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Kyoto University
On the $K(\pi, 1)$-property for rings of integers in the mixed case

By

Alexander SCHMIDT*

Abstract

We investigate the Galois group $G_S(p)$ of the maximal $p$-extension unramified outside a finite set $S$ of primes of a number field in the (mixed) case, when there are primes dividing $p$ inside and outside $S$. We show that the cohomology of $G_S(p)$ is ‘often’ isomorphic to the étale cohomology of the scheme $\text{Spec}(\mathcal{O}_k \setminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

§1. Introduction

Let $Y$ be a connected locally noetherian scheme and let $p$ be a prime number. We denote the étale fundamental group of $Y$ by $\pi_1^{\text{et}}(Y)$ and its maximal pro-$p$ factor group by $\pi_1(Y)(p)$. The Hochschild-Serre spectral sequence induces natural homomorphisms

$$
\phi_i : H^i(\pi_1^{\text{et}}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow H^i_{\text{et}}(Y, \mathbb{Z}/p\mathbb{Z}), \ i \geq 0,
$$

and we call $Y$ a ‘$K(\pi, 1)$ for $p$’ if all $\phi_i$ are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the $K(\pi, 1)$-property. Our main result is the following

Theorem 1.1. Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ does not contain a primitive $p$-th root of unity and that the class number of $k$ is prime to $p$. Then the following holds:

Let $S$ be a finite set of primes of $k$ and let $T$ be a set of primes of $k$ of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $\text{Spec}(\mathcal{O}_k) \setminus (S \cup T_1)$ is a $K(\pi, 1)$ for $p$.  


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*NWF I-Mathematik, Universität Regensburg, 93040 Regensburg, Deutschland.

e-mail: alexander.schmidt@mathematik.uni-regensburg.de

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Remarks. 1. If $S$ contains the set $S_p$ of primes dividing $p$, then Theorem 1.1 holds with $T_1 = \emptyset$ and even without the condition $\zeta_p \notin k$ and $\mathcal{O}(k)(p) = 0$, see [Sch2], Proposition 2.3. In the tame case $S \cap S_p = \emptyset$, the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the ‘mixed’ case $\emptyset \subsetneq S \cap S_p \subsetneq S_p$.

2. For a given number field $k$, all but finitely many prime numbers $p$ satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on $p$.

Let $S$ be a finite set of places of a number field $k$. Let $k_S(p)$ be the maximal $p$-extension of $k$ unramified outside $S$ and put $G_S(p) = \text{Gal}(k_S(p)|k)$. If $S_R$ denotes the set of real places of $k$, then $G_{S\cup S_R}(p) \cong \pi_1(Spec(\mathcal{O}_k) \setminus S)(p)$ (we have $G_S(p) = G_{S\cup S_R}(p)$ if $p$ is odd or $k$ is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

**Theorem 1.2.** The set $T_1 \subset T$ in Theorem 1.1 may be chosen such that

(i) $T_1$ consists of primes $p$ of degree 1 with $N(p) \equiv 1 \mod p$,

(ii) $(k_{S\cup T_1}(p))_p = k_p(p)$ for all primes $p \in S \cup T_1$.

Note that Theorem 1.2 provides nontrivial information even in the case $S \supset S_p$, where assertion (ii) was only known when $k$ contains a primitive $p$-th root of unity (Kuz’min’s theorem, see [Kuz] or [NSW], 10.6.4 or [NSW$^2$], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW$^2$], X §8 exercise, respectively).

By Theorem 3.3 below, Theorem 1.2 provides many examples of $G_S(p)$ being a duality group. If $\zeta_p \notin k$, this is interesting even in the case that $S \supset S_p$, where examples of $G_S(p)$ being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW$^2$], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [Wil], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro-$p$ groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets $S$ and $S'$ are disjoint from $S_p$. This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.
\section{Proof of Theorems 1.1 and 1.2}

We start with the observation that the proofs of Proposition 8.1 and Corollary 8.2 in [Sch2] did not use the assumption that the sets \( S \) and \( S' \) are disjoint from \( S_p \). Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain

**Proposition 2.1.** Let \( k \) be a number field and let \( p \) be a prime number. Assume \( k \) to be totally imaginary if \( p = 2 \). Put \( X = \text{Spec}(\mathcal{O}_k) \) and let \( S \subset S' \) be finite sets of primes of \( k \). Assume that \( X \setminus S \) is a \( K(\pi, 1) \) for \( p \) and that \( G_{S}(p) \neq 1 \). Further assume that each \( \mathfrak{p} \in S' \setminus S \) does not split completely in \( k_S(p) \). Then the following hold.

