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Infinite Pre-dominant Integral Weights for Affine Types

By

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Abstract

We study infinite pre-dominant integral weights for affine Kac-Moody Lie algebras. Furthermore, we determine the W-orbit of an infinite pre-dominant integral weight.

§ 1. Introduction

The purpose of the present paper is to study *infinite pre-dominant* integral weights for affine Kac-Moody Lie algebras. The notion of *pre-dominant* integral weights is introduced in [4]. Our motivation of this study is the following:

Let λ be a partition of d, and Y_λ the Young (or Ferrers) diagram of shape λ. In [6], R. P. Stanley studied the generating fuction $U(Y_\lambda; q)$ for the reverse plane partitions of shape Y_λ, and proved the q-hook formula:

$$U(Y_\lambda; q) = \prod_{v \in Y_\lambda} \frac{1}{1 - q^{h_v}}, \tag{1.1}$$

where h_v denotes the hook-length at $v \in Y$. In a forthcoming paper [5], we have succeeded in proving a generalization of (1.1) by using a root system for a Kac-Moody Lie algebra. Let λ be a *pre-dominant* integral weight, and $\mathrm{D}(\lambda)^\vee$ be the *shape* of λ (a certain subset of positive real coroots). These are defined in section 3. A map $\sigma : \mathrm{D}(\lambda)^\vee \longrightarrow \mathbb{Z}_{\geq 0}$ is called a $\mathrm{D}(\lambda)^\vee$-partition [5] if

1. If $\beta^\vee \lhd \gamma^\vee$ ($\beta^\vee, \gamma^\vee \in \mathrm{D}(\lambda)^\vee$), then $\sigma(\beta^\vee) \geq \sigma(\gamma^\vee)$,

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2. There exists at most finitely many $\beta^\vee \in \mathrm{D}(\lambda)^\vee$ such that $\sigma(\beta^\vee) \neq 0$,

where $\beta^\vee \lhd \gamma^\vee$ means $\beta^\vee < \gamma^\vee$ and $\langle \gamma, \beta^\vee \rangle \geq 1$. The generating function $U(\mathrm{D}(\lambda)^\vee; q)$ for $\mathrm{D}(\lambda)^\vee$-partitions is defined by

$$U(\mathrm{D}(\lambda)^\vee; q) := \sum_{\sigma:\mathrm{D}(\lambda)^\vee-\text{partition}} q^{\sum_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \sigma(\beta^\vee)}.$$

In [5], the following statement is shown:

Theorem 1.1 (q-hook formula [5]). *Let λ be a pre-dominant integral weight satisfying $\#\mathrm{D}(\lambda)^\vee < \infty$. Then we have:*

$$U(\mathrm{D}(\lambda)^\vee; q) = \prod_{\beta^\vee \in \mathrm{D}(\lambda)^\vee} \frac{1}{1 - q^{\mathrm{ht}(\beta)}}, \tag{1.2}$$

where $\mathrm{ht}\,(\beta)$ *denotes the height of* β*.*

We stress that *almost all* part of the proof [5] of the q-hook formula (1.2) is true even if $\#\mathrm{D}(\lambda)^\vee = \infty$. (The condition "$\#\mathrm{D}(\lambda)^\vee < \infty$" is necessary only for certain inductive argument.) The author thinks that, by a suitable modification of formulation of (1.2), the q-hook formula should hold also for *infinite* pre-dominant integral weights.

In the present paper, we study several properties of *infinite* pre-dominant integral weights for affine Kac-Moody Lie algebras. Let $P^{\inf}_{\geq -1}(I)$ denote the set of infinite pre-dominant integral weights. Here, I denotes the index set of simple roots as in TABLE 2. Then, as is shown in Proposition 5.8, we have:

$$P_0(I) \sqcup P^{\inf}_{\geq -1}(I) = P_{\mathrm{sig}}(I), \tag{1.3}$$

where the sets $P_0(I)$ and $P_{\mathrm{sig}}(I)$ are defined in Section 3. An element of $P_{\mathrm{sig}}(I)$ is called a *signature* integral weight. The equation (1.3) shows that classification of infinite pre-dominant integral weights reduces to that of signature integral weights. It is easy to see the set $P_{\mathrm{sig}}(I)$ is closed under the Weyl group action.

In Section 5, we study the set $P_{\mathrm{sig}}(I)$ for a Cartan matrix of affine type and, our main result (Theorem 5.15) gives the classification of signature integral weights in the form of $W(I)$-orbit decomposition of $P_{\mathrm{sig}}(I)$:

$$P_{\mathrm{sig}}(I) = \bigsqcup_{i \in (I)_1} W(I) \cdot P(i, *, I),$$

where $W(I)$ denotes the Weyl group, $*$ is a distiguished index in I, $(I)_1$ is a certain subset of I, $P(i, *, I)$ is a set of integral weights with a certain condition. See Section 5 for unexplained notion and further details.

In Section 4, we study the set $P_{\mathrm{sig}}(I)$ for a Cartan matrix of finite type. In Proposition 4.6, we determine the $W(I)$-orbit decomposition of $P_{\mathrm{sig}}(I)$:

$$P_{\mathrm{sig}}(I) = P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I) \cdot P(i, I).$$

This result is used in the proof of the main result. See Section 4, for the definition of $(I)_0$ and $P(i, I)$.

§ 2. Preliminaries

Let $A = (a_{i,j})_{i,j \in I}$ be a Cartan matrix of a Levi subalgebra $\mathfrak{g}_I$ of a Kac-Moody Lie algebra [2][3] defined over $\mathbb{R}$. We denote the Cartan subalgebra by $\mathfrak{h}$. Then, we have two sets of linearly independent elements $\Pi = \left\{\alpha_i \middle| i \in I\right\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \left\{\alpha_i^\vee \middle| i \in I\right\} \subset \mathfrak{h}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by $P(I)$. An integral weight $\lambda \in P(I)$ is said to be *dominant* if

$$\langle \lambda, \alpha_i^\vee \rangle \geq 0, \quad i \in I.$$

The set of dominant integral weights is denoted by $P_{\geq 0}(I)$. For each $i \in I$, we define $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The group $W = W(I)$ generated by $\left\{s_i \middle| i \in I\right\}$ is called the *Weyl group*, which acts on $\mathfrak{h}$ by:

$$\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle, \quad w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

We define the *real root system* (resp. *real coroot system*) by $\Phi := W\Pi$ (resp. $\Phi^\vee := W\Pi^\vee$) (or, expressing the index set explicitly, $\Phi(I)$ (resp. $\Phi^\vee(I)$)). We denote by Φ_+ and Φ_- the sets of positive and negative roots of Φ, respecively. The *dual* $\beta^\vee \in \Phi^\vee$ of a root $\beta \in \Phi$ is defined by the property

$$w(\beta^\vee) = w(\beta)^\vee, \quad w \in W.$$

For a subset S of Φ, we define a subset $S^\vee$ of $\Phi^\vee$ by:

$$S^\vee := \left\{\beta^\vee \in \Phi^\vee \middle| \beta \in S\right\}.$$

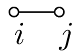
For each $\beta \in \Phi$, we define $s_\beta \in W$ by:

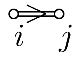
$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*.$$

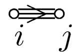
Then, s_β acts on $\mathfrak{h}$ by:

$$s_\beta(h) = h - \langle \beta,\, h \rangle \beta^\vee, \quad h \in \mathfrak{h}.$$

Let $\alpha_i, \alpha_j \in \Pi$.

