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Tensor Representations
for the Quantum Loop Algebras of Type $A$
at Roots of Unity

By
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§ 1. Introduction

Let $q$ be a nonzero complex number which is not a root of unity and let $U_q(\mathfrak{g})$ be the quantum loop algebra over $\mathbb{C}$ associated with a finite-dimensional complex simple Lie algebra $\mathfrak{g}$. It is known that every finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ is a pseudo-highest weight representation and classified by the pseudo-highest weights (see [7]). In 2002, Chari showed the sufficient condition for the tensor product of the finite-dimensional irreducible representations of $U_q(\mathfrak{g})$ to be a pseudo-highest weight representation (see [4]). In particular, by using this result, we obtain the necessary and sufficient conditions for the tensor product of the fundamental representations to be irreducible (see also [2], [13], and [11]). It is known that every finite-dimensional irreducible representation of $U_q(\mathfrak{g})$ is isomorphic to a subquotient of the tensor product of the fundamental representations (see [7]).

In this note, we explain the necessary and sufficient conditions for the tensor product of the fundamental representations for the restricted quantum loop algebras of type $A$ at roots of unity to be irreducible. This is the result in [1].

§ 2. Notations

We fix the following notations:

$\mathbb{N} := \{1, 2, \cdots\}$: the set of natural numbers,

$\mathbb{Z}_+ := \{0, 1, 2, \cdots\}$: the set of non-negative integers,

$I := \{1, 2, \cdots, n\}$, $\tilde{I} := I \sqcup \{0\}$: index sets,
(a_{i,j})_{i,j \in I}: \text{Cartan matrix of type } A_n,
(a_{i,j})_{i,j \in \overline{I}}: \text{generalized Cartan matrix of type } A_{n}^{(1)},
q: \text{indeterminate, } \mathbb{C}(q): \text{rational function field of } q,
[r] := \frac{q^r - q^{-r}}{q - q^{-1}}, \quad [m]! := [m][m-1] \cdots [1], \quad [0]! := 1,
[r \choose m] := \frac{[r][r-1]\cdots[r-m+1]}{[m][m-1]\cdots[1]} \quad (r \in \mathbb{Z}, m \in \mathbb{N}),
l: \text{an odd integer greater than } 2, \quad \varepsilon: \text{a primitive } l\text{-th root of unity},
\mathcal{A} := \mathbb{C}[t, t^{-1}]: \text{Laurent polynomial ring},
\mathbb{C}_\varepsilon: \mathcal{A}-\text{algebra of the set of complex numbers defined by the following formula:}
\quad g(q).c := g(\varepsilon)c \quad \text{for } g(q) \in \mathcal{A}, c \in \mathbb{C}.

§3. \text{The restricted quantum loop algebras}

First, we introduce the generic quantum loop algebras to define the restricted quantum loop algebras.

\textbf{Definition 3.1.} \text{The quantum loop algebra } \widetilde{U}_q \text{ of type } A_n \text{ is the associative } \mathbb{C}(q)-\text{algebra generated by } \{E_i, F_i, K_i^{\pm 1}|i \in I\} \text{ with the following defining relations:}

\begin{align*}
K_i K_i^{-1} & = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_0 = \prod_{p \in I} K_p^{-1}, \\
K_i E_j K_i^{-1} & = q^{\alpha_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q^{-\alpha_{i,j}} F_j, \\
E_i F_j - F_j E_i & = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
\sum_{p=0}^{1-\alpha_{i,j}} (-1)^p E_i^{(p)} E_j E_i^{(1-\alpha_{i,j}-p)} & = \sum_{p=0}^{1-\alpha_{i,j}} (-1)^p F_i^{(p)} F_j F_i^{(1-\alpha_{i,j}-p)} = 0 \quad i \neq j,
\end{align*}

for \(i, j \in \overline{I}\), where

\begin{align*}
E_i^{(m)} & := \frac{1}{[m]!} E_i^m, \quad F_i^{(m)} := \frac{1}{[m]!} F_i^m \quad (m \in \mathbb{Z}_+).
\end{align*}

Next, we introduce the Drinfel’d realization of \(\widetilde{U}_q\) to define the pseudo-highest weight representations of the restricted quantum loop algebras (see §4 in this paper).

