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Kyoto University
Universal Tropical R map of \(\mathfrak{sl}_2\) and Prehomogeneous Geometric Crystals

Dedicated to Professor Masaki Kashiwara on the occasion of his 60th birthday

By

Toshiki Nakashima*

§1. Introduction

The notion of geometric crystal is a sort of geometric lifting of crystal base theory. If a geometric crystal \(\chi\) is "positive", it can be "ultra-discretized" to a crystal \(\mathcal{UD}(\chi)\) (for more details, see below). For a geometric crystal \(\chi = (X, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\})\), if there exists an open dense orbit by \(e_i\)'s, we call \(\chi\) prehomogeneous, where \(e_i : \mathbb{C}^\times \times X \rightarrow X\) defines certain rational \(\mathbb{C}^\times\)-action on \(X\). For isomorphic prehomogeneous geometric crystals \(\chi_1\) and \(\chi_2\) (\(f : \chi_1 \sim \chi_2\)), assume that there exists another isomorphism \(f' : \chi_1 \rightarrow \chi_2\) such that \(f(p) = f'(p)\) for some point \(p \in Z \subset X_1\) where \(Z\) is an open dense orbit in \(X_1\). Then it is easy to see that \(f = f'\) as a rational morphism, which resembles Schur's lemma. In this sense, prehomogeneity of geometric crystals corresponds to irreducibility of modules. In general, it is not easy to show the prehomogeneity of a geometric crystal directly. Here we obtain a sufficient condition for prehomogeneity of a "positive" geometric crystal \(\chi\), which is, indeed, that the ultra-discretized crystal \(\mathcal{UD}(\chi)\) is connected.

Perfect crystals are invented to treat the problem in some physical models from the crystal theoretical point of view ([KMN1],[KMN2]). They possess several remarkable properties; one of the most crucial ones among them is connectedness. Perfect crystals are not only connected themselves but also their tensor product are again connected. Let \(\{B_l\}_{l \geq 1}\) be a coherent family of perfect crystals and \(B_\infty\) its limit([KKM]). Then the crystal \(B_\infty\) holds similar properties to perfect crystals, e.g., connectedness.

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Let \( \{ \chi_{\lambda} \}_{\lambda \in \Lambda} \) be a family of geometric crystals indexed by a set \( \Lambda \). Tropical \( R \) map \( R_{\lambda \mu} \) is an isomorphism \( R_{\lambda \mu} : \chi_{\lambda} \times \chi_{\mu} \rightarrow \chi_{\lambda} \times \chi_{\mu} \) \((\lambda, \mu \in \Lambda)\) satisfying the Yang-Baxter equation. We have not obtained a kind of "universal objects" for geometric crystals just like universal \( R \)-matrix for modules of quantum groups except for the \( \mathfrak{sl}_2 \) case. The universal tropical \( R \) map of \( \mathfrak{sl}_2 \) is introduced in the last section of this article.

In [KNO], [KNO2], we construct the affine geometric crystals \( \mathcal{V}(\mathfrak{g})_L \) \((L \in \mathbb{C}^\times)\) for \( \mathfrak{g} = A_{n}^{(1)}, B_{n}^{(1)}, C_{n}^{(1)}, D_{n+1}^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)} \). Each of them is equipped with a positive structure and then there exists the corresponding affine crystal, which is, indeed, isomorphic to \( \mathfrak{B}_{\infty}(\mathfrak{g}^L) \) the limit of perfect crystal of certain perfect crystals for the Langlands dual affine Lie algebra \( \mathfrak{g}^L \). Since \( \mathfrak{B}_{\infty} \otimes \mathfrak{B}_{\infty} \) is a connected crystal, product of geometric crystals \( \mathcal{V}_L \times \mathcal{V}_M \) \((L, M \in \mathbb{C}^\times)\) is prehomogeneous and then for another tropical \( R \) map \( R'_{LM} \) such that \( R_{LM}(x_0, y_0) = R'_{LM}(x_0, y_0) \), one has \( R_{LM} = R'_{LM} \).

This article is basically a review of [KNO],[KNO2] except the last section, where we introduce the universal tropical \( R \)-map of \( \mathfrak{sl}_2 \).

§ 2. Geometric Crystals and Crystals

The notations and definitions here follow [N1],[N2],[N3],[KNO].

§ 2.1. Geometric Crystals

Fix a symmetrizable generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) with a finite index set \( I \). Let \( (t, \{ \alpha_i \}_{i \in I}, \{ \alpha_i^\vee \}_{i \in I}) \) be the associated root data satisfying \( \alpha_j(\alpha_i^\vee) = a_{ij} \). Let \( \mathfrak{g} = \mathfrak{g}(A) = \langle t, \alpha_i, \alpha_i^\vee \rangle \) be the Kac-Moody Lie algebra associated with \( A \). Let \( P \subset t^* \) be a weight lattice such that \( \mathbb{C} \otimes P = t^* \) and \( P \subset \{ \lambda | \lambda(Q^\vee) \subset \mathbb{Z} \} \), whose element is called a weight.

Define the simple reflections \( s_i \in \text{Aut}(t) \) \((i \in I)\) by \( s_i(h) := h - \alpha_i(h)\alpha_i^\vee \), which generate the Weyl group \( W \). Let \( G \) be the Kac-Moody group associated with \( (\mathfrak{g}, P) \) ([PK]). Let \( U_\alpha := \exp \mathfrak{g}_\alpha \) \((\alpha \in \Delta^{\text{re}})\) be the one-parameter subgroup of \( G \). The group \( G \) is generated by \( U_\alpha \) \((\alpha \in \Delta^{\text{re}})\). For any \( i \in I \), there exists a unique group homomorphism \( \phi_i : SL_2(\mathbb{C}) \rightarrow G \) such that
\[
\phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(tc_i), \quad \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(tf_i) \quad (t \in \mathbb{C}).
\]
Set \( \alpha_i^\vee(c) := \phi_i \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix} \), \( x_i(t) := \exp(tc_i) \), \( y_i(t) := \exp(tf_i) \), \( G_i := \phi_i(SL_2(\mathbb{C})) \), \( T_i := \alpha_i^\vee(\mathbb{C}^\times) \) and \( N_i := N_{G_i}(T_i) \). Let \( T \) be the subgroup of \( G \) with \( P \) as its weight lattice which is called a maximal torus in \( G \), and let \( B^\pm(T) \) be the Borel subgroup of \( G \). The following definition is equivalent to the ones in [N1],[KNO].
**Definition 2.1.** Let $X$ be an ind-variety over $\mathbb{C}$, $\gamma_i$ and $\varepsilon_i$ ($i \in I$) rational functions on $X$, and $e_i : \mathbb{C}^\times \times X \to X$ a rational $\mathbb{C}^\times$-action. A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a $G$ (or $g$)-geometric crystal if

(i) $\{1\} \times X \cap \text{dom}(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$. Here $\text{dom}(e_i)$ is the domain of definition of $e_i : \mathbb{C}^\times \times X \to X$.