If $\langle \alpha_i,\, \alpha_j^\vee \rangle = -1$, and $\langle \alpha_j,\, \alpha_i^\vee \rangle = -1$, then we denote 

If $\langle \alpha_i,\, \alpha_j^\vee \rangle = -2$, and $\langle \alpha_j,\, \alpha_i^\vee \rangle = -1$, then we denote 

If $\langle \alpha_i,\, \alpha_j^\vee \rangle = -3$, and $\langle \alpha_j,\, \alpha_i^\vee \rangle = -1$, then we denote 

§ 3. Pre-dominant Integral Weights and Signature Integral Weights

Definition 3.1. An integral weight λ is said to be *pre-dominant* if:

$$\langle \lambda, \beta^\vee \rangle \geq -1, \text{for each } \beta^\vee \in \Phi_+^\vee.$$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}(I)$.

Definition 3.2. For $\lambda \in P_{\geq -1}$, the set $\mathrm{D}(\lambda)^\vee$ defined by

$$\mathrm{D}(\lambda)^\vee := \left\{ \beta^\vee \in \Phi_+^\vee \,\middle|\, \langle \lambda, \beta^\vee \rangle = -1 \right\}$$

is called the *shape of* λ. We say that a pre-dominant integral weight λ is *finite* if $\#\mathrm{D}(\lambda)^\vee < \infty$, *infinite* otherwise. The set of finite (resp. infinite) pre-dominant integral weights is denoted by $P_{\geq -1}^{\mathrm{fin}}(I)$ (resp. $P_{\geq -1}^{\mathrm{inf}}(I)$).

Lemma 3.3 (See [4]Lemma 4.1)**.** *Let $\lambda \in P_{\geq -1}(I)$ and $\beta^\vee \in \mathrm{D}(\lambda)^\vee$. Then we have $s_\beta(\lambda) \in P_{\geq -1}(I)$.*

Definition 3.4. An integral weight λ is said to be a *signature* integral weight if:

$$\langle \lambda,\, \gamma^\vee \rangle \in \{1, 0, -1\}, \text{ for each } \gamma \in \Phi.$$

The set of signature integral weights is denoted by $P_{\mathrm{sig}}(I)$. We note $P_{\mathrm{sig}}(I) \subseteq P_{\geq -1}(I)$.

The following is clear.

Lemma 3.5. *Let $\lambda \in P_{\mathrm{sig}}(I)$ and $\beta \in \Phi(I)$. Then we have $s_\beta(\lambda) \in P_{\mathrm{sig}}(I)$.*

Since the Weyl group $W(I)$ is generated by $\Pi = \left\{ s_i \,\middle|\, i \in I \right\}$, we get:

Corollary 3.6. *The set $P_{\mathrm{sig}}(I)$ of signature integral weights is closed under the $W(I)$-action.*

Definition 3.7. We define a set $P_0(I)$ by

$$P_0(I) := \left\{ \lambda \in P \,\middle|\, \langle \lambda,\, \alpha_i^\vee \rangle = 0, \text{ for each } i \in I \right\}.$$

We note that $\mathrm{D}(\lambda)^\vee = \varnothing$ for $\lambda \in P_0(I)$ so that we have $P_0(I) \subseteq P_{\geq -1}^{\mathrm{fin}}(I)$ and $P_0(I) \cap P_{\geq -1}^{\mathrm{inf}}(I) = \varnothing$.

Proposition 3.8. *The set $P_0(I)$ is closed under the $W(I)$-action.*

Proof. Let $\lambda \in P_0(I)$, $w \in W(I)$, and $i \in I$. Then we have $\langle w(\lambda),\, \alpha_i^\vee \rangle = \langle \lambda,\, w^{-1}(\alpha_i)^\vee \rangle$. Since $w^{-1}(\alpha_i^\vee)$ is a linear combination of elements of $\Pi^\vee = \left\{ \alpha_j^\vee \,\middle|\, j \in I \right\}$, we have $\langle \lambda,\, w^{-1}(\alpha_i)^\vee \rangle = 0$. Hence, $w(\lambda) \in P_0(I)$. □

§ 4. Classification for Finite Types

Through this section, we suppose the Cartan matrix $A = (a_{i,j})_{i,j \in I}$ is of finite type. The index set I is given as in TABLE 1. The purpose of this section is to determine the $W(I)$-orbit decomposition of $P_{\mathrm{sig}}(I)$.

Definition 4.1. Let $\beta_{hs}^\vee \in \Phi^\vee$ denote the *highest coroot.* The integers $d_i^\vee$ ($i \in I$) are defined by $\beta_{hs}^\vee = \sum_{i \in I} d_i^\vee \alpha_i^\vee$.

Definition 4.2. Let $\beta_{hl}^\vee \in \Phi^\vee$ denote the *dual of the highest root.* The integers $c_i^\vee$ ($i \in I$) are defined by $\beta_{hl}^\vee = \sum_{i \in I} c_i^\vee \alpha_i^\vee$.

We note that if the Cartan matrix A is simply-laced, then we have $\beta_{hs}^\vee = \beta_{hl}^\vee$, and that if multiply-laced, then $\beta_{hs}^\vee > \beta_{hl}^\vee$.

When the Cartan matrix A is of type $B_l, C_l, F_4,$ or G_2, the coroot system $\Phi^\vee$ is decomposed as:

$$\Phi^\vee = \Phi_\ell^\vee \sqcup \Phi_s^\vee, \quad \text{(disjoint union)},$$

where Φ_ℓ is the set of long roots and Φ_s is the set of short roots. We note that we have $\beta_{hl}^\vee \in \Phi_\ell^\vee$ and $\beta_{hs}^\vee \in \Phi_s^\vee$, and that $\Phi_\ell^\vee$ and $\Phi_s^\vee$ are the dual of Φ_ℓ and Φ_s, respectively.

Proposition 4.3. *Let $\lambda \in P(I)$. Then there exists a unique element $\Lambda \in W(I) {\cdot} \lambda$ such that $\Lambda \in P_{\geq 0}(I)$. Furthermore, if $\lambda \in P_{\mathrm{sig}}(I)$, then such a $\Lambda \in W(I) \cdot \lambda$ is also a signature integral weight.*

Proof. Note that A is of finite type in this section. The first part of the proposition is a well-known fact. The second part follows from Corollary 3.6. □

Definition 4.4. We suppose that a Cartan matrix $A = (a_{i,j})_{i,j\in I}$ is of finite type. We define a subset $(I)_0$ of I by:

$$(I)_0 := \left\{ i \in I \,\middle|\, d_i^\vee = 1 \right\}.$$

The highest coroot and the set $(I)_0$ for each finite type are listed in TABLE 1.