\textbf{Theorem 3.2 ([9] and [3]).} \text{As a } \mathbb{C}(q)\text{-algebra, } \widetilde{U}_q \text{ is isomorphic to the algebra with generators } \{X_{i,r}^{\pm}, H_{i,s}, K_i^{\pm 1}|i \in I, r, s \in \mathbb{Z}, s \neq 0\} \text{ and the following defining}
relations:

\[ K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [K_i, K_j] = [K_i, H_{j,s}] = [H_{i,s}, H_{j,s'}] = 0, \]
\[ K_i X_{j,r}^\pm K_i^{-1} = q^{\pm a_{i,j}} X_{j,r}^\pm, \quad [H_{i,s}, X_{j,r}^\pm] = \pm \frac{s a_{i,j}}{s} X_{j,r+s}^\pm, \]
\[ X_{i,r+1}^\pm X_{j,r'}^\pm - q^{\pm a_{i,j}} X_{j,r'}^\pm X_{i,r+1}^\pm = q^{\pm a_{i,j}} X_{i,r}^\pm X_{j,r'+1}^\pm - X_{j,r'+1}^\pm X_{i,r}^\pm, \]
\[ [X_{i,r}^+, X_{j,r'}^-] = \delta_{i,j} \frac{\Psi_{i,r-r'}^+ - \Psi_{i,r-r'}^-}{q - q^{-1}}, \]
\[ \sum_{\pi \in S_m} \sum_{p=0}^m (-1)^p \begin{bmatrix} m \\ p \end{bmatrix} X_{i,r_{\pi(1)}}^\pm \cdots X_{i,r_{\pi(p)}}^\pm X_{j,r_{\pi(p+1)}}^\pm \cdots X_{i,r_{\pi(m)}}^\pm = 0, \quad (i \neq j), \]

for \( r_1, \cdots, r_m \in \mathbb{Z} \), where \( m := 1 - a_{i,j} \), \( S_m \) is the symmetric group on \( m \) letters, and \( \Psi_{i,r}^\pm \) are determined by

\[ \sum_{r=0}^{\infty} \Psi_{i,r}^\pm u^r := K_i^{\pm 1} \exp(\pm(q - q^{-1}) \sum_{s=1}^{\infty} H_{i,s} u^s), \]

and \( \Psi_{i,r}^\pm := 0 \) if \( r < 0 \).

Next, we introduce the restricted quantum loop algebras and their triangular decomposition. Let \( \bar{U}_\mathcal{A}^{\mathrm{res}} \) be the \( \mathcal{A} \)-subalgebra of \( \bar{U}_q \) generated by \( \{E_i^{(m)}, F_i^{(m)}, K_i^{\pm 1} \mid i \in \bar{T}, m \in \mathbb{N}\} \).

**Definition 3.3 ([8] and [12]).** The restricted quantum loop algebra \( \tilde{U}_\varepsilon^{\mathrm{res}} \) is defined as follows:

\[ \tilde{U}_\varepsilon^{\mathrm{res}} := \bar{U}_\mathcal{A}^{\mathrm{res}} \otimes_\mathcal{A} \mathbb{C}_\varepsilon. \]

For \( i \in I, r \in \mathbb{Z} \), and \( m \in \mathbb{N} \), define

\[ \begin{bmatrix} K_i; r \\ m \end{bmatrix} := \prod_{p=1}^{m} \frac{K_i q^{r-p+1} - K_i^{-1} q^{-r+p-1}}{q^p - q^{-p}}, \]
\[ \sum_{m=0}^{\infty} \mathcal{P}_{i,\pm m} u^m := \exp(- \sum_{s=1}^{\infty} q^{s} H_{i,\pm s} u^s) \quad \text{in} \quad \bar{U}_q[[u]]. \]

