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B11 (2009), 43–52

Tensor Representations for the Quantum Loop Algebras of Type A at Roots of Unity

By

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§ 1. Introduction

Let q be a nonzero complex number which is not a root of unity and let $U_q(\widetilde{\mathfrak{g}})$ be the quantum loop algebra over $\mathbb{C}$ associated with a finite-dimensional complex simple Lie algebra $\mathfrak{g}$. It is known that every finite-dimensional irreducible representation of $U_q(\widetilde{\mathfrak{g}})$ is a pseudo-highest weight representation and classified by the pseudo-highest weights (see [7]). In 2002, Chari showed the sufficient condition for the tensor product of the finite-dimensional irreducible representations of $U_q(\widetilde{\mathfrak{g}})$ to be a pseudo-highest weight representation (see [4]). In particular, by using this result, we obtain the necessary and sufficient conditions for the tensor product of the *fundamental representations* to be irreducible (see also [2], [13], and [11]). It is known that every finite-dimensional irreducible representation of $U_q(\widetilde{\mathfrak{g}})$ is isomorphic to a subquotient of the tensor product of the fundamental representations (see [7]).

In this note, we explain the necessary and sufficient conditions for the tensor product of the fundamental representations for the restricted quantum loop algebras of type A at roots of unity to be irreducible. This is the result in [1].

§ 2. Notations

We fix the following notations:
$\mathbb{N} := \{1, 2, \cdots\}$: the set of natural numbers,
$\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$: the set of non-negative integers,
$I := \{1, 2, \cdots, n\}$, $\widetilde{I} := I \sqcup \{0\}$: index sets,

Received January 22, 2008. Accepted January 3, 2009.
2000 Mathematics Subject Classification(s): 17B37
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$(\mathfrak{a}_{i,j})_{i,j\in I}$: Cartan matrix of type A_n,
$(\mathfrak{a}_{i,j})_{i,j\in\widetilde{I}}$: generalized Cartan matrix of type $A_n^{(1)}$,
q: indeterminate, $\mathbb{C}(q)$: rational function field of q,
$[r] := \dfrac{q^r - q^{-r}}{q - q^{-1}}$, $[m]! := [m][m-1]\cdots[1]$, $[0]! := 1$,
$\begin{bmatrix} r \\ m \end{bmatrix} := \dfrac{[r][r-1]\cdots[r-m+1]}{[m][m-1]\cdots[1]}$ $(r \in \mathbb{Z}, m \in \mathbb{N})$,
l: an odd integer greater than 2, ε: a primitive l-th root of unity,
$\mathcal{A} := \mathbb{C}[t, t^{-1}]$: Laurent polynomial ring,
$\mathbb{C}_\varepsilon$: $\mathcal{A}$-algebra of the set of complex numbers defined by the following formula:

$$g(q).c := g(\varepsilon)c \quad \text{for} \quad g(q) \in \mathcal{A}, c \in \mathbb{C}.$$

§ 3. The restricted quantum loop algebras

First, we introduce the generic quantum loop algebras to define the restricted quantum loop algebras.

Definition 3.1. The *quantum loop algebra* $\widetilde{U}_q$ *of type* A_n is the associative $\mathbb{C}(q)$-algebra generated by $\{E_i, F_i, K_i^{\pm 1} \mid i \in \widetilde{I}\}$ with the following defining relations:

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i, \quad K_0 = \prod_{p\in I} K_p^{-1},$$
$$K_iE_jK_i^{-1} = q^{\mathfrak{a}_{i,j}}E_j, \quad K_iF_jK_i^{-1} = q^{-\mathfrak{a}_{i,j}}F_j,$$
$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$
$$\sum_{p=0}^{1-\mathfrak{a}_{i,j}} (-1)^p E_i^{(p)} E_j E_i^{(1-\mathfrak{a}_{i,j}-p)} = \sum_{p=0}^{1-\mathfrak{a}_{i,j}} (-1)^p F_i^{(p)} F_j F_i^{(1-\mathfrak{a}_{i,j}-p)} = 0 \quad i \neq j,$$

for $i, j \in \widetilde{I}$, where

$$E_i^{(m)} := \frac{1}{[m]!}E_i^m, \quad F_i^{(m)} := \frac{1}{[m]!}F_i^m \quad (m \in \mathbb{Z}_+).$$

Next, we introduce the Drinfel'd realization of $\widetilde{U}_q$ to define the pseudo-highest weight representations of the restricted quantum loop algebras (see §4 in this paper).