(ii) The rational function $\gamma_i$ ($i \in I$) satisfies $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$ for any $i, j \in I$.

(iii) $e_i$ and $e_j$ satisfy the following relations:

\[
\begin{align*}
e_i^{c_1}e_j^{c_2} &= e_j^{c_2}e_i^{c_1} & \text{if } a_{ij} = a_{ji} = 0, \\
e_i^{c_1}e_j^{c_2}e_i^{c_1}e_j^{c_2} &= e_j^{c_2}e_i^{c_1}e_j^{c_2}e_i^{c_1} & \text{if } a_{ij} = a_{ji} = -1, \\
e_i^{c_1}e_j^{c_2}e_i^{c_1}e_j^{c_2} &= e_j^{c_2}e_i^{c_1}e_j^{c_2}e_i^{c_1} & \text{if } a_{ij} = -2, a_{ji} = -1, \\
e_i^{c_1}e_j^{c_2}e_i^{c_1}e_j^{c_2}e_i^{c_1}e_j^{c_2} &= e_j^{c_2}e_i^{c_1}e_j^{c_2}e_i^{c_1}e_j^{c_2}e_i^{c_1} & \text{if } a_{ij} = -3, a_{ji} = -1,
\end{align*}
\]

(iv) The rational function $\varepsilon_i$ ($i \in I$) satisfies $\varepsilon_i(e_i^{c}(x)) = c^{-1}\varepsilon_i(x)$.

The relations in (iii) is called Verma relations. If $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies the conditions (i), (ii) and (iv), we call $\chi$ a pre-geometric crystal.

§2.2. Crystals

We recall the notion of crystals.

**Definition 2.2.** A crystal $B$ is a set endowed with the following maps:

\[
\begin{align*}
\text{wt} : B &\to P, \\
\varepsilon_i : B &\to \mathbb{Z} \sqcup \{-\infty\}, \\
\varphi_i : B &\to \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I, \\
\tilde{e}_i : B \sqcup \{0\} &\to B \sqcup \{0\}, \\
\tilde{f}_i : B \sqcup \{0\} &\to B \sqcup \{0\} \quad \text{for } i \in I, \\
\tilde{e}_i(0) &= \tilde{f}_i(0) = 0.
\end{align*}
\]

Those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

\[
\begin{align*}
\varphi_i(b) &= \varepsilon_i(b) + \langle \alpha_i^\vee, \text{wt}(b) \rangle, \\
\text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \\
\text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \\
\tilde{e}_i b_2 = b_1 &\iff \tilde{f}_i b_1 = b_2, \\
\varepsilon_i(b) = -\infty &\implies \tilde{e}_i b = \tilde{f}_i b = 0.
\end{align*}
\]

**Example 2.3.**
(i) If \((L, B)\) is a crystal base, then \(B\) is a crystal.

(ii) For the crystal base \((L(\infty), B(\infty))\) of the subalgebra \(U_q^- (\mathfrak{g})\) of the quantum algebra \(U_q (\mathfrak{g})\), \(B(\infty)\) is a crystal.

(iii) For \(\lambda \in \mathcal{P}\), set \(T_\lambda := \{t_\lambda\}\). We define a crystal structure on \(T_\lambda\) by

\[\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \text{wt}(t_\lambda) = \lambda.\]

**Definition 2.4.**

(i) To a crystal \(B\), a colored oriented graph is associated by

\[b_1 \xrightarrow{i} b_2 \iff \tilde{f}_i b_1 = b_2.\]

We call this graph the *crystal graph* of \(B\).

(ii) A crystal \(B\) is said to be *connected*, if its crystal graph is connected as a graph.

(iii) A crystal \(B\) is free if for any \(b \in B, i \in I\) and \(n > 0\), \(\tilde{e}_i^n (b) \neq 0\) and \(\tilde{f}_i^n (b) \neq 0\).

§ 2.3. Positive structure, Ultra-discretization and Tropicalization

Let us recall the notions of positive structure and ultra-discretization/tropicalization. The setting below is the same as in [KNO2]. Set \(R := \mathbb{C}(c)\) and define

\[v: \mathbb{C}(c) \setminus \{0\} \rightarrow \mathbb{Z}, \quad f(c) \mapsto \deg(f(c)).\]

Here \(\deg\) is the degree of poles at \(c = \infty\). Note that for \(f_1, f_2 \in \mathbb{C}(c) \setminus \{0\}\), we have

\[(2.1) \quad v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2).\]

We say that a non-zero rational function \(f(c) \in \mathbb{C}(c)\) is *positive* if \(f\) can be expressed as a ratio of polynomials with positive coefficients. Note that \(f \in \mathbb{C}(c)\) is positive if and only if any pole of \(f\) is not a positive number and \(f(x) > 0\) for any \(x > 0\).

If \(f_1, f_2 \in R\) are positive, then we have

\[(2.2) \quad v(f_1 + f_2) = \max(v(f_1), v(f_2)).\]

Let \(T \simeq (\mathbb{C}^\times)^l\) be an algebraic torus over \(\mathbb{C}\) and \(X^*(T) := \text{Hom}(T, \mathbb{C}^\times)\) (resp. \(X_*(T) := \text{Hom}(\mathbb{C}^\times, T)\)) be the lattice of characters (resp. co-characters) of \(T\). We denote by \(T_+\) the set of points \(x\) in \(T\) such that \(\chi(x) > 0\) for any character \(\chi\). Then \(((\mathbb{C}^\times)^n)_+ = (\mathbb{R}_{>0})^n\).

A non-zero rational function on an algebraic torus \(T\) is called *positive* if it is written as \(g/h\) where \(g\) and \(h\) are a positive linear combination of characters of \(T\).
Definition 2.5. Let $f : T \to T'$ be a rational mapping between two algebraic tori $T$ and $T'$. We say that $f$ is positive, if $\chi \circ f$ is positive for any character $\chi : T' \to \mathbb{C}$.

Denote by $\text{Mor}^+ (T, T')$ the set of positive rational mappings from $T$ to $T'$.

Note that any $f \in \text{Mor}^+ (T, T')$ induces a real analytic map $f_+ : T_+ \to T'_+$. 

