On the structure of parabolic Humphreys-Verma modules

By

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Through an investigation into the bounded derived category $\mathbb{D}^b(\text{coh}\mathcal{P})$ of the coherent sheaves on a projective homogeneous variety $\mathcal{P}$ we have been led to study the parabolic Humphreys-Verma modules. Although these modules are defined only in positive characteristic, our geometric application appears effective characteristic-free.

Write $\mathcal{P} = G/P$ with $G$ a reductive algebraic group over an algebraically closed field of positive characteristic and $P$ a parabolic subgroup of $G$. Let $G_1$ be the Frobenius kernel of $G$ and let $\nabla_P(\varepsilon)$ be the $G_1P$-module induced from $1$-dimensional trivial $P$-module $\varepsilon$, a parabolic Humphreys-Verma module. We have recently found a way, though verified in only few limited cases yet, to parametrize certain components of the $G_1T$-socle series of $\nabla_P(\varepsilon)$ by the set $W_P$, $w \in W_P$, on $\mathcal{P}$ form a Karoubian complete strongly exceptional poset for $\mathbb{D}^b(\text{coh}\mathcal{P})$. Those sheaves are defined over $\mathbb{Z}$ to verify Catanese’s conjecture [Bö] transferring over to $\mathbb{C}$; in some cases our constructions offer a new evidence to the conjecture in complex algebraic geometry. In this note, however, we will focus on the structure of Humphreys-Verma modules.

In order to be precise in which category the morphisms are taken, we will write $\mathcal{C}(X, Y)$ for the set of morphisms in category $\mathcal{C}$ from object $X$ to $Y$. For an algebraic group $H$ we let $\text{HMod}$ denote the category of rational $H$-modules, and for a variety $X$ the category of modules over the structure sheaf of $X$ will be denoted by $\text{Mod}_X$.

This is an expanded version of the author’s talk at a RIMS meeting under the title of the present volume. Subsequent to the talk I have come up with a description of the structure of Humphreys-Verma modules for projective spaces, which is included in §3; in the talk I could merely exhibit the computations in the cases of $\text{GL}_2$ and $\text{GL}_3$. 

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§ 1. Humphreys-Verma modules

Let $\mathbb{k}$ be an algebraically closed field of positive characteristic $p$. We will assume $p$ is sufficiently large. Let $G$ be a reductive algebraic group over $\mathbb{k}$, $P$ a parabolic subgroup of $G$, and $G_1$ the Frobenius kernel of $G$. We call the functor

$$\text{ind}^{G_1 P}_{P} = \text{Sch}_{\mathbb{k}}(G_1 P,?)^P : \text{PMod} \to G_1 \text{PMod}$$

parabolic Humphreys-Verma induction and write $\hat{\nabla}_P$ for short.

In case $P = B$ a Borel subgroup of $G$ put $\nabla = \hat{\nabla}_B$. Let $\Lambda$ be the character group of $B$, $T$ a maximal torus of $B$, $R \subset \Lambda$ the root system of $G$ relative to $T$. We choose a positive system $R^+$ of $R$ such that the roots of $B$ are $-R^+$, and let $R^s$ be the set of simple roots. Let $W = \text{N}_G(T)/T$ the Weyl group of $G$, $W_p = W \rtimes p\mathbb{Z}R$, $S_p = \{s_i, s_0 \mid \alpha_i \in R^s\}$ with $s_i$ the reflexion associated to $\alpha_i$ and $s_0$ the reflexion in the wall $\{v \in \Lambda \otimes \mathbb{Z}R \mid \langle v, \alpha_i^\vee \rangle = -p\}$, $\alpha_0^\vee$ the highest coroot. Thus $(W_p, S_p)$ forms a Coxeter system. We will consider the dot action of $W_p$ on $\Lambda$ such that $x \bullet \lambda = x(\lambda + \rho) - \rho$, $x \in W_p$, $\lambda \in \Lambda$, $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. One has $\hat{\nabla}(\lambda + p\mu) \simeq \hat{\nabla}(\lambda) \otimes p\mu \forall \mu \in \Lambda$, and each $\hat{\nabla}(\lambda)$ has a $G_1 T$-simple socle $\hat{L}(\lambda)$ of highest weight $\lambda$ with all other composition factors having their highest weights $< \lambda$. If $\Lambda_1 = \{\lambda \in \Lambda \mid \langle \lambda, \alpha_i^\vee \rangle \in [0, p[ \forall \alpha \in R^s\}$ the set of restricted dominant weights, a simple $G_1 T$-module of highest weight $\lambda \in \Lambda_1$ admits a structure of simple $G$-module $L(\lambda)$. Thus, the determination of the composition factors of all $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda_1$, will yield all irreducible characters for $G$ by Steinberg’s tensor product theorem. Moreover, let $\mathcal{A}$ be the set of alcoves on $\Lambda$ with respect to the dot action of $W_p$, $\forall A \in \mathcal{A}$, $\forall w \in W$, $\forall \gamma \in \mathbb{Z}R$, we will write $Aw^{-1}t_{p\gamma}$ for the alcove $(w \bullet A) + p\gamma$. Let $0_A$ be the image of $0 \in \Lambda$ in $A$ under the $W_p$-action. By the translation principle [J, II.7] the structure of $\hat{\nabla}(0_A)$ describes that of all other $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda \cap A$, and the determination of irreducible characters of all $L(0_A)$, $0_A \in \Lambda_1$, will obtain all irreducible characters for $G$, which has now been achieved for indefinitely large $p$ by Andersen, Jantzen and Soergel [AJS] to verify Lusztig’s conjecture. We will thus write $\hat{\nabla}(A)$ for $\hat{\nabla}(0_A)$.