Then we have

\[ \begin{bmatrix} K_i; r \\ m \end{bmatrix}, \quad (X_{i,r}^\pm)^{(m)} = \frac{1}{[m]}(X_{i,r}^\pm)^m \quad \text{for} \quad i \in I, m \in \mathbb{N}, r \in \mathbb{Z}, \quad \mathcal{P}_{i,\pm m} \in \tilde{U}_\varepsilon^{\mathrm{res}}, \]

for \( i \in I, m \in \mathbb{N}, r \in \mathbb{Z} \), and \( s \in \mathbb{Z}^\times \), where \( (X_{i,r}^\pm)^{(m)} := \frac{1}{[m]}(X_{i,r}^\pm)^m \) (see [6, §9.3A] and
Therefore, the following elements are included in $\tilde{U}_\varepsilon^{\text{res}}$:

$$
e^{(m)}_i := E^{(m)}_i \otimes 1, \quad f^{(m)}_i := F^{(m)}_i \otimes 1, \quad k_i := K_i \otimes 1, \quad \left[ \begin{array}{c} k_{j;r}^m \\ m \end{array} \right] := \left[ K_{j;r}^m \right] \otimes 1, \quad (x^{\pm}_{j,r})^{(m)} := (X^{\pm}_{j,r})^{(m)} \otimes 1, \quad h_{j,s} := H_{j,s} \otimes 1, \quad \psi_{j,r}^\pm = \psi_{j,r}^\pm \otimes 1, \quad \mathfrak{p}_{j,r} := \mathcal{P}_{j,r} \otimes 1,$$

where $i \in \tilde{I}$, $j \in I$, $r \in \mathbb{Z}$, $s \in \mathbb{Z}^\times$, and $m \in \mathbb{N}$. Define $(\tilde{U}_\varepsilon^{\text{res}})^+$ (resp. $(\tilde{U}_\varepsilon^{\text{res}})^-$, $(\tilde{U}_\varepsilon^{\text{res}})^0$) to be the $\mathbb{C}$-subalgebra of $\tilde{U}_\varepsilon^{\text{res}}$ generated by $\{(x^{\pm}_{i,r})^{(m)}| i \in I, r \in \mathbb{Z}, m \in \mathbb{N}\}$ (resp. $\{(x^{-}_{i,r})^{(m)}| i \in I, r \in \mathbb{Z}, m \in \mathbb{N}\}$, $\{\mathfrak{p}_{i,r}, k_i, \left[ \begin{array}{c} \mathfrak{p}_{j;r}^m \\ 0 \\ l \end{array} \right]| i \in I, r \in \mathbb{Z}\}$). Then we obtain the triangular decomposition of $\tilde{U}_\varepsilon^{\text{res}}$ (see [8, §6]):

\begin{equation}
(\tilde{U}_\varepsilon^{\text{res}})^- \otimes (\tilde{U}_\varepsilon^{\text{res}})^0 \otimes (\tilde{U}_\varepsilon^{\text{res}})^+ \longrightarrow \tilde{U}_\varepsilon^{\text{res}}.
\end{equation}

The comultiplication $\Delta$ of $\tilde{U}_\varepsilon^{\text{res}}$ is as follows:

$$
\Delta(e^{(m)}_i) = \sum_{p=1}^{m} q^{-p(m-p)} e^{(p)}_i \otimes k_i^p e^{(m-p)}_i, \quad \Delta(f^{(m)}_i) = \sum_{p=1}^{m} q^{p(m-p)} f^{(p)}_i \otimes f^{(m-p)}_i k_i^{-p},
$$

$$
\Delta(k_i) = k_i \otimes k_i,
$$

for any $i \in \tilde{I}$ and $m \in \mathbb{N}$ (see [6, §9.3A]).

By using (3.1) and the results of the generic case (in particular, [4, Proposition 2.2]), we obtain the following lemma:

**Lemma 3.4 ([1, Lemma 6.3]).** Let $i \in I$ and $r \in \mathbb{N}$. Modulo $\tilde{U}_\varepsilon^{\text{res}} \otimes X_+^{\text{res}}$,

$$
\Delta(x^{-}_{i,r}) = x^{-}_{i,r} \otimes k_i + 1 \otimes x^{-}_{i,r} + \sum_{p=1}^{r-1} x^{-}_{i,r-p} \otimes \psi^+_{i,p},
$$

where $X_+^{\text{res}} := \sum_{j \in I, r \in \mathbb{Z}, m \in \mathbb{N}} \tilde{U}_\varepsilon^{\text{res}} (x^+_{j,r})^{(m)}$.

### § 4. Representation theory of the restricted quantum loop algebras

Without loss of generality, we assume that the representation of $\tilde{U}_\varepsilon^{\text{res}}$ is of type 1, that is, $k_i^l = 1$ on the representation for any $i \in I$.

