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Uno's conjecture for the exceptional Iwahori-Hecke algebras

By

Hyohe MIYACHI*

Abstract

In this paper we settle Uno's conjecture [Uno92] on representation types of 1-parameter Iwahori-Hecke algebra for the (crystallographic) exceptional Weyl groups by using mainly two different involutions of these algebras.

§ 1. Introduction

In early 90's Uno [Uno92] determined the representation type of the (1-parameter) Iwahori-Hecke algebras for the Coxeter groups with rank 2 and the finite Weyl groups of type A. Recently, by Ariki-Mathas[AM04] and Ariki[Ari05] we can know all the representation types of classical (1-parameter) Hecke algebras. (In [AM04] the representation type of Iwahori-Hecke algebras of type B with unequal parameters were also discussed.) Motivated by these works, we determine the representation type of the (1-parameter) Iwahori-Hecke algebras for the (crystallographic) exceptional Weyl groups. In particular, we can settle Uno's conjecture (see below Conjecture 1.1).

Let $\mathcal{A} = \mathbb{Z}[v^{\frac{1}{2}}, v^{-\frac{1}{2}}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in an indeterminate $v^{1/2}$. The generic Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{A},v}(W)$ of W over $\mathcal{A}$ with 1-parameter v is a free $\mathcal{A}$-module with (standard) basis $\{T_w |\ w \in W\}$.

We can consider a field $\Bbbk$ as an $\mathcal{A}$-module via the map $\mathcal{A} \to \Bbbk$ sending $v^{1/2}$ to a square root of $q \in \Bbbk$. For $0 \neq q \in \Bbbk$, let e be the least natural number such that $[e]_q = 1 + q + q^2 + \cdots + q^{e-1} = 0$ in $\Bbbk$. Set $e = \infty$, if no such number exists. We consider the case $\mathcal{H}_{\Bbbk,q}(W)$ is not semisimple.

Uno's conjecture [Uno92, p.288 Question] on representation types of Hecke algebras is as follows:

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*Graduate School of Mathematics, Nagoya University Chikusa-ku, Nagoya 464-8602, Japan.
e-mail: miyachi@math.nagoya-u.ac.jp

Conjecture 1.1 (**K.Uno**).

Assume that $\Bbbk$ *is a splitting field of* $\mathcal{H}_{\Bbbk,q}(W)$. $\mathcal{H}_{\Bbbk,q}(W)$ *is of finite type and not semisimple if and only if* q *is a simple root of the Poincaré polynomial equation* $P_W(v) = 0$ *of* $\mathcal{H}_{\mathcal{A},v}(W)$. *Here, the Poincaré polynomial* $P_W(v)$ *is defined by*

$$P_W(v) = \sum_{w \in W} v^{l(w)}$$

where $l(w)$ *is the length function on* W.

The main purpose of this paper is to settle Uno's conjecture above.

For simplicity and becasue the number of the page given to the author is limited, we assume the following, to prove Uno's conjecture for crystallographic exceptional Hecke algebras:

Assumption 1.2.

(i) If $e = 4, 6$ *and* $ch(\Bbbk) \neq 2, 3$, *then the decomposition matrix of the principal* Φ_e*- block of* $\mathcal{H}_{\Bbbk,q}(F_4)$ *doesn't depend upon* $ch(\Bbbk)$ *and depends only upon* e.

(ii) If $e = 6$ *and* $ch(\Bbbk) \neq 2, 3$, *then the decomposition matrix of the principal* Φ_6*- block of* $\mathcal{H}_{\Bbbk,q}(E_6)$ *doesn't depend upon* $ch(\Bbbk)$ *and depends only upon* e.

(iii) If $e \in \{5, 8, 10, 12\}$ *and* $ch(\Bbbk) \neq 2, 3, 5$, *then the decomposition matrix of the principal* Φ_e*- block of* $\mathcal{H}_{\Bbbk,q}(E_8)$ *doesn't depend upon* $ch(\Bbbk)$ *and depends only upon* e.

Now, we can state the main result in this paper as follows:

Theorem 1.3. *Suppose that the characteristic of* $\Bbbk$ *is zero or good for a finite (crystallographic) exceptional Weyl group* W. *Then, under Assumption 1.2, Uno's conjecture is true for the* 1*-parameter Iwahori-Hecke algebras for* W.

To show the above Theorem 1.3, we shall show the following theorem, which will cover all the cases of finite representation type:

Theorem 1.4. *Let* W *be Weyl group.*

(i) Suppose that $ch(\Bbbk)$ *is not bad for* W. *Then, all the decomposition numbers in any* Φ_e*-block* B *of* $\mathcal{H}_{\Bbbk,q}(W)$ *with* Φ_e*-defect* 1 *doesn't depend upon* $ch(\Bbbk)$ *and depends only upon* e.

(ii) In the case of (i), the block B *is a Brauer line tree algebra without any multiplicities.*

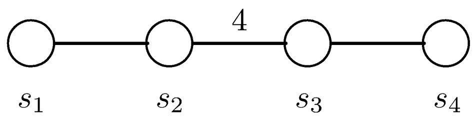

TABLE F_4

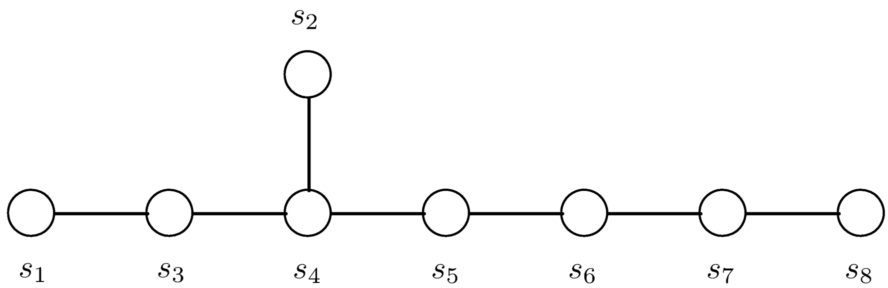

TABLE E_8

Since for the main result we will mainly investigate the structure of Iwahori-Hecke algebras of type F_4, E_6 and E_8, we choose the labellings of simple reflections of the Weyl groups of type F_4 and type E_8 as in the Dynkin diagrams in TABLE F_4 and TABLE E_8 respectively.

For type E_6, we choose the simple reflections $\{s_1, s_2, \cdots, s_6\}$ of type E_8 to give the set of simple reflections of type E_6. Moreover, $W(X)$ (resp. $\mathcal{H}_{R,v}(X)$) denotes the Weyl group (resp. the 1-parameter Iwahori-Hecke algebra over R) of type X.

The paper is structured as follows: In Section 2 we review some general notions and notation on finite dimensional symmetric algebras and Iwahori-Hecke algebras. In Section 3, for Iwahori-Hecke algebras we mimic representation theory of finite groups in positive characteristics. There, we review the $\Bbbk$-dual and its effects on induced modules. The "fixed" modules by this duality will be used to find indecomposable modules M' and M'' such that $\mathrm{Soc}(M') \cong \mathrm{Soc}(M'')$ and $M' \not\cong M''$. gluing M' and M'' at their socles will be used later to find representation types. In Section 4 we recall certain criteria for determining the representation types of finite dimensional algebras. In Section 5, we recall some properties on Goldman involution of Iwahori-Hecke algebras. Then, combining the results in Section 3 and Section 4 with the effects by Goldman involution, we find a condition giving two indecomposable modules M' and M'' with the above properties. In Section 6 we recall (mainly) Geck's theorems on decomposition numbers for 1-parameter Iwahori-Hecke algebras in not bad characteristics. In Section 7 we recall known results on representation types for 1-parameter Iwahori-Hecke algebras and prove Theorem 1.3 by mainly finding two such indecomposables M' and M''.

The result of this paper was stated in LMS Durham Symposium: Representations of Finite Groups and Related Algebras (1-11 July 2002). The first draft of this paper was much longer, like 63 pages. This included some explicit calculations and matrix representations, and also included Scott's permutation module theory and a solution for James's conjecture for some Φ_e-weight 2 blocks in exceptional Hecke algebras. The result about a solution of James's conjecture for the Φ_e-weight 2 blocks is probably contained in [GM] and its forthcomming second version as a minor part of them. So, in this paper we shall take those results as Assumption 1.2.

§ 2. Symmetric algebras

Let A be a finite dimensional associative symmetric algebra over a field $\Bbbk$. For an A-module M we denote the $\Bbbk$-dual of M (*i.e.* $\mathrm{Hom}_{\Bbbk}(M, \Bbbk)$) via the given symmetrizing form by M^*. Suppose that A-modules M and N are such that $\mathrm{Soc}(M) \cong \mathrm{Soc}(N)$. Let $V_{M,N}$ be the sum $M + N$ as a submodule of the injective hull of $\mathrm{Soc}(M)$. $R_0(A)$ denotes the Grothendieck group of the category of finite dimensional A-modules. The class of an A-module V in the Grothendieck group will be denoted by $[V]$. We denote by $[M : S]$ the multiplicity of a simple module S in M as composition factors.