Theorem 3.2 (**[9]** and **[3]**). *As a* $\mathbb{C}(q)$*-algebra,* $\widetilde{U}_q$ *is isomorphic to the algebra with generators* $\{X_{i,r}^{\pm}, H_{i,s}, K_i^{\pm 1} \mid i \in I, r, s \in \mathbb{Z}, s \neq 0\}$ *and the following defining* *relations:*

$$\begin{aligned}
&K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad [K_i, K_j] = [K_i, H_{j,s}] = [H_{i,s}, H_{j,s'}] = 0,\\
&K_iX_{j,r}^{\pm}K_i^{-1} = q^{\pm\mathfrak{a}_{i,j}}X_{j,r}^{\pm}, \qquad [H_{i,s}, X_{j,r}^{\pm}] = \pm\frac{[s\mathfrak{a}_{i,j}]}{s}X_{j,r+s}^{\pm},\\
&X_{i,r+1}^{\pm}X_{j,r'}^{\pm} - q^{\pm\mathfrak{a}_{i,j}}X_{j,r'}^{\pm}X_{i,r+1}^{\pm} = q^{\pm\mathfrak{a}_{i,j}}X_{i,r}^{\pm}X_{j,r'+1}^{\pm} - X_{j,r'+1}^{\pm}X_{i,r}^{\pm},\\
&[X_{i,r}^{+}, X_{j,r'}^{-}] = \delta_{i,j}\frac{\Psi_{i,r+r'}^{+} - \Psi_{i,r+r'}^{-}}{q-q^{-1}},\\
&\sum_{\pi\in\mathcal{S}_{\boldsymbol{m}}}\sum_{p=0}^{\boldsymbol{m}}(-1)^p\begin{bmatrix}\boldsymbol{m}\\ p\end{bmatrix}X_{i,r_{\pi(1)}}^{\pm}\cdots X_{i,r_{\pi(p)}}^{\pm}X_{j,r'}^{\pm}X_{i,r_{\pi(p+1)}}^{\pm}\cdots X_{i,r_{\pi(m)}}^{\pm} = 0, \quad (i \neq j),
\end{aligned}$$

for $r_1, \cdots, r_{\boldsymbol{m}} \in \mathbb{Z}$, *where* $\boldsymbol{m} := 1 - \mathfrak{a}_{i,j}$, $\mathcal{S}_{\boldsymbol{m}}$ *is the symmetric group on* $\boldsymbol{m}$ *letters, and* $\Psi_{i,r}^{\pm}$ *are determined by*

$$\sum_{r=0}^{\infty}\Psi_{i,\pm r}^{\pm}u^{\pm r} := K_i^{\pm 1}\exp(\pm(q-q^{-1})\sum_{s=1}^{\infty}H_{i,\pm s}u^{\pm s}),$$

and $\Psi_{i,\pm r}^{\pm} := 0$ *if* $r < 0$.

Next, we introduce the restricted quantum loop algebras and their triangular decomposition. Let $\widetilde{U}_{\mathcal{A}}^{\mathrm{res}}$ be the $\mathcal{A}$-subalgebra of $\widetilde{U}_q$ generated by $\{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1} \mid i \in \widetilde{I},\ m \in \mathbb{N}\}$.