For any $\pi = (\pi_i(t))_{i \in I} \in (\mathbb{C}[t])^n$ ($\mathbb{C}[t] := \{ \pi(t) \in \mathbb{C}[t] | \pi(0) = 1 \}$), there exists a unique $\mathbb{C}$-algebra homomorphism $\Lambda_\pi : (\tilde{U}_\varepsilon^{\text{res}})^0 \longrightarrow \mathbb{C}$ such that

$$
\Lambda_\pi(k_i^{\pm l}) = \epsilon^{p_i^{(l)}}_i, \quad \Lambda_\pi\left( \left[ \begin{array}{c} k_i; 0 \\ l \end{array} \right] \right) = p_i^{(1)}(t), \quad \sum_{m=1}^{\infty} \Lambda_\pi(\mathfrak{p}_{i,\pm m}) t^m = \pi_{i,\pm}(t),
$$
where

$$
\pi_i^+(t) := \pi_i(t), \quad \pi_i^-(t) := \frac{t^{\deg \pi_i(t)}\pi_i(t^{-1})}{(t^{\deg \pi_i(t)}\pi_i(t^{-1}))|_{t=0}},
$$

$$
\deg(\pi_i(t)) = p_i^{(0)} + lp_i^{(1)} \quad (0 \leq p_i^{(0)} < l).
$$

**Definition 4.1.** Let $V$ be a representation of $\tilde{U}_\varepsilon^{\text{res}}$ and $\pi = (\pi_i(t))_{i \in I}$ be an element in $(\mathbb{C}_0[t])^n$. We assume that there exists a nonzero vector $v_\Lambda \in V$ such that $(x_{i,r}^+)^{(m)}v_\Lambda = 0$ for all $i \in I$, $r \in \mathbb{Z}$, and $m \in \mathbb{N}$ and $uv_\Lambda = \Lambda_\pi(u)v_\Lambda$ for all $u \in (\tilde{U}_\varepsilon^{\text{res}})^0$. If $V$ is generated by $v_\Lambda$ as a representation of $\tilde{U}_\varepsilon^{\text{res}}$, we call $V$ a pseudo-highest weight representation of $\tilde{U}_\varepsilon^{\text{res}}$ with the pseudo-highest weight $\pi$ generated by a pseudo-highest weight vector $v_\Lambda$.

**Theorem 4.2 ([8, §8]).** For any $\pi \in (\mathbb{C}_0[t])^n$, there exists a finite-dimensional irreducible pseudo-highest weight $\tilde{U}_\varepsilon^{\text{res}}$-representation $\tilde{V}_\varepsilon^{\text{res}}(\pi)$ with the pseudo-highest weight $\pi$. Moreover, for any finite-dimensional irreducible $\tilde{U}_\varepsilon^{\text{res}}$-representation $V$, there exists a unique $\pi \in (\mathbb{C}_0[t])^n$ such that $V$ is isomorphic to $\tilde{V}_\varepsilon^{\text{res}}(\pi)$.

**Theorem 4.3 ([8, Proposition 8.3]).** Let $\pi, \pi' \in (\mathbb{C}_0[t])^n$ and let $v_\pi$ (resp. $v_\pi'$) be a pseudo-highest weight vector in $\tilde{V}_\varepsilon^{\text{res}}(\pi)$ (resp. $\tilde{V}_\varepsilon^{\text{res}}(\pi')$). $\tilde{V}_\varepsilon^{\text{res}}(\pi \pi')$ is isomorphic to a quotient of the $\tilde{U}_\varepsilon^{\text{res}}$-subrepresentation of $\tilde{V}_\varepsilon^{\text{res}}(\pi) \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi')$ generated by $v_\pi \otimes v_\pi'$. In particular, if $\tilde{V}_\varepsilon^{\text{res}}(\pi) \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi')$ is irreducible, $\tilde{V}_\varepsilon^{\text{res}}(\pi) \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi')$ is isomorphic to $\tilde{V}_\varepsilon^{\text{res}}(\pi \pi')$.

