im}(f_2 \circ f_1) = \bigoplus_{(\lambda,\mu) \in \Lambda_H(S_{m+1}) \times \Lambda_H(S_{n+1})} R_{\lambda,\mu}.$$ 

On the other hand, we have

$$\text{End}_C(C_{q;\lambda,\mu}^{A(m,n)}) = \bigoplus_{d_1, d_2 \in D_{m+1|n+1}} (t_{d_1}^{\lambda,\mu} \circ P_{d_2}^{\lambda,\mu}) R_{\lambda,\mu} (t_{d_{\lambda}}^{\lambda,\mu} \circ P_{d_2}^{\lambda,\mu}).$$

Hence by (2.36) we can easily see that $f_2$ is surjective. In particular, we have

$$\dim H_q(A(m,n)) \geq \sum_{(\lambda,\mu) \in \Lambda_H(S_{m+1}) \times \Lambda_H(S_{n+1})} |D_{m+1|n+1}|^2 \dim R_{\lambda,\mu}$$

$$= |D_{m+1|n+1}|^2 \sum_{(\lambda,\mu) \in \Lambda_H(S_{m+1}) \times \Lambda_H(S_{n+1})} \dim R_{\lambda,\mu}$$

$$= \left(\frac{(m+n+2)!}{(m+1)!(n+1)!}\right)^2 \sum_{(\lambda,\mu) \in \Lambda_H(S_{m+1}) \times \Lambda_H(S_{n+1})} (\dim V_{H_q(S_{m+1}),\lambda})^2 (\dim V_{H_q(S_{n+1}),\mu})^2$$

$$= \left(\frac{(m+n+2)!}{(m+1)!(n+1)!}\right)^2 \dim H_q(S_{m+1}) \dim H_q(S_{n+1})$$

$$= \left(\frac{(m+n+2)!}{(m+1)!(n+1)!}\right)^2 \dim H_q(S_{m+1}) \dim H_q(S_{n+1})$$

Hence by (2.35), we have (2.42). Hence $f_2$ is an isomorphism. Then the rest of the statement follows from well-known facts concerning semisimple algebras (cf. [CR, (25.22) and (27.4)]).  

\section{Iwahori-Hecke type algebra associated with the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$}

Let $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. Let $\ell := m + n$. For $1 \leq i \leq \ell$, define $\hat{\sigma}_i \in S_\ell$ by $\hat{\sigma}_i := \sigma_i$ ($1 \leq i \leq \ell - 1$) and $\hat{\sigma}_\ell := \text{id}$. 

Let $W$ be the Coxeter groupoid associated with $(R, N, A, \triangleright) \in \mathcal{R}$ such that $N = \{1, 2, \ldots, \ell\}$, $A = D_{m|n}$, $i \triangleright d = \hat{\sigma}_i \triangleright d$, $V_0 = V_0^{(m+n)}$, $R^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i - \varepsilon_j|1 \leq i < j \leq m+n\}$.
Figure 4. Dynkin diagrams of the Lie superalgebra $B(1,2)$

\[
\begin{align*}
  d = (0,0,1) & \quad \begin{array}{c}
  \circ - \bigcirc - \bigcirc \\
  2
  \end{array} \\
  d = (0,1,0) & \quad \begin{array}{c}
  \bigcirc - \bigcirc = \bullet \\
  1
  \end{array} \\
  d = (1,0,0) & \quad \begin{array}{c}
  \bigcirc - \bigcirc - \bullet \\
  \end{array}
\end{align*}
\]

\[
\ell \cup \{\epsilon_i | 1 \leq i \leq \ell\}, \alpha_{i,d} = \epsilon_i - \epsilon_{i+1} (1 \leq i \leq \ell - 1), \alpha_{\ell,d} = \epsilon_{\ell}
\]