The Lusztig conjecture on the irreducible $G$-characters, in turn, determines the $G_1 T$-socle series of each $\hat{\nabla}(A)$; let $\text{soc}_i \hat{\nabla}(A)$ be the $i$-th $G_1 T$-socle layer of $\hat{\nabla}(A)$ and let $\hat{L}(C)$ be the simple $G_1 T$-module of highest weight $0_C$; by the linkage principle [J, II.9.15] the composition factors of $\hat{\nabla}(A)$ are of the form $\hat{L}(C)$. Let $w_0 \in W$ be the longest element of $W$ with respect to $\{s_i \mid \alpha_i \in R^s\}$. Then

**Theorem 1.1** ([AK]/[RIMS]). Let $A \in \mathcal{A}$.

(i) The Loewy length, that is the length of the socle series, of each $\hat{\nabla}(A)$ is $\ell(w_0) + 1$. 

(ii) If $0_C \in \Lambda_1 + p(\xi - \rho), \xi \in \Lambda$, then with $w_\xi = w_0 t_p(\xi - w_0 \xi)$

$$Q_{A,Cw_\xi} = \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A,Cw_\xi)+i-\ell(w_0)-1\}} \text{soc}_i \hat{\nabla}(A) : \hat{L}(C),$$

where $Q_{A,C}$ is the periodic inverse Kazhdan-Lusztig polynomial [L] associated to $A, C \in \mathcal{A}$ and $d(A,Cw_\xi)$ is the distance from $A$ to $Cw_\xi$.

### §2. Parabolic Humphreys-Verma modules

We wish to obtain a formula to describe the socle series for general parabolic $P$. Let $\Lambda_P = \text{Grp}_k(P, \text{GL}_1)$, $L$ the standard Levi subgroup of $P$, $U_L$ the unipotent radical of $B \cap L$, $\text{Dist}(U_L)$ the algebra of distributions on $U_L$, $R_L \subseteq R$ the root system of $L$, and $W_P = \langle s_\alpha \mid \alpha \in R_L \rangle \leq W$ the Weyl group of $P$ and also of $L$. $\forall \nu \in \Lambda_P$, regarded as $G_1 T$-modules

$$\hat{\nabla}_P(\nu) = \text{ind}_{P_1}^{G_1 P}(\nu) \simeq \text{ind}_{P_1 T}^{G_1 T}(\nu),$$

where $P_1$ is the Frobenius kernel of $P$. We call $\hat{\nabla}_P(\nu)$ the parabolic Humphreys-Verma module of highest weight $\nu$; if $\rho_P = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_L} \alpha$,

$$\hat{\nabla}_P(\nu) \simeq \text{ind}_{P_1}^{G_1}(\nu) \simeq \text{Dist}(G_1) \otimes_{\text{Dist}(P_1)} (\nu - 2(p-1)\rho_P) \quad \text{by \ [J, II.9.2].}$$