(i) \( X \setminus S' \) is a \( K(\pi, 1) \) for \( p \).

(ii) \( k_{S'}(p)_\mathfrak{p} = k_{\mathfrak{p}}(p) \) for all \( \mathfrak{p} \in S' \setminus S \).

Furthermore, the arithmetic form of Riemann’s existence theorem holds, i.e., setting \( K = k_S(p) \), the natural homomorphism

\[
\begin{array}{c}
\bigotimes_{\mathfrak{p} \in S' \setminus S(K)} T(K_p(p)|K_p) \longrightarrow \text{Gal}(k_{S'}(p)|K)
\end{array}
\]

is an isomorphism. Here \( T(K_p(p)|K_p) \) is the inertia group and \( \bigotimes \) denotes the free pro-\( p \)-product of a bundle of pro-\( p \)-groups, cf. [NSW], Ch. IV, §3. In particular, the group \( \text{Gal}(k_{S'}(p)|k_S(p)) \) is a free pro-\( p \)-group.

**Proof.** The \( K(\pi, 1) \)-property implies

\[
H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H^{i+1}_{et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for} \quad i \geq 4,
\]

hence \( cd \ G_S(p) \leq 3 \). Let \( \mathfrak{p} \in S' \setminus S \). Since \( \mathfrak{p} \) does not split completely in \( k_S(p) \) and since \( cd \ G_S(p) < \infty \), the decomposition group of \( \mathfrak{p} \) in \( k_S(p)|k \) is a non-trivial and torsion-free quotient of \( \mathbb{Z}_p \cong \text{Gal}(k_{\mathfrak{p}}^{nr}(p)|k_{\mathfrak{p}}) \). Therefore \( k_S(p)_\mathfrak{p} \) is the maximal unramified \( p \)-extension of \( k_{\mathfrak{p}} \). We denote the normalization of an integral normal scheme \( Y \) in an algebraic extension \( L \) of its function field by \( Y_L \). Then \( (X \setminus S)_{ks(p)} \) is the universal pro-\( p \)-covering of \( X \setminus S \). We consider the étale excision sequence for the pair \( ((X \setminus S)_{ks(p)}, (X \setminus S')_{ks(p)}) \). By assumption, \( X \setminus S \) is a \( K(\pi, 1) \) for \( p \), hence

\[
H^i_{et}((X \setminus S)_{k_S(p)}; \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for} \quad i \geq 1 \quad \text{by} \quad [\text{Sch2}], \text{Proposition 2.1}.
\]

Omitting the coefficients \( \mathbb{Z}/p\mathbb{Z} \) from the notation, this implies isomorphisms

\[
\bigotimes_{\mathfrak{p} \in S' \setminus S(k_S(p))} H^{i+1}_p(((X \setminus S)_{k_S(p)})_\mathfrak{p}) 
\]

for \( i \geq 1 \). Here (and in variants also below) we use the notational convention

\[
\bigotimes_{\mathfrak{p} \in S' \setminus S(k_S(p))} H^{i+1}_p(((X \setminus S)_{k_S(p)})_\mathfrak{p}) := \lim_{\longrightarrow} \bigoplus_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S' \setminus S(K)} H^{i+1}_p(((X \setminus S)_K)_\mathfrak{p}),
\]
where \( K \) runs through the finite extensions of \( k \) inside \( k_S(p) \). As \( k_S(p) \) realizes the maximal unramified \( p \)-extension of \( k \) for all \( p \in S' \setminus S \), the schemes \( ((X \setminus S)_{k_S(p)})_p, p \in S' \setminus S(k_S(p)) \), have trivial cohomology with values in \( \mathbb{Z}/p\mathbb{Z} \) and we obtain isomorphisms