§ 5. The fundamental representations of the restricted quantum loop algebras

Now, we fix the following notations. We define $\hat{I} := I \sqcup \{n+1\} = \{1,2,\ldots,n+1\}$ and $J_\xi := \{J = \{j_1,j_2,\ldots,j_\xi\} \subset \hat{I} \mid j_1 < j_2 < \cdots < j_\xi\}$, where $\xi \in I$. Let $L_\xi$ be the $\#(J_\xi)$-dimensional $\mathbb{C}$-vector space having $J_\xi$ as a $\mathbb{C}$-basis. For $a \in \mathbb{C}^\times$, we define

$$
(5.1) \quad \pi_{\xi,j}^a(t) := \begin{cases} 
1 - at, & \text{if } j = \xi, \\
1, & \text{if } j \neq \xi,
\end{cases} \quad \pi_{\xi}^a := (\pi_{\xi,j}^a(t))_{j \in I},
$$

and call $\tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi}^a)$ a fundamental representation of $\tilde{U}_\varepsilon^{\text{res}}$.

By using the result of the generic case (see [10, §2.2] and [2, B.1]), we obtain the following proposition:

**Proposition 5.1.** For $\xi \in I$ and $a \in \mathbb{C}^\times$, the following formula gives a $U_\varepsilon^{\text{res}}$-
representation structure on $L_{\xi}$: for $i \in \overline{I}$, $m \in \mathbb{N}$ ($m \geq 2$), and $J \in \mathcal{J}_{\xi}$,

\[
e_i J = \begin{cases} \tilde{a}_{\delta,0} (J \setminus \{i + 1\}) \cup \{i\}, & \text{if } i + 1 \in J \text{ and } i \notin J, \\ 0, & \text{otherwise}, \end{cases}
\]

\[
f_i J = \begin{cases} \tilde{a}^{-\delta,0} (J \setminus \{i\}) \cup \{i + 1\}, & \text{if } i \in J \text{ and } i + 1 \notin J, \\ 0, & \text{otherwise}, \end{cases}
\]

\[
k_i J = \begin{cases} \varepsilon J, & \text{if } i \in J \text{ and } i + 1 \notin J, \\ \varepsilon^{-1} J, & \text{if } i \notin J \text{ and } i + 1 \in J, \\ J, & \text{otherwise}, \end{cases}
\]

\[
e_i^{(m)} J = f_i^{(m)} J = 0,
\]

where $\tilde{a} := a \varepsilon^{n-1} (-1)^{\xi}$ and we regard 0 in $J$ as $(n + 1)$. Moreover, $L_{\xi}$ is isomorphic to $\tilde{V}_{\varepsilon}^{\text{res}}(\pi_{\xi}^a)$ as a representation of $\overline{U}_{\varepsilon}^{\text{res}}$.

For $1 \leq i \leq j \leq n$, let $J_{i,j}^{\xi}$ be the extremal vector in $\tilde{V}_{\varepsilon}^{\text{res}}(\pi_{\xi}^a)$ (see [2]). In this case, $J_{i,j}^{\xi}$ is given by the following formula (see [1, Proposition 4.11]):

\[
J_{i,j}^{\xi} = \begin{cases} \{1, 2, \cdots, \xi\}, & \text{if } j < \xi, \\ \{j + 2 - \xi, j + 3 - \xi, \cdots, j + 1\}, & \text{if } \xi \leq j \text{ and } i \leq j + 1 - \xi, \\ \{j + 1 - \xi, \cdots, i - 1, i + 1, \cdots, j + 1\}, & \text{if } \xi \leq j \text{ and } j + 1 - \xi < i. \end{cases}
\]

In particular, $J_{1,1}^{\xi}$ (resp. $J_{1,n}^{\xi}$) is a pseudo-highest (resp. lowest) weight vector in $\tilde{V}_{\varepsilon}^{\text{res}}(\pi_{\xi}^a)$.

In a similar way to the proof of Lemma 4.2 in [4], we obtain the following lemma.

**Lemma 5.2 ([1, Lemma 6.12]).** Let $\xi \in I$, $a \in \mathbb{C}^\times$, and $\pi \in (\mathbb{C}_0[t])^n$. Let $V$ be a pseudo-highest weight representation of $\tilde{U}_{\varepsilon}^{\text{res}}$ with the pseudo-highest weight $\pi$ and let $v_\pi$ be a pseudo-highest weight vector in $V$. We assume $J_{1,n}^{\xi} \otimes v_\pi \in \tilde{U}_{\varepsilon}^{\text{res}}(J_{1,1}^{\xi} \otimes v_\pi)$. Then $\tilde{V}_{\varepsilon}^{\text{res}}(\pi_{\xi}^a) \otimes V$ is a pseudo-highest weight representation of $\tilde{U}_{\varepsilon}^{\text{res}}$ with the pseudo-highest weight $\pi_{\xi}^a \pi$.