Denote $H_q(W)$ by $H_q(B(m, n))$. Then $H_q(B(m, n))$ is the $\mathbb{C}$-algebra (with 1) generated by

\[
\{E_d | d \in \mathcal{D}_{m|n}\} \cup \{T_{i,d} | 1 \leq i \leq \ell, d \in \mathcal{D}_{m|n}\}
\]

and defined by the relations (2.6)-(2.11) and the relations

\[
T_{\ell-1,\sigma_{\ell-1}vd}T_{\ell,\sigma_{\ell-1}vd}T_{\ell-1,d} = T_{\ell,d}T_{\ell-1,\sigma_{\ell-1}vd}T_{\ell,\sigma_{\ell-1}vd}T_{\ell-1,d}
\]

if $1 \leq i \leq \ell - 1$,

\[
T_{i,\sigma_{i+1}vd}T_{i+1,\sigma_{i+1}vd}T_{i,d} = T_{i+1,\sigma_{i+1}vd}T_{i,\sigma_{i+1}vd}T_{i+1,d}
\]

if $|i - j| \geq 2$.

Recall \( \rho \) and \( d_e \in \mathcal{D}_{m|n} \) from Theorem 1.5 and (2.31) respectively. Then

\[
P_{d_e, \rho}(e_{d_e}W e_{d_e})_{t_{d_e}} \subset \{(\sum_{i=1}^{m+n} z_i E_{\sigma(i)i}) | \sigma \in S_{m+n}, z_i \in \{-1, 1\}, \sigma(\{1, \ldots, m\}) = \{1, \ldots, m\}\}.
\]

Hence $|e_{d_e}W e_{d_e}| \leq 2^{m+n}m!n!$ by Theorem 1.5, so $|W \setminus \{0\}| = |\mathcal{D}_{m|n}|e_{d_e}W e_{d_e}| \leq \frac{2^{m+n}((m+n)!)^2}{m!n!}$. Hence by (2.16), we conclude

\[
\dim H_q(B(m, n)) \leq \frac{2^{m+n}((m+n)!)^2}{m!n!}.
\]

**Proposition 2.6.** Let $V_1$ and $V_\tau$ be finite dimensional $\mathbb{C}$-linear spaces, and let

\[
1 : H_q(W(B_m)) \to \text{End}_\mathbb{C}(V_1) \quad \text{and} \quad \tau : H_q(W(B_n)) \to \text{End}_\mathbb{C}(V_\tau)
\]

be $\mathbb{C}$-algebra homomorphisms, i.e., representations. Let $1 \otimes \tau : H_q(W(B_m)) \otimes H_q(W(B_n)) \to \text{End}_\mathbb{C}(V_1 \otimes V_\tau)$ denote the tensor representation of $1$ and $\tau$ in the ordinary sense. Let $C_{V_1 \otimes V_\tau; d}$ be copies
of the \( \mathbb{C} \)-linear space \( V_i \otimes V_r \), indexed by \( d \in D_{m|n} \). Let \( C_{V_i \otimes V_r} := \oplus_{d \in D_{m|n}} C_{V_i \otimes V_r; d} \). Let \( P_d : C_{V_i \otimes V_r} \to C_{V_i \otimes W; d} \) and \( \iota_d : C_{V_i \otimes W; d} \to C_{V_i \otimes V_r} \) denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique \( \mathbb{C} \)-algebra homomorphism \( \mathbb{I} \otimes \mathbb{r} = \mathbb{I} \otimes^{B(m,n)} \mathbb{r} : H_q(B(m,n)) \to \text{End}_\mathbb{C}(C_{V_i \otimes V_r}) \) satisfying the following conditions:

(i) For each \( d \in D_{m|n} \), one has \( (\mathbb{I} \otimes \mathbb{r})(E_d) = \iota_d \circ P_d \),

(ii) For each \( i \in \{1, \ldots, \ell = m+n \} \) and each \( d = (p_1, \ldots, p_\ell) \in D_{m|n} \), one has

\[
(\mathbb{I} \otimes \mathbb{r})(T_{i,d}) = \begin{cases} 
P_{d,\phi,d} \circ \iota_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i \neq p_{i+1}, \\
\iota_d \circ (1(T_{r+1,d}^{-1}(q)) \otimes \text{id}_{V_r}) \circ P_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\
\iota_d \circ (\text{id}_{V_1} \otimes (1(T_{r-1,d}^{-1}(q))) \otimes \text{id}_{V_r}) \circ P_d & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 1, \\
\iota_d \circ (\iota_d \circ \text{id}_{V_1} \otimes (T_{r,d}(q)) \otimes \text{id}_{V_r}) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 0, \\
\iota_d \circ (\iota_d \circ \text{id}_{V_1} \otimes \text{id}_{V_r}) \circ P_d & \text{if } i = \ell \text{ and } p_\ell = 1, 
\end{cases}
\]

where \( \tau_{\pm,d} \) are the ones of (2.32).