Definition 4.5. Let $i \in I$. We define a subset $P(i;I)$ of $P(I)$ by:

$$P(i;I) := \left\{ \lambda \in P(I) \,\middle|\, \langle \lambda,\, \alpha_j^\vee \rangle = \delta_{i,j}\ (j \in I) \right\},$$

where $\delta_{i,j}$ denotes the Kronecker's delta. An element of $P(i;I)$ is called an *i-th fundamental weight (over I).*

Remark. If $\dim \mathfrak{h} = \#I$, then the set $P(i;I)$ contains a unique element. In Section 5, we deal with a Cartan matrix $A = (a_{i,j})_{i,j\in I}$ of affine type, and we restrict the index set I to a subset J, which is of finite type. In such a situation, we have $\dim \mathfrak{h} > \#J$. Hence, we have $\#P(i;J) \neq 1$.

Remark. When a Cartan matrix $A = (a_{i,j})_{i,j\in I}$ is of finite type, a dominant integral weight Λ is said to be *minuscule* if $\langle \Lambda,\, \beta^\vee \rangle \leq 1$ for all $\beta \in \Phi_+$ and there exists $i \in I$ such that $\langle \Lambda,\, \alpha_j^\vee \rangle = \delta_{i,j}$ for $j \in I$. Equivalently, Λ is minuscule if

$$\text{there exists an index } i \in (I)_0 \text{ such that } \Lambda \in P(i;I).$$

Let Λ be a minuscule weight. By Corollary 3.6, if $\lambda \in W(I)\cdot\Lambda$, then $\lambda \in P_{\text{sig}}(I)$.

The following proposition shows the converse of the previous remark.

Proposition 4.6. *We have:*

$$P_{\text{sig}}(I) = P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I)\cdot P(i;I) \quad \textit{(disjoint union)},$$

Proof. Let $\lambda \in P_{\text{sig}}(I)$. By Proposition 4.3, we may assume $\lambda \in P_{\text{sig}}(I)$ is dominant. Since $\langle \lambda,\, \beta_{hs}^\vee \rangle \in \{0,1\}$, we have $\sum_{i\in I} d_i^\vee \langle \lambda,\, \alpha_i^\vee \rangle \in \{0,1\}$. Since $\langle \lambda,\, \alpha_i^\vee \rangle \in \{0,1\}$, we have either:

1. $\langle \lambda,\, \alpha_i^\vee \rangle = 0\,(i \in I)$, or
2. there exists a unique index $i \in (I)_0$ such that $\langle \lambda,\, \alpha_j^\vee \rangle = \delta_{i,j}\,(j \in I)$.

Hence, we have:

$$P_{\text{sig}}(I) \subseteq P_0(I) \sqcup \bigsqcup_{i \in (I)_0} W(I)\cdot P(i;I). \tag{4.1}$$

The opposite inclusion is by the previous remark. ∎

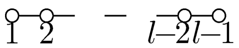
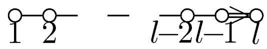
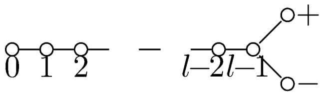
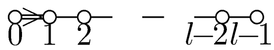
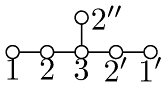
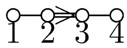
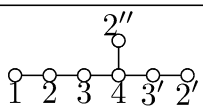
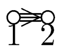
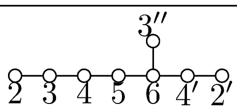
$A_{l-1} \quad (l \geq 2)$	$B_l \quad (l \geq 2)$
	
$\alpha_1^\vee + \cdots + \alpha_{l-1}^\vee$	$\alpha_1^\vee{+}2\alpha_2^\vee{+}\cdots{+}2\alpha_{l-1}^\vee{+}\alpha_l^\vee$, $2\alpha_1^\vee{+}\cdots{+}2\alpha_{l-1}^\vee{+}\alpha_l^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee, \cdots, \alpha_{l-1}^\vee\}$, $\Phi_s^\vee := W\{\alpha_l^\vee\}$
$I = \{1, \cdots, l{-}1\}$	$\{l\}$
${}_lC_i \quad (1 \leq i \leq l{-}1)$	2^l
$D_{l+2} \quad (l \geq 2)$	$C_l \quad (l \geq 2)$
	
$\alpha_0^\vee{+}2\alpha_1^\vee{+}\cdots{+}2\alpha_{l-1}^\vee{+}\alpha_+^\vee{+}\alpha_-^\vee$	$\alpha_0^\vee{+}\alpha_1^\vee{+}\cdots{+}\alpha_{l-2}^\vee{+}\alpha_{l-1}^\vee$, $2\alpha_0^\vee{+}2\alpha_1^\vee{+}\cdots{+}2\alpha_{l-2}^\vee{+}\alpha_{l-1}^\vee$ $\Phi_\ell^\vee := W\{\alpha_0^\vee\}$, $\Phi_s^\vee := W\{\alpha_1^\vee, \cdots, \alpha_{l-1}^\vee\}$
$\{0, +, -\}$	$\{l{-}1\}$
$2(l{+}2), \quad 2^{l+1} \quad , \quad 2^{l+1}$	$2{\cdot}l$
E_6	F_4
	
$\alpha_1^\vee{+}2\alpha_2^\vee{+}3\alpha_3^\vee$ ${+}2\alpha_{2'}^\vee{+}\alpha_{1'}^\vee{+}2\alpha_{2''}^\vee$	$2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$, $2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee, \alpha_2^\vee\}$, $\Phi_s^\vee := W\{\alpha_3^\vee, \alpha_4^\vee\}$
$\{1, 1'\}$	$\varnothing$
$27, 27$	–
E_7	G_2
	
$\alpha_1^\vee{+}2\alpha_2^\vee{+}3\alpha_3^\vee{+}4\alpha_4^\vee$ ${+}3\alpha_{3'}^\vee{+}2\alpha_{2'}^\vee{+}2\alpha_{2''}^\vee$	$2\alpha_1^\vee + \alpha_2^\vee$, $3\alpha_1^\vee + 2\alpha_2^\vee$ $\Phi_\ell^\vee := W\{\alpha_1^\vee\}$, $\Phi_s^\vee := W\{\alpha_2^\vee\}$
$\{1\}$	$\varnothing$
56	–
E_8	The type
	Dynkin diagram with indexes of vertices
$2\alpha_2^\vee{+}3\alpha_3^\vee{+}4\alpha_4^\vee{+}5\alpha_5^\vee{+}6\alpha_6^\vee$ ${+}4\alpha_{4'}^\vee{+}2\alpha_{2'}^\vee{+}3\alpha_{3''}^\vee$	the dual of the highest root , the highest coroot definition of $\Phi_\ell^\vee$, definition of $\Phi_s^\vee$
$\varnothing$	$(I)_0$
–	$\#(W \cdot \lambda) \qquad (\lambda \in (i; I)\,,\ i \in (I)_0)$