§ 6. Tensor product of the fundamental representations for the restricted quantum loop algebras

§ 6.1. Irreducibility

In a similar way to the proof of the generic case (see [4] and [5]), we obtain the following theorem:
Theorem 6.1 ([1, Theorem 6.14]). Let $\xi_1, \cdots, \xi_m \in I$ and $a_1, \cdots, a_m \in \mathbb{C}^\times$. If
\[
\frac{a_{k'}}{a_k} \neq \varepsilon^{2t - \xi_k - \xi_{k'} + 2},
\]
for any $1 \leq k \neq k' \leq m$ and $\max(\xi_k, \xi_{k'}) \leq t \leq \min(\xi_k + \xi_{k'} - 1, n)$, $\tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_1}^{a_1}) \otimes \cdots \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_m}^{a_m})$ is a pseudo-highest weight representation of $\tilde{U}_\varepsilon^{\text{res}}$ with the pseudo-highest weight $\pi_{\xi_1}^{a_1} \cdot \cdots \cdot \pi_{\xi_m}^{a_m}$ generated by a pseudo-highest weight vector $J_{1,1}^{\xi_1} \otimes \cdots \otimes J_{1,1}^{\xi_m}$.

Sketch of Proof. If we can prove the following formula, we obtain this theorem from Proposition 7.4 in [8]:
\[
\tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_1}^{a_1}) \otimes \cdots \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_m}^{a_m}) = \tilde{V}_\varepsilon^{\text{res}}(J_{1,1}^{\xi_1} \otimes \cdots \otimes J_{1,1}^{\xi_m}).
\]
We shall prove this formula by the induction on $m$. If $m = 1$, this follows from the definition of the pseudo-highest representation of $\tilde{U}_\varepsilon^{\text{res}}$. Now, we assume $m > 1$ and the case of $(m-1)$ holds. We set
\[
V' := \tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_2}^{a_2}) \otimes \cdots \otimes \tilde{V}_\varepsilon^{\text{res}}(\pi_{\xi_m}^{a_m}), \quad J' := J_{1,1}^{\xi_2} \otimes \cdots \otimes J_{1,1}^{\xi_m}.
\]
From the assumption of the induction on $m$, $V'$ is a pseudo-highest weight representation of $\tilde{U}_\varepsilon^{\text{res}}$ with the pseudo-highest weight $\pi_{\xi_2}^{a_2} \cdot \cdots \cdot \pi_{\xi_m}^{a_m}$ generated by a pseudo-highest weight vector $J'$. Hence, from Lemma 5.2, it is enough to prove that
\[
J_{1,n}^{\xi_1} \otimes J' \in \tilde{U}_\varepsilon^{\text{res}}(J_{1,1}^{\xi_1} \otimes J').
\]
We shall prove that
\[
J_{i,j}^{\xi_1} \otimes J' \in \tilde{U}_\varepsilon^{\text{res}}(J_{1,1}^{\xi_1} \otimes J'),
\]
for any $1 \leq i \leq j \leq n$ by the induction on $(i, j)$, where
\[
(i-1, j) > (i, j) \quad \text{and} \quad (j + 1, j + 1) > (1, j).
\]
If $(i, j) = (1, 1)$, we have nothing to prove. We assume that the case of $(i, j)$ holds. As an example, let us give a proof for the case that $i \neq 1, j - i + 2 \leq \xi_1 \leq j$, and $\xi_2 = \cdots = \xi_m = \xi_1 - 1$. Similarly we can prove the other case (see Case 2 of the proof of Theorem 5.3 in [1]).

