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Universal Tropical R map of $\mathfrak{sl}_2$ and Prehomogeneous Geometric Crystals

Dedicated to Professor Masaki Kashiwara on the occasion of his 60th birthday

By

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§ 1. Introduction

The notion of geometric crystal is a sort of geometric lifting of crystal base theory. If a geometric crystal χ is "positive", it can be "ultra-discretized" to a crystal $\mathcal{UD}(\chi)$ (for more details, see below). For a geometric crystal $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$, if there exists an open dense orbit by e_i^c's, we call χ *prehomogeneous*, where $e_i : \mathbb{C}^\times \times X \to X$ ($e_i^c(x) = e_i(c, x)$) defines certain rational $\mathbb{C}^\times$-action on X. For isomorphic prehomogeneous geometric crystals χ_1 and χ_2 ($f : \chi_1 \xrightarrow{\sim} \chi_2$), assume that there exists another isomorphism $f' : \chi_1 \to \chi_2$ such that $f(p) = f'(p)$ for some point $p \in Z \subset X_1$ where Z is an open dense orbit in X_1. Then it is easy to see that $f = f'$ as a rational morphism, which resembles Schur's lemma. In this sense, prehomogeneity of geometric crystals corresponds to irreducibility of modules. In general, it is not easy to show the prehomogeneity of a geometric crystal directly. Here we obtain a sufficient condition for prehomogeneity of a "positive" geometric crystal χ, which is, indeed, that the ultra-discretized crystal $\mathcal{UD}(\chi)$ is connected.

Perfect crystals are invented to treat the problem in some physical models from the crystal theoretical point of view ([KMN1],[KMN2]). They possess several remarkable properties; one of the most crucial ones among them is connectedness. Perfect crystals are not only connected themselves but also their tensor product are again connected. Let $\{B_l\}_{l \geq 1}$ be a coherent family of perfect crystals and B_∞ its limit([KKM]). Then the crystal B_∞ holds similar properties to perfect crystals, *e.g.,* connectedness.

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Let $\{\chi_\lambda\}_{\lambda\in\Lambda}$ be a family of geometric crystals indexed by a set Λ. Tropical R map $R_{\lambda\mu}$ is an isomorphism $R_{\lambda\mu}:\chi_\lambda\times\chi_\mu\to\chi_\mu\times\chi_\lambda$ $(\lambda,\mu\in\Lambda)$ satisfying the Yang-Baxter equation. We have not obtained a kind of "universal objects" for geometric crystals just like universal R-matrix for modules of quantum groups except for the $\mathfrak{sl}_2$- case. The universal tropical R map of $\mathfrak{sl}_2$ is introduced in the last section of this article.

In [KNO],[KNO2], we construct the affine geometric crystals $\mathcal{V}(\mathfrak{g})_L$ $(L\in\mathbb{C}^\times)$ for $\mathfrak{g}=A_n^{(1)}$, $B_n^{(1)}$, $(C_n^{(1)})$, $D_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$. Each of them is equipped with a positive structure and then there exists the corresponding affine crystal, which is, indeed, isomorphic to $B_\infty(\mathfrak{g}^L)$ the limit of perfect crystal of certain perfect crystals for the Langlands dual affine Lie algebra $\mathfrak{g}^L$. Since $B_\infty\otimes B_\infty$ is a connected crystal, product of geometric crystals $\mathcal{V}_L\times\mathcal{V}_M$ $(L,M\in\mathbb{C}^\times)$ is prehomogeneous and then for another tropical R map R'_{LM} such that $R_{LM}(x_0,y_0)=R'_{LM}(x_0,y_0)$, one has $R_{LM}=R'_{LM}$.

This article is basically a review of [KNO],[KNO2] except the last section, where we introduce the universal tropical R-map of $\mathfrak{sl}_2$.

§ 2. Geometric Crystals and Crystals

The notations and definitions here follow [N1],[N2],[N3],[KNO].

§ 2.1. Geometric Crystals

Fix a symmetrizable generalized Cartan matrix $A=(a_{ij})_{i,j\in I}$ with a finite index set I. Let $(\mathfrak{t},\{\alpha_i\}_{i\in I},\{\alpha_i^\vee\}_{i\in I})$ be the associated root data satisfying $\alpha_j(\alpha_i^\vee)=a_{ij}$. Let $\mathfrak{g}=\mathfrak{g}(A)=\langle\mathfrak{t},e_i,f_i(i\in I)\rangle$ be the Kac-Moody Lie algebra associated with A. Let $P\subset\mathfrak{t}^*$ be a weight lattice such that $\mathbb{C}\otimes P=\mathfrak{t}^*$ and $P\subset\{\lambda\,|\,\lambda(Q^\vee)\subset\mathbb{Z}\}$, whose element is called a weight.

Define the simple reflections $s_i\in\mathrm{Aut}(\mathfrak{t})$ $(i\in I)$ by $s_i(h):=h-\alpha_i(h)\alpha_i^\vee$, which generate the Weyl group W. Let G be the Kac-Moody group associated with $(\mathfrak{g},P)$ ([PK]). Let $U_\alpha:=\exp\mathfrak{g}_\alpha$ $(\alpha\in\Delta^{\mathrm{re}})$ be the one-parameter subgroup of G. The group G is generated by U_α $(\alpha\in\Delta^{\mathrm{re}})$. For any $i\in I$, there exists a unique group homomorphism $\phi_i\colon SL_2(\mathbb{C})\to G$ such that

$$\phi_i\left(\begin{pmatrix}1&t\\0&1\end{pmatrix}\right)=\exp(te_i),\ \phi_i\left(\begin{pmatrix}1&0\\t&1\end{pmatrix}\right)=\exp(tf_i)\qquad(t\in\mathbb{C}).$$

Set $\alpha_i^\vee(c):=\phi_i\left(\left(\begin{smallmatrix}c&0\\0&c^{-1}\end{smallmatrix}\right)\right)$, $x_i(t):=\exp(te_i)$, $y_i(t):=\exp(tf_i)$, $G_i:=\phi_i(SL_2(\mathbb{C}))$, $T_i:=\alpha_i^\vee(\mathbb{C}^\times)$ and $N_i:=N_{G_i}(T_i)$. Let T be the subgroup of G with P as its weight lattice which is called a *maximal torus* in G, and let $B^\pm(\supset T)$ be the Borel subgroup of G. The following definition is equivalent to the ones in [N1],[KNO].