If $U_P^+$ is the subgroup of $G$ generated by the root subgroups $U_\alpha$, $\alpha \in R^+ \setminus R_L$, and if $U_{P,1}^+$ is the Frobenius kernel of $U_P^+$, then $\text{Dist}(G_1) \simeq \text{Dist}(U_{P,1}^+) \otimes_k \text{Dist}(P_1)$, and hence

$$\hat{\nabla}(\nu) \simeq \text{Dist}(U_{P,1}^+) \otimes_k (\nu - 2(p-1)\rho_P)$$

with $\text{Dist}(U_{P,1}^+) \simeq \otimes_{\alpha \in R^+ \setminus R_L} \text{Dist}(U_{\alpha,1})$. To relate a parabolic Humphreys-Verma module to ordinary Humphreys-Verma modules, one has at the character level

$$\chi(\hat{\nabla}_P(\nu)) \simeq \chi(\text{Dist}(U_{P,1}^+) \otimes_k (\nu - 2(p-1)\rho_P)),$$

where $\chi$ denotes the character.
Proposition 2.1 ([KY]). \( \forall \nu \in \Lambda, \)
\[
\operatorname{ch}_{\nu} \hat{\nabla}_{P}(\nu) = e^{\nu} \prod_{\alpha \in R_{+} \setminus R_{L}} \frac{1 - e^{-\rho \alpha}}{1 - e^{-\alpha}} = \sum_{w \in W_{P}} (-1)^{\ell(w)} \dim(\operatorname{Dist}(U_{L_{\gamma}})) \operatorname{ch}_{w \bullet \nu + p \gamma} \hat{\nabla}.
\]

Now let \( A \in \mathcal{A} \) with \( 0_{A} \in \Lambda_{P} \). As \( \hat{\nabla}_{P}(A) \leq \hat{\nabla}(A) \) as \( G_{1}B \)-modules, a naïve speculation on the \( G_{1}T \)-socle series of \( \hat{\nabla}_{P}(A) \) would be that \( \forall C \in \mathcal{A} \) with \( 0_{C} \in \Lambda_{1} + p(\xi - \rho), \xi \in \Lambda, \)
\[
(1) \quad \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A, Cw_{\xi}) + i - \ell(w_{0}) - 1\}} [\operatorname{soc}_{i} \hat{\nabla}_{P}(A) : \hat{L}(C)] = \sum_{w \in W_{P}} (-1)^{\ell(w)} \dim(\operatorname{Dist}(U_{L_{\gamma}})) i \in \mathbb{N} w \in W_{P}
\]
\[
= \sum_{w \in W_{P}, \gamma \in \mathbb{Z}R_{L}} (-1)^{\ell(w)} \dim(\operatorname{Dist}(U_{L_{\gamma}})) Q_{Aw^{-1}t_{p \gamma}, Cw_{\xi}}.
\]

By design (1) holds under the specialization \( q \rightsquigarrow 1 \), and also in case \( P = B \).

Proposition 2.2 ([KY]). (1) holds for \( G \) of rank \( \leq 2 \).

If \( W_{P} = \{ w \in W \mid wR_{L}^{+} \subseteq R^{+} \} \), \( W = \bigsqcup_{w \in W_{P}} wW_{P} \). Let \( w_{0,P} \in W_{P} \) with \( w_{0,P}R_{L}^{+} = -R_{L}^{+} \), and set \( w_{0}^{P} = w_{0}w_{0,P} \in W_{P} \). \( \forall x \in W_{P}, \exists! w \in W: x \bullet 0 \equiv w \bullet 0 \mod p\Lambda \). \( \forall w \in W \), choose \( 0_{P,w} \in \Lambda_{1} \) such that \( 0_{P,w} \equiv w_{0}ww_{0,P} \bullet 0 \mod p\Lambda \). Then \( W_{P} \bullet 0 + p\Lambda = \bigsqcup_{w \in W_{P}} (0_{P,w} + p\Lambda) \).

Theorem 2.3 ([KY]). Assume \( \text{rk} G \leq 2 \). Let \( A \in \mathcal{A} \) with \( 0_{A} \in \Lambda_{P} \).
(i) The Loewy length of \( \hat{\nabla}_{P}(A) \) is \( \ell(w_{0}^{P}) + 1 \).
(ii) \( \forall i \in [1, \ell(w_{0}^{P}) + 1] \), there is a decomposition as \( G_{1}P \)-modules
\[
\operatorname{soc}_{i} \hat{\nabla}_{P}(A) = \prod_{w \in W_{P}} L(0_{P,w}) \otimes G_{1}\text{Mod}(L(0_{P,w}), \operatorname{soc}_{i} \hat{\nabla}_{P}(A)).
\]
(iii) In case \( A = A^{+} \) the bottom dominant alcove,
\[
G_{1}\text{Mod}(L(0_{P,w}), \operatorname{soc}_{\ell(w_{0}^{P})+1-\ell(w)} \hat{\nabla}_{P}(A^{+})) \neq 0 \quad \forall w \in W_{P}.
\]

Remark 2.4. (i) In case \( P = B \), each \( L(0_{B,w}), w \in W \), appears as a \( G_{1} \)-composition factor of any \( \hat{\nabla}(A) \).
To see that, \( \forall x, y \in W, \forall \mu \in \Lambda \),

\[
[\hat{\nabla}(A^+ x) : L(0_{B,y}) \otimes p \mu] = [\hat{\nabla}(x^{-1} \cdot 0) : \hat{L}(0_{B,y} + px^{-1} x \cdot \mu)]
\]

\[
= [\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,y} + px \cdot \mu)]
\]

by [J, II.9.16.4] with \( \varepsilon \) denoting \( 0 \in \Lambda \).