It is known that any Iwahori-Hecke algebra $\mathcal{H}$ of a finite Weyl group with invertible parameters is symmetric (see for example [GR97]). The elements in the dual basis of $\{T_w\}$ are given by $\{T_w^\vee\}$, where $T_w^\vee = \mathrm{ind}(T_w)^{-1}T_{w^{-1}}$. Moreover, any block ideal of Iwahori-Hecke algebra is also symmetric thanks to the restriction of symmetrizing form to the block ideal. Moreover, any simple $\mathcal{H}$-module is selfdual since any irreducible character of a finite Weyl group is selfdual, any character corresponding to some projective indecomposable module is a linear combination of some irreducible characters and so any projective indecomposable module is selfdual. Let $\mathcal{H}'$ be a subalgebra of $\mathcal{H}$ corresponding to a parabolic subgroup. Then, it is known that $\mathcal{H}$ is free as a $\mathcal{H}'$-module. If V is a $\mathcal{H}'$-module, then $V \uparrow^{\mathcal{H}} := V \otimes_{\mathcal{H}'} \mathcal{H}$ is defined to be the induced module. For a block ideal B of $\mathcal{H}$, we denote $V \uparrow^{\mathcal{H}} \otimes_{\mathcal{H}} B$ by $V \uparrow^{B}$. We also use this kind of notation for the restriction. We denote by $P(S)$ the projective indecomposable module corresponding to a simple module S over a finite dimensional associative symmetric algebra. Let ζ_e be a primitive e-th root of unity in $\mathbb{C}$. Let $\Bbbk_0$ be a field such that $\Bbbk_0$ has characteristic 0 and contains ζ_{2e}. Let $\overline{\cdot}$ be a specialization $\mathcal{A} \to \mathbb{C}$ defined by $\overline{v^{\frac{1}{2}}} = \zeta_{2e}$. Let a triple $(K, \mathcal{O}, \Bbbk_0 \subset \mathbb{C})$ be a Φ_e-modular system associated with $\overline{\cdot}$. Here, $\mathcal{O}$ is c.d.v.r and $\Bbbk_0 \subset \mathbb{C}$. We may assume that this Φ_e-modular system is large enough for $\mathcal{H}_{\Bbbk_0,\zeta_e}(W)$. We say that a set $\mathcal{B}$ of irreducible characters ϕ's of $\mathcal{H}_{\mathbb{Q}(u),u}(W)$ is a Φ_e- *block* of $\mathcal{H}_{\mathbb{Q}(u),u}(W)$ if $\mathcal{B}$ is the minimal subset which satisfies the condition that if $\phi \in \mathcal{B}$ and $\bar{\phi}$ and $\bar{\phi}'$ have a common composition factor then $\phi' \in \mathcal{B}$. We also call a centrally primitive orthogonal idempotent $\mathbf{e}$ of $\mathcal{H}_{\Bbbk,q}(W)$ a Φ_e-*block idempotent* of $\mathcal{H}_{\Bbbk,q}(W)$. We mean that a Φ_e-*block ideal* of $\mathcal{H}_{\Bbbk,q}(W)$ is $\mathcal{H}_{\Bbbk,q}(W)\mathbf{e}$ for some Φ_e-block idempotent $\mathbf{e}$ of $\mathcal{H}_{\Bbbk,q}(W)$. The *principal* Φ_e-block idempotent (resp. ideal) of $\mathcal{H}_{\Bbbk,q}(W)$ is the block idempotent (resp. ideal) which dose not annihilate the index representation of $\mathcal{H}_{\Bbbk,q}(W)$. And we denote the principal Φ_e-block ideal by $B_0(\mathcal{H}_{\Bbbk,q}(W))$. We should mention that we also use some terminologies similar to the above for a general modular system $(K', \mathcal{O}', \Bbbk)$ by replacing Φ_e- by $\mathfrak{p}$ which is given by the kernel of a specialization of $\mathcal{A} \to \Bbbk$. We will denote more precise setting in 6.2. Denote by $\langle \lambda, \mu \rangle$ the usual Hermitian product for characters λ, μ of $\mathcal{H}_{\mathbb{Q}(v),v}(W) \cong \mathbb{Q}(v)[W]$. Moreover, for a $\mathcal{H}$-module V we often identify the character χ_V of V with $[V]$ if $\mathcal{H}$ is semisimple.

§ 3. Induced modules and duality

In this section we shall mimic some arguments in Alperin's text book [Alp86]. The consequences will be useful for considering the (Loewy) structures of modules over any Hecke algebras for finite Weyl groups with invertible parameters. In this section we always assume the following setting up:

(i) Let W be a finite Weyl group with a set of simple reflections $\mathcal{S}$.

(ii) Let $\mathcal{H}$ be the Iwahori-Hecke algebra for W over a field $\Bbbk$ with parameters $\{q^{c_s}\}_{s \in \mathcal{S}}$.

(iii) We assume that q is a unit in $\Bbbk$.

(iv) Let $\mathcal{S}'$ be a subset of $\mathcal{S}$, $W' := \langle \mathcal{S}' \rangle$ be the corresponding standard parabolic subgroup of W, and $\mathcal{H}'$ be the subalgebra of $\mathcal{H}$ corresponding to W'.

Remark. Note that we need the condition that $\mathcal{H}'$ is not only parabolic but also standard parabolic for a suitable Mackey system.

§ 3.1. Duality

For a vector space V over $\Bbbk$ let V^* be the dual space $\mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ (the space of linear maps from V to $\Bbbk$). We can regard V^* as a $\mathcal{H}$-module if V is a $\mathcal{H}$-module by defining the action of $T_w, w \in W$, on $\psi \in V^*$ by

$$(\psi \cdot T_w)(x) = \psi(x \cdot T_{w^{-1}})$$

for any $x \in V$. It is one of the strongest and group like properties of the Hecke algebra for Coxeter groups that there exists an anti-involution $\alpha : T_w \to T_{w^{-1}}$. Then, we know a *reflexive* property $V \cong V^{**}$ for any finitely generated $\mathcal{H}$-modules V. (Note that $\mathcal{H}$ is a finite dimensional symmetric algebra over $\Bbbk$.)

Definition 3.1. We say that an $\mathcal{H}$-module V is *selfdual* if $V \cong V^*$ as an $\mathcal{H}$-module.

Now, we would like to consider selfdual modules. First, we consider the most typical cases. Since $\mathcal{H}$ is symmetric, all the injective indecomposable modules are projective. Let D be a simple $\mathcal{H}$-module, and $P \to D \to 0$ be the projective cover of D. It is known that P is selfdual. Hence, so is D.

Thanks to *Frobenius reciprocity* and the characterization of induced modules in terms of relatively free modules, we can easily prove the following proposition, which will be a useful tool of investigating some structures of induced modules.

Proposition 3.2. *Let V be an $\mathcal{H}'$-module. Then,*

$$(V^*)\uparrow^{\mathcal{H}} \cong (V\uparrow^{\mathcal{H}})^*$$

as an $\mathcal{H}$-module.

Lemma 3.3. *Let V, V_1, V_2 be $\mathcal{H}'$-modules.*

(i) If V is free (projective) then $V\uparrow^{\mathcal{H}}$ is free (projective).

(ii) $(V_1 \oplus V_2)\uparrow^{\mathcal{H}} \cong V_1\uparrow^{\mathcal{H}} \oplus V_2\uparrow^{\mathcal{H}}$.

Proof. Use Proposition 3.2 and copy [Alp86, pp.54-58]. □

Lemma 3.4. *Let $\mathbf{e}$ be a block idempotent of $\mathcal{H}$. M be a selfdual $\mathcal{H}$-module. Then, $M\mathbf{e}$ is selfdual.*

Proof. Clear by $\mathbf{e}^* = \mathbf{e}$. □

Proposition 3.5. *Any simple $\mathcal{H}$-module is selfdual.*

Proof. Take a simple $\mathcal{H}$-module S and its projective cover $P \to S \to 0$. Consier a suitable modular system $(K, \mathcal{O}, \Bbbk)$ where $\mathcal{O}$ is a complete discrete valuation ring. P is uniquely lifted over $\mathcal{H}_{\mathcal{O}}$, say $P_{\mathcal{O}}$. Write P_K for $P_{\mathcal{O}} \otimes_{\mathcal{O}} K$. Note that taking $\Bbbk$-dual $(-)^*$ and K-dual $\mathrm{Hom}_K(-, K)$ (we still use $*$ for this operation with abuse) are compatible with taking modular reductions at least for the unique lifed module of any projective $\mathcal{H}$-module. So, it suffices to show that $P_K^* = \mathrm{Hom}_K(P_K, K) \cong P_K$ as an $\mathcal{H}_K$-module, which is equivalent to the character multiplicity condition

$$\langle [P_K^*], \chi\rangle = \langle [P_K], \chi\rangle \text{ for any } \chi \in \mathrm{Irr}(KW).$$

Since W is a Coxeter group, the character value of any irreducible character χ of KW is real, which leads $\chi = \chi^*$ where χ^* is the K-dual of χ. So, we are done. □

Conjecture 3.6. *Let M be a selfdual simple $\mathcal{H}'$-module. Then, any direct summand of $M\uparrow^{\mathcal{H}}$ is selfdual.*

§ 4. Fundamental lemmas

In this section we always assume that A is a symmetric algebra over a field $\Bbbk$.

Definition 4.1. We say that A is of *finite representation type* if there are at most finitely many isomorphism classes of indecomposable A-modules. Also, we say that A is of *infinite representation type* if A is not of finite representation type.