\[
H^i((k_S(p))_p) \cong H^{i+1}((X \setminus S)_{k_S(p)}(p))
\]

for \( i \geq 1 \). These groups vanish for \( i \geq 2 \). This implies

\[
H^i_{et}((X \setminus S')_{k_S(p)}) = 0
\]

for \( i \geq 2 \). Since the scheme \( (X \setminus S')_{k_S'(p)}(p) \) is the universal pro-\( p \) covering of \( (X \setminus S')_{k_S(p)}(p) \), the Hochschild-Serre spectral sequence

\[
E_2^{ij} = H^i(Gal(k_S'(p)|k_S(p)), H^{j}_{et}((X \setminus S')_{k_S(p)}(p))) \Rightarrow H^{i+j}_{et}((X \setminus S')_{k_S(p)}(p))
\]

yields an inclusion

\[
H^2(Gal(k_S'(p)|k_S(p))) \hookrightarrow H^2_{et}((X \setminus S')_{k_S(p)}(p)) = 0.
\]

Hence \( Gal(k_S'(p)|k_S(p)) \) is a free pro-\( p \)-group and

\[
H^1(Gal(k_S'(p)|k_S(p))) \cong H^1_{et}((X \setminus S')_{k_S(p)}(p)) \cong \bigoplus_{p \in S' \setminus S(k_S(p))} H^1(k_S(p)_p).
\]

We set \( K = k_S(p) \) and consider the natural homomorphism

\[
\phi : T(K_p(p)|K_p(p)) \longrightarrow Gal(k_S'(p)|K).
\]

By the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), \( \phi \) is a homomorphism between free pro-\( p \)-groups which induces an isomorphism on mod \( p \) cohomology. Therefore \( \phi \) is an isomorphism. In particular, \( k_S'(p)_p = k_p(p) \) for all \( p \in S' \setminus S \). Using that \( Gal(k_S'(p)|k_S(p)) \) is free, the Hochschild-Serre spectral sequence induces an isomorphism

\[
0 = H^2_{et}((X \setminus S')_{k_S(p)}) \cong H^2_{et}((X \setminus S')_{k_S'(p)}(p))^{Gal(k_S'(p)|k_S(p))}.
\]

Hence \( H^2_{et}((X \setminus S')_{k_S'(p)}) = 0 \), since \( Gal(k_S'(p)|k_S(p)) \) is a pro-\( p \)-group. Now [Sch2], Proposition 2.1 implies that \( X \setminus S' \) is a \( K(\pi, 1) \) for \( p \). \( \square \)

In order to prove Theorem 1.1, we first provide the following lemma. For an extension field \( K|k \) and a set of primes \( T \) of \( k \), we write \( T(K) \) for the set of prolongations of primes in \( T \) to \( K \) and \( \delta_K(T) \) for the Dirichlet density of the set of primes \( T(K) \) of \( K \).
Lemma 2.2. Let $k$ be a number field, $p$ a prime number and $S$ a finite set of nonarchimedean primes of $k$. Let $T$ be a set of primes of $k$ with $\delta_{k(\mu_p)}(T) = 1$. Then there exists a finite subset $T_0 \subset T$ such that all primes $\mathfrak{p} \in S$ do not split completely in the extension $k_{T_0}(p)|k$.

Proof. By [NSW], 9.2.2 (ii) or [NSW$^2$], 9.2.3 (ii), respectively, the restriction map

$$H^1(G_{T \cup S \cup S_p \cup S_R}(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow \prod_{\mathfrak{p} \in S \cup S_p \cup S_R} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class in $\alpha \in H^1(G_{T \cup S \cup S_p \cup S_R}(p), \mathbb{Z}/p\mathbb{Z})$ which restricts to an unramified class $\alpha_{\mathfrak{p}} \in H^1_{nr}(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$ for all $\mathfrak{p} \in S \cup S_p \cup S_R$ is contained in $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$. Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^1(G_{T \cup S \cup S_p \cup S_R}(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow \prod_{\mathfrak{p} \in S} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

contains the subgroup $\prod_{\mathfrak{p} \in S} H^1_{nr}(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$. As this group is finite, it is already contained in the image of $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$ for some finite subset $T_0 \subset T$. We conclude that no prime in $S$ splits completely in the maximal elementary abelian $p$-extension of $k$ unramified outside $T_0$. $\square$