By using Lemma 3.4, we obtain a matrix $A = (A_{r,s})_{r,s=1}^m$ such that
\[
x_{i-1,r}(J_{i,j}^{\xi_1} \otimes J') = A_{r,1}(J_{i-1,j}^{\xi_1} \otimes J') + \sum_{s=2}^m A_{r,s}(J_{i,j}^{\xi_1} \otimes J_{1,1}^{\xi_2} \otimes \cdots \otimes (f_{i-1}J_{1,1}^{\xi_s}) \otimes \cdots \otimes J_{1,1}^{\xi_m}),
\]
\[
\det(A) = \prod_{k=2}^m (a_k - a_1 \varepsilon^{2j-\xi_1-\xi_k+2}) \prod_{2 \leq k < k' \leq m} (a_k' - a_k \varepsilon^2).
\]
(see Case 1 of the proof of Theorem 5.3 in [1]). Since \( x_{i-1,j} \) \((\mathcal{J}_{1,1}^{\xi_1} \otimes \mathcal{J}') \) \( \in \mathcal{U}_{\mathbb{C}^\times}^{\text{res}}(\mathcal{J}_{1,1}^{\xi_1} \otimes \mathcal{J}') \) from the assumption of the induction on \((i, j)\), we obtain
\[
\det(A)(\mathcal{J}_{i-1,j}^{\xi_1} \otimes \mathcal{J}') \in \mathcal{U}_{\mathbb{C}^\times}^{\text{res}}(\mathcal{J}_{1,1}^{\xi_1} \otimes \mathcal{J}').
\]
Since \( \det(A) \neq 0 \) from the assumption of this theorem, we obtain
\[
\mathcal{J}_{i-1,j}^{\xi_1} \otimes \mathcal{J}' \in \mathcal{U}_{\mathbb{C}^\times}^{\text{res}}(\mathcal{J}_{1,1}^{\xi_1} \otimes \mathcal{J}').
\]

Corollary 6.2 ([1, Corollary 6.15]). Let \( \xi_1, \ldots, \xi_m \in I \) and \( a_1, \ldots, a_m \in \mathbb{C}^\times \). If
\[
\frac{a_{k'}}{a_k} \neq \varepsilon^{\pm(2t-\xi_k-\xi_{k'}+2)},
\]
for any \( 1 \leq k \neq k' \leq m \) and max(\( \xi_k, \xi_{k'} \)) \( \leq t \leq \min(\xi_k + \xi_{k'} - 1, n) \), \( \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_1}) \otimes \cdots \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_m}) \) is an irreducible representation of \( \mathcal{U}_{\mathbb{C}^\times}^{\text{res}} \).

Sketch of Proof. We set \( V := \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_1}) \otimes \cdots \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_m}) \). Let \( V^* \) be the dual representation of \( V \). It is enough to prove that \( V \) and \( V^* \) are the pseudo-highest weight representations. From Proposition 6.11 in [1], there exists an integer \( c \in \mathbb{Z} \) such that
\[
V^* \cong \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_1}^{c}) \otimes \cdots \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi_m}^{c}),
\]
(see also [1, Proposition 4.9]). Hence, this corollary follows from Theorem 6.1 and the assumption of this corollary. \( \square \)

§ 6.2. Reducibility

Proposition 6.3 ([1, Proposition 6.16]). Let \( \xi, \zeta \in I \) and \( a, b \in \mathbb{C}^\times \). If there exists \( 1 \leq t \leq \min(\xi, \zeta, n+1-\xi, n+1-\zeta) \) such that \( b = \varepsilon^{2t+|\xi-\zeta|} a \) or \( \varepsilon^{-(2t+|\xi-\zeta|)} a \), then \( \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi}) \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\zeta}) \) is reducible as a representation of \( \mathcal{U}_{\mathbb{C}^\times}^{\text{res}} \).