As in [AM04, Sect.2 Lemma 2.2], in order to show that A is of infinite type, we just find a suitable projective indecomposable module whose endomorphism ring is isomorphic to none of truncated polynomial rings over $\Bbbk$. More precisely, we recall the following:

Lemma 4.2. *Let P be a projective indecomposable A-module.*

(i) If $\mathrm{End}_A(P)$ is of infinite representation type, then so is A.

(ii) $\mathrm{End}_A(P)$ is of finite representation type if, and only if $\mathrm{End}_A(P) \cong \Bbbk[x]/\langle x^m \rangle$ for some non-negative integer m.

To find such a projective indecomposable module with the property (ii) above, the following lemma is fundamental:

Lemma 4.3. *Let M and N be non-projective, non-simple and indecomposable A-modules satisfying the following:*

(i) $S := \mathrm{Soc}(M) \cong \mathrm{Soc}(N)$, which is simple,

(ii) $S \cong \mathrm{Top}(M) \cong \mathrm{Top}(N)$,

(iii) $M \ncong N$,

(iv) Both M and N have the same Loewy length.

Then, the following holds:

(1) $\mathrm{Soc}(V_{M,N}) \cong S$.

(2) $\mathrm{Top}(V_{M,N}) \cong S \oplus S$.

(3) $\mathrm{End}_A(V^*_{M,N}) \ncong \Bbbk[x]/(x^n)$ *for any* n.

(4) $\mathrm{End}_A(P(S)) \ncong \Bbbk[x]/(x^n)$ *for any* n.

Proof. By definition, (1) is clear. Suppose that $\mathrm{Top}(V_{M,N}) \cong S$. We prove that M and N are proper submodules of $V_{M,N}$. Since the argument is the same, it suffices to show that M is a proper submodule of $V_{M,N}$. Assume to the contrary that $M = V_{M,N}$. Then $N \subset M$. Since $M \not\cong N$ and M has a simple top, we have that $N \subset \mathrm{Rad}(M)$, which contradicts the assumption that M and N have the same Loewy length. Therefore, M and N are proper submodules of $V_{M,N}$. Then, $V_{M,N}$ has the unique maximal submodule $\mathrm{Rad}(V_{M,N})$. So, we get $M + N \subset \mathrm{Rad}(V_{M,N})$, a contradiction. Hence, $\mathrm{Top}(V_{M,N}) \cong S \oplus S$. (3) and (4) are clear by considering maps from $\mathrm{Top}((V_{M,N})^*) \cong \mathrm{Soc}(V_{M,N})$ to $\mathrm{Soc}((V_{M,N})^*) \cong \mathrm{Top}(V_{M,N})$. □

As the above theorem says, we need to investigate some Loewy structures of projective indecomposable modules. In particular, we want to know extension groups among simple A-modules. For this aim we need the following lemma, which is related to the structure of the hearts of indecomposable projective A-modules:

Lemma 4.4. *Let A be a finite dimensional symmetric algebra over a field. Let M be an A-module. Let S be a simple submodule of M. If $[\mathrm{Top}(M) : S] \neq 0$ and $[\mathrm{Top}(M/S) : S] = 0$, then S is a direct summand of M.*

Proof. Suppose that S is a submodule of $\mathrm{Rad}(M)$. Then, $\mathrm{Rad}(M/S) \cong \mathrm{Rad}(M)/S$. So, $\mathrm{Top}(M/S) \cong \mathrm{Top}(M)$. However, this contradicts the property :$[\mathrm{Top}(M) : S] \neq 0$. Hence, S is not contained in $\mathrm{Rad}(M)$ as a submodule. Therefore, $S \cap \mathrm{Rad}(M) = 0$. Let M' be $S \oplus \mathrm{Rad}(M)$. So, there exists an A-submodule M'' of M satisfying the following:

$$(M'/\mathrm{Rad}(M)) \oplus (M''/\mathrm{Rad}(M)) \cong M/\mathrm{Rad}(M),$$

$$M' + M'' = M, M' \cap M'' = \mathrm{Rad}(M).$$

Hence, $M = S + \mathrm{Rad}(M) + M''$. Moreover, $\mathrm{Rad}(M)$ is contained in M''. So, $M = S + M''$.

On the other hand,

$$S \cap M'' = (S \cap M') \cap M'' = S \cap (M' \cap M'') = S \cap \mathrm{Rad}(M) = 0.$$

Therefore, $M = S \oplus M''$. □

Corollary 4.5. *Let A be a finite dimensional symmetric algebra over a field. Let M be a selfdual A-module. Let S be a simple selfdual submodule of M. Let $s := [M : S]$ be the composition multiplicity of S in M. Suppose that $S^{\oplus s}$ is a direct summand of $\mathrm{Soc}(M)$. Then, $S^{\oplus s}$ is a direct summand of M.*

§ 5. Goldman involution

Let $\mathcal{H}$ be the Iwahori-Hecke algebra of a finite Weyl group W over $\Bbbk$. There is an automorphism σ of $\mathcal{H}$ defined by $\sigma(T_s) := -u^{c_s}T_s^{-1}, s \in \mathcal{S}$ (see [CR87, (67.4)] to check the defining relations, and [Gec92, (8.4)] for the operations on block idempotents). σ is an algebras automorphism, which is called Goldman involution[1] , in particular, a self Morita equivalence. For any $\mathcal{H}$-module M, we can get an $\mathcal{H}$-module M^σ by the following compositions of algebras homomorphisms:

$$\mathcal{H} \xrightarrow{\sigma} \mathcal{H} \longrightarrow \mathrm{End}(M).$$

Since σ is an equivalence, we know the following lemma:

Lemma 5.1.

$$[P : S] = [P^\sigma : S^\sigma]$$

for any P and S such that P is a projective $\mathcal{H}_\Bbbk$-module (resp. is a unique $\mathcal{O}$-lift for a projective $\mathcal{H}_\Bbbk$-module) and S is a simple $\mathcal{H}_\Bbbk$-(resp. $\mathcal{H}_K$-) module.

Let $\mathcal{H}'$ be a subalgebra of $\mathcal{H}$ corresponding to a parabolic subgroup.

Lemma 5.2. *Let $\mathcal{H}'$ be a parabolic subalgebra of $\mathcal{H}$. Let P be a projective indecomposable $\mathcal{H}'$-module. Let T be the top of P* (i.e. *, T is also isomorphic to the socle of P). Let B be a block ideal of $\mathcal{H}$.*

(i) If $P\uparrow^B$ is a projective indecomposable module, then

  (a) $\mathrm{Top}(T\uparrow^B) \cong \mathrm{Soc}(T\uparrow^B)$,

  (b) $\mathrm{Top}(T\uparrow^B)$ is simple.

(ii) Let Q and Q' be projective indecomposable $\mathcal{H}$-modules. If the character of Q is coincident with that of Q', then $Q \cong Q'$.

(iii) $(M\uparrow^{\mathcal{H}})^\sigma \cong M^\sigma\uparrow^{\mathcal{H}}$ for any $\mathcal{H}'$-module M.

(iv) Suppose that P is not simple, $P\uparrow^B$ is indecomposable, $P\uparrow^B \cong (P\uparrow^B)^\sigma$ and $T\uparrow^B \ncong T^\sigma\uparrow^B$. Put $M := T\uparrow^B$ and $S := \mathrm{Top}(M)$.

  (a) M and M^σ are selfdual.

  (b) The projective cover of ${V_{M,M^\sigma}}^*$ is $P\uparrow^B$.

  (c) $\mathrm{Top}({V_{M,M^\sigma}}^*) \cong S$.

[1] The naming is due to N. Iwahori

(d) $\mathrm{Soc}(V_{M,M^{\sigma}}{}^{*}) \cong S \oplus S$.

(e) $\mathrm{End}_{\mathcal{H}}(V_{M,M^{\sigma}}{}^{*}) \ncong \Bbbk[x]/(x^n)$ *for any non-negative integer* n.

Proof. (i) (a)(b): By Lemma 3.3, $P\uparrow^{B}$ is surely projective. Clear by the property that any projective indecomposable module has a unique top and socle and $T\uparrow^{B}$ is a submodule and a quotient of $P\uparrow^{B}$. (ii): This follows from the linear independence of the characters corresponding to the projective indecomposable modules for Hecke algebras. (iii):(Let $\sigma_{\mathcal{H}}$ (resp.$\sigma_{\mathcal{H}'}$) be the Goldman involution of $\mathcal{H}$ (resp. $\mathcal{H}'$). Clearly, the restriction of $\sigma_{\mathcal{H}}$ to $\mathcal{H}'$ is $\sigma_{\mathcal{H}'}$. So, we need not care the subindex of σ.) For any $\mathcal{H}$-module U, U^{σ} is an $\mathcal{H}'$-module as well. Hence, by Frobenius reciprocity and the fact that σ is an equivalence, we get the following equation:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{H}'}(V^{\sigma}, U)\\ \cong\ &\mathrm{Hom}_{\mathcal{H}'}(V, U^{\sigma})\\ \cong\ &\mathrm{Hom}_{\mathcal{H}}(V\uparrow^{\mathcal{H}}, U^{\sigma})\\ \cong\ &\mathrm{Hom}_{\mathcal{H}}((V\uparrow^{\mathcal{H}})^{\sigma}, U). \end{aligned}$$

Therefore, by the characterization of induced module, namely $(V\uparrow^{\mathcal{H}})^{\sigma}$ is relatively free modules with respect to $\mathcal{H}$ and $\mathcal{H}'$ as $\dim(V\uparrow^{\mathcal{H}})^{\sigma} = [\mathcal{H} : \mathcal{H}'] \dim V^{\sigma}$, we deduce that $(V\uparrow^{\mathcal{H}})^{\sigma} \cong (V\uparrow^{\sigma})^{\mathcal{H}}$. (iv): By Proposition 3.2 and the property σ is an equivalence, we get (a). The others are clear by Lemma 4.3 and (i). □

Definition 5.3. We call an $\mathcal{H}$-module M σ-stable if $M \cong M^{\sigma}$.