Lemma 2.6 ([BK]). For any $f \in \text{Mor}^+ (T_1, T_2)$ and $g \in \text{Mor}^+ (T_2, T_3)$, the composition $g \circ f$ is well-defined and belongs to $\text{Mor}^+(T_1, T_3)$.

By Lemma 2.6, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational mappings. The category $\mathcal{T}_+$ admits products. For two algebraic tori $T$ and $T'$, their product in $\mathcal{T}_+$ coincides with the usual product of $T$ and $T'$.

Note that $T \mapsto T_+$ gives a functor from $\mathcal{T}_+$ to the category of real analytic manifolds.

Let $f : T \to T'$ be a positive rational mapping of algebraic tori $T$ and $T'$. We define a map $\hat{f} : X_*(T) \to X_*(T')$ by

$$\langle \chi, \hat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$. Note that $\chi \circ f \circ \xi$ is a rational map from $\mathbb{C}^\times$ to itself.

Lemma 2.7 ([BK]). For any algebraic tori $T_1$, $T_2$, $T_3$, and positive rational mappings $f \in \text{Mor}^+ (T_1, T_2)$, $g \in \text{Mor}^+ (T_2, T_3)$, we have $g \circ f = \hat{g} \circ \hat{f}$.

By this lemma, we obtain a functor

$$UD : \mathcal{T}_+ \to \text{Set}$$

$$T \mapsto X_*(T)$$

$$(f : T \to T') \mapsto (\hat{f} : X_*(T) \to X_*(T')).$$

Let us come back to the situation in §2.1. Hence $G$ is a Kac-Moody group and $T$ is its Cartan subgroup.

Definition 2.8 ([BK]). Let $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$ be a $G$ (or $\mathfrak{g}$)-geometric crystal, $T'$ an algebraic torus and $\theta : T' \to X$ a birational mapping. The mapping $\theta$ is called a positive structure on $\chi$ if it satisfies

(i) for any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\epsilon_i \circ \theta : T' \to \mathbb{C}$ are positive,

(ii) for any $i \in I$, the rational mapping $e_{i, \theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i, \theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.
Let $\theta: T' \rightarrow X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$. Applying the functor $\mathcal{U}D$ to the positive rational mappings $e_{i, \theta}: \mathbb{C}^\times \times T' \rightarrow T'$ and $\gamma_i \circ \theta, \epsilon_i \circ \theta: T' \rightarrow \mathbb{C}^\times$, we obtain

$$\tilde{e}_i := \mathcal{U}D(e_{i, \theta}): \mathbb{Z} \times X_*(T') \rightarrow X_*(T')$$

$$\mathrm{wt}_i := \mathcal{U}D(\gamma_i \circ \theta), \epsilon_i := \mathcal{U}D(\epsilon_i \circ \theta): X_*(T') \rightarrow \mathbb{Z}.$$

Hence the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\mathrm{wt}_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$ is a free pre-crystal structure (see [BK, 2.2]) and we denote it by $\mathcal{U}D_{\theta, T'}(\chi)$. As for the definition of crystal, see 3.4 or [KKM],[K1],[K2]. We have thus the following theorem:

**Theorem 2.9 ([BK][N1]).** For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$ and a positive structure $\theta: T' \rightarrow X$, the associated pre-crystal $\mathcal{U}D_{\theta, T'}(\chi)$ is a crystal (see [BK, 2.2]).

Now, let $\mathcal{GC}^+(\mathfrak{g})$ be the category whose object is a triplet $(\chi, T', \theta)$ where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\})$ is a $\mathfrak{g}$-geometric crystal and $\theta: T' \rightarrow X$ is a positive structure on $\chi$, and morphism $f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)$ is given by a morphism $\varphi: X_1 \rightarrow X_2$ of geometric crystals such that

$$f := \theta_2^{-1} \circ \varphi \circ \theta_1: T'_1 \rightarrow T'_2,$$

is a positive rational mapping. Let $\mathcal{CR} (\mathfrak{g})$ be the category of $\mathfrak{g}$-crystals. Then by the theorem above, we have

**Corollary 2.10.** $\mathcal{U}D$ defines a functor

$$\mathcal{U}D: \mathcal{GC}^+(\mathfrak{g}) \rightarrow \mathcal{CR} (\mathfrak{g}^L)$$

$$(\chi, T', \theta) \mapsto \mathcal{U}D_{\theta, T'}(\chi) = (X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\mathrm{wt}_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$$

$$(f: (\chi_1, T'_1, \theta_1) \rightarrow (\chi_2, T'_2, \theta_2)) \mapsto (\tilde{f}: X_*(T'_1) \rightarrow X_*(T'_2)),$$

where $\mathfrak{g}^L$ is the Langlands dual for $\mathfrak{g}$.

We call the functor $\mathcal{U}D$ “ultra-discretization” as in [N1],[N2]. While for a crystal $B$, if there exists a geometric crystal $\chi$ and a positive structure $\theta: T' \rightarrow X$ on $\chi$ such that $\mathcal{U}D(\chi, T', \theta) \simeq B$ as crystals, an object $(\chi, T', \theta)$ in $\mathcal{GC}^+(\mathfrak{g})$ is called a tropicalization of $B$.

§ 3. Perfect Crystals

§ 3.1. Affine weights

Let $\mathfrak{g}$ be an affine Lie algebra. The sets $t, \{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ be as in 2.1. We take $t$ so that $\dim t = \# I + 1$. Let $\delta \in Q_+$ be a unique element satisfying

$$\{\lambda \in Q | \langle \alpha_i^\vee, \lambda \rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta,$$
and let $c \in \sum_{i} \mathbb{Z}_{\geq 0} \alpha_{i} \subset g$ be a unique central element satisfying

$$\{ h \in Q^\vee \mid \langle h, \alpha_{i} \rangle = 0 \text{ for any } i \in I \} = Zc.$$ 

We write ([Kac, 6.1])

$$c = \sum_{i} a_{i} \alpha_{i}^\vee, \quad \delta = \sum_{i} a_{i} \alpha_{i}.$$

Let $(\ , \ )$ be the non-degenerate $W$-invariant symmetric bilinear form on $t^*$ normalized by $(\delta, \lambda) = \langle c, \lambda \rangle$ for $\lambda \in t^*$. Let us set $t^*_c := t^*/\mathbb{C} \delta$ and let $cl : t^* \rightarrow t^*_c$ be the canonical projection. Then we have $t^*_c \cong \bigoplus_{i} (\mathbb{C} \alpha_{i}^\vee)^*$. Set $t^*_0 := \{ \lambda \in t^* | \langle c, \lambda \rangle = 0 \}$, $(t^*_c)_0 := cl(t^*_0)$. Then we have a positive-definite symmetric form on $(t^*_c)_0$ induced by the one on $t^*$.