Proof. We can check out this directly in a way similar to that for Proof of Proposition 2.4. \( \square \)

For \( \lambda \in \Lambda_{H_q(W(B_m))} \) and \( \mu \in \Lambda_{H_q(W(B_n))} \), we denote \( \rho_{H_q(W(B_m)), \lambda} \otimes^{B(m,n)} \rho_{W(B_n)), \mu} \) by \( \rho_{B(m,n)}^{q; \lambda, \mu} \) and we denote \( C_{V \otimes W} \) for \( V = V_{H_q(W(B_m)), \lambda} \) and \( W = V_{H_q(W(B_n)), \mu} \) by

\[
C_{q; \lambda, \mu}^{B(m,n)}.
\]

**Theorem 2.7.** Let \( q \in \mathbb{C} \) and assume that

\[
qP_{W(B_m)}(q)P_{W(B_n)}(q) \neq 0.
\]

Then the \( \mathbb{C} \)-algebra homomorphism

\[
\bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \rho_{B(m,n)}^{q; \lambda, \mu} : H_q(B(m,n)) \to \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}} \text{End}_\mathbb{C}(C_{q; \lambda, \mu}^{B(m,n)})
\]

is an isomorphism. Further we have

\[
\dim H_q(B(m,n)) = \frac{2^{m+n}((m+n)!)^2}{m!n!}.
\]

Moreover \( H_q(B(m,n)) \) is a semisimple \( \mathbb{C} \)-algebra and a complete set of non-equivalent irreducible representations of \( H_q(B(m,n)) \) is given by \( \{\rho_{B(m,n)}^{q; \lambda, \mu} | (\lambda, \mu) \in \Lambda_{H_q(W(B_m))} \times \Lambda_{H_q(W(B_n))}\} \).
Proof. Let $1 : H_q(W(B_m)) \rightarrow \End_{\mathbb{C}}(V_1)$ and $r : H_q(W(B_n)) \rightarrow \End_{\mathbb{C}}(V_r)$ be irreducible representations. Further, let $1 \boxtimes r : H_q(B(m,n)) \rightarrow \End_{\mathbb{C}}(V_{1 \boxtimes V_r})$ be the representative introduced in Proposition 2.6 for these $1$ and $r$. By (2.48), we can easily see that

$$\forall d', \forall d'' \in \mathcal{D}_{m|n}, \quad P_{d'} \circ \iota_{d''} \in \text{Im}(1 \boxtimes r).$$

Define the representation $f_1 : H_q(W(B_m)) \otimes H_q(W(B_n)) \rightarrow \End_{\mathbb{C}}(C_{V_1 \otimes V_r})$ by $f_1(T_i \otimes 1) = (P_{d_o} \circ \iota_{d_e})((1 \otimes r)(T_{n+i,d_o}))(P_{d_e} \circ \iota_{d_o})$ and $f_1(1 \otimes T_j) = (1 \otimes r)(T_{m+j,d_e})$. The condition (2.49) implies that $f_1$ is an irreducible representation of $H_q(W(B_m)) \otimes H_q(W(B_n))$. Moreover, using (2.52), we can easily see that $1 \boxtimes r$ is an irreducible representation of $H_q(B(m,n))$.

By the above argument, together with (2.47), in the same way as that for Proof of Theorem 2.5, we can complete the proof of this theorem. \(\square\)

§ 2.7. Iwahori-Hecke type algebra associated with the Lie superalgebra $osp(2m|2n)$

Let $m, n \in \mathbb{N}$. Define the set $\mathcal{D}_{m|n}^{CD}$ by

$$\mathcal{D}_{m|n}^{CD} := \{d^D | d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n}, p_{m+n} = 0\} \cup \{d_C^+, d_C^- | d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n}, p_{m+n} = 1\},$$

so that

$$|\mathcal{D}_{m|n}^{CD}| = \frac{(m+n-1)!}{(m-1)!n!} + 2\frac{(m+n-1)!}{m!(n-1)!} = \frac{(m+n-1)!(m+2n)}{m!n!}.$$  