Lemma 5.4. *Suppose that there exists an* $\mathcal{H}$*-module* M *such that*

(1) M *is selfdual,*

(2) $\mathrm{Top}(M)$ *is* σ*-stable,*

(3) $\mathrm{Top}(M)$ *is a direct sum of non-isomorphic simple modules* S *and* S'*,*

(4) The Cartan invariant $c_{S,S'}$ *is not more than* 1*,*

(5) $[\mathrm{Rad}(M)]$ *is multiplicity-free,*

(6) $[M]$ *is not* σ*-stable,*

(7) $[M : S] = [M : S'] = 2$*,*

(8) $[M] \neq [S] + [P(S')], [S'] + [P(S)]$*,*

Then, there exist indecomposable $\mathcal{H}$ *-modules* M' *and* M'' *such that*

(i) $\mathrm{Top}(M') \cong \mathrm{Soc}(M') \cong \mathrm{Soc}(M'') \cong \mathrm{Top}(M'')$ *is simple,*

(ii) $M' \not\cong M''$,

(iii) *Both* M' *and* M'' *have Loewy length* 3.

Proof. By (3) we know that M is indecomposable or M is a direct sum of two non-isomorphic indecomposable modules.

Suppose that M is decomposable. For $X \in \{S, S'\}$ and $L \in \{M, M^\sigma\}$ let L_X be the unique direct summand of L such that $\mathrm{Soc}(L_X) \cong X$. Then, by selfduality of M, M_S is isomorphic to either $(M_S)^*$ or $(M_{S'})^*$. Suppose that M_S is isomorphic to $(M_{S'})^*$. So, $M_{S'}$ is isomorphic to $(M_S)^*$ and $\mathrm{Top}(M_S) \cong S'$. By (6), We know that either $(M^\sigma)_S \not\cong M_S$ or $(M^\sigma)_{S'} \not\cong M_{S'}$. Since σ preserves the Loewy structure of modules, $\mathrm{Top}((M^\sigma)_S) \cong \mathrm{Top}(M_S)$. Considering $V_{(M^\sigma)_S, M_S}$, we get $c_{S,S'} \geq 2$. This contradicts $c_{S,S'} \leq 1$. Hence, both M_S and $M_{S'}$ are selfdual. Suppose that M_S is simple. Then, using the property (5) and (7), and applying Corollary 4.5 to $\mathrm{Rad}(M_{S'})/\mathrm{Soc}(M_{S'})$, we know that $M_{S'}$ has Loewy length 3 and $S \mid \mathrm{Rad}(M_{S'})/\mathrm{Soc}(M_{S'})$. Since $M_{S'}$ is a submodule of $P(S')$, by (8) we get S is a direct summand of $\mathrm{Soc}^2(P(S'))/\mathrm{Soc}(P(S'))$. By Corollary 4.5, S is a direct summand of $H(P(S'))$. In particular, by $c_{S,S'} = 1$, if there exists a submodule Z of $H(P(S'))$ such that $\mathrm{Soc}(Z)$ contains S, then S is a direct summand of Z. On the other hand, since $M_{S'} \not\cong P(S')$ by (8), $M_{S'}/S'$ is a submodule of $H(P(S'))$ and has the unique maximal submodule $\mathrm{Rad}(M_{S'}/S')$, which contains S. However, S is not a direct summand of $M_{S'}/S'$. We get a contradiction. Hence, we deduce that both M_S and $M_{S'}$ are not simple. Therefore, both M_S and $M_{S'}$ have Loewy length 3, unique top and socle. By (2) and (6) we have either $(M^\sigma)_S \not\cong M_S$ or $(M^\sigma)_{S'} \not\cong M_{S'}$. So, we are done.

Suppose that M is indecomposable. By the selfduality and indecomposablity of M, and Corollary 4.5, we know that $\mathrm{Soc}(M)$ is a submodule of $\mathrm{Rad}(M)$. Moreover, by using the property $\mathrm{Top}(M) \cong \mathrm{Top}(M)^* \cong \mathrm{Soc}(M)$ and the dual argument taking the heart of M, we know that $H(M) := \mathrm{Rad}(M)/\mathrm{Soc}(M) \cong (\mathrm{Rad}(M)/\mathrm{Soc}(M))^*$. Moreover, by the condition (6) we know that M can not have Loewy length 2. Hence, the Loewy length of M is greater than or equal to 3. By applying Lemma 4.4 to $H(M)$ repeatedly due to (5), we know that $H(M)$ is semisimple. Hence, we know that $\mathrm{Soc}(M) = \mathrm{Rad}^2(M)$ and M has Loewy length 3.

Let N be M^σ. Then, N satisfies the same assumption from (1) to (8) with M. Put $\mathrm{Rad}(M)/\mathrm{Rad}^2(M) = \bigoplus_{i=1}^m M_i$ where each M_i is simple. Let $I := \{1, 2, \dots, m\}$. For $L \in \{M, N\}$ and $X \in \{S, S'\}$ let $\pi_{L,X}$ be a canonical epimorphism from M to X. (If both $\mathrm{Ker}(\pi_{M,S})/S$ and $\mathrm{Ker}(\pi_{N,S})/S$ are indecomposable, then we are done. Suppose that $\mathrm{Ker}(\pi_{M,S})/S$ is decomposable.) We may write

$$\mathrm{Ker}(\pi_{M,S})/S = V \oplus \bigoplus_{i \in J} M_i.$$

with $\mathrm{Top}(V) \cong \mathrm{Soc}(V) \cong S'$ for some subset J of I, and

$$\mathrm{Ker}(\pi_{N,S})/S = U \oplus \bigoplus_{i \in J'} M_i^\sigma.$$

with $\mathrm{Top}(U) \cong \mathrm{Soc}(U) \cong S'$ for some subset J' of I. If $U \not\cong V$, then we are done.

So, we may assume that $U \cong V$. By (6) we deduce that both $\bigoplus_{i \in J} M_i$ and $\bigoplus_{i \in J'} M_i^\sigma$ are not σ-stable. In particular, there exists $j \in J$ such that $M_j^\sigma \not\cong M_i$ for any $i \in I$. So, there exists a uniserial submodule T of $\mathrm{Rad}(M)$ such that $\mathrm{Top}(T) \cong M_j$ and $\mathrm{Soc}(T) \cong S$. Moreover, we emphasize a property of M_j as follows:

$$[M : M_j] = 1, [N : M_j] = 0. \tag{1}$$

Clearly, T is a submodule of $\mathrm{Ker}(\pi_{M,S'})$.

Now we consider the projection $\mathrm{Ker}(\pi_{N,S'}) \to \mathrm{Ker}(\pi_{N,S'})/S'$ and denote the image of T by T'. As $\mathrm{Soc}(T) = S \not\cong S'$, T' is isomorphic to T.

Let M' (resp. M'') be the unique indecomposable direct summand of $\mathrm{Ker}(\pi_{M,S'})/S'$ (resp. $\mathrm{Ker}(\pi_{N,S'})/S'$) such that $\mathrm{Top}(M') \cong \mathrm{Soc}(M') \cong S$ (resp. $\mathrm{Top}(M'') \cong \mathrm{Soc}(M'') \cong S$). Note that M' must have a submodule T'.Clearly, both M' and M'' have Loewy length 3.

Since we have $[M' : M_j] = 1$,$[M'' : M_j] = 0$ by 1, we deduce that $M' \not\cong M''$ as desired. □

§ 6. Decomposition matrices and Geck's theorem

In this section we will discuss decomposition matrices of Hecke algebras.

§ 6.1. The a-functions and Geck's theorem

First we recall Geck's results [Gec98a] on labellings of the simple modules over Hecke algebras by a-functions and their property on decomposition numbers.

Definition 6.1. Let $\mathcal{H}$ be $\mathcal{H}_{\mathbb{Z}[v^{\frac{1}{2}},v^{-\frac{1}{2}}],v}(W)$ for a finite Weyl group. Let $\{C_w \mid w \in W\}$ be Kazhdan-Lusztig basis of $\mathcal{H}$. Namely,

$$C_w = \sum_{y \leq w} (-1)^{l(w)-l(y)} v^{\frac{l(w)-2l(y)}{2}} P_{y,w}(v^{-1}) T_y \quad (w \in W).$$

Let $h_{x,y,z}$ be the structure constant defined by $C_x C_y = \sum_z h_{x,y,z} C_z$.