Definition 2.1. Let X be an ind-variety over $\mathbb{C}$, γ_i and ε_i $(i \in I)$ rational functions on X, and $e_i : \mathbb{C}^\times \times X \to X$ a rational $\mathbb{C}^\times$-action. A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i,\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a G (or $\mathfrak{g}$)-*geometric crystal* if

(i) $\{1\} \times X \cap dom(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$. Here $dom(e_i)$ is the domain of definition of $e_i \colon \mathbb{C}^\times \times X \to X$.

(ii) The rational function γ_i $(i \in I)$ satisfies $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$ for any $i, j \in I$.

(iii) e_i and e_j satisfy the following relations:

$$
\begin{array}{ll}
e_i^{c_1}e_j^{c_2} = e_j^{c_2}e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0,\\
e_i^{c_1}e_j^{c_1c_2}e_i^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} & \text{if } a_{ij} = a_{ji} = -1,\\
e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1} & \text{if } a_{ij} = -2,\ a_{ji} = -1,\\
e_i^{c_1}e_j^{c_1^3c_2}e_i^{c_1^2c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^2c_2}e_j^{c_1^3c_2}e_i^{c_1} & \text{if } a_{ij} = -3,\ a_{ji} = -1,
\end{array}
$$

(iv) The rational function ε_i $(i \in I)$ satisfies $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$.

The relations in (iii) is called *Verma relations.* If $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies the conditions (i), (ii) and (iv), we call χ a *pre-geometric crystal.*

§ 2.2. Crystals

We recall the notion of crystals.

Definition 2.2. A *crystal* B is a set endowed with the following maps:

$$
\begin{aligned}
&\mathrm{wt}\colon B \longrightarrow P,\\
&\varepsilon_i\colon B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i\colon B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for} \quad i \in I,\\
&\tilde{e}_i\colon B \sqcup \{0\} \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i\colon B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I,\\
&\tilde{e}_i(0) = \tilde{f}_i(0) = 0.
\end{aligned}
$$

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$
\begin{aligned}
&\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle,\\
&\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B,\\
&\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B,\\
&\tilde{e}_i b_2 = b_1 \Longleftrightarrow \tilde{f}_i b_1 = b_2,\\
&\varepsilon_i(b) = -\infty \Longrightarrow \tilde{e}_i b = \tilde{f}_i b = 0.
\end{aligned}
$$

Example 2.3.

(i) *If* (L,B) *is a crystal base, then* B *is a crystal.*

(ii) *For the crystal base* $(L(\infty),B(\infty))$ *of the subalgebra* $U_q^-(\mathfrak{g})$ *of the quantum algebra* $U_q(\mathfrak{g})$, $B(\infty)$ *is a crystal.*

(iii) *For* $\lambda\in P$, *set* $T_\lambda:=\{t_\lambda\}$. *We define a crystal structure on* T_λ *by*

$$\tilde{e}_i(t_\lambda)=\tilde{f}_i(t_\lambda)=0,\quad \varepsilon_i(t_\lambda)=\varphi_i(t_\lambda)=-\infty,\quad \mathrm{wt}(t_\lambda)=\lambda.$$

Definition 2.4.

(i) To a crystal B, a colored oriented graph is associated by

$$b_1\overset{i}{\longrightarrow}b_2\Longleftrightarrow \tilde{f}_ib_1=b_2.$$

We call this graph the *crystal graph* of B.

(ii) A crystal B is said to be *connected*, if its crystal graph is connected as a graph.

(iii) A crystal B is free if for any $b\in B$, $i\in I$ and $n>0$, $\tilde{e}_i^n(b)\neq 0$ and $\tilde{f}_i^n(b)\neq 0$.

§ 2.3. Positive structure, Ultra-discretization and Tropicalization

Let us recall the notions of positive structure and ultra-discretization/tropicalization. The setting below is the same as in [KNO2]. Set $R:=\mathbb{C}(c)$ and define

$$\begin{array}{rccc} v\colon & R\setminus\{0\} & \longrightarrow & \mathbb{Z} \\ & f(c) & \mapsto & \deg(f(c)). \end{array}$$

Here deg is the degree of poles at $c=\infty$. Note that for $f_1,f_2\in R\setminus\{0\}$, we have

$$v(f_1f_2)=v(f_1)+v(f_2),\quad v\left(\frac{f_1}{f_2}\right)=v(f_1)-v(f_2). \tag{2.1}$$

We say that a non-zero rational function $f(c)\in\mathbb{C}(c)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients. Note that $f\in\mathbb{C}(c)$ is positive if and only if any pole of f is not a positive number and $f(x)>0$ for any $x>0$.

If $f_1,\ f_2\in R$ are positive, then we have

$$v(f_1+f_2)=\max(v(f_1),v(f_2)). \tag{2.2}$$

Let $T\simeq(\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T)$:=Hom$(T,\mathbb{C}^\times)$ (resp. $X_*(T)$:= Hom$(\mathbb{C}^\times,T)$) be the lattice of characters (resp. co-characters) of T. We denote by T_+ the set of points x in T such that $\chi(x)>0$ for any character χ. Then $\big((\mathbb{C}^\times)^n\big)_+=(\mathbb{R}_{>0})^n$.

A non-zero rational function on an algebraic torus T is called *positive* if it is written as g/h where g and h are a positive linear combination of characters of T.

Definition 2.5. Let $f\colon T \to T'$ be a rational mapping between two algebraic tori T and T'. We say that f is *positive*, if $\chi \circ f$ is positive for any character $\chi\colon T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational mappings from T to T'.

Note that any $f \in \mathrm{Mor}^+(T, T')$ induces a real analytic map $f_+\colon T_+ \to T'_+$.