Let $\ell := m+n$ and $N = \{1, \ldots, \ell\}$. Define the action $\triangleright$ of $F_2(N)$ on $\mathcal{D}_{m|n}^{CD}$ by

$$i \triangleright a = \begin{cases} 
(\sigma_i \triangleright d)^D & \text{if } a = d^D, 1 \leq i \leq \ell - 2 \text{ and } p_i \neq p_{i+1}, \\
(\sigma_i \triangleright d)_C^+ & \text{if } a = d^D, i = \ell - 1 \text{ and } p_i \neq p_{i+1}, \\
(\sigma_i \triangleright d)_C^- & \text{if } a = d^D, i = \ell \text{ and } p_{i-1} \neq p_i, \\
(\sigma_i \triangleright d)_C^+ & \text{if } a = d_C^+, 1 \leq i \leq \ell - 2 \text{ and } p_i \neq p_{i+1}, \\
(\sigma_i \triangleright d)_C^- & \text{if } a = d_C^-, i = \ell - 1 \text{ and } p_i \neq p_{i+1}, \\
(\sigma_i \triangleright d)^D & \text{if } a = d_C^-, i = \ell \text{ and } p_{i-1} \neq p_i, \\
(\sigma_i \triangleright d)^D & \text{otherwise.}
\end{cases}$$

Now we define $R = (R, N, \mathcal{D}_{m|n}^{CD}, \triangleright) \in \mathcal{R}$ as follows. Let $N$ be as above. Let $A = \mathcal{D}_{m|n}^{CD}$. Let $V_0 = V_0^{(\ell)}$. Let $a = d^D, d_C^+$ or $d_C^- \in \mathcal{D}_{m|n}^{CD}$ with $d = (p_1, \ldots, p_{m+n}) \in \mathcal{D}_{m|n}$. Let
$R^+_a$ be the subset of $V_0$ formed by the elements $\varepsilon_i - \varepsilon_j$, $\varepsilon_i + \varepsilon_j$, $(1 \leq i < j \leq \ell)$ and $(\pm 1)^{k\ell}2\varepsilon_k$ $(1 \leq k \leq \ell$ and $p_k = 1$, and $\pm$ is the one of $d^C_\pm$). Define

$$\alpha_{i,a} := \begin{cases} 
\varepsilon_i - \varepsilon_{i+1} & \text{if } a = d^D \text{ or } d^C_+ \text{ and } 1 \leq i \leq \ell - 1, \\
\varepsilon_i - \varepsilon_{i+1} & \text{if } a = d^C_- \text{ and } 1 \leq i \leq \ell - 2, \\
\varepsilon_{\ell-1} + \varepsilon_\ell & \text{if } a = d^D \text{ and } i = \ell, \\
2\varepsilon_\ell & \text{if } a = d^C_+ \text{ and } i = \ell, \\
-2\varepsilon_\ell & \text{if } a = d^C_- \text{ and } i = \ell - 1, \\
\varepsilon_{\ell-1} + \varepsilon_\ell & \text{if } a = d^C_- \text{ and } i = \ell.
\end{cases}
$$

Define $\sigma_{i,a} := \overline{\sigma}_{\alpha_{i,a}}$. Let $W$ be the Coxeter groupoid associated with $R$. Recall $\rho$ and $d_e \in \mathcal{D}_{m|n}$ from Theorem 1.5 and (2.31) respectively. It is easy to show that

$$\rho(e_{(d_e)^D}W_e_{(d_e)^D}) = \{ \sum_{j=1}^{\ell} z_j E_{\sigma(j)j} | \sigma \in S_\ell, z_j \in \{-1, 1\}, \prod_{j=n+1}^{\ell} z_j = 1, \sigma(\{1, \ldots, n\}) = \{1, \ldots, n\} \},$$

so $|e_{(d_e)^D}W_e_{(d_e)^D}| \leq m!n!2^{\ell-1}$ by Theorem 1.5. Hence $|W \setminus \{0\}| \leq |\mathcal{D}^{CD}_{m|n}|^2m!n!2^{\ell-1}$.