Definition 6.2 (Lusztig). We call the function $a(z) := \min\{i \in \mathbb{N} \mid v^{\frac{i}{2}} h_{x,y,z} \in \mathbb{Z}[v^{\frac{1}{2}}]\}$ on W the *a-function* for W. For an $\mathcal{H}$-module V let a_V be $\max\{a(w) \mid C_w \cdot V \neq 0\}$.

Theorem 6.3 (Geck). [*Gec98a, p.*270 *Theorem* 3.3] *Suppose that the characteristic of* $\Bbbk$ *is 0 or a good prime for* W*. Then, there exist a* **unique** *subset* $\mathcal{B}_{\text{basic}}$ *of* $\mathrm{Irr}(\mathcal{H}_K)$ *such that there exists a bijection* $\mathbf{g}$ *between* $\mathcal{B}_{\text{basic}}$ *and* $\mathrm{Irr}(\mathcal{H}_\Bbbk)$ *such that for any* $M \in \mathrm{Irr}(\mathcal{H}_\Bbbk)$

(a) $\mathbf{g}(V) = M$ *for* $\exists! V \in \mathcal{B}_{\text{basic}}$.

(b) $a_V = a_M$.

(c)

$$[\overline{V}] - [M] \in \sum_U \mathbb{Z}_{\geq 0} \cdot [U] \tag{2}$$

Here, in the sum U *runs over simple* $\mathcal{H}_\Bbbk$*-modules satisfying* $a_U < a_V$.

Put $d_{V,M} := [V : M]$*, which is called a decomposition number.*

Corollary 6.4 (Geck).

$$a_M = \min\{\ a_V \mid d_{V,M} \geq 1, V \in \mathcal{B}_{\text{basic}}\}.$$

The following lemma is taken from [Gec98a, Lemma 4.2].

Lemma 6.5 (Geck). *Suppose that* $d_{V_1,M} \geq 1$*,* $d_{V_2,M} \geq 1$ *and* $a_{V_1} = a_M$*. Then,*

(i) $a_{V_1} \leq a_{V_2}$ *and* $a_{\sigma(V_2)} \leq a_{\sigma(V_1)}$.

(ii) $a_{V_1} < a_{V_2}$ *if and only if* $a_{\sigma(V_2)} < a_{\sigma(V_1)}$.

Definition 6.6. For any union $\mathcal{C}$ of Φ_e- or $\mathfrak{p}$-blocks, we denote by $\mathcal{C}_{\text{basic}}$ the restriction $\mathcal{C} \cap \mathcal{B}_{\text{basic}}$. For a finite Weyl group W, a splitting field $\Bbbk$ of $\mathcal{H}_{\Bbbk,q}(W)$ with a good or zero characteristic for W and an invertible element q of $\Bbbk$ with multiplicative order e. We call the map $\mathbf{g} = \mathbf{g}_{e,W}$ between $\mathcal{B}_{\text{basic}}$ and the totality of simple $\mathcal{H}_{\Bbbk,q}(W)$-modules Geck bijection. The Geck bijection depends only on e and W if $\Bbbk$ is a splitting field for $\mathcal{H}_{\Bbbk,q}(W)$ and dose not have a bad characteristic for W.

§ 6.2. The numbers of simple modules of $\mathcal{H}_\Bbbk$ and decomposition maps

Let us recall some known facts on the numbers of non-isomorphic simple modules of Iwahori-Hecke algebras. (see [Gec00, Theorem 1.1] and references there for the details on the following results.)

Theorem 6.7 (Dipper-James/Ariki-Mathas/Geck-Rouquier/Geck). *Suppose that the characteristic* ℓ *of* $\Bbbk$ *is zero or a good prime for* W*. Then, the number of non-isomorphic simple modules of* $\mathcal{H}_\Bbbk$ *only depends upon* e*. Namely,*

$$|\mathrm{Irr}(\mathcal{H}_{\Bbbk,q}(W))| = |\mathrm{Irr}(\mathcal{H}_{\mathbb{Q}(\zeta_e),\zeta_e}(W))|$$

where ζ_e *is a primitive* e*-th root of unity in* $\mathbb{C}$.

Let $\Bbbk^-$ be an algebraically closed field with characteristic $\ell > 0$. Fix a $q \in \Bbbk^- \backslash \{0\}$. Assume that e is equal to the multiplicative order of $q \in \Bbbk^-$. As in Introduction, let $\theta : \mathcal{A} \to \Bbbk^-$ be the unique ring homomorphism such that $\theta(v) = q^{\frac{1}{2}} \in \Bbbk^-$. Following [Gec00, §2] let us recall some canonical decomposition maps. Let $\Bbbk$ be the field of fractions of the image of θ (which contains the square root of q in order to keep $\mathcal{H}_\Bbbk$ split). Let $\mathfrak{p}$ be the kernel of θ. We may regard $\Bbbk$ as the field of fractions of $\mathcal{A}/\mathfrak{p}$ and may also regard θ as the canonical map $\mathcal{A} \to \mathcal{A}/\mathfrak{p} \subset \Bbbk$. We can assume that $\Phi_{2e}(v) \in \mathfrak{p}$. Let $\mathfrak{q}$ be $(\Phi_{2e}(v))$. So, we get the following coefficient rings:

(i) $0 \neq \mathfrak{q} \subset \mathfrak{p} \subset \mathcal{A}$,

(ii) $F := \mathbb{Q}[\zeta_{2e}] \subset \mathbb{C}$,

(iii) $\mathcal{A}/\mathfrak{q} = \mathbb{Z}[\zeta_{2e}]$ is the ring of algebraic integers in F.

Then, the natural map $\mathcal{A} \to \mathcal{A}/\mathfrak{p} \subset \Bbbk$ (resp. $\mathcal{A} \to \mathcal{A}/\mathfrak{q} \subset F$) defines a decomposition map $\mathbf{d}_\mathfrak{p} : R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_\Bbbk)$ (resp. $\mathbf{d}_e : R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_F)$). Then, we have the following factorization of these decomposition maps thanks to the property that $\mathcal{A}/\mathfrak{q}$ is integrally closed:

$$\begin{tikzcd} R_0(\mathcal{H}_K) \arrow[dd, "\mathbf{d}_\mathfrak{p}"'] \arrow[dr, "\mathbf{d}_e"] & \\ & R_0(\mathcal{H}_F) \arrow[dl, "\mathbf{d}_\mathfrak{p}^e"] \\ R_0(\mathcal{H}_\Bbbk) & \end{tikzcd}$$

Here, $\mathbf{d}_\mathfrak{p}^e$ is the decomposition map defined by the canonical map $\mathcal{A}/\mathfrak{q} \to \mathcal{A}/\mathfrak{p}$.

In this setting up, the matrix $\mathbf{D}_\mathfrak{p}^e$ of the decomposition map $\mathbf{d}_\mathfrak{p}^e$ is called the adjustment matrix for $\mathcal{H}_\Bbbk$ (see, [Gec92],[Gec98b]). By Theorem 6.3 and Theorem 6.7, we know that the adjustment matrix $\mathbf{D}_\mathfrak{p}^e$ is a square $k \times k$ lower unitriangular matrix with the indices $\mathcal{B}_{\text{basic}}$ and the factorization $\mathbf{D}_\mathfrak{p} = \mathbf{D}_e \cdot \mathbf{D}_\mathfrak{p}^e$ where $k = |\mathrm{Irr}(\mathcal{H}_F)| = |\mathrm{Irr}(\mathcal{H}_{\mathbb{Q}(\zeta_e),\zeta_e})(W)|$ and $\mathbf{D}_\mathfrak{p} = (d_{i,j}^\mathfrak{p})$ (resp. $\mathbf{D}_e = (d_{i,j})$) is the decomposition matrix of $\mathbf{d}_\mathfrak{p}$ (resp. $\mathbf{d}_e$). Moreover, each entry of $\mathbf{D}_\mathfrak{p}^e$ is a non-negative integer thanks to Geck's argument in [Gec92]. (The name "adjustment matrix" is first given by James [Jam90].) In this subsection we consider the possible adjustment matrix appearing in $\mathcal{H}_{\Bbbk,q}(W)$ when the characteristic $\ell > 0$ of $\Bbbk$ is a good prime for W. From now on we always assume that the Brauer trees appearing in Φ_e-blocks of $\mathcal{H}_{\mathbb{Q}(\zeta_e),\zeta_e}(W)$ are straight lines with multiplicity 1. (See [Alp86] for the definition of Brauer graph algebras.) When q is in $\mathbb{C}$ or in $\mathbb{F}_\ell$, W is one of E_6, E_7 and E_8, and q is a simple root of the Poincaré polynomial $P_W(u)$ of W, the Brauer trees appearing in $\mathcal{H}_{\Bbbk,q}(W)$ is completely determined in [Gec92]. When q is in $\mathbb{C}$ or in $\mathbb{F}_\ell$, the Brauer trees appearing in $\mathcal{H}_{\Bbbk,q}(F_4)$ is completely determined in [GL91]. In particular we know the following:

Theorem 6.8 (**Geck-Lux/Geck**). *Suppose that W is one of the Weyl groups of type F_4, E_6, E_7 and E_8 and $\Phi_e(u)$ divides $P_W(u)$ exactly once. Then, any Φ_e-block ideal of $\mathcal{H}_{\mathbb{Q}(\zeta_e),\zeta_e}(W)$ is either Φ_e-defect zero or a Brauer line tree algebra with multiplicity* 1*.*

Our assumption and setting up in this subsection are as follows:

(i) $\mathrm{Irr}W = \coprod_i \coprod_j \mathcal{B}_{\phi,j}$ is the Φ_e-block partitions of the irreducible characters of W where the defect of $\mathcal{B}_{\phi,j}$ is j and ϕ has the minimal a-values in $\mathcal{B}_{\phi,j}$.