Let $\Lambda_{i} \in t^*_c$ $(i \in I)$ be a weight such that $\langle \alpha_{i}^\vee, \Lambda_{j} \rangle = \delta_{i,j}$, which is called a fundamental weight. We choose $P$ so that $P_c := cl(P)$ coincides with $\bigoplus_{i \in I} \mathbb{Z} \Lambda_{i}$ and we call $P_c$ the classical weight lattice.

### §3.2. Affinization

Let $U_q(g) = \langle e_{i}, f_{i}, q^{h} | i \in I, h \in P \rangle$ be the quantum affine algebra associated with $P$ and $U'_q(g) = \langle e_{i}, f_{i}, q^{h} | i \in I, h \in (P_c)^* \rangle$ its subalgebra associated with $(P_c)^*$. Set $Mod^f(g, P_c)$ the category of a finite dimensional $U'_q(g)$-module $M$ with a weight decomposition $M = \bigoplus_{\lambda \in P_c} M_{\lambda}$.

Let $M$ be an object in $Mod^f(g, P_c)$. For $l \in \mathbb{C}^\times$, define the $U'_q(g)$-module $M_l$ as follows: There exists a $\mathbb{C}$-linear bijective homomorphism $\Phi_l : M \rightarrow M$ such that

$$q^{h} \Phi_l(u) = \Phi_l(q^{h}u) \quad \text{for } h \in P^*_c,$$

$$e_{i} \Phi_l(u) = l^{\delta_{i,0}} \Phi_l(e_{i}u),$$

$$f_{i} \Phi_l(u) = l^{-\delta_{i,0}} \Phi_l(f_{i}u).$$

The module $M_l := \Phi_l(M)$ is said to be an affinization of $M$ ([KMN1],[K1]).

### §3.3. Fundamental representation $W(\varpi_{1})_l$

Here we consider the following affine Lie algebras $g = A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, A^{(2)}_{2n-1}, D^{(2)}_{n+1}, A^{(2)}_{2n}$. Let $\{ \Lambda_{i} | i \in I \}$ be the set of fundamental weights as in §3.1. Let $\varpi_{1} := \Lambda_{1} - \alpha_{1}^\vee \Lambda_{0}$ be the (level 0) fundamental weight, where $i = 1$ is the node of the Dynkin diagram as in [KNO][KNO2].

Let $W(\varpi_{1})$ be the fundamental representation of $U'_q(g)$ (see [K1, Sect 5.]). By [K1, Theorem 5.17], $W(\varpi_{1})$ is a finite-dimensional irreducible integrable $U'_q(g)$-module, an object in $Mod^f(g, P_c)$ and has a global basis with a simple crystal. Thus, we can consider its affinization $W(\varpi_{1})_l$ ($l \in \mathbb{C}^\times$) specialized at $q = 1$. Then we obtain a
finite-dimensional $\mathfrak{g}$-module denoted also by $W(\varpi_1)_l$ in which we shall construct affine geometric crystals in the next section.

Note that $W(\varpi_1)$ is an irreducible $\mathfrak{g}$-module for any $\mathfrak{g}$ as above. But, for some $i$ and $\mathfrak{g}$, $W(\varpi_i)$ is not necessarily irreducible.

§ 3.4. Limit of perfect crystals

We review the limits of perfect crystals following [KKM]. (See also [KMN1],[KMN2].) Let $\mathfrak{g}$ be an affine Lie algebra, $P_{\text{cl}}$ the classical weight lattice as above and for $l \in \mathbb{Z}_{>0}$ set $(P_{\text{cl}})_l^+ := \{ \lambda \in P_{\text{cl}} | \langle c, \lambda \rangle = l, \langle \alpha_i^\vee, \lambda \rangle \geq 0 \}$.

**Definition 3.1.** We say that a crystal $B$ is perfect of level $l$ if

(i) $B \otimes B$ is connected as a crystal graph.

(ii) There exists $\lambda_0 \in P_{\text{cl}}$ such that

$$\text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \text{cl}(\alpha_i), \quad \sharp B_{\lambda_0} = 1$$

(iii) There exists a finite-dimensional $U'_q(\mathfrak{g})$-module $V$ with a crystal pseudo-base $B_{ps}$ such that $B \cong B_{ps}/\pm 1$

(iv) The maps $\varepsilon, \varphi: B^{\text{min}} := \{ b \in B | \langle c, \varepsilon(b) \rangle = l \} \rightarrow (P_{\text{cl}})_l^+$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b) \Lambda_i$.

**Theorem 3.2 ([KMN1],[KMN2]).** For each affine Lie algebra $\mathfrak{g}$ as above and for any positive integer $l$, there exists a perfect crystal $B_l(\mathfrak{g})$ of level $l$.

For an affine Lie algebra $\mathfrak{g}$, let $\{B_l(\mathfrak{g})\}_{l \geq 1}$ be a family of perfect crystals of level $l$ and set $J := \{ (l, b) | l > 0, b \in B_l^{\text{min}} \}$.

**Definition 3.3.** A crystal $B_\infty = B_\infty(\mathfrak{g})$ with an element $b_\infty$ is called the limit of $\{B_l\}_{l \geq 1}$ if

(i) $\text{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$.

(ii) For any $(l, b) \in J$, there exists an embedding of crystals:

$$f(l, b): T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty$$

$$t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty$$

(iii) $B_\infty = \bigcup_{(l, b) \in J} \text{Im} f(l, b)$. 
As for the crystal $T_\lambda$, see Example 2.3 (iii). If the limit of a family $\{B_l\}$ exists, we say that $\{B_l\}$ is a **coherent family** of perfect crystals.

**Proposition 3.4 ([KKM]).** For each affine Lie algebra $\mathfrak{g}$ as above there exists a coherent family of perfect crystals $\{B_l(\mathfrak{g})\}_{l>0}$ and its limit $B_\infty(\mathfrak{g})$.

**§ 4. Affine Geometric Crystals**

Following the method in [KNO2], we shall see how to construct the affine geometric crystal $\mathcal{V}(\mathfrak{g})_l (l \in \mathbb{C}^\times)$ in $W(\varpi_1)_l$.

**§ 4.1. Translation $t(\tilde{\varpi}_1)$**

For $\xi_0 \in (\mathfrak{t}^*_c)_0$, let $t(\xi_0)$ be as in [K1, Sect.4]:

$$t(\xi_0)(\lambda) := \lambda + (\delta, \lambda)\xi - (\xi, \lambda)\delta - \frac{(\xi, \xi)}{2}(\delta, \lambda)\delta$$

for $\xi \in \mathfrak{t}^*$ such that $\text{cl}(\xi) = \xi_0$. Then $t(\xi_0)$ does not depend on the choice of $\xi$, and it is well-defined.