Definition 3.3 (**[8] and [12]**). The *restricted quantum loop algebra* $\widetilde{U}_{\varepsilon}^{\mathrm{res}}$ is defined as follows:

$$\widetilde{U}_{\varepsilon}^{\mathrm{res}} := \widetilde{U}_{\mathcal{A}}^{\mathrm{res}} \otimes_{\mathcal{A}} \mathbb{C}_{\varepsilon}.$$

For $i \in I$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$, define

$$\begin{aligned}
&\begin{bmatrix}K_i; r\\ m\end{bmatrix} := \prod_{p=1}^{m}\frac{K_iq^{r-p+1} - K_i^{-1}q^{-r+p-1}}{q^p - q^{-p}},\\
&\sum_{m=0}^{\infty}\mathcal{P}_{i,\pm m}u^m := \exp(-\sum_{s=1}^{\infty}\frac{q^s}{[s]}H_{i,\pm s}u^s) \quad \text{in} \quad \widetilde{U}_q[[u]].
\end{aligned}$$

Then we have

$$\begin{bmatrix}K_i; r\\ m\end{bmatrix}, \quad (X_{i,r}^{\pm})^{(m)}, \quad \frac{1}{[s]}H_{i,s}, \quad \mathcal{P}_{i,r} \in \widetilde{U}_{\mathcal{A}}^{\mathrm{res}},$$

for $i \in I$, $m \in \mathbb{N}$, $r \in \mathbb{Z}$, and $s \in \mathbb{Z}^{\times}$, where $(X_{i,r}^{\pm})^{(m)} := \frac{1}{[m]!}(X_{i,r}^{\pm})^m$ (see [6, §9.3A] and

[8, §3.1]). Therefore, the following elements are included in $\widetilde{U}_\varepsilon^{\mathrm{res}}$:

$$e_i^{(m)} := E_i^{(m)} \otimes 1, \quad f_i^{(m)} := F_i^{(m)} \otimes 1, \quad k_i := K_i \otimes 1, \quad \begin{bmatrix} k_j; r \\ m \end{bmatrix} := \begin{bmatrix} K_j; r \\ m \end{bmatrix} \otimes 1,$$
$$(x_{j,r}^{\pm})^{(m)} := (X_{j,r}^{\pm})^{(m)} \otimes 1, \quad h_{j,s} := H_{j,s} \otimes 1, \quad \psi_{j,r}^{\pm} = \Psi_{j,r}^{\pm} \otimes 1, \quad \mathfrak{p}_{j,r} := \mathcal{P}_{j,r} \otimes 1,$$

where $i \in \widetilde{I}$, $j \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}^\times$, and $m \in \mathbb{N}$. Define $(\widetilde{U}_\varepsilon^{\mathrm{res}})^+$ (resp. $(\widetilde{U}_\varepsilon^{\mathrm{res}})^-$, $(\widetilde{U}_\varepsilon^{\mathrm{res}})^0$) to be the $\mathbb{C}$-subalgebra of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ generated by $\{(x_{i,r}^+)^{(m)} \mid i \in I, r \in \mathbb{Z}, m \in \mathbb{N}\}$ (resp. $\{(x_{i,r}^-)^{(m)} \mid i \in I, r \in \mathbb{Z}, m \in \mathbb{N}\}$, $\{\mathfrak{p}_{i,r}, k_i, \begin{bmatrix} k_i; 0 \\ l \end{bmatrix} \mid i \in I, r \in \mathbb{Z}\}$). Then we obtain the triangular decomposition of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ (see [8, §6]):

$$(\widetilde{U}_\varepsilon^{\mathrm{res}})^- \otimes (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \otimes (\widetilde{U}_\varepsilon^{\mathrm{res}})^+ \xrightarrow{\sim} \widetilde{U}_\varepsilon^{\mathrm{res}}. \tag{3.1}$$

The comultiplication Δ of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ is as follows:

$$\Delta(e_i^{(m)}) = \sum_{p=1}^{m} q^{-p(m-p)} e_i^{(p)} \otimes k_i^p e_i^{(m-p)}, \quad \Delta(f_i^{(m)}) = \sum_{p=1}^{m} q^{p(m-p)} f_i^{(p)} \otimes f_i^{(m-p)} k_i^{-p},$$
$$\Delta(k_i) = k_i \otimes k_i,$$

for any $i \in \widetilde{I}$ and $m \in \mathbb{N}$ (see [6, §9.3A]).