Denote $H_q(W)$ by $H_q(\mathfrak{osp}(2m|2n))$. By (2.16) and (2.54), we have

$$\dim H_q(\mathfrak{osp}(2m|2n)) \leq \frac{2^{m+n-1}((m+n-1)!(m+2n))^2}{m!n!}.$$

Recall that $H_q(\mathfrak{osp}(2m|2n))$ is the $\mathbb{C}$-algebra (with 1) generated by

$$(2.58) \quad \{ E_a | a \in \mathcal{D}^{CD}_{m|n} \} \cup \{ T_{i,a} | 1 \leq i \leq m+n, a \in \mathcal{D}^{CD}_{m|n} \}$$

and defined by the relations (2.6)-(2.11) and the relations

$$\text{(2.59)} \quad (T_{i,a}T_{j,a})^2 = (T_{j,a}T_{i,a})^2 \text{ if } a = d^C_\pm, p_{\ell-1} = p_\ell \text{ and } i = \ell - 1, j = \ell,$$

$$\text{(2.60)} \quad T_{i,a}T_{j,a} = T_{j,a}T_{i,a} \text{ if } a = d^D, p_{\ell-1} = p_\ell \text{ and } i = \ell - 1, j = \ell,$$

$$\text{(2.61)} \quad T_{i,j}d_{\ell}T_{j,i}d_{\ell} = T_{j,i}d_{\ell}T_{i,j}d_{\ell} \text{ if } p_{\ell-1} \neq p_\ell \text{ and } i = \ell - 1, j = \ell,$$

$$\text{(2.62)} \quad T_{i,j}d_{\ell}T_{j,i}d_{\ell} = T_{j,i}d_{\ell}T_{i,j}d_{\ell} \text{ if } 1 \leq i \leq \ell - 3, j = i + 1,$$

$$\text{(2.63)} \quad T_{i,j}d_{\ell}T_{j,i}d_{\ell} = T_{j,i}d_{\ell}T_{i,j}d_{\ell} \text{ if } a = d^C_+ \text{ and } i = \ell - 2, j = \ell - 1,$$

$$\text{(2.64)} \quad T_{i,j}d_{\ell}T_{j,i}d_{\ell} = T_{j,i}d_{\ell}T_{i,j}d_{\ell} \text{ if } a = d^C_- \text{ and } i = \ell - 2, j = \ell,$$

$$\text{(2.65)} \quad T_{i,j}d_{\ell}T_{j,i}d_{\ell} = T_{j,i}d_{\ell}T_{i,j}d_{\ell} \text{ if } a = d^D \text{ and } i = \ell - 2, j \in \{\ell - 1, \ell\},$$

$$\text{(2.66)} \quad T_{j,i}d_{\ell}T_{i}d_{\ell} = T_{i,j}d_{\ell}T_{j}d_{\ell} \text{ if } i < j, \text{ and } i, j \text{ are not the ones in (2.64)-(2.65).}$$

Recall that $W(C_k) = W(B_k)$ and $H_q(W(C_k)) = H_q(W(B_k))$.
where 1) $(1, 0, 0, 0)^D$, 2) $(0, 1, 0, 0)^D$, 3) $(0, 0, 1, 0)^D$, 4) $(0, 0, 0, 1)^C$, 5) $(0, 0, 0, 1)^C$

Figure 5. Dynkin diagrams of the Lie superalgebra $D(3, 1)$
**Proposition 2.8.** Let $V_1$ and $V_r$ be finite dimensional $\mathbb{C}$-linear spaces, and let $1 : H_q(W(D_m)) \to \text{End}_{\mathbb{C}}(V_1)$ and $r : H_q(W(C_n)) \to \text{End}_{\mathbb{C}}(V_r)$ be $\mathbb{C}$-algebra homomorphisms, i.e., representations. Let $1 \otimes r : H_q(W(D_m)) \otimes H_q(W(C_n)) \to \text{End}_{\mathbb{C}}(V_1 \otimes V_r)$ denote the tensor representation of $1$ and $r$ in the ordinary sense. Let $C_{V_1 \otimes V_r; a}$ be copies of the $\mathbb{C}$-linear space $V_1 \otimes V_r$, indexed by $a \in D_{m|n}^{CD}$. Let $C_{V_1 \otimes V_r} := \bigoplus_{d \in D_{m|n}^{CD}} C_{V_1 \otimes V_r; d}$. Let $P_a : C_{V_1 \otimes V_r} \to C_{V_1 \otimes V_r; a}$ and $\iota_a : C_{V_1 \otimes V_r; a} \to C_{V_1 \otimes V_r}$ denote the canonical projection and the canonical inclusion map respectively. Then there exists a unique $\mathbb{C}$-algebra homomorphism $1 \otimes r = 1 \otimes^{CD} r : H_q(\mathfrak{osp}(2m|2n)) \to \text{End}_{\mathbb{C}}(C_{V_1 \otimes V_r})$ satisfying the following conditions:

(i) For each $a \in D_{m|n}^{CD}$, one has $(1 \otimes r)(E_a) = \iota_a \circ P_a$.

(ii) For each $i \in \{1, \ldots, \ell = m+n\}$ and each $a \in D_{m|n}^{CD}$ with $d = (p_1, \ldots, p_\ell) \in D_{m|n}$ such that $a = d^+_{\ell+1}$, $d^-_{\ell+1}$ or $d^D_{\ell+1}$, one has

$$(2.67) (1 \otimes r)(T_{i,a}) = \begin{cases} P_{\beta a} \circ \iota_a & \text{if } 1 \leq i \leq \ell \text{ and } i \triangleright a \neq a, \\
\iota_a \circ (l(T_{r+, a}(i)) \otimes \text{id}_{V_1}) \circ P_a & \text{if } 1 \leq i \leq \ell - 1 \text{ and } p_i = p_{i+1} = 0, \\
\iota_a \circ (l(T_{\ell}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell+1} = 0, \\
\iota_a \circ (l(T_{r+, a}(i-1)) \otimes \text{id}_{V_1}) \circ P_a & \text{if } 1 \leq i \leq \ell - 2 \text{ and } p_i = p_{i+1} = 1, \\
\iota_a \circ (l(T_{r-1}) \otimes \text{id}_{V_1} \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_{\ell-1} = p_\ell = 1, a = d^{+}_{\ell+1}, \\
\iota_a \circ (l(T_{\ell}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell - 1 \text{ and } p_{\ell} = 1, a = d^{-}_{\ell+1}, \\
\iota_a \circ (l(T_{r}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell \text{ and } p_\ell = 1, a = d^{C}_{\ell+1}, \\
\iota_a \circ (l(T_{r-1}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell-1} = p_\ell = 1, a = d^{D}_{\ell+1}, \\
\iota_a \circ (l(T_{\ell}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell \text{ and } p_{\ell} = 1, a = d^{C}_{\ell+1}, \\
\iota_a \circ (l(T_{r}) \otimes \text{id}_{V_r}) \circ P_a & \text{if } i = \ell \text{ and } p_\ell = 1, a = d^{D}_{\ell+1}, \\ 
\end{cases}$$

where $r_{\pm, a} \in S_{m+n}$ are the ones of $(2.32)$.

**Proof.** We can check out this directly in a way similar to that for Proof of Proposition 2.4. \qed

For $\lambda \in \Lambda_H(W(D_m))$ and $\mu \in \Lambda_H(W(C_n))$, we denote $\rho_{\lambda, \mu} : H_q(W(D_m)) \otimes H_q(W(C_n)) \to \text{End}_{\mathbb{C}}(V_{\lambda} \otimes W_{\mu})$ by $\rho^{CD}_{\lambda, \mu}$ and we denote $C_{V \otimes W}$ for $V = V_{\lambda}$ and $W = V_{\mu}$ by $C_{\lambda, \mu}^{CD}$.  