Let $c_i^\vee$ be as follows:

\[
(4.1) \quad c_i^\vee := \max(1, \frac{2}{(\alpha_i, \alpha_i)}).
\]

Then $t(m\varpi_i)$ belongs to the extended Weyl group $\widetilde{W}$ if and only if $m \in c_i^\vee \mathbb{Z}$. Setting $\tilde{\varpi}_i := c_i^\vee \varpi_i (i \in I)$ ([K2]), $t(\tilde{\varpi}_1)$ is expressed as follows:

$$t(\tilde{\varpi}_1) = \begin{cases} 
\iota(s_n s_{n-1} \cdots s_2 s_1) & A_n^{(1)} \text{ case}, \\
\iota(s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & B_n^{(1)}, A_{2n-1}^{(2)} \text{ cases,} \\
(s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & C_n^{(1)}, D_{n+1}^{(2)} \text{ cases,} \\
\iota(s_1 \cdots s_n)(s_{n-2} \cdots s_2 s_1) & D_n^{(1)} \text{ case,} \\
(s_0 s_1 \cdots s_n)(s_{n-1} \cdots s_2 s_1) & A_{2n}^{(2)} \text{ case,} 
\end{cases}$$

where $\iota$ is certain Dynkin diagram automorphism. Now, we know that each $t(\tilde{\varpi}_1)$ is in the form $w_1$ or $\iota \cdot w_1$ for some $w_1 \in W$, e.g., $w_1 = s_n \cdots s_1$ for $A_n^{(1)}$, $w_1 = (s_1 \cdots s_n)(s_{n-1} \cdots s_1)$ for $B_n^{(1)}$, etc., ....

In the case $A_{2n}^{(2)}$, $\eta := \text{wt}(v_n) = \Lambda_{n-1} - \Lambda_n$ (resp. $\Lambda_{n-1} - 2\Lambda_n$) is a unique weight of $W(\varpi_1)_l$ which satisfies $\langle \alpha_i^\vee, \eta \rangle \geq 0$ for $i \neq n$. For this $\eta$ we have

\[
(4.2) \quad t(\eta) = (s_n s_{n-1} \cdots s_1)(s_0 s_1 \cdots s_{n-1}) =: w_2.
\]
§ 4.2. Affine geometric crystals in $W(\varpi_1)_l$

Let $\sigma$ be the Dynkin diagram automorphism as in [KNO2] and $w_1 = s_{i_1} \cdots s_{i_k}$ be as in the previous subsection. Let $H \in \mathfrak{t}$ be an element as in [KNO2].

$$
\alpha_i(H) = \begin{cases} 
1 & \text{if } i = 1 \text{ and } \mathfrak{g} \neq D_{n+1}^{(2)}, A_{2n}^{(2)} \\
2 & \text{if } i = 1 \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)} \\
-1 & \text{if } i = 0 \text{ and } \mathfrak{g} \neq D_{n+1}^{(2)}, A_{2n}^{(2)} \\
-2 & \text{if } i = 0 \text{ and } \mathfrak{g} = D_{n+1}^{(2)}, A_{2n}^{(2)} \\
0 & \text{otherwise.} 
\end{cases}
$$

Set

\begin{align*}
\mathcal{V}(\mathfrak{g})_l := \{v(x_1, \ldots, x_k) := Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) l^H v_1 \mid x_1, \ldots, x_k \in \mathbb{C}^\times \} \subset W(\varpi_1)_l
\end{align*}

where the vector $v_1$ is a highest weight vector in $W(\varpi_1)_l$ as a $\mathfrak{g}_0$-module and then $\mathcal{V}(\mathfrak{g})$ has a $G_0$-geometric crystal structure, where $\mathfrak{g}_0 \subset \mathfrak{g}$ (resp. simple algebraic group) corresponding to the index set $I_0 := I \setminus \{0\}$. Moreover $(\mathbb{C}^\times)^k \to \mathcal{V}(\mathfrak{g})_l$ given by $(x_1, \ldots, x_k) \mapsto v(x_1, \ldots, x_k)$ is a birational morphism. We shall define a $G$-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by using the Dynkin diagram automorphism $\sigma$ except for $A_{2n}^{(2)}$. This $\sigma$ induces an automorphism of $W(\varpi_1)_l$, which is denoted by $\sigma_I : W(\varpi_1)_l \to W(\varpi_1)_l$.

**Theorem 4.1 ([KNO2]).**

(i) Case $\mathfrak{g} \neq A_{2n}^{(2)}$. For $x = (x_1 \cdots, x_k) \in (\mathbb{C}^\times)^k$, there exist a unique $y = (y_1, \ldots, y_k) \in (\mathbb{C}^\times)^k$ and a positive rational function $a(x)$ such that

\begin{align*}
v(y) = a(x) \sigma_I(v(x)), \quad \varepsilon_{\sigma(i)}(v(y)) = \varepsilon_i(v(x)) \quad \text{if } i, \sigma(i) \neq 0.
\end{align*}

(ii) Case $\mathfrak{g} = A_{2n}^{(2)}$. Associated with $w_1$ and $w_2$ as in the previous subsection, we define

\begin{align*}
\mathcal{V}(\mathfrak{g})_l := \{v_1(x) = Y_0(x_0)Y_1(x_1) \cdots Y_n(x_n)x_{n-1}(\bar{x}_{n-1}) \cdots Y_1(\bar{x}_1) l^H v_1 \mid x_i, \bar{x}_i \in \mathbb{C}^\times \}, \\
\mathcal{V}_2(\mathfrak{g})_l := \{v_2(y) = Y_n(y_n) \cdots Y_1(y_1)Y_0(y_0)Y_1(\bar{y}_1) \cdots Y_{n-1}(\bar{y}_{n-1}) l^{H'} v_{n} \mid y_i, \bar{y}_i \in \mathbb{C}^\times \},
\end{align*}

where $\alpha_0(H') = 2$, $\alpha_n(H') = -4$ and $\alpha_i(H') = 0$ otherwise. (Note that $\text{wt}(v_1)(H) = \text{wt}(v_{n})(H')$. For any $x \in (\mathbb{C}^\times)^{2n}$ there exist a unique $y \in (\mathbb{C}^\times)^{2n}$ and a rational function $a(x)$ such that $v_2(y) = a(x)v_1(x)$.