By using (3.1) and the results of the generic case (in particular, [4, Proposition 2.2]), we obtain the following lemma:

Lemma 3.4 ([1, Lemma 6.3]). *Let $i \in I$ and $r \in \mathbb{N}$. Modulo $\widetilde{U}_\varepsilon^{\mathrm{res}} \otimes X_+^{\mathrm{res}}$,*

$$\Delta(x_{i,r}^-) = x_{i,r}^- \otimes k_i + 1 \otimes x_{i,r}^- + \sum_{p=1}^{r-1} x_{i,r-p}^- \otimes \psi_{i,p}^+,$$

where $X_+^{\mathrm{res}} := \sum_{j \in I, r \in \mathbb{Z}, m \in \mathbb{N}} \widetilde{U}_\varepsilon^{\mathrm{res}} (x_{j,r}^+)^{(m)}$.

§ 4. Representation theory of the restricted quantum loop algebras

Without loss of generality, we assume that the representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ is of type 1, that is, $k_i^l = 1$ on the representation for any $i \in I$.

For any $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}_0[t])^n$ ($\mathbb{C}_0[t] := \{\pi(t) \in \mathbb{C}[t] \mid \pi(0) = 1\}$), there exists a unique $\mathbb{C}$-algebra homomorphism $\Lambda_\pi : (\widetilde{U}_\varepsilon^{\mathrm{res}})^0 \longrightarrow \mathbb{C}$ such that

$$\Lambda_\pi(k_i^{\pm 1}) = \varepsilon^{p_i^{(0)}}, \quad \Lambda_\pi(\begin{bmatrix} k_i; 0 \\ l \end{bmatrix}) = p_i^{(1)}, \quad \sum_{m=1}^{\infty} \Lambda_\pi(\mathfrak{p}_{i,\pm m}) t^m = \pi_i^{\pm}(t),$$

where

$$\pi_i^+(t) := \pi_i(t), \quad \pi_i^-(t) := \frac{t^{\deg \pi_i(t)}\pi_i(t^{-1})}{(t^{\deg \pi_i(t)}\pi_i(t^{-1}))|_{t=0}},$$
$$\deg(\pi_i(t)) = p_i^{(0)} + lp_i^{(1)} \quad (0 \leq p_i^{(0)} < l).$$

Definition 4.1. Let V be a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ and $\pi = (\pi_i(t))_{i \in I}$ be an element in $(\mathbb{C}_0[t])^n$. We assume that there exists a nonzero vector $v_\Lambda \in V$ such that $(x_{i,r}^+)^{(m)}v_\Lambda = 0$ for all $i \in I$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$ and $uv_\Lambda = \Lambda_\pi(u)v_\Lambda$ for all $u \in (\widetilde{U}_\varepsilon^{\mathrm{res}})^0$. If V is generated by v_Λ as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$, we call V a *pseudo-highest weight representation* of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the *pseudo-highest weight* π generated by a *pseudo-highest weight vector* v_Λ.

Theorem 4.2 ([8, §8]). *For any $\pi \in (\mathbb{C}_0[t])^n$, there exists a finite-dimensional irreducible pseudo-highest weight $\widetilde{U}_\varepsilon^{\mathrm{res}}$-representation $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ with the pseudo-highest weight π. Moreover, for any finite-dimensional irreducible $\widetilde{U}_\varepsilon^{\mathrm{res}}$-representation V, there exists a unique $\pi \in (\mathbb{C}_0[t])^n$ such that V is isomorphic to $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$.*

Theorem 4.3 ([8, Proposition 8.3]). *Let $\pi, \pi' \in (\mathbb{C}_0[t])^n$ and let v_π (resp. $v'_{\pi'}$) be a pseudo-highest weight vector in $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi)$ (resp. $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi')$). $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi\pi')$ is isomorphic to a quotient of the $\widetilde{U}_\varepsilon^{\mathrm{res}}$-subrepresentation of $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi')$ generated by $v_\pi \otimes v'_{\pi'}$. In particular, if $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi')$ is irreducible, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi')$ is isomorphic to $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi\pi')$.*