Lemma 2.6 ([BK]). *For any $f \in \mathrm{Mor}^+(T_1, T_2)$ and $g \in \mathrm{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is well-defined and belongs to $\mathrm{Mor}^+(T_1, T_3)$.*

By Lemma 2.6, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational mappings. The category $\mathcal{T}_+$ admits products. For two algebraic tori T and T', their product in $\mathcal{T}_+$ coincides with the usual product of T and T'.

Note that $T \mapsto T_+$ gives a functor from $\mathcal{T}_+$ to the category of real analytic manifolds.

Let $f\colon T \to T'$ be a positive rational mapping of algebraic tori T and T'. We define a map $\widehat{f}\colon X_*(T) \to X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$. Note that $\chi \circ f \circ \xi$ is a rational map from $\mathbb{C}^\times$ to itself.

Lemma 2.7 ([BK]). *For any algebraic tori T_1, T_2, T_3, and positive rational mappings $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{array}{rccc} \mathcal{UD}\colon & \mathcal{T}_+ & \longrightarrow & \mathrm{Set} \\ & T & \mapsto & X_*(T) \\ & (f\colon T \to T') & \mapsto & (\widehat{f}\colon X_*(T) \to X_*(T')). \end{array}$$

Let us come back to the situation in §2.1. Hence G is a Kac-Moody group and T is its Cartan subgroup.

Definition 2.8 ([BK]). Let $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ be a G (or $\mathfrak{g}$)-geometric crystal, T' an algebraic torus and $\theta\colon T' \to X$ a birational mapping. The mapping θ is called a *positive structure* on χ if it satisfies

(i) for any $i \in I$ the rational functions $\gamma_i \circ \theta\colon T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta\colon T' \to \mathbb{C}$ are positive,

(ii) for any $i \in I$, the rational mapping $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta\colon T' \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to the positive rational mappings $e_{i,\theta}\colon \mathbb{C}^\times \times T' \to T'$ and $\gamma_i \circ \theta,\ \varepsilon_i \circ \theta\colon T' \to \mathbb{C}^\times$, we obtain

$$\tilde{e}_i := \mathcal{UD}(e_{i,\theta})\colon \mathbb{Z} \times X_*(T') \to X_*(T')$$
$$\mathrm{wt}_i := \mathcal{UD}(\gamma_i \circ \theta),\ \varepsilon_i := \mathcal{UD}(\varepsilon_i \circ \theta)\colon X_*(T') \to \mathbb{Z}.$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a free pre-crystal structure (see [BK, 2.2]) and we denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. As for the definition of crystal, see 3.4 or [KKM],[K1], [K2]. We have thus the following theorem:

Theorem 2.9 ([BK][N1]). *For any geometric crystal* $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ *and a positive structure* $\theta\colon T' \to X$*, the associated pre-crystal* $\mathcal{UD}_{\theta,T'}(\chi)$ *is a crystal* (see [BK, 2.2]).

Now, let $\mathcal{GC}^+(\mathfrak{g})$ be the category whose object is a triplet (χ, T', θ) where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a $\mathfrak{g}$-geometric crystal and $\theta\colon T' \to X$ is a positive structure on χ, and morphism $f\colon (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)$ is given by a morphism $\varphi\colon X_1 \longrightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1\colon T_1' \longrightarrow T_2',$$

is a positive rational mapping. Let $\mathcal{CR}(\mathfrak{g})$ be the category of $\mathfrak{g}$-crystals. Then by the theorem above, we have

Corollary 2.10. *$\mathcal{UD}$ defines a functor*

$$\begin{array}{rcl} \mathcal{UD} : \mathcal{GC}^+(\mathfrak{g}) & \longrightarrow & \mathcal{CR}(\mathfrak{g}^L) \\ (\chi, T', \theta) & \mapsto & \mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I}) \\ (f\colon (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)) & \mapsto & (\widehat{f}\colon X_*(T_1') \to X_*(T_2')), \end{array}$$

where $\mathfrak{g}^L$ is the Langlands dual for $\mathfrak{g}$.

We call the functor $\mathcal{UD}$ *"ultra-discretization"* as in [N1],[N2]. While for a crystal B, if there exists a geometric crystal χ and a positive structure $\theta\colon T' \to X$ on χ such that $\mathcal{UD}(\chi, T', \theta) \simeq B$ as crystals, an object (χ, T', θ) in $\mathcal{GC}^+(\mathfrak{g})$ is called a *tropicalization* of B.

§ 3. Perfect Crystals

§ 3.1. Affine weights

Let $\mathfrak{g}$ be an affine Lie algebra. The sets $\mathfrak{t}$, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ be as in 2.1. We take $\mathfrak{t}$ so that $\dim \mathfrak{t} = \sharp I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$

and let $\mathbf{c} \in \sum_i \mathbb{Z}_{\geq 0}\alpha_i^\vee \subset \mathfrak{g}$ be a unique central element satisfying

$$\{h \in Q^\vee \mid \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}.$$

We write ([Kac, 6.1])

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i. \tag{3.1}$$

Let $(\ ,\ )$ be the non-degenerate W-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda \rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\mathrm{cl}} := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl}\colon \mathfrak{t}^* \longrightarrow \mathfrak{t}^*_{\mathrm{cl}}$ be the canonical projection. Then we have $\mathfrak{t}^*_{\mathrm{cl}} \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}^*_0 := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda \rangle = 0\}$, $(\mathfrak{t}^*_{\mathrm{cl}})_0 := \mathrm{cl}(\mathfrak{t}^*_0)$. Then we have a positive-definite symmetric form on $(\mathfrak{t}^*_{\mathrm{cl}})_0$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}^*_{\mathrm{cl}}$ $(i \in I)$ be a weight such that $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{i,j}$, which is called a *fundamental weight*. We choose P so that $P_{\mathrm{cl}} := \mathrm{cl}(P)$ coincides with $\oplus_{i \in I} \mathbb{Z}\Lambda_i$ and we call P_{cl} the *classical weight lattice*.