Now, using this theorem, we define the rational mapping

\begin{align*}
\tilde{\sigma} : \mathcal{V}(\mathfrak{g})_l &\to \mathcal{V}(\mathfrak{g})_l, \\
v(x) &\mapsto v(y) \quad (\mathfrak{g} \neq A_{2n}^{(2)}), \\
\tilde{\sigma} : \mathcal{V}(\mathfrak{g})_l &\to \mathcal{V}_2(\mathfrak{g})_l, \\
v_1(x) &\mapsto v_2(y) \quad (\mathfrak{g} = A_{2n}^{(2)}),
\end{align*}

Associated with an element $w \in W$, let $B_w^{-}$ be the geometric crystal as in [N1],[KNO], which is isomorphic to the geometric crystal on the Schubert cell $X_w$(see [N1]).
\textbf{Theorem 4.2 ([KNO2]).} The rational mapping $\overline{\sigma}$ is birational. If we define a $\mathfrak{g}_0$-geometric crystal structure on $\mathcal{V}(\mathfrak{g})_l$ by the one on $B_{w_1}^{-} \cdot l^H$ and a rational $\mathbb{C}^\times$-action $e_0 : \mathbb{C}^\times \times \mathcal{V}(\mathfrak{g})_l \to \mathcal{V}(\mathfrak{g})_l$ and rational functions $\varepsilon_0$ and $\varepsilon_0$ on $\mathcal{V}(\mathfrak{g})_l$ by

\begin{equation}
\begin{cases}
e_0^c := \overline{\sigma}^{-1} \circ e_{\sigma(0)}^c \circ \overline{\sigma}, & \varepsilon_0 := \varepsilon_{\sigma(0)} \circ \overline{\sigma}, \quad \gamma_0 := \gamma_{\sigma(0)} \circ \overline{\sigma}, \quad \text{for } \mathfrak{g} \neq A_{2n}^{(2)}, \\
e_0^c := \overline{\sigma}^{-1} \circ e_0^c \circ \overline{\sigma}, & \varepsilon_0 := \varepsilon_0 \circ \overline{\sigma}, \quad \gamma_0 := \gamma_0 \circ \overline{\sigma}, \quad \text{for } \mathfrak{g} = A_{2n}^{(2)}.
\end{cases}
\end{equation}

then $(\mathcal{V}(\mathfrak{g})_l, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\epsilon_i\}_{i \in I})$ turns out to be a positive affine $\mathfrak{g}$-geometric crystal.

\textbf{Remark.} In the case $\mathfrak{g} = A_{2n}^{(2)}$, $\mathcal{V}_2(\mathfrak{g})$ has a $\mathfrak{g}_{I\setminus\{n\}}$-geometric crystal structure. Thus, $e_0, \gamma_0, \varepsilon_0$ are well-defined on $\mathcal{V}_2(\mathfrak{g})$.

\section*{4.3. Product structure on affine geometric crystals}

In general, if $\chi_1$ and $\chi_2$ are geometric crystals induced from unipotent crystals, the product $\chi_1 \times \chi_2$ possesses a geometric crystal structure ([BK]). More precisely, let $\chi_1 = (X, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\})$ and $\chi_2 = (Y, \{e_i\}, \{\gamma_i\}, \{\epsilon_i\})$ be geometric crystals induced from unipotent crystals. For $x \in X$ and $y \in Y$ set

\begin{equation}
\gamma_i(x, y) := \gamma_i(x)\gamma_i(y),
\end{equation}

\begin{equation}
\varepsilon_i(x, y) := \varepsilon_i(x) + \frac{\varepsilon_i(x)\varepsilon_i(y)}{\varphi_i(x)} \quad (\varphi_i(x) = \gamma(x)\epsilon_i(x)),
\end{equation}

\begin{equation}
e_i^c(x, y) := (e_i^{c_1}(x), e_i^{c_2}(y)) \quad \text{where } c_1 := \frac{c \varphi_i(x) + \varepsilon_i(y)}{\varphi_i(x) + \varepsilon_i(y)}, \quad c_2 := \frac{c}{c_1}.
\end{equation}

Then, (4.7)–(4.9) endow the product $X \times Y$ with a structure of a geometric crystal.

As for the affine geometric crystal $\mathcal{V}_l$ in the previous subsection, its data $e_i, \gamma_i, \varepsilon_i$ ($i = 1, \ldots, n$) are obtained from the ones of the geometric crystal $B_i^{-} \cdot l^H$ which is induced from the unipotent crystal on some $X_w \times l^H$ where $i$ is a reduced word for $w$ and $X_w$ is the Schubert cell associated with $w \in W$ and note that $\{l^H\}$ holds trivial unipotent crystal structure. We can check the $i = 0$-case directly and then obtain:

\textbf{Theorem 4.3 ([KNO2]).} For any $k \in \mathbb{Z}_{\geq 0}$ and $L_1 \cdots, L_k \in \mathbb{C}^\times$, the product $\mathcal{V}_{L_1} \times \cdots \times \mathcal{V}_{L_k}$ has an affine geometric crystal structure.

\section*{4.4. Ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$}

Let us see the ultra-discretization of $\mathcal{V}(\mathfrak{g})_l$.

By Theorem 4.2 on $\mathcal{V}(\mathfrak{g})_l$, for $l > 0$ it holds a natural positive structure $\theta_l : (\mathbb{C}^\times)^m \to \mathcal{V}(\mathfrak{g})_l (x \mapsto v(x))$ where $m = \dim \mathcal{V}(\mathfrak{g})_l$. Then we have the following theorem:

\textbf{Theorem 4.4 ([KNO2]).} For $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, D_n^{(1)}, D_{n+1}^{(2)}, A_{2n-1}^{(2)}$ and $A_{2n}^{(2)}$, suppose that $l > 0$. Then the ultra-discretization $UD_{\theta_l}(\mathcal{V}(\mathfrak{g})_l)$ associated with the positive structure $\theta_l$ is isomorphic to the crystal $B_\infty(\mathfrak{g}^L)$. 
§ 5. Tropical R Maps