(ii) $\mathrm{Irr}W = \coprod_i \coprod_j \mathcal{B}^{\mathfrak{p}}_{i,j}$ is the $\mathfrak{p}$-block partitions of the irreducible characters of W.

Definition 6.9. Let $\mathrm{Irr}W = \{\phi_1, \phi_2, \cdots, \phi_k\}$. We call ϕ initial (resp. terminal) if ϕ lies in a Φ_e-block $\mathcal{B}$ of $\mathrm{Irr}W$, ϕ has the minimal (resp. maximal) a-value in $\mathcal{B}$ and a Φ_e-reduction of ϕ is irreducible. Conversely, for each Φ_e-block $\mathcal{B}$ of $\mathrm{Irr}W$, there exists (at least) an initial (resp. terminal) character ϕ. (see, Theorem 6.3.) Let $\mathcal{B}_i$ be the union of Φ_e-blocks with Φ_e-defect i. Let I be the subset of $\{1, 2, \ldots, |\mathrm{Irr}W|\}$ consisting of those ϕ_i with Φ_e-defect 1. By Theorem 6.3, we can choose a total order $\leq$ on $\mathrm{Irr}W$ by the a-function. So, we can attach an element in I to each character in $\mathcal{B}_1$ such that $a(\chi_i) \leq a(\chi_{i+1})$ and $\chi_i, \chi_{i+1} \in \mathcal{B}_1$ in the Brauer tree graph of some Φ_e-defect 1 block for $i = 1, \ldots, k_1 - 1$. Let I_{ini} (resp. I_{ter}) be the subset of I corresponding to the set of initial (resp. terminal) characters in I. Namely, $\mathcal{B}_1 = \coprod_{i \in I_{\mathrm{ini}}} \mathcal{B}_{\phi_i,1}$.

Define a bijection Ω from $I - I_{\mathrm{ter}}$ to $I - I_{\mathrm{ini}}$ by

$$\Omega(i) = j \text{ if } \phi_j \text{ is connected to } \phi_i \text{ with } a(\phi_i) < a(\phi_j)$$

Before investigating decomposition numbers, we will recall the block partitions for Φ_e-weight 1.

Lemma 6.10. *Suppose that q is a simple root of the Poincaré polynomial $P_W(u)$. Then, q is a simple root of $\Phi_e(u)$ and the block distribution of* $\mathrm{Irr}W$ *dose not depend upon the characteristic of* $\Bbbk$*.*

Proof. Let B a Φ_e-block of $\mathrm{Irr}W$. Since $\Phi_e(u)$ divides $\sum_{\phi \in B} D_\phi(u)\mathrm{ind}(T_w)\phi(T_{w^{-1}})$ exactly once,

$$\frac{\sum_{\phi \in B} D_\phi(u)\mathrm{ind}(T_w)\phi(T_{w^{-1}})}{P_W(u)} \Big|_{u=q} \tag{3}$$

is an element of $\mathcal{O}$. Here, $D_\phi(u)$ is the generic degree polynomial corresponding to ϕ. On the other hand, the primitive central orthogonal idempotent $\mathbf{e}_\phi$ in $\mathcal{H}_{\mathbb{Q}(u),u}(W)$ corresponding to $\phi \in \mathrm{Irr}\mathcal{H}_{\mathbb{Q}(u),u}(W)$ is given by

$$\mathbf{e}_\phi = \frac{D_\phi(u)}{P_W(u)} \sum_{w\in W} \mathrm{ind}(T_w)^{-1}\phi(T_{w^{-1}})T_w$$

and $(P_W(u)/\Phi_e(u))\mid_{u=q}$ is an invertible element of $\mathcal{O}$. Hence, the block distribution of $\mathrm{Irr}W$ dose not depend upon the characteristic of $\Bbbk$. $\square$

Lemma 6.11.

(i) *Suppoese that $\mathcal{B}_{\phi,1}$ is coincident with only a $\mathfrak{p}$-block of* $\mathrm{Irr}W$. *(In general some $\mathfrak{p}$-block is a union of Φ_e-blocks.) Then, the adjustment matrix $\mathbf{D}^e_{\mathfrak{p}} = (d^{e,\mathfrak{p}}_{\lambda,\mu})$ satisfy the following:*

$$d^{e,\mathfrak{p}}_{\lambda,\mu} = \begin{cases} 1 & \text{if } \phi = \chi \\ 0 & \text{otherwise.} \end{cases}$$

for any $\lambda \in \mathcal{B}_{\mathrm{basic}}$ and $\mu \in \mathcal{B}_{\phi,1}$. Namely, the part of adjustment matrix corresponding to $\mathcal{B}_{\phi,1}$ is identity.

(ii) $\mathbf{D}_e = \mathbf{D}_{\mathfrak{p}}$ *if q is a simple root of the Poincaré polynomial $P_W(u)$ of W.*

(iii) *If* $\mathrm{Irr}W = \coprod_{i=0}^2 \mathcal{B}_i$ *with $\mathcal{B}_2 = \mathcal{B}_{\phi,2}$ for some initial character ϕ, then $\mathcal{B}_{\phi,2} = \mathcal{B}^{\mathfrak{p}}_{\phi,2}$.*

Proof. (i): Suppose that $\mathcal{B}_{\phi_i,1}$ is coincident with only a $\mathfrak{p}$-block of $\mathrm{Irr}W$ and $\{i, \Omega(i), \ldots, \Omega^n(i)\} \subset I$ is the subset of indices corresponding to $\mathcal{B}_{\phi_i,1}$. Note that $i \in I_{\mathrm{ini}}$ and $\Omega^n(i) \in I_{\mathrm{ter}}$. Moreover, by Geck's Theorem 6.3 we can label all the projective indecomposable $\mathcal{H}_{\Bbbk}$-modules lying in $\mathcal{B}_{\phi_i,1}$ as $P_i, P_{\Omega(i)}, \ldots, P_{\Omega^{n-1}(i)}$ such that for any $0 \le j \le n-1$

$$\langle\ [(P_{\Omega^j(i)})_{\mathcal{O}}],\ \phi_{\Omega^m(i)}\ \rangle = 0 \text{ if } m < j$$

and

$$\langle\ [(P_{\Omega^j(i)})_{\mathcal{O}}],\ \phi_{\Omega^j(i)}\ \rangle = 1.$$

Then, we know that the p.i.m. $P_{\Omega^{n-1}(i)}$ satisfies

$$[P_{\Omega^{n-1}(i)}] = \phi_{\Omega^{n-1}(i)} + \phi_{\Omega^n(i)}. \tag{4}$$

Let $\sigma\mathcal{B}_{\phi_i,1}$ be $\{\sigma(\chi) \mid \chi \in \mathcal{B}_{\phi_i,1}\}$. Then, the above observation also holds for $\sigma(\mathcal{B}_{\phi_i,1})$. Moreover, by the property of σ, we have already known that if $s < t$ then $a(\sigma(\phi_s)) > a(\sigma(\phi_t))$.(see [Gec98a].) Since $\phi_{\Omega^n(i)}$ takes the maximal a-value in $\mathcal{B}_{\phi_i,1}$, $\sigma(\phi_{\Omega^n(i)})$ is the minimal a-value in $\sigma(\mathcal{B}_{\phi_i,1})$. Note that σ induces a bijection $\tilde{\sigma}$ of I defined by $\tilde{\sigma}(j) = l$

if $\sigma(\phi_j) = \phi_l$. (In particular, σ induces a bijection between I_{ini} and I_{ter}.) $\sigma(\phi_i)$ takes the maximal a-value in $\sigma(\mathcal{B}_{\phi_i,1})$. More precisely, we know that

$$a(\sigma(\phi_{\Omega^n(i)})) < a(\sigma(\phi_{\Omega^{n-1}(i)})) < \cdots < a(\sigma(\phi_{\Omega(i)})) < a(\sigma(\phi_i)). \tag{5}$$

By (4) and the above observation we know that $\sigma(\phi_{\Omega(i)}) + \sigma(\phi_i)$ is a character of projective indecomposable $\mathcal{H}_{\Bbbk}$-module lying in $\sigma(\mathcal{B}_{\phi_i,1})$. Hence, by Lemma 5.1, we know that $[(P_i)_{\mathcal{O}}] = \phi_i + \phi_{\Omega(i)}$.