Sketch of Proof. Without loss of generality, we assume \( b = \varepsilon^{2t+|\xi-\zeta|} a \). If \( \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi}) \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\zeta}) \) is irreducible as a representation of \( \mathcal{U}_{\mathbb{C}^\times}^{\text{res}} \), we obtain
\[
\mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi}) \otimes \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\zeta}) \cong \mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi}^{a} \pi_{\zeta}^{b}),
\]
from Theorem 4.3. Here, for \( \pi \in (\mathbb{C}_0[t])^n \), let \( \mathcal{V}_q(\pi) \) be the finite-dimensional irreducible representation of \( \mathcal{U}_q \) with the pseudo-highest weight \( \pi \) and, for \( c \in (\mathbb{C}(q))^\times \) and \( \gamma \in I \), let \( \mathcal{V}_q(\pi_{\xi}) \) be the fundamental representation of \( \mathcal{U}_q \), where \( \pi_{\xi} \) is as in (5.1). It is known that \( \dim_{\mathbb{C}(q)}(\mathcal{V}_q(\pi_{\xi})) = J_{\gamma} \). Hence, we have
\[
\dim_{\mathbb{C}}(\mathcal{V}_{\mathbb{C}^\times}^{\text{res}}(\pi_{\xi}^{a} \pi_{\zeta}^{b})) = \dim_{\mathbb{C}(q)}(\mathcal{V}_q(\pi_{\xi}^{a})) \times \dim_{\mathbb{C}(q)}(\mathcal{V}_q(\pi_{\zeta}^{b})).
\]
where $b_q := q^{2t + |\xi - \zeta|} a$. On the other hand, we have
\[
\dim_{\mathbb{C}}(\tilde{\mathcal{V}}_{\epsilon}^{\text{res}}(\pi_{\xi}^{a} \pi_{\zeta}^{b})) \leq \dim_{\mathbb{C}(q)} \tilde{V}_q(\pi_{\xi}^{a} \pi_{\zeta}^{b}),
\]
(see [1, §6.4]). Thus, we have
\[
\dim_{\mathbb{C}(q)}(\tilde{V}_q(\pi_{\xi}^{a})) \times \dim_{\mathbb{C}(q)}(\tilde{V}_q(\pi_{\zeta}^{b})) \leq \dim_{\mathbb{C}(q)}(\tilde{V}_q(\pi_{\xi}^{a} \pi_{\zeta}^{b})).
\]
Hence, by the generic case of Theorem 4.3 (see [7]), we obtain
\[
\tilde{V}_q(\pi_{\xi}^{a}) \otimes \tilde{V}_q(\pi_{\zeta}^{b}) \cong \tilde{V}_q(\pi_{\xi}^{a} \pi_{\zeta}^{b}).
\]
In particular, $\tilde{V}_q(\pi_{\xi}^{a}) \otimes \tilde{V}_q(\pi_{\zeta}^{b})$ is irreducible as a representation of $\tilde{U}_q$. However, from the result of the generic case (see [4] or [1, Proposition 5.7]), $\tilde{V}_q(\pi_{\xi}^{a}) \otimes \tilde{V}_q(\pi_{\zeta}^{b})$ is reducible. This is a contradiction. Therefore $\tilde{V}_\epsilon^{\text{res}}(\pi_{\xi}^{a}) \otimes \tilde{V}_\epsilon^{\text{res}}(\pi_{\zeta}^{b})$ is reducible as a representation of $\tilde{U}_\epsilon^{\text{res}}$. \qed

§ 6.3. The necessary and sufficient condition

From Corollary 6.2 and Proposition 6.3, we obtain the following theorem.

**Theorem 6.4 ([1, Theorem 6.17]).** Let $\xi_1, \ldots, \xi_m \in I$ and $a_1, \ldots, a_m \in \mathbb{C}^\times$. $\tilde{V}_\epsilon^{\text{res}}(\pi_{\xi_1}^{a_1}) \otimes \cdots \otimes \tilde{V}_\epsilon^{\text{res}}(\pi_{\xi_m}^{a_m})$ is an irreducible representation of $\tilde{U}_\epsilon^{\text{res}}$ if and only if
\[
\frac{a_k'}{a_k} \neq \epsilon^{\pm(2t + |\xi_k - \xi_{k'}|)},
\]
for any $1 \leq k \neq k' \leq m$ and $1 \leq t \leq \min(\xi_k, \xi_{k'}, n + 1 - \xi_k, n + 1 - \xi_{k'})$.

**Example 6.5.** We consider the case of $n = 4$ and $l = 7$. Then, for $a, b \in \mathbb{C}^\times$, $\tilde{V}_\epsilon^{\text{res}}(\pi_2^{a}) \otimes \tilde{V}_\epsilon^{\text{res}}(\pi_3^{b})$ is irreducible if and only if
\[
b \neq a\epsilon^2, a\epsilon^3, a\epsilon^4, a\epsilon^5.
\]

**References**