§ 3.2. Affinization

Let $U_q(\mathfrak{g}) = \langle e_i, f_i, q^h | i \in I,\ h \in P \rangle$ be the quantum affine algebra associated with P and $U'_q(\mathfrak{g}) = \langle e_i, f_i, q^h | i \in I,\ h \in (P_{\mathrm{cl}})^* \rangle$ its subalgebra associated with $(P_{\mathrm{cl}})^*$. Set $\mathrm{Mod}^f(\mathfrak{g}, P_{\mathrm{cl}})$ the category of a finite dimensional $U'_q(\mathfrak{g})$-module M with a weight decomposition $M = \oplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda$.

Let M be an object in $\mathrm{Mod}^f(\mathfrak{g}, P_{\mathrm{cl}})$. For $l \in \mathbb{C}^\times$, define the $U'_q(\mathfrak{g})$-module M_l as follows: There exists a $\mathbb{C}$-linear bijective homomorphism $\Phi_l : M \longrightarrow M$, such that

$$\begin{aligned} q^h \Phi_l(u) &= \Phi_l(q^h u) \quad \text{for } h \in P^*_{\mathrm{cl}}, \\ e_i \Phi_l(u) &= l^{\delta_{i,0}} \Phi_l(e_i u), \\ f_i \Phi_l(u) &= l^{-\delta_{i,0}} \Phi_l(f_i u). \end{aligned}$$

The module $M_l := \Phi_l(M)$ is said to be an *affinization* of M ([KMN1],[K1]).

§ 3.3. Fundamental representation $W(\varpi_1)_l$

Here we consider the following affine Lie algebras $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, D_{n+1}^{(2)}, A_{2n}^{(2)}$. Let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights as in §3.1. Let $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as in [KNO][KNO2].

Let $W(\varpi_1)$ be the fundamental representation of $U'_q(\mathfrak{g})$ (see [K1, Sect 5.]). By [K1, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U'_q(\mathfrak{g})$-module, an object in $\mathrm{Mod}^f(\mathfrak{g}, P_{\mathrm{lc}})$ and has a global basis with a simple crystal. Thus, we can consider its affinization $W(\varpi_1)_l$ $(l \in \mathbb{C}^\times)$ specialized at $q = 1$. Then we obtain a finite-dimensional $\mathfrak{g}$-module denoted also by $W(\varpi_1)_l$ in which we shall construct affine geometric crystals in the next section.

Note that $W(\varpi_1)$ is an irreducible $\mathfrak{g}$-module for any $\mathfrak{g}$ as above. But, for some i and $\mathfrak{g}$, $W(\varpi_i)$ is not necessarily irreducible.

§ 3.4. Limit of perfect crystals

We review the limits of perfect crystals following [KKM]. (See also [KMN1],[KMN2].)

Let $\mathfrak{g}$ be an affine Lie algebra, $P_{\rm cl}$ the classical weight lattice as above and for $l \in \mathbb{Z}_{>0}$ set $(P_{\rm cl})_l^+ := \{\lambda \in P_{\rm cl} \mid \langle \mathbf{c}, \lambda\rangle = l,\ \langle \alpha_i^\vee, \lambda\rangle \geq 0\}$.

Definition 3.1. We say that a crystal B is *perfect* of level l if

(i) $B \otimes B$ is connected as a crystal graph.

(ii) There exists $\lambda_0 \in P_{\rm cl}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i\neq 0} \mathbb{Z}_{\leq 0}\mathrm{cl}(\alpha_i), \qquad \sharp B_{\lambda_0} = 1$$

(iii) There exists a finite-dimensional $U'_q(\mathfrak{g})$-module V with a crystal pseudo-base B_{ps} such that $B \cong B_{ps}/\pm 1$

(iv) The maps $\varepsilon, \varphi \colon B^{min} := \{b \in B \mid \langle c, \varepsilon(b)\rangle = l\} \longrightarrow (P_{\rm cl}^+)_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b)\Lambda_i$.

Theorem 3.2 ([KMN1],[KMN2]). *For each affine Lie algebra $\mathfrak{g}$ as above and for any positive integer l, there exists a perfect crystal $B_l(\mathfrak{g})$ of level l.*

For an affine Lie algebra $\mathfrak{g}$, let $\{B_l(\mathfrak{g})\}_{l\geq 1}$ be a family of perfect crystals of level l and set $J := \big\{(l,b) \mid l > 0,\ b \in B_l^{min}\big\}$.

Definition 3.3. A crystal $B_\infty = B_\infty(\mathfrak{g})$ with an element b_∞ is called the *limit of* $\{B_l\}_{l\geq 1}$ if

(i) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.

(ii) For any $(l,b) \in J$, there exists an embedding of crystals:

$$\begin{aligned} f_{(l,b)} \colon\ T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} &\hookrightarrow B_\infty \\ t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} &\mapsto b_\infty \end{aligned}$$

(iii) $B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im} f_{(l,b)}$.

As for the crystal T_λ, see Example 2.3 (iii). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

Proposition 3.4 ([KKM]). *For each affine Lie algebra* $\mathfrak{g}$ *as above there exists a coherent family of perfect crystals* $\{B_l(\mathfrak{g})\}_{l>0}$ *and its limit* $B_\infty(\mathfrak{g})$.

§ 4. Affine Geometric Crystals

Following the method in [KNO2], we shall see how to construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l$ ($l \in \mathbb{C}^\times$) in $W(\varpi_1)_l$.

§ 4.1. Translation $t(\widetilde{\varpi}_1)$

For $\xi_0 \in (\mathfrak{t}^*_{\mathrm{cl}})_0$, let $t(\xi_0)$ be as in [K1, Sect.4]:

$$t(\xi_0)(\lambda) := \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\mathrm{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of ξ, and it is well-defined.