§ 5. The fundamental representations of the restricted quantum loop algebras

Now, we fix the following notations. We define $\widehat{I} := I \sqcup \{n+1\} = \{1, 2, \cdots, n+1\}$ and $\mathcal{J}_\xi := \{J = \{j_1, j_2, \cdots, j_\xi\} \subset \widehat{I} \,|\, j_1 < j_2 < \cdots < j_\xi\}$, where $\xi \in I$. Let L_ξ be the $\#(\mathcal{J}_\xi)$-dimensional $\mathbb{C}$-vector space having $\mathcal{J}_\xi$ as a $\mathbb{C}$-basis. For $\mathbf{a} \in \mathbb{C}^\times$, we define

$$\pi_{\xi,j}^{\mathbf{a}}(t) := \begin{cases} 1 - \mathbf{a}t, & \text{if} \quad j = \xi, \\ 1, & \text{if} \quad j \neq \xi, \end{cases} \qquad \pi_\xi^{\mathbf{a}} := (\pi_{\xi,j}^{\mathbf{a}}(t))_{j \in I}. \tag{5.1}$$

and call $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ a *fundamental representation* of $\widetilde{U}_\varepsilon^{\mathrm{res}}$.

By using the result of the generic case (see [10, §2.2] and [2, B.1]), we obtain the following proposition:

Proposition 5.1. *For $\xi \in I$ and $\mathbf{a} \in \mathbb{C}^\times$, the following formula gives a $U_\varepsilon^{\mathrm{res}}$-*

representation structure on L_ξ*: for* $i \in \widetilde{I}$*,* $m \in \mathbb{N}$ $(m \geq 2)$*, and* $J \in \mathcal{J}_\xi$*,*

$$e_i J = \begin{cases} \widetilde{\mathbf{a}}^{\delta_{i,0}}(J \backslash \{i+1\}) \sqcup \{i\}, & \text{if } i+1 \in J \text{ and } i \notin J, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_i J = \begin{cases} \widetilde{\mathbf{a}}^{-\delta_{i,0}}(J \backslash \{i\}) \sqcup \{i+1\}, & \text{if } i \in J \text{ and } i+1 \notin J, \\ 0, & \text{otherwise}, \end{cases}$$

$$k_i J = \begin{cases} \varepsilon J, & \text{if } i \in J \text{ and } i+1 \notin J, \\ \varepsilon^{-1} J, & \text{if } i \notin J \text{ and } i+1 \in J, \\ J, & \text{otherwise}, \end{cases}$$

$$e_i^{(m)} J = f_i^{(m)} J = 0,$$

where $\widetilde{\mathbf{a}} := \mathbf{a}\varepsilon^{n-1}(-1)^\xi$ *and we regard* 0 *in* J *as* $(n+1)$*. Moreover,* L_ξ *is isomorphic to* $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ *as a representation of* $\widetilde{U}_\varepsilon^{\mathrm{res}}$*.*

For $1 \leq i \leq j \leq n$, let $J_{i,j}^\xi$ be the extremal vector in $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$ (see [2]). In this case, $J_{i,j}^\xi$ is given by the following formula (see [1, Proposition 4.11]):

$$J_{i,j}^\xi = \begin{cases} \{1, 2, \cdots, \xi\}, & \text{if } j < \xi, \\ \{j+2-\xi, j+3-\xi, \cdots, j+1\}, & \text{if } \xi \leq j \text{ and } i \leq j+1-\xi, \\ \{j+1-\xi, \cdots, i-1, i+1, \cdots, j+1\}, & \text{if } \xi \leq j \text{ and } j+1-\xi < i. \end{cases}$$

In particular, $J_{1,1}^\xi$ (resp. $J_{1,n}^\xi$) is a pseudo-highest (resp. lowest) weight vector in $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}})$.

In a similar way to the proof of Lemma 4.2 in [4], we obtain the following lemma.