It is therefore enough to show that \( \forall w \in W, \exists \mu \in \Lambda : [\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p \mu)] \neq 0 \). Write \( 0_{B,w} = w_0 w \cdot 0 + p \eta \) for some \( \eta \in \Lambda \). Then

\[
[\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p \mu)] = [\hat{\nabla}(\varepsilon) : \hat{L}(w_0 w \cdot 0 + p(\eta + \mu))] = [\hat{\nabla}(-p(\eta + \mu)) : \hat{L}(w_0 w \cdot 0)].
\]

Thus we have only to check \( \forall C \in A, \exists \gamma \in \mathbb{Z}R : [\hat{\nabla}(A^+ t_{p\gamma}) : \hat{L}(C)] \neq 0 \). If \( C \subseteq \Lambda_1 + p(\xi - \rho) \) for some \( \xi \in \Lambda \), then by [Y] \( \forall w \in W, [\hat{\nabla}(A^+ wt_{p\xi}) : \hat{L}(C)] \neq 0 \). Write \( A^+ t_{p\xi} = A^+ t_{p\gamma} y, \gamma \in \mathbb{Z}R, y \in W \). Then

\[
A^+ y^{-1} t_{p\xi} = A^+ t_{p\gamma} y t_{-p\xi} y^{-1} t_{p\xi} = A^+ t_{p\gamma} t_{-py\xi} t_{p\xi} = A^+ t_{p(\gamma - y\xi + \xi)}.
\]

If \( y = y_1 y_2 \) with \( y_1, y_2 \in W \), then \( y_1 \xi - \xi = y_1 y_2 \xi - \xi = y_1 (y_2 \xi - \xi) + y_1 \xi - \xi \), and \( \forall \alpha \in R^+, s_{\alpha} \xi - \xi = \xi - \langle \xi, \alpha^\vee \rangle \alpha - \xi = -\langle \xi, \alpha^\vee \rangle \alpha \in \mathbb{Z}R \). Thus \( \gamma - y\xi + \xi \in \mathbb{Z}R \), as desired.

(ii) In application to the study of \( D^b(\operatorname{coh}\mathcal{P}), \mathcal{P} = G/P \), the \( P \)-module structure on each \( G_1 \mathcal{M}od(L(0_{P,w}), \operatorname{soc}_{\ell(w_{0P}^P) + 1 - \ell(w)} \hat{\nabla}_{P}(A^+)), w \in W^P \), appears to play an important role: as \( G_1 \) acts trivially on those, untwisting the Frobenius, put \( \operatorname{soc}^1_{P,w} = G_1 \mathcal{M}od(L(0_{P,w}), \operatorname{soc}_{\ell(w_{0P}^P) + 1 - \ell(w)} \hat{\nabla}_{P}(A^+))^{-1} \). It appears from [KY] that each \( \operatorname{soc}^1_{P,w} \) admits a direct summand \( E_w \) such that, writing \( \mathcal{L}_\mathcal{P}(E_w) \) for the locally free sheaf on \( \mathcal{P} \) associated to \( E_w \), \( \{\mathcal{L}_\mathcal{P}(E_w) \mid w \in W^P\} \) forms a Karoubian complete strongly exceptional poset such that \( \operatorname{Mod}_\mathcal{P}(\mathcal{L}_\mathcal{P}(E_x), \mathcal{L}_\mathcal{P}(E_y)) \neq 0 \) iff \( x \leq y \). Moreover, those \( E_w \) are defined over \( \mathbb{Z} \) to yield also a Karoubian complete strongly exceptional poset in characteristic \( 0 \).