Table 1. Finite Types

Remark (How to read TABLE 1). In TABLE 1, we list the following information from the top:

- the type of Cartan matrix A,
- the Dynkin and our indexing of the nodes,
- the dual $\beta_{hl}^{\vee}$ of the highest root and the highest coroot $\beta_{hs}^{\vee}$,
- the set $(I)_0$,
- the cardinality $\#(W(I)\cdot\lambda)$ of $W(I)$-orbit for $\lambda \in P(i;I)$ $(i \in (I)_0)$,

as indicated at the end of the table. The cardinality of $W(I)$-orbit for a minuscule λ is equal to the dimension of the irreducible highest weight module $V(\lambda)$ so that the Weyl dimension formula gives the cardinality.

§ 5. Classification for Affine Types

In this section, we suppose a Cartan matrix $A = (a_{i,j})_{i,j\in I}$ is of affine type. We choose the index set I as in TABLE 2. In particular, the distinguished index $*$ is specified as in the table. The purpose of this section is to determine the $W(I)$-orbit decomposition of $P_{\mathrm{sig}}(I)$.

§ 5.1. Coroots

Definition 5.1. Let K denote the *null coroot* (the *canonical central element*) defined by

$$\langle \alpha_i, K\rangle = 0, \quad i \in I.$$

The integers $a_i^{\vee}$ $(i \in I)$ are defined by $K = \sum_{i\in I} a_i^{\vee}\alpha_i^{\vee}$.

Remark. Put $J := I \setminus \{*\}$. Note that we have $a_*^{\vee} = 1$. Through this section, we always fix this subset J of I. Then we note that the Cartan matrix $A(J) := (a_{i,j})_{i,j\in J}$ is of finite type.

If the Cartan matrix A is of type $A_{l-1}^{(1)}$ (resp. $D_{l+2}^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $D_{l+1}^{(2)}$, $A_{2l-1}^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$), then the Cartan matrix $A(J) := (a_{i,j})_{i,j\in J}$ is of type A_{l-1} (resp. D_{l+2}, E_6, E_7, E_8, B_l, C_l, F_4, G_2). We note that we have $a_i^{\vee} = d_i^{\vee}$ for each $i \in J$.

If the Cartan matrix A is of type $B_l^{(1)}$, $C_l^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$, then the Cartan matrix $A(J) := (a_{i,j})_{i,j\in J}$ is of type B_l, C_l, F_4, G_2. We note that we have $a_i^{\vee} = c_i^{\vee}$ for each $i \in J$.

Observe that $1 \le c_i^{\vee} \le d_i^{\vee}$ for $i \in I$. Thus, if $d_i^{\vee} = 1$, then $a_i^{\vee} = 1$ for all types except $A_{2l}^{(2)}$.

If the Cartan matrix A is of type $A_{2l}^{(2)}$, then the Cartan matrix $A(J) := (a_{i,j})_{i,j\in J}$ is of type B_l. In this case, we have $a_i^{\vee} = 2$ for each $i \in J$.

Lemma 5.2 (See[2]). *The set $\Phi_+^\vee(I)$ is decomposed as follows:*

1. *If I is of type $A_{l-1}^{(1)}, D_{l+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n\geq 1} \left(\Phi^\vee(J) + nK\right), \quad \textit{(disjoint union).}$$

2. *If I is of type $B_l^{(1)}, C_l^{(1)}, F_4^{(1)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n\geq 1} \left(\Phi_\ell^\vee(J) + 2nK\right) \sqcup \bigsqcup_{m\geq 1} \left(\Phi_s^\vee(J) + mK\right).$$

3. *If I is of type $G_2^{(1)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n\geq 1} \left(\Phi_\ell^\vee(J) + 3nK\right) \sqcup \bigsqcup_{m\geq 1} \left(\Phi_s^\vee(J) + mK\right).$$

4. *If I is of type $A_{2l}^{(2)}$, then:*

$$\Phi_+^\vee(I) = \Phi_+^\vee(J) \sqcup \bigsqcup_{n\geq 1} \frac{1}{2}\left(\Phi_\ell^\vee(J) + (2n-1)K\right) \sqcup \bigsqcup_{n\geq 1} \left(\Phi_s^\vee(J) + nK\right) \sqcup \bigsqcup_{n\geq 1} \left(\Phi_\ell^\vee(J) + 2nK\right).$$

Lemma 5.3. *We have:*

1. *If A is not of type $A_{2l}^{(2)}$ $(l \geq 1)$, then:*

$$\Phi_+^\vee(I) \subseteq \bigsqcup_{n\geq 0} \left(\Phi^\vee(J) + nK\right).$$

2. *If A is of type $A_{2l}^{(2)}$ $(l \geq 1)$, then:*

$$\Phi_+^\vee(I) \subseteq \bigsqcup_{n\geq 0} \frac{1}{2}\left(\Phi^\vee(J) + (2n-1)K\right) \sqcup \bigsqcup_{n\geq 0} \left(\Phi^\vee(J) + nK\right).$$

Proof. This follows from Lemma 5.2. □

Definition 5.4. We define an integer $r^\vee$ by:

$$r^\vee := \begin{cases} 1 & \text{if } I \text{ is of type } A_{l-1}^{(1)}, D_{l+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, D_{l+1}^{(2)}, A_{2l-1}^{(2)}, E_6^{(2)}, D_4^{(3)} \\ 2 & \text{if } I \text{ is of type } B_l^{(1)}, C_l^{(1)}, F_4^{(1)}, A_{2l}^{(2)} \\ 3 & \text{if } I \text{ is of type } G_2^{(1)} \end{cases}$$

Proposition 5.5. *Let $\beta^\vee \in \Phi_+^\vee(I)$ and $n \geq 0$. Then we have $\beta^\vee + r^\vee nK \in \Phi_+^\vee(I)$.*

Proof. This follows from Lemma 5.2. □

Proposition 5.6. *Let $\lambda \in P_{\geq -1}$. Then we have:*

1. $\langle \lambda, K \rangle \geq 0$.
2. *If $\langle \lambda, K \rangle \geq 1$, then we have $\lambda \in P_{\geq -1}^{\mathrm{fin}}(I) \setminus P_0(I)$.*
3. *If $\langle \lambda, K \rangle = 0$, then we have $\lambda \in P_{\geq -1}^{\mathrm{inf}}(I) \sqcup P_0(I)$.*

Proof. (1) Fix an arbitrary $\beta^\vee \in \Phi_+^\vee(I)$. By Proposition 5.5, for each $n \geq 0$, we have:

$$\beta^\vee + r^\vee nK \in \Phi_+^\vee(I).$$

If $\langle \lambda, K \rangle \leq -1$, then, for a sufficiently large n, we have $\langle \lambda, \beta^\vee + r^\vee nK \rangle \leq -2$. This contradicts $\lambda \in P_{\geq -1}$. This proves Part (1).