Let $c_i^\vee$ be as follows:

$$c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)}). \tag{4.1}$$

Then $t(m\varpi_i)$ belongs to the extended Weyl group $\widetilde{W}$ if and only if $m \in c_i^\vee\mathbb{Z}$. Setting $\widetilde{\varpi}_i := c_i^\vee \varpi_i$ ($i \in I$) ([K2]), $t(\widetilde{\varpi}_1)$ is expressed as follows:

$$t(\widetilde{\varpi}_1) = \begin{cases} \iota(s_n s_{n-1} \cdots s_2 s_1) & A_n^{(1)} \text{ case,} \\ \iota(s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & B_n^{(1)},\ A_{2n-1}^{(2)} \text{ cases,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & C_n^{(1)},\ D_{n+1}^{(2)} \text{ cases,} \\ \iota(s_1 \cdots s_n)(s_{n-2} \cdots s_2 s_1) & D_n^{(1)} \text{ case,} \\ (s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & A_{2n}^{(2)},\ \text{case,} \end{cases}$$

where ι is certain Dynkin diagram automorphism. Now, we know that each $t(\widetilde{\varpi}_1)$ is in the form w_1 or $\iota \cdot w_1$ for some $w_1 \in W$, *e.g.*, $w_1 = s_n \cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1 \cdots s_n)(s_{n-1} \cdots s_1)$ for $B_n^{(1)}$, *etc.*, ...

In the case $A_{2n}^{(2)}$, $\eta := \mathrm{wt}(v_{\overline{n}}) = \Lambda_{n-1} - \Lambda_n$ (resp. $\Lambda_{n-1} - 2\Lambda_n$) is a unique weight of $W(\varpi_1)_l$ which satisfies $\langle \alpha_i^\vee, \eta \rangle \geq 0$ for $i \neq n$. For this η we have

$$t(\eta) = (s_n s_{n-1} \cdots s_1)(s_0 s_1 \cdots s_{n-1}) =: w_2. \tag{4.2}$$

§ 4.2. Affine geometric crystals in $W(\varpi_1)_l$

Let σ be the Dynkin diagram automorphism as in [KNO2] and $w_1 = s_{i_1} \cdots s_{i_k}$ be as in the previous subsection. Let $H \in \mathfrak{t}$ be an element as in [KNO2].

$$\alpha_i(H) = \begin{cases} 1 & \text{if } i = 1 \text{ and } \mathfrak{g} \neq D^{(2)}_{n+1}, A^{(2)}_{2n} \\ 2 & \text{if } i = 1 \text{ and } \mathfrak{g} = D^{(2)}_{n+1}, A^{(2)}_{2n}, \\ -1 & \text{if } i = 0 \text{ and } \mathfrak{g} \neq D^{(2)}_{n+1}, A^{(2)}_{2n}, \\ -2 & \text{if } i = 0 \text{ and } \mathfrak{g} = D^{(2)}_{n+1}, A^{(2)}_{2n} \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\mathcal{V}(\mathfrak{g})_l := \{v(x_1, \ldots, x_k) := Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) l^H v_1 \,\big|\, x_1, \ldots, x_k \in \mathbb{C}^\times\} \subset W(\varpi_1)_l \tag{4.3}$$

where the vector v_1 is a highest weight vector in $W(\varpi_1)_l$ as a $\mathfrak{g}_0$-module and then $\mathcal{V}(\mathfrak{g})$ has a G_0-geometric crystal structure, where $\mathfrak{g}_0 \subset \mathfrak{g}$ (resp. $G_0 \subset G$) is a simple Lie algebra (resp. simple algebraic group) corresponding to the index set $I_0 := I \setminus \{0\}$. Moreover $(\mathbb{C}^\times)^k \to \mathcal{V}(\mathfrak{g})_l$ given by $(x_1, \ldots, x_k) \mapsto v(x_1, \ldots, x_k)$ is a birational morphism. We shall define a G-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by using the Dynkin diagram automorphism σ except for $A^{(2)}_{2n}$. This σ induces an automorphism of $W(\varpi_1)_l$, which is denoted by $\sigma_l \colon W(\varpi_1)_l \longrightarrow W(\varpi_1)_l$.

Theorem 4.1 ([KNO2]).

(i) *Case* $\mathfrak{g} \neq A^{(2)}_{2n}$. *For* $x = (x_1 \cdots, x_k) \in (\mathbb{C}^\times)^k$, *there exist a unique* $y = (y_1, \ldots, y_k) \in (\mathbb{C}^\times)^k$ *and a positive rational function* $a(x)$ *such that*

$$v(y) = a(x)\sigma_l(v(x)), \quad \varepsilon_{\sigma(i)}(v(y)) = \varepsilon_i(v(x)) \quad \text{if } i, \sigma(i) \neq 0. \tag{4.4}$$

(ii) *Case* $\mathfrak{g} = A^{(2)}_{2n}$. *Associated with* w_1 *and* w_2 *as in the previous subsection, we define*

$$\mathcal{V}(\mathfrak{g})_l := \{v_1(x) = Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)Y_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) l^H v_1 \;\big|\; x_i, \bar{x}_i \in \mathbb{C}^\times\},$$
$$\mathcal{V}_2(\mathfrak{g})_l := \{v_2(y) = Y_n(y_n) \cdots Y_1(y_1)Y_0(y_0)Y_1(\overline{y}_1) \cdots Y_{n-1}(\overline{y}_{n-1}) l^{H'} v_{\overline{n}} \,\big|\, y_i, \overline{y}_i \in \mathbb{C}^\times\},$$

where $\alpha_0(H') = 2$, $\alpha_n(H') = -4$ *and* $\alpha_i(H') = 0$ *otherwise. (Note that* $\mathrm{wt}(v_1)(H) = \mathrm{wt}(v_{\overline{n}})(H')$.*) For any* $x \in (\mathbb{C}^\times)^{2n}$ *there exist a unique* $y \in (\mathbb{C}^\times)^{2n}$ *and a rational function* $a(x)$ *such that* $v_2(y) = a(x)v_1(x)$.