§ 3. Projective spaces

Let \( E \) be a \( k \)-linear space of basis \( e_1, \ldots, e_{n+1}, G = \operatorname{GL}(E) \), and \( P = N_G(ke_{n+1}) \). Thus \( \mathcal{P} = G/P \simeq \mathbb{P}^{n}_{\mathbb{F}_k} \). If \( F_{\mathcal{P}} \) (resp. \( F_k \)) is the absolute Frobenius morphism on \( \mathcal{P} \) (resp. \( \operatorname{Spec}(k) \)) and if \( q : G/P \rightarrow G/G_1 P \) is a natural morphism, one has a commutative diagram of schemes

\[
\begin{array}{ccc}
P \ar[r] & \mathcal{P} \ar[r] & \operatorname{Spec}(k) \\
q \ar[r] & \phi \ar[u] & \\
G/G_1 P \ar[r]_{\sim} & \mathcal{P}^{(1)} \ar[u]_{F_k} & \\
& & \\ 
\end{array}
\]
If $A^+$ is the bottom dominant alcove, one has from [Haa]

$$F_{P*} \mathcal{O}_P \simeq \phi_* \mathcal{L}_{G/G_1 P}(\hat{\nabla}_P(A^+)).$$

On the other hand, we know from [HKR]

$$F_{P*} \mathcal{O}_P \simeq \prod_{i=0}^{n} \mathcal{O}_P(-i) \otimes_{\mathbb{k}} V_i,$$

where $V_i = \prod_{j \in [0, p^{n+1}]} \mathbb{k} x^j$ in the polynomial algebra $\mathbb{k}[x_1, \ldots, x_{n+1}]$ with $x^j = \prod_{i=1}^{n+1} x_i^{j_i}$ and $|j| = \sum_{i=1}^{n+1} j_i$ if $j = (j_1, \ldots, j_{n+1})$. Regarding $x_1, \ldots, x_{n+1}$ as the dual basis of $e_1, \ldots, e_{n+1}$, let $G$ act on $\mathbb{k}[x_1, \ldots, x_{n+1}]$ and also on $\mathbb{k}[x_1, \ldots, x_{n+1}]/(x_1^p, \ldots, x_{n+1}^p)$ contragrediently. Then one can equip $V_i$ with a structure of $G$-module by identifying it with its image in $\mathbb{k}[x_1, \ldots, x_{n+1}]/(x_1^p, \ldots, x_{n+1}^p)$.

Now let $B$ be a Borel subgroup of $P$ consisting of lower triangular matrices and $T$ a maximal torus of $B$ consisting of diagonal matrices. Identify $A$ with $\mathbb{Z}^{\oplus n+1}$ via the basis $\epsilon_i : \text{diag}(a_1, \ldots, a_{n+1}) \mapsto a_i, i \in [1, n+1]$, and $W$ with the symmetric group $\mathfrak{S}_{n+1}$ permuting the $\epsilon_i, i \in [1, n+1]$. Then $W^P = \{(i (i+1 \ldots n+1) | i \in [1, n+1]\}$, and Serre’s twisted sheaf on $\mathbb{P}^n$ is given by $\mathcal{O}_P(1) = L_P(-\epsilon_{n+1}). \forall i \in [1, n+1]$, set $\lambda_{(i (i+1 \ldots n+1)} = (i-1)\epsilon_{n+1}$ and $0_P(i (i+1 \ldots n+1) = -(i-1)\epsilon_{n+2-i} - (p-1)(\epsilon_{n+3-i} + \cdots + \epsilon_{n+1}) \in \Lambda_1$, which we agree to be $0$ in case $i = 1$. Then $V_{i-1} \simeq L(0_P(i (i+1 \ldots n+1) \forall i \in [1, n+1]$, and hence

$$F_{P*} \mathcal{O}_P \simeq \prod_{i=1}^{n+1} \mathcal{L}_P(\lambda_{(i (i+1 \ldots n+1)}) \otimes_{\mathbb{k}} L(0_P(i (i+1 \ldots n+1)).$$

Confirming the pattern in Theorem 2.3, it holds that

**Theorem 3.1** ([K]). Assume $p \geq n + 1$.

(i) The Loewy length of $\hat{\nabla}_P(A^+)$ is $n + 1 = \ell(w_0^P) + 1$.

(ii) $\forall i \in [1, n+1]$, $\text{soc}_i \hat{\nabla}_P(A^+) \simeq L(0_P(i (i+1 \ldots n+1)) \otimes \lambda_{(i (i+1 \ldots n+1)}^{[1]}.$

**Remark 3.2.** Regardless of characteristic $\{\mathcal{L}_P(\lambda_w) | w \in W^P\}$ forms a complete strongly exceptional poset on $\mathbb{P}^n$ such that $\text{Mod}_{\mathbb{P}^n}(\mathcal{L}_P(\lambda_x), \mathcal{L}_P(\lambda_y)) \neq 0$ iff $x \leq y$ [HKR] / [K08].

### References

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[KY] Kaneda M. and Ye J.-C., *Some observations on Karoubian complete strongly exceptional posets on the projective homogeneous varieties*, to appear