(2) If $\lambda \in P_0(I)$, then we have $\langle \lambda, K \rangle = \sum_{i \in I} a_i^\vee \langle \lambda, \alpha_i^\vee \rangle = 0$. Hence, we have $\lambda \notin P_0(I)$. If the Cartan matrix A is not of type $A_{2l}^{(2)}$ $(l \geq 1)$, then, by Lemma 5.3 (1), we have:

$$\mathrm{D}(\lambda)^\vee \subseteq \bigsqcup_{n \geq 0} \left(\mathrm{D}(\lambda)^\vee \cap (\Phi^\vee(J) + nK)\right),$$

Since $\langle \lambda, K \rangle \geq 1$, we have $\mathrm{D}(\lambda)^\vee \cap (\Phi^\vee(J) + nK) = \varnothing$ for $n \gg 0$. Since the set $\Phi^\vee(J)$ is finite, $\mathrm{D}(\lambda)^\vee$ is finite.

If the Cartan matrix A is of type $A_{2l}^{(2)}$ $(l \geq 1)$, then, by Lemma 5.3 (2), we have:

$$\mathrm{D}(\lambda)^\vee \subseteq \bigsqcup_{n \geq 0} \left(\mathrm{D}(\lambda)^\vee \cap \frac{1}{2}\left(\Phi^\vee(J) + (2n-1)K\right)\right) \sqcup \bigsqcup_{n \geq 0} \left(\mathrm{D}(\lambda)^\vee \cap (\Phi^\vee(J) + nK)\right).$$

Since $\langle \lambda, K \rangle \geq 1$, we have $\mathrm{D}(\lambda)^\vee \cap \frac{1}{2}\left(\Phi^\vee(J) + (2n-1)K\right) = \varnothing$ and $\mathrm{D}(\lambda)^\vee \cap (\Phi^\vee(J) + nK) = \varnothing$ for $n \gg 0$. Since the set $\Phi^\vee(J)$ is finite, $\mathrm{D}(\lambda)^\vee$ is finite.

(3) Suppose $\lambda \notin P_0(I)$. Then, there exists an index $i \in I$ such that $\langle \lambda, \alpha_i^\vee \rangle \neq 0$. Since $\langle \lambda, K \rangle = 0$, there exists an index $i_0 \in I$ such that $\langle \lambda, \alpha_{i_0}^\vee \rangle < 0$. Since $\langle \lambda, \alpha_{i_0}^\vee \rangle \geq -1$, we have $\langle \lambda, \alpha_{i_0}^\vee \rangle = -1$. Hence, we have $\alpha_{i_0}^\vee \in \mathrm{D}(\lambda)^\vee$. By Proposition 5.5, for each $n \geq 0$, we have $\alpha_{i_0}^\vee + r^\vee nK \in \Phi_+^\vee(I)$. Since

$$\langle \lambda, \alpha_{i_0}^\vee + r^\vee nK \rangle = \langle \lambda, \alpha_{i_0}^\vee \rangle + r^\vee n \langle \lambda, K \rangle = -1 + r^\vee n \cdot 0 = -1,$$

we have $\alpha_{i_0}^\vee + r^\vee nK \in \mathrm{D}(\lambda)^\vee$, for each $n \geq 0$. Hence, we have $\#\mathrm{D}(\lambda)^\vee = \infty$. □

Proposition 5.7. *Let $\lambda \in P^{\inf}_{\geq -1}(I)$. Then we have $\lambda \in P_{\rm sig}(I)$.*

Proof. Let $\lambda \in P_{\geq -1}$ and $\beta \in \Phi$. By Proposition 5.6, we have $\langle \lambda, K \rangle = 0$. First, suppose $\beta \in \Phi_+$. Then we have $\langle \lambda, \beta^\vee \rangle \geq -1$. Since there exists a positive integer m such that $\beta^\vee < r^\vee mK$, we have $r^\vee mK - \beta^\vee \in \Phi^\vee_+$. Since $\lambda \in P_{\geq -1}$, we have:

$$-1 \leq \langle \lambda, r^\vee mK - \beta^\vee \rangle = r^\vee m \langle \lambda, K \rangle - \langle \lambda, \beta^\vee \rangle = -\langle \lambda, \beta^\vee \rangle.$$

Hence, we have $\langle \lambda, \beta^\vee \rangle \leq 1$. This proves $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$ for $\beta \in \Phi_+$.
Suppose, on the other hand, $\beta \in \Phi_-$, then we have $-\beta \in \Phi_+$. By the above argument, we have $\langle \lambda, (-\beta)^\vee \rangle \in \{1, 0, -1\}$. This proves $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$ for $\beta \in \Phi_-$.
Thus, we always have $\langle \lambda, \beta^\vee \rangle \in \{1, 0, -1\}$, for $\beta \in \Phi$. □

Proposition 5.8. *We have $P_0(I) \sqcup P^{\inf}_{\geq -1}(I) = P_{\rm sig}(I)$.*

Proof. It is trivial that $P_0(I) \subseteq P_{\rm sig}(I)$. By Proposition 5.7, we have $P^{\inf}_{\geq -1}(I) \subseteq P_{\rm sig}(I)$. Hence, we have $P_0(I) \sqcup P^{\inf}_{\geq -1}(I) \subseteq P_{\rm sig}(I)$. Now, we prove the converse inclusion. Let $\lambda \in P_{\rm sig}(I)$. Suppose $\langle \lambda, K \rangle \neq 0$. If $\beta^\vee \in \Phi^\vee_+$, then, by Propositon 5.5, we have $\beta^\vee + r^\vee nK \in \Phi^\vee$, for each $n \geq 0$. Hence, for a sufficient large n, we have $\langle \lambda, \beta^\vee + r^\vee nK \rangle \notin \{1, 0, -1\}$. This contradicts $\lambda \in P_{\rm sig}(I)$. Hence, we have $\langle \lambda, K \rangle = 0$. By Proposition 5.6 (3), we have $P_0(I) \sqcup P^{\inf}_{\geq -1}(I) \supseteq P_{\rm sig}(I)$. □

§ 5.2. An invariant of W-Orbit

Definition 5.9. Let Γ be an abelian group. A map $\gamma : I \to \Gamma$ is said to *satisfy Condition (A)* if:

$$\prod_{j \in I} \gamma(j)^{\langle \alpha_i, \alpha_j^\vee \rangle} = 1, \text{ for each } i \in I.$$