Now, using this theorem, we define the rational mapping

$$\begin{aligned} &\overline{\sigma} \colon \mathcal{V}(\mathfrak{g})_l \longrightarrow \mathcal{V}(\mathfrak{g})_l, && \overline{\sigma} \colon \mathcal{V}(\mathfrak{g})_l \longrightarrow \mathcal{V}_2(\mathfrak{g})_l, \\ &\quad v(x) \mapsto v(y) \;\; (\mathfrak{g} \neq A^{(2)}_{2n}), && \quad v_1(x) \mapsto v_2(y) \;\; (\mathfrak{g} = A^{(2)}_{2n}), \end{aligned} \tag{4.5}$$

Associated with an element $w \in W$, let B^-_w be the geometric crystal as in [N1],[KNO], which is isomorphic to the geometric crystal on the Schubert cell X_w(see [N1]).

Theorem 4.2 ([KNO2]). *The rational mapping $\overline{\sigma}$ is birational. If we define a $\mathfrak{g}_0$-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by the one on $B^-_{w_1}\cdot l^H$ and a rational $\mathbb{C}^\times$-action $e_0:\mathbb{C}^\times\times\mathcal{V}(\mathfrak{g})_l\to\mathcal{V}(\mathfrak{g})_l$ and rational functions* wt_0 *and ε_0 on $\mathcal{V}(\mathfrak{g})_l$ by*

$$(4.6)\qquad \begin{cases} e_0^c:=\overline{\sigma}^{-1}\circ e^c_{\sigma(0)}\circ\overline{\sigma}, & \varepsilon_0:=\varepsilon_{\sigma(0)}\circ\overline{\sigma}, \quad \gamma_0:=\gamma_{\sigma(0)}\circ\overline{\sigma}, \quad \textit{for } \mathfrak{g}\neq A^{(2)}_{2n},\\ e_0^c:=\overline{\sigma}^{-1}\circ e^c_0\circ\overline{\sigma}, & \varepsilon_0:=\varepsilon_0\circ\overline{\sigma}, \quad \gamma_0:=\gamma_0\circ\overline{\sigma}, \quad \textit{for } \mathfrak{g}=A^{(2)}_{2n}.\end{cases}$$

then $(\mathcal{V}(\mathfrak{g})_l,\{e_i\}_{i\in I},\{\gamma_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ turns out to be a positive affine $\mathfrak{g}$-geometric crystal.

Remark. In the case $\mathfrak{g}=A^{(2)}_{2n}$, $\mathcal{V}_2(\mathfrak{g})$ has a $\mathfrak{g}_{I\setminus\{n\}}$-geometric crystal structure. Thus, $e_0,\gamma_0,\varepsilon_0$ are well-defined on $\mathcal{V}_2(\mathfrak{g})$.

§ 4.3. Product structure on affine geometric crystals

In general, if χ_1 and χ_2 are geometric crystals induced from unipotent crystals, the product $\chi_1\times\chi_2$ possesses a geometric crystal structure ([BK]). More precisely, let $\chi_1=(X,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ and $\chi_2=(Y,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ be geometric crystals induced from unipotent crystals. For $x\in X$ and $y\in Y$ set

$$(4.7)\qquad \gamma_i(x,y):=\gamma_i(x)\gamma_i(y),$$

$$(4.8)\qquad \varepsilon_i(x,y):=\varepsilon_i(x)+\frac{\varepsilon_i(x)\varepsilon_i(y)}{\varphi_i(x)}\quad(\varphi_i(x)=\gamma(x)\varepsilon_i(x)),$$

$$(4.9)\qquad e_i^c(x,y):=(e_i^{c_1}(x),e_i^{c_2}(y))\qquad\text{where } c_1:=\frac{c\varphi_i(x)+\varepsilon_i(y)}{\varphi_i(x)+\varepsilon_i(y)},\quad c_2:=\frac{c}{c_1}.$$

Then, (4.7)–(4.9) endow the product $X\times Y$ with a structure of a geometric crystal.

As for the affine geometric crystal $\mathcal{V}_l$ in the previous subsection, its data $e_i,\gamma_i,\varepsilon_i$ $(i=1,\dots,n)$ are obtained from the ones of the geometric crystal $B^-_{\mathbf{i}}\cdot l^H$ which is induced from the unipotent crystal on some $X_w\times l^H$ where $\mathbf{i}$ is a reduced word for w and X_w is the Schubert cell associated with $w\in W$ and note that $\{l^H\}$ holds trivial unipotent crystal structure. We can check the $i=0$-case directly and then obtain:

Theorem 4.3 ([KNO2]). *For any $k\in\mathbb{Z}_{\geq 0}$ and $L_1\cdots,L_k\in\mathbb{C}^\times$, the product $\mathcal{V}_{L_1}\times\cdots\times\mathcal{V}_{L_k}$ has an affine geometric crystal structure.*

§ 4.4. Ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$

Let us see the ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$.

By Theorem 4.2 on $\mathcal{V}(\mathfrak{g})_l$, for $l>0$ it holds a natural positive structure $\theta_l\colon(\mathbb{C}^\times)^m\to\mathcal{V}(\mathfrak{g})_l$ $(x\mapsto v(x))$ where $m=\dim\mathcal{V}(\mathfrak{g})_l$. Then we have the following theorem:

Theorem 4.4 ([KNO2]). *For $\mathfrak{g}=A^{(1)}_n,B^{(1)}_n,D^{(1)}_n,D^{(2)}_{n+1},A^{(2)}_{2n-1}$ and $A^{(2)}_{2n}$, suppose that $l>0$. Then the ultra-discretization $\mathcal{UD}_{\theta_l}(\mathcal{V}(\mathfrak{g})_l)$ associated with the positive structure θ_l is isomorphic to the crystal $B_\infty(\mathfrak{g}^L)$.*

§ 5. Tropical R Maps

§ 5.1. Definition of tropical R maps

Definition 5.1. Let $\{(X_\lambda, \{e_i^\lambda\}, \{\gamma_i^\lambda\}, \{\varepsilon_i^\lambda\})\}_{\lambda\in\Lambda}$ be a family of geometric crystals equipped with the product structures, where Λ is a certain index set and its element is called a *spectral parameter*. A birational isomorphism $\mathcal{R}_{\lambda\mu} : X_\lambda \times X_\mu \longrightarrow X_\mu \times X_\lambda$ $(\lambda, \mu \in \Lambda)$ is said to be a *tropical R map* if it satisfies the following conditions:

$$(e_i^{X_\mu\times X_\lambda})^c \circ \mathcal{R}_{\lambda\mu} = \mathcal{R}_{\lambda\mu} \circ (e_i^{X_\lambda\times X_\mu})^c, \tag{5.1}$$

$$\varepsilon_i^{X_\lambda\times X_\mu} = \varepsilon_i^{X_\mu\times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{5.2}$$

$$\gamma_{X_\lambda\times X_\mu} = \gamma_{X_\mu\times X_\lambda} \circ \mathcal{R}_{\lambda\mu}, \tag{5.3}$$

$$\mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)} = \mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)} \quad \text{on } X_\lambda \times X_\mu \times X_\nu, \tag{5.4}$$

for any $i \in I$ and any $\lambda, \mu, \nu \in \Lambda$. Here $\mathcal{R}^{(ij)}$ means that it acts on i-th and j-th components of the product.