Lemma 5.2 ([1, Lemma 6.12]). *Let* $\xi \in I$*,* $\mathbf{a} \in \mathbb{C}^\times$*, and* $\pi \in (\mathbb{C}_0[t])^n$*. Let* V *be a pseudo-highest weight representation of* $\widetilde{U}_\varepsilon^{\mathrm{res}}$ *with the pseudo-highest weight* π *and let* v_π *be a pseudo-highest weight vector in* V*. We assume* $J_{1,n}^\xi \otimes v_\pi \in \widetilde{U}_\varepsilon^{\mathrm{res}}(J_{1,1}^\xi \otimes v_\pi)$*. Then* $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}) \otimes V$ *is a pseudo-highest weight representation of* $\widetilde{U}_\varepsilon^{\mathrm{res}}$ *with the pseudo-highest weight* $\pi_\xi^{\mathbf{a}}\pi$*.*

§ 6. Tensor product of the fundamental representations for the restricted quantum loop algebras

§ 6.1. Irreducibility

In a similar way to the proof of the generic case (see [4] and [5]), we obtain the following theorem:

Theorem 6.1 ([1, Theorem 6.14]). *Let $\xi_1, \cdots, \xi_m \in I$ and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. If*

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{2t-\xi_k-\xi_{k'}+2},$$

for any $1 \le k \neq k' \le m$ and $\max(\xi_k, \xi_{k'}) \le t \le \min(\xi_k + \xi_{k'} - 1, n)$, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m})$ is a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\pi_{\xi_1}^{\mathbf{a}_1} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ generated by a pseudo-highest weight vector $J_{1,1}^{\xi_1} \otimes \cdots \otimes J_{1,1}^{\xi_m}$.

Sketch of Proof. If we can prove the following formula, we obtain this theorem from Proposition 7.4 in [8]:

$$\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m}) = \widetilde{U}_\varepsilon^{\mathrm{res}}(J_{1,1}^{\xi_1} \otimes \cdots \otimes J_{1,1}^{\xi_m}).$$

We shall prove this formula by the induction on m. If $m = 1$, this follows from the definition of the pseudo-highest representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$. Now, we assume $m > 1$ and the case of $(m-1)$ holds. We set

$$V' := \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_2}^{\mathbf{a}_2}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m}), \quad J' := J_{1,1}^{\xi_2} \otimes \cdots \otimes J_{1,1}^{\xi_m}.$$

From the assumption of the induction on m, V' is a pseudo-highest weight representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ with the pseudo-highest weight $\pi_{\xi_2}^{\mathbf{a}_2} \cdots \pi_{\xi_m}^{\mathbf{a}_m}$ generated by a pseudo-highest weight vector J'. Hence, from Lemma 5.2, it is enough to prove that

$$J_{1,n}^{\xi_1} \otimes J' \in \widetilde{U}_\varepsilon^{\mathrm{res}}(J_{1,1}^{\xi_1} \otimes J').$$

We shall prove that

$$J_{i,j}^{\xi_1} \otimes J' \in \widetilde{U}_\varepsilon^{\mathrm{res}}(J_{1,1}^{\xi_1} \otimes J'),$$

for any $1 \le i \le j \le n$ by the induction on (i, j), where

$$(i-1, j) > (i, j) \quad \text{and} \quad (j+1, j+1) > (1, j).$$

If $(i, j) = (1, 1)$, we have nothing to prove. We assume that the case of (i, j) holds. As an example, let us give a proof for the case that $i \neq 1$, $j - i + 2 \le \xi_1 \le j$, and $\xi_2 = \cdots = \xi_m = i - 1$. Similarly we can prove the other case (see Case 2 of the proof of Theorem 5.3 in [1]).