Let $\lambda \in P(I)$. We define an element $a(\lambda) \in \Gamma$ as:

$$a(\lambda) := \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle}.$$

Proposition 5.10. *Suppose that $\gamma : I \longrightarrow \Gamma$ satisfies Condition (A). Then, for $\lambda \in P(I)$ and $w \in W(I)$, we have $a(\lambda) = a(w(\lambda))$.*

Proof. We may assume that w is a simple reflection s_i. Then we have:

$$a(s_i(\lambda)) = \prod_{j \in I} \gamma(j)^{\langle s_i(\lambda), \alpha_j^\vee \rangle} = \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle - \langle \lambda, \alpha_i^\vee \rangle \langle \alpha_i, \alpha_j^\vee \rangle}$$

$$= \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle} \cdot \left(\prod_{j \in I} \gamma(j)^{\langle \alpha_i, \alpha_j^\vee \rangle} \right)^{-\langle \lambda, \alpha_i^\vee \rangle} = \prod_{j \in I} \gamma(j)^{\langle \lambda, \alpha_j^\vee \rangle} = a(\lambda).$$

This proves the statement. □

This proposition shows that the map $a : P(I) \ni \lambda \longmapsto a(\lambda) \in \Gamma$ is a $W(I)$-invariant function on $P(I)$.

Recall that $J = I \setminus \{*\}$ and $(J)_0 = \left\{ i \in J \,\middle|\, d_i^\vee = 1 \right\}$. We explained that if $i \in (J)_0$ then $a_i^\vee = 1$. Note that if A is of type $A_{2l}^{(2)}$ then $(J)_0 = \varnothing$.

Definition 5.11. We define a subset $(I)_1$ of I by:

$$(I)_1 := \begin{cases} \{*\} \sqcup (J)_0 & \text{if } A \text{ is not of type } A_{2l}^{(2)}, \\ \{*\} & \text{if } A \text{ is of type } A_{2l}^{(2)}. \end{cases}$$

Definition 5.12. A map map $\gamma : I \longrightarrow \Gamma$ is said to *satisfy Condition (B)* if

$$\gamma(i) \neq \gamma(j), \quad \text{for } i, j \ (i \neq j) \in (I)_1.$$

We shall construct $\gamma_1 : I \longrightarrow \Gamma_1$ which satisfies Condition (A) and (B).

5.2.1. Case of $A_{l-1}^{(1)}$ $(l \geq 2)$ Suppose that the Cartan matrix A is of type $A_{l-1}^{(1)}$. We denote the cyclic group $C_l = \{1, g, \cdots, g^{l-1}\}$ of degree l by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} * &\longmapsto 1, \\ i \ (1 \leq i \leq l-1) &\longmapsto g^i. \end{aligned}$$

5.2.2. Case of $D_{l+2}^{(1)}$ $(l \geq 2, l$: even$)$ Suppose that the Cartan matrix A is of type $D_{l+2}^{(1)}(l : \text{even})$.
We denote the Klein four-group $K_4 = \{1, x, y, xy\}$ by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1, i : \text{odd}) &\longmapsto 1, \\ + &\longmapsto x, \\ - &\longmapsto y, \\ 0, i \ (1 \leq i \leq l-1, i : \text{even}) &\longmapsto xy. \end{aligned}$$

5.2.3. Case of $D_{l+2}^{(1)}$ $(l \geq 2, l$: odd$)$ Suppose that the Cartan matrix A is of type $D_{l+2}^{(1)}(l : \text{odd})$.
We denote the cyclic group $C_4 = \{1, g, g^2, g^3\}$ by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1, i : \text{odd}) &\longmapsto 1, \\ + &\longmapsto g, \\ 0, i \ (1 \leq i \leq l-1, i : \text{even}) &\longmapsto g^2, \\ - &\longmapsto g^3. \end{aligned}$$

5.2.4. Case of $E_6^{(1)}$ Suppose that the Cartan matrix A is of type $E_6^{(1)}$. We denote the cyclic group $C_3 = \{1, g, g^2\}$ by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, 2'', 3 &\longmapsto 1, \\ 1', 2 &\longmapsto g, \\ 1, 2' &\longmapsto g^2. \end{aligned}$$

5.2.5. Case of $E_7^{(1)}$ Suppose that the Cartan matrix A is of type $E_7^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, 2', 3', 4, 2 &\longmapsto 1, \\ 1, 3, 2'' &\longmapsto g. \end{aligned}$$

5.2.6. Case of $B_l^{(1)}$ Suppose that the Cartan matrix A is of type $B_l^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1) &\longmapsto 1, \\ l &\longmapsto g. \end{aligned}$$

5.2.7. Case of $D_{l+1}^{(2)}$ Suppose that the Cartan matrix A is of type $D_{l+1}^{(2)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (1 \leq i \leq l-1) &\longmapsto 1, \\ l &\longmapsto g. \end{aligned}$$

5.2.8. Case of $C_l^{(1)}$ Suppose that the Cartan matrix A is of type $C_l^{(1)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (0 \leq i \leq l-1, l-i : \text{even}) &\longmapsto 1, \\ i \ (0 \leq i \leq l-1, l-i : \text{ odd }) &\longmapsto g. \end{aligned}$$

5.2.9. Case of $A_{2l-1}^{(2)}$ Suppose that the Cartan matrix A is of type $A_{2l-1}^{(2)}$. We denote the cyclic group $C_2 = \{1, g\}$ of degree 2 by Γ_1. We define a map $\gamma_1 : I \longrightarrow \Gamma_1$ by:

$$\begin{aligned} *, i \ (0 \leq i \leq l-1, i : \text{even}) &\longmapsto 1, \\ i \ (0 \leq i \leq l-1, i : \text{ odd }) &\longmapsto g. \end{aligned}$$

In each case written above, it is straightforward to see that γ_1 satisfies Condition (A) and (B).

5.2.10. Other Cases : $E_8^{(1)}, F_4^{(1)}, E_6^{(2)}, G_2^{(1)}, D_4^{(3)}$, or $A_{2l}^{(2)}$ We do not need construct γ_1 because $(I)_1 = \{*\}$.

Definition 5.13. Let $i, j \in I$. We define a set $P(i, j; I)$ by:

$$P(i, j; I) := \left\{ \lambda \in P(I) \,\middle|\, \langle \lambda, \alpha_k^\vee \rangle = \delta_{i,k} - \delta_{j,k}, \text{ for each } k \in I \right\}.$$

Note that if $i = j$ then $P(i, j; I) = P_0(I)$.

Proposition 5.14. *Let γ_1 as above, and let $a : P(I) \longrightarrow \Gamma_1$ be the invariant function on $P(I)$ associated with γ_1. Let $i \in (I)_1$. Then, for any $\lambda \in W(I) \cdot P(i, *; I)$, we have $a(\lambda) = \gamma_1(i)$.*

Proof. By our assumption, there exists $w \in W(I)$ such that $w^{-1}(\lambda) \in P(i, *; I)$. Hence, by Proposition 5.10, we have $a(\lambda) = a(w^{-1}(\lambda))$. Then, since $w^{-1}(\lambda) \in P(i, *; I)$, we have $a(w^{-1}(\lambda)) = \gamma_1(i)\gamma_1(*)^{-1}$. Observe that $\gamma_1(*) = 1$ in all cases. Then we get $a(\lambda) = \gamma_1(i)$. □

§ 5.3. Main Theorem

We now state the main result of this paper.