In [KNO2], we obtain tropical R maps $\{\overline{R}_{LM}\}_{L,M>0}$ of the family of affine geometric crystals $\{\mathcal{V}_L\}_{L>0}$ for the above affine Lie algebras.

§ 5.2. Prehomogeneous Geometric Crystal

Definition 5.2. Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a geometric crystal. We say that χ is *prehomogeneous* if there exists a Zariski open dense subset $\Omega \subset X$ which is an orbit by the actions of the e_i^c's.

The following lemma is obvious.

Lemma 5.3 ([KNO2]). *Let $\chi_j = (X_j, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ $(j = 1, 2)$ be prehomogeneous geometric crystals. Let $\Omega_1 \subset X_1$ be an open dense orbit in X_1. For isomorphisms of geometric crystals ϕ, $\phi'\colon \chi_1 \to \chi_2$, suppose that there exists $p_1 \in \Omega_1$ such that $\phi(p_1) = \phi'(p_1) \in X_2$. Then, we have $\phi = \phi'$ as rational morphisms.*

The following is the key of this article:

Theorem 5.4 ([KNO2]). *Let $\chi = (X, \{e_i^c\}, \{\gamma_i\}, \{\varepsilon_i\})$ be a finite-dimensional positive geometric crystal with the positive structure $\theta\colon T \to X$ and $B := \mathcal{UD}_\theta(\chi)$ the crystal obtained as the ultra-discretization of χ. χ is prehomogeneous if B is a connected crystal.*

§ 5.3. Uniqueness of the birational R-maps

Theorem 5.5 ([KNO2]). *Let $\overline{\mathcal{R}}_{LM}$ be the tropical R map as above. Set $z_0 := \overline{\mathcal{R}}(\mathbf{1}, \mathbf{1})$ where $\mathbf{1} := v(1, \cdots, 1) \in \mathcal{V}_L$. Let $\mathcal{R}'$ be a tropical R map such that $\mathcal{R}'(\mathbf{1}, \mathbf{1}) = z_0$. Then we have $\overline{\mathcal{R}} = \mathcal{R}'$ as birational morphisms.*

Proof. Let $\mathcal{V}_l, \mathcal{V}_m$ $(l, m \in \mathbb{C}^\times)$ be the affine geometric crystals constructed in Sect.4 and $\mathcal{R}$ the tropical R map as in this section and $\mathcal{R}' : \mathcal{V}_l \times \mathcal{V}_m \to \mathcal{V}_m \times \mathcal{V}_l$ other tropical R map such that $\mathcal{R}(\mathbf{1}, \mathbf{1}) = \mathcal{R}'(\mathbf{1}, \mathbf{1})$. By Theorem 4.4, we have that $\mathcal{UD}_{(\Theta_l, \Theta_m)}(\mathcal{V}_l(\mathfrak{g}) \times \mathcal{V}_m(\mathfrak{g}))$ is isomorphic to the crystal $B_\infty(\mathfrak{g}^L) \otimes B_\infty(\mathfrak{g}^L)$.

It follows from that for any positive integer m and any affine Lie algebra $\mathfrak{g}$ appearing in this article tensor product of crystals $\bigotimes_{i=1}^m \mathcal{UD}_{\Theta_{l_i}}(\mathcal{V}(\mathfrak{g})_{l_i}) \cong B_\infty(\mathfrak{g}^L)^{\otimes m}$ is connected. By Theorem 5.4 we obtain that the geometric crystal $\mathcal{V}_l \times \mathcal{V}_m$ is prehomogeneous. Therefore, by Lemma 5.3 we obtain $\mathcal{R} = \mathcal{R}'$, which completes the proof of Theorem 5.5. □

§ 6. Universal tropical R map of $\mathfrak{sl}_2$

Only for $\mathfrak{sl}_2$-case, we found a kind of "universal" tropical R map. For arbitrary $\mathfrak{sl}_2$-geometric crystals $\mathbb{X} = (X, e_X, \gamma_X, \varepsilon_X)$ and $\mathbb{Y} = (Y, e_Y, \gamma_Y, \varepsilon_Y)$, the morphism $\mathcal{R} : X \times Y \to Y \times X$ is defined by

(6.1)
$$\mathcal{R}(x, y) := (e_Y^{a(x,y)} y, e_X^{b(x,y)} x), \quad \text{where} \quad a(x, y) := \frac{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)}}, \qquad b(x, y) = \frac{1}{a(x, y)}.$$

Theorem 6.1. *The morphism $\mathcal{R}$ satisfies the following:*

$$\gamma_{Y\times X}(\mathcal{R}(x, y)) = \gamma_{X\times Y}(x, y), \tag{6.2}$$
$$\varepsilon_{Y\times X}(\mathcal{R}(x, y)) = \varepsilon_{X\times Y}(x, y), \tag{6.3}$$
$$\mathcal{R} \circ e^c = e^c \circ \mathcal{R}, \tag{6.4}$$
$$\mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)} = \mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)} \quad \textit{(Yang-Baxter eq.)} \tag{6.5}$$
$$\mathcal{R} \circ \mathcal{R} = \mathrm{id}_{X\times Y} \tag{6.6}$$

This implies that there exists the canonical morphism with the same properties as a universal R-matrix for modules of quantum groups. Furthermore, the morphism $\mathcal{R}$ is an isomorphism of geometric crystals.