**Theorem 2.9.** Let $q \in \mathbb{C}$ and assume that

$$(2.68) q P_{W(D_m)}(q) P_{W(C_n)}(q) \neq 0.$$
Then the \( \mathbb{C} \)-algebra homomorphism
\[
\bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))}} \rho_{q; \lambda, \mu}^{CD} : H_q(\mathfrak{osp}(2m|2n)) \rightarrow \bigoplus_{(\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))}} \text{End}_{\mathbb{C}}(C_{q; \lambda, \mu}^{CD})
\]
is an isomorphism. Further we have
\[
\dim H_q(\mathfrak{osp}(2m|2n)) = \frac{2^{m+n-1}((m+n-1)!)(m+2n)^2}{m!n!}.
\]
Moreover \( H_q(\mathfrak{osp}(2m|2n)) \) is a semisimple \( \mathbb{C} \)-algebra and a complete set of non-equivalent irreducible representations of \( H_q(\mathfrak{osp}(2m|2n)) \) is given by \( \{ \rho_{q; \lambda, \mu}^{CD} | (\lambda, \mu) \in \Lambda_{H_q(W(D_m))} \times \Lambda_{H_q(W(C_n))} \} \).

**Proof.** Note that \( W(D_m) \times W(C_n) = 2^{m+n-1}m!n! \). Then we can prove this theorem in the same way as that for Proof of Theorem 2.7.

**Remark 1.** Now, by (2.3), (2.4) and Theorems 2.5, 2.7 and 2.9, it has turned out that if \( q \) is non-zero and not any primitive root of unity, then as a \( \mathbb{C} \)-algebra, \( H_q(\mathfrak{g}) = H_q(W) \) introduced in this section for the Lie superalgebra \( \mathfrak{g} = A(m, n) \) or \( \mathfrak{osp}(m|2n) \) is very similar to the Iwahori-Hecke algebra \( H_q(W_0) \) associated with the Weyl group \( W_0 \) of the Lie algebra \( \mathfrak{g}(0) \) given as the even part of \( \mathfrak{g} \), that is, Morita equivalence.

**Remark 2.** Assume \( q \) to be an element of \( \mathbb{C} \) transcendental over \( \mathbb{Q} \). Then the \( \mathbb{Z} \)-subalgebra (with identity) of \( \mathbb{C} \) generated by \( q \) can also be regarded as the polynomial ring \( \mathbb{Z}[q] \) in the variable \( q \) over \( \mathbb{Z} \). Let \( W \) be one of the Coxeter groupoids treated in Subsections 2.5, 2.6 and 2.7. By Lemma 2.2 and (2.42), (2.51), (2.70), one see that \( \{ f(w) | w \in W \setminus \{0\} \} \) is a \( \mathbb{C} \)-basis of \( H_q(W) \), that is, \( H_q(W) = \bigoplus_{w \in W \setminus \{0\}} \mathbb{C}f(w) \). Define \( H_{\mathbb{Z}[q], q}(W) \) to be the \( \mathbb{Z}[q] \)-submodule generated by \( \{ f(w) | w \in W \setminus \{0\} \} \). Using Theorem 1.7 and Corollary 1.8, one see that \( H_{\mathbb{Z}[q], q}(W) \) is also a \( \mathbb{Z}[q] \)-subalgebra of \( H_q(W) \). Let \( A \) be any commutative ring (with identity). Let \( \zeta \) be any element of \( A \). Regard \( A \) as a \( \mathbb{Z}[q] \)-algebra via the \( \mathbb{Z} \)-algebra homomorphism \( \mathbb{Z}[q] \rightarrow A \) that takes \( q \) to \( \zeta \). Let \( H_{A, \zeta}(W) \) be the \( A \)-algebra (with identity) defined by \( H_{A, \zeta}(W) := H_{\mathbb{Z}[q], q}(W) \otimes_{\mathbb{Z}[q]} A \). For \( X \in H_{\mathbb{Z}[q], q}(W) \), we also denote the element \( X \otimes 1 \) of \( H_{A, \zeta}(W) \) by \( X \). Then \( H_{A, \zeta}(W) \) is a free \( A \)-module with an \( A \)-basis \( \{ f(w) | w \in W \setminus \{0\} \} \), that is,
\[
\text{rank}_A H_{A, \zeta}(W) = |W| - 1.
\]
Using Theorem 1.7 and Corollary 1.8 again, one see that \( H_{A, \zeta}(W) \) can also be defined by the same generators as (2.5) and the same relations as (2.6)-(2.13) with \( \zeta \) in place of \( q \).
The same properties as above seem to be true for many Coxeter groupoids, which might be able to be proved in a way similar to that of the proof of [L, Proposition 3.3].

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