§ 5.1. Definition of tropical R maps

Definition 5.1. Let \( \{ (X_{\lambda}, \{ e_{i}^{\lambda} \}, \{ \gamma_{i}^{\lambda} \}, \{ \epsilon_{i}^{\lambda} \}) \}_{\lambda \in \Lambda} \) be a family of geometric crystals equipped with the product structures, where \( \Lambda \) is a certain index set and its element is called a spectral parameter. A birational isomorphism \( \mathcal{R}_{\lambda \mu} : X_{\lambda} \times X_{\mu} \to X_{\mu} \times X_{\lambda} \) (\( \lambda, \mu \in \Lambda \)) is said to be a tropical R map if it satisfies the following conditions:

\[
\begin{align*}
(5.1) \quad (e_{i}^{X_{\mu} \times X_{\lambda}})^{c} \circ \mathcal{R}_{\lambda \mu} &= \mathcal{R}_{\lambda \mu} \circ (e_{i}^{X_{\lambda} \times X_{\mu}})^{c}, \\
(5.2) \quad \epsilon_{i}^{X_{\lambda} \times X_{\mu}} &= \epsilon_{i}^{X_{\mu} \times X_{\lambda}} \circ \mathcal{R}_{\lambda \mu}, \\
(5.3) \quad \gamma_{X_{\lambda} \times X_{\mu}} &= \gamma_{X_{\mu} \times X_{\lambda}} \circ \mathcal{R}_{\lambda \mu}, \\
(5.4) \quad \mathcal{R}^{(12)} \mathcal{R}^{(23)} \mathcal{R}^{(12)} &= \mathcal{R}^{(23)} \mathcal{R}^{(12)} \mathcal{R}^{(23)} \text{ on } X_{\lambda} \times X_{\mu} \times X_{\nu},
\end{align*}
\]

for any \( i \in I \) and any \( \lambda, \mu, \nu \in \Lambda \). Here \( \mathcal{R}^{(ij)} \) means that it acts on \( i \)-th and \( j \)-th components of the product.

In [KNO2], we obtain tropical R maps \( \{ \overline{R}_{LM} \}_{L, M > 0} \) of the family of affine geometric crystals \( \{ V_{L} \}_{L > 0} \) for the above affine Lie algebras.

§ 5.2. Prehomogeneous Geometric Crystal

Definition 5.2. Let \( \chi = (X, \{ e_{i}^{c} \}, \{ \gamma_{i} \}, \{ \epsilon_{i} \}) \) be a geometric crystal. We say that \( \chi \) is prehomogeneous if there exists a Zariski open dense subset \( \Omega \subset X \) which is an orbit by the actions of the \( e_{i}^{c} \)'s.

The following lemma is obvious.

Lemma 5.3 ([KNO2]). Let \( \chi_{j} = (X_{j}, \{ e_{i}^{j} \}, \{ \gamma_{i} \}, \{ \epsilon_{i} \}) \) \( (j = 1, 2) \) be prehomogeneous geometric crystals. Let \( \Omega_{1} \subset X_{1} \) be an open dense orbit in \( X_{1} \). For isomorphisms of geometric crystals \( \phi, \phi' : \chi_{1} \to \chi_{2} \), suppose that there exists \( p_{1} \in \Omega_{1} \) such that \( \phi(p_{1}) = \phi'(p_{1}) \in X_{2} \). Then, we have \( \phi = \phi' \) as rational morphisms.

The following is the key of this article:

Theorem 5.4 ([KNO2]). Let \( \chi = (X, \{ e_{i}^{c} \}, \{ \gamma_{i} \}, \{ \epsilon_{i} \}) \) be a finite-dimensional positive geometric crystal with the positive structure \( \theta : T \to X \) and \( B := UD_{\theta}(\chi) \) the crystal obtained as the ultra-discretization of \( \chi \). \( \chi \) is prehomogeneous if \( B \) is a connected crystal.

§ 5.3. Uniqueness of the birational R-maps

Theorem 5.5 ([KNO2]). Let \( \overline{R}_{LM} \) be the tropical R map as above. Set \( z_{0} := \overline{R}(1, 1) \) where \( 1 := v(1, \cdots, 1) \in V_{L} \). Let \( \mathcal{R}' \) be a tropical R map such that \( \mathcal{R}'(1, 1) = z_{0} \). Then we have \( \overline{R} = \mathcal{R}' \) as birational morphisms.
Proof. Let $V_l, V_m \ (l, m \in \mathbb{C}^\times)$ be the affine geometric crystals constructed in Sect.4 and $\mathcal{R}$ the tropical R map as in this section and $\mathcal{R}': V_l \times V_m \to V_m \times V_l$ other tropical R map such that $\mathcal{R}(1,1) = \mathcal{R}'(1,1)$. By Theorem 4.4, we have that $UD_{(\Theta_l, \Theta_m)}(V_l(\mathfrak{g}) \times V_m(\mathfrak{g}))$ is isomorphic to the crystal $B_\infty(\mathfrak{g}^L) \otimes B_\infty(\mathfrak{g}^L)$.

It follows from that for any positive integer $m$ and any affine Lie algebra $\mathfrak{g}$ appearing in this article tensor product of crystals $\bigotimes_{i=1}^{m} UD_{\Theta_{l_i}}(V(\mathfrak{g})_{l_i}) \cong B_\infty(\mathfrak{g}^L)^\otimes m$ is connected. By Theorem 5.4 we obtain that the geometric crystal $V_l \times V_m$ is prehomogeneous. Therefore, by Lemma 5.3 we obtain $\mathcal{R} = \mathcal{R}'$, which completes the proof of Theorem 5.5. 

\[\square\]

§6. Universal tropical $R$ map of $\mathfrak{sl}_2$

Only for $\mathfrak{sl}_2$-case, we found a kind of "universal" tropical R map. For arbitrary $\mathfrak{sl}_2$-geometric crystals $X = (X, e_X, \gamma_X, \epsilon_X)$ and $Y = (Y, e_Y, \gamma_Y, \epsilon_Y)$, the morphism $\mathcal{R}: X \times Y \to Y \times X$ is defined by

\begin{equation}
\mathcal{R}(x, y) := (e_Y^{a(x,y)} y, e_X^{b(x,y)} x), \quad \text{where} \quad a(x, y) := \frac{\epsilon_Y(y) + \frac{\epsilon_X(x)}{\gamma_Y(y)}}{\epsilon_X(x) + \frac{\epsilon_Y(y)}{\gamma_X(x)}}, \quad b(x, y) = \frac{1}{a(x, y)}.
\end{equation}

Theorem 6.1. The morphism $\mathcal{R}$ satisfies the following:

\begin{align}
\gamma_{Y \times X}(\mathcal{R}(x, y)) &= \gamma_{X \times Y}(x, y), \\
\epsilon_{Y \times X}(\mathcal{R}(x, y)) &= \epsilon_{X \times Y}(x, y), \\
\mathcal{R} \circ e^c &= e^c \circ \mathcal{R}, \\
\mathcal{R}^{(12)} \mathcal{R}^{(23)} \mathcal{R}^{(12)} &= \mathcal{R}^{(23)} \mathcal{R}^{(12)} \mathcal{R}^{(23)} \quad \text{(Yang-Baxter eq.)} \\
\mathcal{R} \circ \mathcal{R} &= \text{id}_{X \times Y}
\end{align}

This implies that there exists the canonical morphism with the same properties as a universal R-matrix for modules of quantum groups. Furthermore, the morphism $\mathcal{R}$ is an isomorphism of geometric crystals.