Theorem 5.15. *We suppose that the Cartan matrix A is of affine type. Then we have*

$$P_{\text{sig}}(I) = \bigsqcup_{i \in (I)_1} W(I) \cdot P(i, *; I).$$

Theorem 5.15 follows from Proposition 5.17 and Proposition 5.18 below.

First, we suppose that A is not of type $A_{2l}^{(2)}$.

Lemma 5.16. *We have:*

1. $P_0(I) = P_0(J) \cap P_{\text{sig}}(I)$.
2. $P(i, *; I) = P(i; J) \cap P_{\text{sig}}(I)$, *for each* $i \in (J)_0$.

Proof. (1) It is trivial that we have $P_0(I) \subseteq P_0(J) \cap P_{\text{sig}}(I)$.
Now, we prove $P_0(I) \supseteq P_0(J) \cap P_{\text{sig}}(I)$. Let $\lambda \in P_0(J) \cap P_{\text{sig}}(I)$. Since $\lambda \in P_0(J)$, we have $\langle \lambda, \alpha_j^\vee \rangle = 0$ $(j \in J)$. Since $\lambda \in P_{\text{sig}}(I)$, we have $\langle \lambda, K \rangle = 0$. (This follows from Proposition 5.6 and Proposition 5.8.) Since

$$0 = \langle \lambda, K \rangle = \sum_{i \in I} a_i^\vee \cdot \langle \lambda, \alpha_i^\vee \rangle = 1 \cdot \langle \lambda, \alpha_*^\vee \rangle,$$

we have $\langle \lambda, \alpha_*^\vee \rangle = 0$. Hence, we have $\lambda \in P_0(I)$.
(2) We show that the required equality holds when $a_i^\vee = 1$. First, we prove $P(i, *; I) \subseteq$

$P(i; J) \cap P_{\rm sig}(I)$. Let $i \in (J)_0$ and $\lambda \in P(i,*;I)$. Since $i \in J$ and $* \notin J$, we have $\lambda \in P(i; J)$. Since

$$\langle\lambda,\, K\rangle = \sum_{j\in I} a_j^\vee \cdot \langle\lambda,\, \alpha_j^\vee\rangle = a_i^\vee \cdot 1 + a_*^\vee \cdot (-1) + \sum_{j\in I\setminus\{i,*\}} a_j^\vee \cdot 0 = 0,$$

we have $\langle\lambda,\, K\rangle = 0$. By Proposition 5.6 and Proposition 5.7, we have $\lambda \in P_{\rm sig}(I)$.
Next, we prove $P(i,*;I) \supseteq P(i;J) \cap P_{\rm sig}(I)$. Let $\lambda \in P(i;J) \cap P_{\rm sig}(I)$. Since $\lambda \in P(i;J)$, we have $\langle\lambda,\, \alpha_i^\vee\rangle = 1$ and $\langle\lambda,\, \alpha_j^\vee\rangle = 0$ $(j \in J\backslash\{i\})$. Since $\lambda \in P_{\rm sig}(I)$, we have $\langle\lambda,\, K\rangle = 0$. Since

$$0 = \langle\lambda,\, K\rangle = \sum_{j\in I} a_j^\vee \cdot \langle\lambda,\, \alpha_j^\vee\rangle = 1 \cdot \langle\lambda,\, \alpha_*^\vee\rangle + 1 \cdot 1,$$

we have $\langle\lambda,\, \alpha_*^\vee\rangle = -1$. Hence, we have $\lambda \in P(i,*;I)$. □

Proposition 5.17. *We have*

$$P_{\rm sig}(I) = \bigsqcup_{i\in(I)_1} W(I) \cdot P(i,*;I).$$

Proof. First, we prove $P_{\rm sig}(I) \supseteq \bigcup_{i\in(I)_1} W(I) \cdot P(i,*;I)$. Let $\lambda \in W(I) \cdot P(i,*;I)$ for $i \in (I)_1$. If $i = *$, then we have $\lambda \in P_0(I) \subseteq P_{\rm sig}(I)$. If $i \in (J)_0$, then, by Lemma 5.16 (2), we have $\lambda \in W(I) \cdot (P(i;J) \cap P_{\rm sig}(I)) \subseteq W(I) \cdot P_{\rm sig}(I) = P_{\rm sig}(I)$. Thus, we always have $\lambda \in P_{\rm sig}(I)$.
Next, we prove $P_{\rm sig}(I) \subseteq \bigcup_{i\in(I)_1} W(I) \cdot P(i,*;J)$. Let $\lambda \in P_{\rm sig}(I)$. Then we have $\lambda \in P_{\rm sig}(J)$. Hence, by Proposition 4.6, we have $\lambda \in P_0(J) \sqcup \bigsqcup_{i\in(J)_0} W(J) \cdot P(i;J)$. Since $\lambda \in P_{\rm sig}(I)$, we have $\lambda \in (P_0(J) \cap P_{\rm sig}(I)) \sqcup \bigsqcup_{i\in(J)_0} W(J) \cdot (P(i;J) \cap P_{\rm sig}(I))$. By Lemma 5.16 (1) and (2), we have $\lambda \in P_0(I) \sqcup \bigsqcup_{i\in(J)_0} W(J) \cdot P(i,*;I)$. Hence, we have $\lambda \in P_0(I) \cup \bigcup_{i\in(J)_0} W(I) \cdot P(i,*;I) = \bigcup_{i\in(I)_1} W(I) \cdot P(i,*;I)$.
Finally, we prove the disjointness of the decomposition. Let $i, j \in (I)_1$ $(i \neq j)$. Let $\lambda \in W(I) \cdot P(i,*;I)$ and $\mu \in W(I) \cdot P(j,*;I)$. Then, by Proposition 5.14, we have $a(\lambda) = \gamma_1(i)$ and $a(\mu) = \gamma_1(j)$. Since γ_1 satisfies Condition (B), we have $\gamma_1(i) \neq \gamma_1(j)$. Hence, we have $(W(I) \cdot P(i,*;I)) \cap (W(I) \cdot P(j,*;I)) = \varnothing$. This proves the statement. □

Next, we suppose that A is of type $A_{2l}^{(2)}$.

Proposition 5.18. *We have* $P_{\rm sig}(I) = P_0(I)$.