We would like to proceed on induction on $(n, <)$. We assume the following:

Assumption I:

Fix m_0. For any s and m with $0 \leq m < m_0$

$$d^{\mathfrak{p}}_{\Omega^s(i),\Omega^m(i)} = 0 \text{ unless } s = m, m+1, \tag{6}$$

and

$$d^{\mathfrak{p}}_{\Omega^m(i),\Omega^m(i)} = d^{\mathfrak{p}}_{\Omega^{m+1}(i),\Omega^m(i)} = 1. \tag{7}$$

By Lemma 5.1 the above assumption also means the following: for any s and m with $1 \leq m < m_0$

$$d^{\mathfrak{p}}_{\Omega^{-s}(\tilde{\sigma}(i)),\Omega^{-m}(\tilde{\sigma}(i))} = 0 \text{ unless } s = m, m-1.$$

Suppose (for a contradiction) that there exists t with $t < m_0 - 1$ such that $d^{\mathfrak{p}}_{\Omega^{-t}(\tilde{\sigma}(i)),\Omega^{-m_0}(\tilde{\sigma}(i))} > 0$ and $\phi_{\Omega^{-t}(\tilde{\sigma}(i))}$ takes the maximal a-value in $\{\phi_l \in \sigma\mathcal{B}_{\phi,1} \mid d_{\Omega^{-t}(\tilde{\sigma}(i)),l} > 0\}$.

By Lemma 5.1 and the equation 5, we know that

$$[(\sigma P_{\Omega^{-t}(\tilde{\sigma}(i))})_{\mathcal{O}}] = \sigma(\phi_{\Omega^{-t}(\tilde{\sigma}(i))}) + \sigma(\phi_{\Omega^{-m_0}(\tilde{\sigma}(i))}) + \sigma(\phi_{\Omega^{-m_0+1}(\tilde{\sigma}(i))}) + \cdots.$$

This contradicts Assumption I. Hence, the equation (6) holds for $m = m_0$. Moreover, note that the maximality condition forces the coefficient of $\sigma(\phi_{\Omega^{-t}(\tilde{\sigma}(i))})$ to be 1 by the lower unitriangular shape of the decomposition matrix $\mathbf{D}_{\mathfrak{p}}$.

From these facts and Lemma 5.1 the equation (7) follows for $m = m_0$.

(ii): Clear by Lemma 6.10 and (i).

(iii): Clear by (ii). $\square$

§ 7. Representation type

In this section we will show that Uno's conjecture is true for Iwahori-Hecke algebras of exceptional Weyl groups. First, we recall some known facts on Uno's conjecture, which will be used for some reductions of our proof. The following theorem is the starting point on this problem.

Theorem 7.1 (**Uno[Uno92]**).

(i) $\mathcal{H}_{\Bbbk,q}(A_{n-1})$ *is of finite type and not semisimple if and only if* q *is a simple root of the Poincaré polynomial* $P_{W(A_{n-1})}(v)$ *of* $\mathcal{H}_{\mathcal{A},v}(A_{n-1})$.

(ii) $\mathcal{H}_{\Bbbk,q}(I_2(m))$ *is of finite type and not semisimple if and only if* q *is a simple root of the Poincaré polynomial* $P_{I_2(m)}(v)$ *of* $\mathcal{H}_{\mathcal{A},v}(I_2(m))$.

Moreover, Uno[Uno92, Theorem 3.4] also proved the following (see also [AM04, Theorem 2.5] and remarks there):

Theorem 7.2 (**Uno**). *Suppose that* A *is a symmetric indecomposable algebra such that the decomposition matrix of* A *can be written in the following shape:*

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then A *is of finite representation type.*

Theorem 7.3 (**Ariki-Mathas[AM04]**). $\mathcal{H}_{\Bbbk,q}(B_n)$ *is of finite type and not semisimple if and only if* q *is a simple root of the Poincaré polynomial* $P_{W(B_n)}(v)$ *of* $\mathcal{H}_{\mathcal{A},v}(B_n)$.

One of the difficulties to show this theorem is that we can not find any suitable subalgebras like defect groups in $\mathcal{H}_{\Bbbk,q}(B_n)$. This gives an essential difference from type A. Ariki and Mathas overcame this by using Dipper-James-Murphy's Specht theory, Ariki's classification of simple modules, James-Mathas' Jantzen-Schaper theorem, and so on.

Using the above theorem, an embedding of $\mathcal{H}_{\Bbbk,q}(D_{2e})$ into some Hecke algebras of type A and B, and so on, Ariki showed the following:

Theorem 7.4 (**Ariki[Ari05]**). $\mathcal{H}_{\Bbbk,q}(D_n)$ *is of finite type and not semisimple if and only if* q *is a simple root of the Poincaré polynomial* $P_{W(D_n)}(v)$ *of* $\mathcal{H}_{\mathcal{A},v}(D_n)$.

In Lemma 6.11 we have already seen that the decomposition matrix of a Φ_e-block with Φ_e-defect 1 completely controls that of the corresponding $\mathfrak{p}$-block. So, by Theorem 7.2 we deduce that if q is a simple root of $P_W(v)$ then $\mathcal{H}_{\Bbbk,q}(W)$ is of finite type.

On the other hand, in order to verify Uno's conjecture, we can use some reductions on the Mackey system of the Iwahori-Hecke algebras for the Weyl groups as in Ariki-Mathas [AM04, Corollary 3.2]. By Theorem 7.1, Theorem 7.3, and Theorem 7.4, it is enough to consider the following cases:

(i) $F_4, e = 3, 4, 6$.

(ii) $E_6, e = 6$.

(iii) $E_8, e = 5, 8, 10, 12$.

The case $F_4, e = 3$ will be well-understood by [Miy01] as follows:

Theorem 7.5. *Let $\Bbbk$ be a field such that $1+q+q^2 = 0$ in $\Bbbk$ and the characteristic of $\Bbbk$ is not 2. Then, the principal block of $\mathcal{H}_{\Bbbk,q}(F_4)$ is isomorphic to $\mathcal{H}_{\Bbbk,q}(A_2) \otimes_{\Bbbk} \mathcal{H}_{\Bbbk,q}(A_2)$.*

From now on $\mathcal{H}(W)$ always means a specialized Hecke algebra $\mathcal{H}_{\Bbbk,q}(W)$ with 1-parameter $q \in \Bbbk$ with the minimal positive integer e such that $[e]_q = 0$ in $\Bbbk$, unless stated otherwise. Moreover, B will always mean the principal Φ_e-block $\mathcal{H} = \mathcal{H}(W)$ with Φ_e-defect 2 for given e and $\mathcal{H}'$ will mean a suitable parabolic subalgebra of $\mathcal{H}$ in the rest of sections.

The rest of the proof for Uno's conjecture, namely, Theorem 1.3, goes as follows, but we shall avoid writing notes of computations in this paper.

(i) For the cases $(W, e) = (F_4, 4)$ and $(E_8, 5)$, we can find some σ-stable projective indecomposable $\mathcal{H}$-modules as in Lemma 5.2. 4 induced from some parabolic subalgebra $\mathcal{H}'$. Let P be such an indecomposable projective $\mathcal{H}'$-module with $P\!\uparrow^B \cong (P\!\uparrow^B)^\sigma$ indecomposable. Let T be the top of P (equivalently the socle of P). Then, we will see that T has the properties:

   (a) $T\!\uparrow^B$ is selfdual by Proposition 3.2 and Lemma 3.4,

   (b) $\mathrm{Top}(T\!\uparrow^B)$ is simple by Lemma 5.2 (i) (b),

   (c) $T\!\uparrow^B$ is not σ-stable, which we will prove in the later sections by checking the composition factors of $T\!\uparrow^B$.

   The explicit choices for $(W, e) = (F_4, 4)$ and $(E_8, 5)$ are:

$(F_4, 4)$: Set $T := D^{(\emptyset,21)}$ (resp. $T^\sigma := D^{(21,\emptyset)}$) to be the simple $\mathcal{H}(B_3)$-module corresponding to $\phi_{(\emptyset,21)}$ (resp. $\phi_{(21,\emptyset)}$). Set P to be the projective cover of $D^{(\emptyset,21)}$.

$(E_8, 5)$: By [Gec92, Theorem 12.6] and Lemma 6.11 we know that the character $\phi_{84,12}+\phi_{216,16}$ corresponds to a projective indecomposable module over $\mathcal{H}(E_7)$. Set P to be this module. Set T (resp. T^σ) to be the simple module corresponding the character $\phi_{84,12} - \phi_{1,0}$ (resp. $\phi_{84,15} - \phi_{1,63}$). T and T^σ belong to different blocks.

(ii) For the cases $(W, e) = (F_4, 6), (E_8, 8), (E_8, 10)$, and $(E_8, 12)$, we can find a projective indecomposable $\mathcal{H}'$-module, say, P such that P is not σ-stable, $P\uparrow^B$ is σ-stable and is isomorphic to a direct sum of two non-isomorphic projective indecomposable B-modules. Let T be the top of P. Then, we will see that $T\uparrow^B$ has the properties (1) to (6) in Lemma 5.4 as M does.

(a) $T\uparrow^B$ is selfdual by Proposition 3.2 and Lemma 3.4.

(b) The top of $T\uparrow^B$ will be isomorphic to a direct sum $S \oplus S'$ of certain two non-isomorphic simple B-modules S and S' by known decomposition matrices, Frobenius reciprocity and some direct calculations.

(c) $\mathrm{Top}(T\uparrow^B)$ is σ-stable since $P\uparrow^B$ is so.

(d) The condition on the Cartan invariant $c_{S,S'}$ is easily checked by the decomposition matrices in [GL91],[Gec93],[Mül01].

(e) The multiplicity-free property of $\mathrm{Rad}(T\uparrow^B)$ will be checked by the condition on the top of $T\uparrow^B$ by using Frobenius reciprocity and the computation for the composition factors of M.

(f) $[T\uparrow^B]$ will not be σ-stable by the same calculation.