Proof. Let us show (6.2). We have

\begin{align}
\gamma_{Y \times X}(\mathcal{R}(x, y)) &= \gamma_Y(e^{a(x,y)} y) \gamma_X( e^{b(x,y)} x) = a(x, y)^2 b(x, y)^2 \gamma_X(x) \gamma_Y(y) = \gamma_{X \times Y}(x, y).
\end{align}

Next, let us see (6.3).

\begin{align}
\epsilon_{Y \times X}(\mathcal{R}(x, y)) &= \epsilon_Y(e^{a(x,y)} y) + \frac{\epsilon_X(e^{b(x,y)} x)}{\gamma_Y(e^{a(x,y)} y)} = a(x, y)^{-1} \epsilon_Y(y) + \frac{b(x, y)^{-1} \epsilon_X(x)}{a(x, y)^2 \gamma_Y(y)} \\
&= a(x, y)^{-1} \left( \frac{\epsilon_Y(y) + \frac{\epsilon_X(x)}{\gamma_X(x)}}{\gamma_Y(y)} \right) = \epsilon_X(x) + \frac{\epsilon_Y(y)}{\gamma_X(x)} = \epsilon_{X \times Y}(x, y).
\end{align}
Let us show (6.4). For $c \in \mathbb{C}^\times$ and $(x, y) \in X \times Y$, let $c_1 := c_1(c, x, y)$, $c_2 := c_2(c, x, y)$ be as in (4.9). We have

\begin{align*}
(6.7) \quad \mathcal{R}(e^c(x, y)) &= (e^{a(e^{c_1}x, e^{c_2}y)c_2}y, e^{b(e^{c_1}x, e^{c_2}y)c_1}x), \\
(6.8) \quad e^c\mathcal{R}(x, y) &= (e^{c'_1(x,y)}y, e^{c'_2(x,y)}x),
\end{align*}

where $c'_1 := c_1(c, e^{a(x,y)}y, e^{b(x,y)}x)$ and $c'_2 := c_2(c, e^{a(x,y)}y, e^{b(x,y)}x)$. By simple calculations, one has

\begin{align*}
(6.9) \quad a(e^{c_1}x, e^{c_2}y) &= \frac{c_1}{c_2} \frac{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}, \quad c'_1 = \frac{c_1}{c_2} \frac{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}}{\varepsilon_Y(y) + \frac{\varepsilon_X(x)}{\gamma_Y(y)}},
\end{align*}

which means $a(e^{c_1}x, e^{c_2}y)c_2 = c'_1a(x, y)$ and then $b(e^{c_1}x, e^{c_2}y)c_1 = c'_2a(x, y)$. We obtained (6.4).

Let us show (6.5). Denote

\begin{align*}
\mathcal{R}^{(12)}\mathcal{R}^{(23)}\mathcal{R}^{(12)}(x, y, z) &= (e^A z, e^B y, e^C x), \\
\mathcal{R}^{(23)}\mathcal{R}^{(12)}\mathcal{R}^{(23)}(x, y, z) &= (e^{A'} z, e^{B'} y, e^{C'} x).
\end{align*}

It is easy to see that $ABC = 1 = A'B'C'$ and then we may show that $A = A'$ and $C = C'$. By simple calculations, we have

\begin{align*}
A &= a(e^{a(x,y)}y, e^{a(e^{b(x,y)}x,z)}z)a(e^{b(x,y)}x, z) = \frac{\varepsilon_Z(z) + \varepsilon_X(x)}{\frac{\varepsilon_X(x)}{b(x,y)^2}\gamma_X(x)} \\
&= \frac{\varepsilon_Z(z) \left( \varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_Z(z)} \right) + \varepsilon_X(x) \left( \varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)}{\varepsilon_X(x) \left( \varepsilon_Z(z) + \frac{\varepsilon_Y(y)}{\gamma_X(z)} \right) + \varepsilon_X(x) \left( \varepsilon_Y(y) + \frac{\varepsilon_Z(z)}{\gamma_Y(y)} \right)},
\end{align*}

\begin{align*}
A' &= a(x, e^{a(y,z)}z)a(y, z) = \frac{\varepsilon_Z(z) + \frac{\varepsilon_X(x)}{a(y,z)\gamma_Z(z)}}{\varepsilon_X(x) + \frac{\varepsilon_Z(z)}{a(y,z)\gamma_X(x)}},
\end{align*}

which means $A = A'$.

Next, we have

\begin{align*}
C &= b(e^{b(x,y)}x, z)b(x, y), \\
C' &= b(e^{b(x,e^{a(y,z)}z)}x, e^{b(y,z)}y)b(x, e^{a(y,z)}z).
\end{align*}

It follows from the explicit forms of $a(x, y)$ and $b(x, y)$ that $a(x, y) = b(y, x)$. Thus we
obtain

\[
C_{|x\leftrightarrow z} = b(e^{b(z,y)}z, x)b(z, y) = a(x, e^{a(y,z)}z)a(y, z) = A'
\]

\[
C_{|x\leftrightarrow z}' = b(e^{b(z,e^{a(y,x)}x)}z, e^{b(y,x)}y)b(z, e^{a(y,x)}x)
\]

\[
= a(e^{a(x,y)}y, e^{a(e^{b(x,y)}x,z)}z)a(e^{b(x,y)}x, z) = A,
\]

which implies \(C = C'\). Thus, we also obtain \(B = B'\).

Finally, let us show (6.6). Set \((y', x') = \mathcal{R}(x, y)\). Then we have

\[
\mathcal{R} \circ \mathcal{R}(x, y) = (e_X^{a(y',x')}e_Y^{b(x,y)}x, e_X^{b(y',x')}e_Y^{a(x,y)}y).
\]

By the formula (6.9) and \(a(x, y)b(x, y) = 1\) we obtain

\[
a(y', x') = a(e_X^{a(x,y)}y, e_X^{b(x,y)}x) = \frac{a(x,y)}{b(x,y)}a(y, x) = \frac{a(x,y)}{b(x,y)}b(x, y) = a(x, y),
\]

and also \(b(y', x') = b(x, y)\). Thus, we have \(\mathcal{R} \circ \mathcal{R}(x, y) = (x, y)\) and completed the proof.

\[\square\]

References