Proof. Let us show (6.2). We have

$$\gamma_{Y\times X}(\mathcal{R}(x, y)) = \gamma_Y(e^{a(x,y)} y)\gamma_X(e^{b(x,y)} x) = a(x, y)^2 b(x, y)^2 \gamma_X(x)\gamma_Y(y) = \gamma_{X\times Y}(x, y).$$

Next, let us see (6.3).

$$\varepsilon_{Y\times X}(\mathcal{R}(x, y)) = \varepsilon_Y(e^{a(x,y)} y) + \frac{\varepsilon_X(e^{b(x,y)} x)}{\gamma_Y(e^{a(x,y)} y)} = a(x, y)^{-1}\varepsilon_Y(y) + \frac{b(x, y)^{-1}\varepsilon_X(x)}{a(x, y)^2\gamma_Y(y)}$$
$$= a(x, y)^{-1}\left(\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}\right) = \varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)} = \varepsilon_{X\times Y}(x, y).$$

Let us show (6.4). For $c \in \mathbb{C}^\times$ and $(x,y) \in X \times Y$, let $c_1 := c_1(c,x,y), c_2 := c_2(c,x,y)$ be as in (4.9). We have

$$\mathcal{R}(e^c(x,y)) = (e^{a(e^{c_1}x,e^{c_2}y)c_2}y, e^{b(e^{c_1}x,e^{c_2}y)c_1}x), \tag{6.7}$$
$$e^c\mathcal{R}(x,y) = (e^{c_1'a(x,y)}y, e^{c_2'b(x,y)}x), \tag{6.8}$$

where $c_1' := c_1(c, e^{a(x,y)}y, e^{b(x,y)}x)$ and $c_2' := c_2(c, e^{a(x,y)}y, e^{b(x,y)}x)$. By simple calculations, one has

$$a(e^{c_1}x, e^{c_2}y) = \frac{c_1}{c_2}\frac{c\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{c\varepsilon_X(x) + \frac{\varepsilon_Y(y)}{\gamma_X(x)}}, \qquad c_1' = \frac{c\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}, \tag{6.9}$$

which means $a(e^{c_1}x, e^{c_2}y)c_2 = c_1'a(x,y)$ and then $b(e^{c_1}x, e^{c_2}y)c_1 = c_2'a(x,y)$. We obtained (6.4).

Let us show (6.5). Denote

$$\mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)}(x,y,z) = (e^A z, e^B y, e^C x),$$
$$\mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)}(x,y,z) = (e^{A'} z, e^{B'} y, e^{C'} x).$$

It is easy to see that $ABC = 1 = A'B'C'$ and then we may show that $A = A'$ and $C = C'$. By simple calculations, we have

$$A = a(e^{a(x,y)}y, e^{a(e^{b(x,y)}x,z)}z)a(e^{b(x,y)}x,z) = \frac{\varepsilon_Z(z) + \frac{\varepsilon_X(x)}{b(x,y)\gamma_Z(z)}}{\frac{\varepsilon_X(x)}{b(x,y)} + \frac{\varepsilon_Z(z)}{b(x,y)^2\gamma_X(x)}}$$
$$= \frac{\varepsilon_Z(z)\left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)}\right) + \frac{\varepsilon_X(x)}{\gamma_Z(z)}\left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)}\right)}{\varepsilon_X(x)\left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)}\right) + \frac{\varepsilon_Z(z)}{\gamma_X(x)}\left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)}\right)},$$
$$A' = a(x, e^{a(y,z)}z)a(y,z) = \frac{\varepsilon_Z(z) + \frac{\varepsilon_X(x)}{a(y,z)\gamma_Z(z)}}{\varepsilon_X(x) + \frac{\varepsilon_Z(z)}{a(y,z)\gamma_X(x)}}$$
$$= \frac{\varepsilon_Z(z)\left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)}\right) + \frac{\varepsilon_X(x)}{\gamma_Z(z)}\left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)}\right)}{\varepsilon_X(x)\left(\varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)}\right) + \frac{\varepsilon_Z(z)}{\gamma_X(x)}\left(\varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)}\right)},$$

which means $A = A'$.

Next, we have

$$C = b(e^{b(x,y)}x, z)b(x,y),$$
$$C' = b(e^{b(x,e^{a(y,z)}z)}x, e^{b(y,z)}y)b(x, e^{a(y,z)}z)$$

It follows from the explicit forms of $a(x,y)$ and $b(x,y)$ that $a(x,y) = b(y,x)$. Thus we obtain

$$
\begin{aligned}
C_{|x\leftrightarrow z} &= b(e^{b(z,y)}z,x)b(z,y) = a(x,e^{a(y,z)}z)a(y,z) = A' \\
C'_{|x\leftrightarrow z} &= b(e^{b(z,e^{a(y,x)}x)}z,e^{b(y,x)}y)b(z,e^{a(y,x)}x) \\
&= a(e^{a(x,y)}y,e^{a(e^{b(x,y)}x,z)}z)a(e^{b(x,y)}x,z) = A,
\end{aligned}
$$

which implies $C = C'$. Thus, we also obtain $B = B'$.

Finally, let us show (6.6). Set $(y',x') = \mathcal{R}(x,y)$. Then we have

$$\mathcal{R}\circ\mathcal{R}(x,y) = (e_X^{a(y',x')}e_X^{b(x,y)}x, e_Y^{b(y',x')}e_Y^{a(x,y)}y).$$

By the formula (6.9) and $a(x,y)b(x,y) = 1$ we obtain

$$a(y',x') = a(e_y^{a(x,y)}y, e_X^{b(x,y)}x) = \frac{a(x,y)}{b(x,y)}a(y,x) = \frac{a(x,y)}{b(x,y)}b(x,y) = a(x,y),$$

and also $b(y',x') = b(x,y)$. Thus, we have $\mathcal{R}\circ\mathcal{R}(x,y) = (x,y)$ and completed the proof.
□

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