The explicit choices of T and P for $(W, e) = (F_4, 6), (E_8, 8), (E_8, 10)$ and $(E_8, 12)$ are:

$(F_4, 6)$: Set $T := D^{(2,1)}$ to be the simple $\mathcal{H}(B_3)$-module corresponding to $(\phi_{(2,1)} - \phi_{(3,\emptyset)})$. Set P to be the projective cover of T.

$(E_8, 8)$: Set P to be the projective indecomposable module corresponding to the character $\phi_{216,6} + \phi_{280,8}$. Set T to be the top of P.

$(E_8, 10)$: Set P to be the projective indecomposable module corresponding to the character $\phi_{420,10} + \phi_{405,15}$. Set T to be the top of P.

$(E_8, 12)$: Set P to be the projective indecomposable module corresponding to the character $\phi_{336,11} + \phi_{280,18}$. Set T to be the top of P.

(iii) The case $E_6, e = 6$, unlike the previous cases, will be done by showing an existence of two non-isomorphic indecomposable modules M and N as in Lemma 4.3. Write $\{\nu_0, \nu_1, \nu_2, \nu_3, \nu_3', \nu_5, \nu_6, \nu_7, \nu_7', \nu_{11}, \nu_{15}\}$ for a complete set of simple B-modules. Here, those correspond to

$$\{\phi_{1,0}, \phi_{6,1}, \phi_{20,2}, \phi_{30,3}, \phi_{15,4}, \phi_{60,5}, \phi_{24,6}, \phi_{80,7}, \phi_{60,8}, \phi_{60,11}, \phi_{30,15}\}$$

by Geck's a-value order, which is consistent with [Gec93].

Since the branching rule between B and $\mathcal{H}(D_5)$ and the decomposition matrix for B and $\mathcal{H}(D_5)$ are known, we can get the following result:

Lemma 7.6.

(a) Let $P^{(1,211)}$ be the projective indecomposable $\mathcal{H}(D_5)$-module corresponding to $\phi_{(1,211)} + \phi_{(\emptyset,2111)}$. Then $P^{(1,211)} \uparrow^B \cong P(\nu_7)$.

(b) Let $P^{(11,21)}$ be the projective indecomposable $\mathcal{H}(D_5)$-module corresponding to $\phi_{(11,21)} + \phi_{(11,111)}$. Then, $P^{(11,21)} \uparrow^B \cong P(\nu_7) \oplus P(\nu_7')$.

Let S be the simple $\mathcal{H}(D_5)$-module corresponding to $\phi_{(11,111)} - \phi_{(\emptyset,11111)}$. So, $P^{(11,21)}$ is the projective cover of S.

Lemma 7.7.

$$S \uparrow^B \leftrightarrow \nu_5 + \nu_6 + 2\nu_7 + 2\nu_7' + \nu_{11}.$$

Proof.

$$\begin{aligned}(\phi_{(11,111)} - \phi_{(\emptyset,11111)}) \uparrow^B &= \phi_{60,11} + \phi_{30,15} + \phi_{15,16} - \phi_{6,25} - \phi_{1,36} \\ &\leftrightarrow (\nu_5 + \nu_7 + \nu_7' + \nu_{11}) + (\nu_6 + \nu_7 + \nu_{15}) + (\nu_7' + \nu_{11}) \\ &\quad -(\nu_{15}) - (\nu_{11}) \\ &= \nu_5 + \nu_6 + 2\nu_7 + 2\nu_7' + \nu_{11}.\end{aligned}$$

□

Proposition 7.8. $\mathrm{End}_{\mathcal{H}(E_6)}(P(\nu_7)) \ncong \Bbbk[x]/(x^n)$ *for any n. In particular, by Lemma 4.2, Uno's conjecture is true for $\mathcal{H}(E_6)$.*

Proof. Note that $\mathrm{Ext}^1(\nu_6, \nu_{11}) = \mathrm{Ext}^1(\nu_{11}, \nu_6) = 0$ since the Cartan invariant $c_{\nu_6,\nu_{11}} = 0$ by the decomposition matrix in [Gec93]. By Frobenius reciprocity, we know

$$\dim \mathrm{Hom}_{\mathcal{H}(E_6)}(S \uparrow^B, \nu_7) = \dim \mathrm{Hom}_{\mathcal{H}(E_6)}(S \uparrow^B, \nu_7') = 1.$$

Suppose that $S \uparrow^B$ is decomposable. Put $S \uparrow^B = V_7 \oplus V_7'$ such that $\mathrm{Top}(V_7) = \nu_7$ and $\mathrm{Top}(V_7') = \nu_7'$. Looking at the composition factors of $S \uparrow^B$, we deduce $\mathrm{Soc}(V_7) = \nu_7$ and $\mathrm{Soc}(V_7') = \nu_7'$. By the selfduality of $S \uparrow^B$, we know that the Loewy length of $S \uparrow^B$ is 3. In order to show that $B = B_0(\mathcal{H}(E_6))$ has infinite representation type we just check that $V_7 \ncong \nu_7$ by Lemma 4.3 since we have already known that there exists a B-module

$$\phi_{(\emptyset,2111)} \uparrow^B = \begin{pmatrix} & \nu_7 & \\ \nu_3 & \nu_{11} & \nu_{15} \\ & \nu_7 & \end{pmatrix} \leftrightarrow \phi_{24,12} + \phi_{20,20}.$$

Suppose that $V_7 = \nu_7$. So, V_7' has the following Loewy structure:

$$V_7' = \begin{pmatrix} \nu_7' \\ \nu_5 \ \nu_6 \ \nu_7 \ \nu_{11} \\ \nu_7' \end{pmatrix}.$$

However, $\mathrm{Ext}^1(\nu_7', \nu_6) = 0 = \mathrm{Ext}^1(\nu_6, \nu_7')$. Hence $V_7 \not\cong \nu_7$. Next, we assume that $S \uparrow^B$ is indecomposable. Then, $S \uparrow^B$ has the following Loewy structure:

$$S \uparrow^B = \begin{pmatrix} \nu_7 \ \nu_7' \\ \nu_5 \ \nu_6 \ \nu_{11} \\ \nu_7 \ \nu_7' \end{pmatrix}.$$

Let π_8 be a canonical epimorphism from $S \uparrow^B$ to ν_7'. The top and socle of $\mathrm{Ker}(\pi_8)/\nu_7'$ contain ν_7. Moreover, $\phi_{(\emptyset,2111)} \uparrow^B$ is not isomorphic to any direct summand of $\mathrm{Ker}(\pi_8)/\nu_7'$. By Lemma 4.3 we are done. □

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References

[Alp86] J. L. Alperin. *Local representation theory*, volume 11 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[AM04] S. Ariki and A. Mathas. The representation type of Hecke algebras of type B. *Adv. Math.*, 181(1):134–159, 2004.

[Ari05] S. Ariki. Hecke algebras of classical type and their representation type. *Proc. London Math. Soc. (3)*, 91(2):355–413, 2005.

[CR87] C. W. Curtis and I. Reiner. *Methods of Representation Theory*, volume II. Wiley-Interscience, 1987.

[Gec92] M. Geck. Brauer trees of Hecke algebras. *Comm. Algebra.*, 20:2937–2973, 1992.

[Gec93] M. Geck. The decomposition numbers of the Hecke algebra of type E_6^*. *Math. Comp.*, 61:889–899, 1993.

[Gec98a] M. Geck. Kazhdan-Lusztig cells and decomposition numbers. *Represent. theory*, 2:264–277, 1998.

[Gec98b] M. Geck. Representations of Hecke algebras at roots of unity,Séminaire Bourbaki, 50 ème année,1997-98. *Astérisque*, 252:33–55, 1998.

[Gec00] M. Geck. On the number of simple modules of Iwahori-Hecke algebras of finite Weyl groups. *Bul. Stiit. Univ. Baia Mare, Ser B*, 16:235–246, 2000.

[GL91] M. Geck and K. Lux. The decomposition numbers of the Hecke algebra of type F_4. *Manuscripta Math.*, 70:285–306, 1991.

[GM] M. Geck and J. Müller. James' Conjecture for Hecke algebras of exceptional type, I, arXiv:0712.1620.

[GP00] M. Geck and G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*, volume 21 of *London Math. Soc. Monographs New Series.* Oxford University Press, New York, 2000. xvi+446 pp.,ISBN: 0-19-850250-8.

[GR97] M. Geck and R. Rouquier. Centers and simple modules for Iwahori-Hecke algebras. In Marc Cabanes, editor, *Finite Reductive Groups: Related Structure and Representations*, volume 141, pages 251–272, Boston, 1997. Birkhäuser.

[Jam90] G. D. James. The decomposition matrices of $GL_n(q)$ for $n \leq 10$. *Proc. London Math. Soc.*, 60(3):225–265, 1990.

[Miy01] H. Miyachi. On the Iwahori-Hecke algebra of type F_4 at a 3-rd root of unity. *preprint*, 2001.

[Mül01] J. Müller. Decomposition numbers for generic Iwahori-Hecke algebras of type E_8. *preprint*, 2001.

[S+95] Martin Schönert et al. *GAP – Groups, Algorithms, and Programming.* Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995.

[Uno92] K. Uno. On representations of non-semisimple specialized Hecke algbras. *J. Algebra*, 149:287–312, 1992.