By using Lemma 3.4, we obtain a matrix $A = (A_{r,s})_{r,s=1}^m$ such that

$$x_{i-1,r}^-(J_{i,j}^{\xi_1} \otimes J') = A_{r,1}(J_{i-1,j}^{\xi_1} \otimes J') + \sum_{s=2}^m A_{r,s}(J_{i,j}^{\xi_1} \otimes J_{1,1}^{\xi_2} \otimes \cdots \otimes (f_{i-1} J_{1,1}^{\xi_s}) \otimes \cdots \otimes J_{1,1}^{\xi_m}),$$

$$\det(A) = (\prod_{k=2}^m (\mathbf{a}_k - \mathbf{a}_1 \varepsilon^{2j-\xi_1-\xi_k+2}))(\prod_{2 \le k < k' \le m} (\mathbf{a}_{k'} - \mathbf{a}_k \varepsilon^2)),$$

(see Case 1 of the proof of Theorem 5.3 in [1]). Since $x^-_{i-1,r}(J^{\xi_1}_{i,j} \otimes J^{'}) \in \widetilde{U}^{\mathrm{res}}_{\varepsilon}(J^{\xi_1}_{1,1} \otimes J^{'})$ from the assumption of the induction on (i, j), we obtain

$$\det(A)(J^{\xi_1}_{i-1,j} \otimes J^{'}) \in \widetilde{U}^{\mathrm{res}}_{\varepsilon}(J^{\xi_1}_{1,1} \otimes J^{'}).$$

Since $\det(A) \neq 0$ from the assumption of this theorem, we obtain

$$J^{\xi_1}_{i-1,j} \otimes J^{'} \in \widetilde{U}^{\mathrm{res}}_{\varepsilon}(J^{\xi_1}_{1,1} \otimes J^{'}).$$

□

Corollary 6.2 (**[1, Corollary 6.15]**). *Let* $\xi_1, \cdots, \xi_m \in I$ *and* $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^{\times}$. *If*

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{\pm(2t-\xi_k-\xi_{k'}+2)},$$

for any $1 \leq k \neq k^{'} \leq m$ *and* $\max(\xi_k, \xi_{k'}) \leq t \leq \min(\xi_k + \xi_{k'} - 1, n)$, $\widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}_1}_{\xi_1}) \otimes \cdots \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}_m}_{\xi_m})$ *is an irreducible representation of* $\widetilde{U}^{\mathrm{res}}_{\varepsilon}$.

Sketch of Proof. We set $V := \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}_1}_{\xi_1}) \otimes \cdots \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}_m}_{\xi_m})$. Let V^* be the dual representation of V. It is enough to prove that V and V^* are the pseudo-highest weight representations. From Proposition 6.11 in [1], there exists an integer $c \in \mathbb{Z}$ such that

$$V^* \cong \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\varepsilon^c \mathbf{a}_m}_{\xi_m}) \otimes \cdots \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\varepsilon^c \mathbf{a}_1}_{\xi_1}),$$

(see also [1, Proposition 4.9]). Hence, this corollary follows from Theorem 6.1 and the assumption of this corollary. □

§ 6.2. Reducibility

Proposition 6.3 (**[1, Proposition 6.16]**). *Let* $\xi, \zeta \in I$ *and* $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{\times}$. *If there exists* $1 \leq t \leq \min(\xi, \zeta, n + 1 - \xi, n + 1 - \zeta)$ *such that* $\mathbf{b} = \varepsilon^{2t+|\xi-\zeta|}\mathbf{a}$ *or* $\varepsilon^{-(2t+|\xi-\zeta|)}\mathbf{a}$, *then* $\widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}}_{\xi}) \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{b}}_{\zeta})$ *is reducible as a representation of* $\widetilde{U}^{\mathrm{res}}_{\varepsilon}$.