Proof. We prove $P_{\rm sig}(I) \subseteq P_0(I)$. The opposite inclusion is clear. Let $\lambda \in P_{\rm sig}(I)$. Then the Cartan matrix $A(J) := (a_{i,j})_{i,j\in J}$ is of type B_l. We have $\lambda \in P_{\rm sig}(J)$. By Proposition 4.3, we have $w(\lambda) \in P_{\rm sig}(J) \cap P_{\geq 0}(J)$ for some $w \in W(J)$. Hence, by Proposition 4.6, we have either:

(a) $w(\lambda) \in P_0(J)$.

(b) $w(\lambda) \in P(l; J)$.

Suppose that the case (a) holds. Since $w(\lambda) \in P_{\mathrm{sig}}(I)$, we have $\langle w(\lambda), K\rangle = 0$. Since

$$0 = \langle w(\lambda), K\rangle = a_*^\vee \cdot \langle w(\lambda), \alpha_*^\vee\rangle + \sum_{i\in J} a_i^\vee \cdot \langle w(\lambda), \alpha_i^\vee\rangle = \langle w(\lambda), \alpha_*^\vee\rangle,$$

we have $\langle w(\lambda), \alpha_*^\vee\rangle = 0$. Since $\langle w(\lambda), \alpha_i^\vee\rangle = 0$ for each $i \in I$, we have $w(\lambda) \in P_0(I)$. Hence, we have $\lambda \in P_0(I)$. Suppose, on the other hand, the case (b) holds. Since $w(\lambda) \in P_{\mathrm{sig}}(I)$, we have $\langle w(\lambda), K\rangle = 0$. Since

$$\begin{aligned} 0 &= \langle w(\lambda), K\rangle = a_*^\vee \cdot \langle w(\lambda), \alpha_*^\vee\rangle + \sum_{i\in J} a_i^\vee \cdot \langle w(\lambda), \alpha_i^\vee\rangle = \langle w(\lambda), \alpha_*^\vee\rangle + 2\langle w(\lambda), \alpha_l^\vee\rangle \\ &= \langle w(\lambda), \alpha_*^\vee\rangle + 2, \end{aligned}$$

we have $\langle w(\lambda), \alpha_*^\vee\rangle = -2$. This contradicts $\lambda \in P_{\geq -1}(I)$. Hence, we get $P_{\mathrm{sig}}(I) = P_0(I)$. □

References

[1] N. Bourbaki, "Eléments de Mathématique, t.XXXVIII, Groupes et Algĕbres de Lie," Hermann, Paris, 1975, Chap.VIII.

[2] V. G. Kac, "Infinite Dimentional Lie Algebras," Cambridge Univ. Press, Cambridge, UK, 1990.

[3] R. V. Moody and A. Pianzola, "Lie Algebras With Triangular Decompositions," Canadian Mathematical Society Series of Monograph and Advanced Text, 1995.

[4] K. Nakada, *Colored hook formula for a generalized Young diagram*, Osaka J. of Math. **Vol. 54 No. 4** (2008), 1085-1120.

[5] K. Nakada, *q-Hook formula for a generalized Young diagram*, preprint.

[6] R. P. Stanley, *Ordered Structures and Partitions,* Ph.D thesis, Harvard University, 1971.

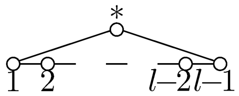
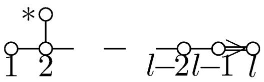
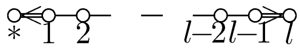
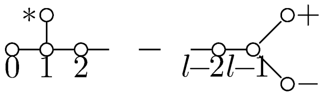
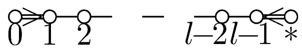
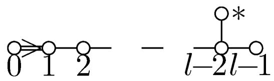
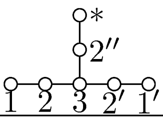
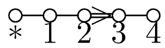
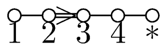
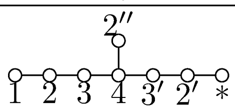
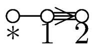
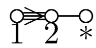
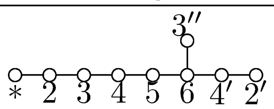
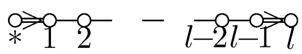
$A^{(1)}_{l-1}$ $(l \geq 2)$	$B^{(1)}_{l}$ $(l \geq 3)$	$D^{(2)}_{l+1}$ $(l \geq 2)$
		
$\alpha_*^\vee + \alpha_1^\vee + \cdots + \alpha_{l-1}^\vee$	$\alpha_*^\vee +$ $\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_l^\vee$	$\alpha_*^\vee +$ $2\alpha_1^\vee + \cdots + 2\alpha_{l-2}^\vee + 2\alpha_{l-1}^\vee + \alpha_l^\vee$
I C_l	$\{*, l\}$ C_2	$\{*, l\}$ C_2
$D^{(1)}_{l+2}$ $(l \geq 2)$	$C^{(1)}_{l}$ $(l \geq 2)$	$A^{(2)}_{2l-1}$ $(l \geq 3)$
		
$\alpha_*^\vee +$ $\alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + \alpha_+^\vee + \alpha_-^\vee$	$\alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_{l-2}^\vee + \alpha_{l-1}^\vee$ $+\alpha_*^\vee$	$2\alpha_0^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-2}^\vee + \alpha_{l-1}^\vee$ $+\alpha_*^\vee$
$\{*, 0, +, -\}$ $K_4(l : \text{even})$ or $C_4(l : \text{odd})$	$\{*, l-1\}$ C_2	$\{*, l-1\}$ C_2
$E^{(1)}_{6}$	$F^{(1)}_{4}$	$E^{(2)}_{6}$
		
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee$ $+2\alpha_{2'}^\vee + \alpha_{1'}^\vee + 2\alpha_{2''}^\vee + \alpha_*^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$	$2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee + \alpha_*^\vee$
$\{*, 1, 1'\}$ C_3	$\{*\}$ e	$\{*\}$ e
$E^{(1)}_{7}$	$G^{(1)}_{2}$	$D^{(3)}_{4}$
		
$\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee$ $+3\alpha_{3'}^\vee + 2\alpha_{2'}^\vee + \alpha_*^\vee + 2\alpha_{2''}^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + \alpha_2^\vee$	$3\alpha_1^\vee + 2\alpha_2^\vee + \alpha_*^\vee$
$\{*, 1\}$ C_2	$\{*\}$ e	$\{*\}$ e
$E^{(1)}_{8}$	$A^{(2)}_{2l}$ $(l \geq 1)$	The type
		Dynkin diagram with indexes of vertices
$\alpha_*^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 5\alpha_5^\vee + 6\alpha_6^\vee$ $+4\alpha_{4'}^\vee + 2\alpha_{2'}^\vee + 3\alpha_{3''}^\vee$	$\alpha_*^\vee + 2\alpha_1^\vee + \cdots + 2\alpha_{l-1}^\vee + 2\alpha_l^\vee$	the null coroot
$\{*\}$ e	$\{*\}$ e	$(I)_1$ Γ_1

Table 2. Affine Types