Sketch of Proof. Without loss of generality, we assume $\mathbf{b} = \varepsilon^{2t+|\xi-\zeta|}\mathbf{a}$. If $\widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}}_{\xi}) \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{b}}_{\zeta})$ is irreducible as a representation of $\widetilde{U}^{\mathrm{res}}_{\varepsilon}$, we obtain

$$\widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}}_{\xi}) \otimes \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{b}}_{\zeta}) \cong \widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}}_{\xi}\pi^{\mathbf{b}}_{\zeta}),$$

from Theorem 4.3. Here, for $\pi \in (\mathbb{C}_0[t])^n$, let $\widetilde{V}_q(\pi)$ be the finite-dimensional irreducible representation of $\widetilde{U}_q$ with the pseudo-highest weight π and, for $\mathbf{c} \in (\mathbb{C}(q))^{\times}$ and $\gamma \in I$, let $\widetilde{V}_q(\pi^{\mathbf{c}}_{\gamma})$ be the fundamental representation of $\widetilde{U}_q$, where $\pi^{\mathbf{c}}_{\gamma}$ is as in (5.1). It is known that $\dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi^{\mathbf{c}}_{\gamma})) = \mathcal{J}_{\gamma}$. Hence, we have

$$\dim_{\mathbb{C}}(\widetilde{V}^{\mathrm{res}}_{\varepsilon}(\pi^{\mathbf{a}}_{\xi}\pi^{\mathbf{b}}_{\zeta})) = \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi^{\mathbf{a}}_{\xi})) \times \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi^{\mathbf{b}_q}_{\zeta})),$$

where $\mathbf{b}_q := q^{2t+|\xi-\zeta|}\mathbf{a}$. On the other hand, we have

$$\dim_{\mathbb{C}}(\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}})) \leq \dim_{\mathbb{C}(q)} \widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q}),$$

(see [1, §6.4]). Thus, we have

$$\dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi_\xi^{\mathbf{a}})) \times \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi_\zeta^{\mathbf{b}_q})) \leq \dim_{\mathbb{C}(q)}(\widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q})).$$

Hence, by the generic case of Theorem 4.3 (see [7]), we obtain

$$\widetilde{V}_q(\pi_\xi^{\mathbf{a}}) \otimes \widetilde{V}_q(\pi_\zeta^{\mathbf{b}_q}) \cong \widetilde{V}_q(\pi_\xi^{\mathbf{a}}\pi_\zeta^{\mathbf{b}_q}).$$

In particular, $\widetilde{V}_q(\pi_\xi^{\mathbf{a}})\otimes\widetilde{V}_q(\pi_\zeta^{\mathbf{b}_q})$ is irreducible as a representation of $\widetilde{U}_q$. However, from the result of the generic case (see [4] or [1, Proposition 5.7]), $\widetilde{V}_q(\pi_\xi^{\mathbf{a}}) \otimes \widetilde{V}_q(\pi_\zeta^{\mathbf{b}_q})$ is reducible. This is a contradiction. Therefore $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\xi^{\mathbf{a}}) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_\zeta^{\mathbf{b}})$ is reducible as a representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$. □

§ 6.3. The necessary and sufficient condition

From Corollary 6.2 and Proposition 6.3, we obtain the following theorem.

Theorem 6.4 (**[1, Theorem 6.17]**). *Let $\xi_1, \cdots, \xi_m \in I$ and $\mathbf{a}_1, \cdots, \mathbf{a}_m \in \mathbb{C}^\times$. $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_1}^{\mathbf{a}_1}) \otimes \cdots \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_{\xi_m}^{\mathbf{a}_m})$ is an irreducible representation of $\widetilde{U}_\varepsilon^{\mathrm{res}}$ if and only if*

$$\frac{\mathbf{a}_{k'}}{\mathbf{a}_k} \neq \varepsilon^{\pm(2t+|\xi_k-\xi_{k'}|)},$$

for any $1 \leq k \neq k' \leq m$ and $1 \leq t \leq \min(\xi_k, \xi_{k'}, n+1-\xi_k, n+1-\xi_{k'})$.

Example 6.5. We consider the case of $n = 4$ and $l = 7$. Then, for $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^\times$, $\widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_2^{\mathbf{a}}) \otimes \widetilde{V}_\varepsilon^{\mathrm{res}}(\pi_3^{\mathbf{b}})$ is irreducible if and only if

$$\mathbf{b} \neq \mathbf{a}\varepsilon^2, \mathbf{a}\varepsilon^3, \mathbf{a}\varepsilon^4, \mathbf{a}\varepsilon^5.$$

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