Proof of Theorems 1.1 and 1.2. As $p \neq 2$, we may ignore archimedean primes. Furthermore, we may remove the primes in $S \cup S_p$ and all primes of degree greater than 1 from $T$. In addition, we remove all primes $\mathfrak{p}$ with $N(\mathfrak{p}) \not\equiv 1 \mod p$ from $T$. After these changes, we still have $\delta_{k(\mu_p)}(T) = 1$.

By Lemma 2.2, we find a finite subset $T_0 \subset T$ such that no prime in $S$ splits completely in $k_{T_0}(p)|k$. Put $X = \text{Spec}(\mathcal{O}_k)$. By [Sch2], Theorem 6.2, applied to $T_0$ and $T \setminus T_0$, we find a finite subset $T_2 \subset T \setminus T_0$ such that $X \setminus (T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Then Proposition 2.1 applied to $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$, shows that also $X \setminus (S \cup T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Now put $T_1 = T_0 \cup T_2 \subset T$.

It remains to show Theorem 1.2. Assertion (i) holds by construction of $T_1$. Again by construction, $X \setminus T_1$ is a $K(\pi, 1)$ for $p$. By [Sch2], Theorem 3, the field $k_{T_1}(p)$ realizes $k_{p}(p)$ for $p \in T_1$, showing (ii) for these primes. Finally, assertion (ii) for $p \in S$ follows from Proposition 2.1. $\square$

§ 3. Duality

We start by investigating the relation between the $K(\pi, 1)$-property and the universal norms of global units.
Let us first remove redundant primes from $S$: If $p \nmid p$ is a prime with $\zeta_p \notin k_p$, then every $p$-extension of the local field $k_p$ is unramified (see [NSW], 7.5.1 or [NSW$^2$], 7.5.9, respectively). Therefore primes $p \notin S_p$ with $N(p) \not\equiv 1 \mathrm{mod} p$ cannot ramify in a $p$-extension. Removing all these redundant primes from $S$, we obtain a subset $S_{\min} \subset S$, which has the property that $G_S(p) = G_{S_{\min}}(p)$. Furthermore, by [Sch2], Lemma 4.1, $X \backslash S$ is a $K(\pi, 1)$ for $p$ if and only if $X \backslash S_{\min}$ is a $K(\pi, 1)$ for $p$.

**Theorem 3.1.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite set of nonarchimedean primes of $k$. Then any two of the following conditions (a) – (c) imply the third.

(a) $\text{Spec}(\mathcal{O}_k) \backslash S$ is a $K(\pi, 1)$ for $p$.

(b) $\lim_{K \subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p = 0$.

(c) $(k_S(p))_p = k_p(p)$ for all primes $p \in S_{\min}$.

The limit in (b) runs through all finite extensions $K$ of $k$ inside $k_S$. If (a)–(c) hold, then also

$$\lim_{K \subset k_S(p)} \mathcal{O}_{K,S_{\min}}^\times \otimes \mathbb{Z}_p = 0.$$

**Remarks:**

1. Assume that $\zeta_p \in k$ and $S \supset S_p$. Then (a) holds and condition (c) holds for $p > 2$ if $0 < \mathcal{O}_c = S_{\min} \subset S$ (see [NSW$^2$], Remark 2 after 10.9.3). In the case $k = \mathbb{Q}(\zeta_p)$, $S = S_p$, condition (c) holds if and only if $p$ is an irregular prime number.

2. Assume that $S \cap S_p = \emptyset$ and $S_{\min} \neq \emptyset$. If condition (a) holds, then either $G_S(p) = 1$ (which only happens in very special situations, see [Sch2], Proposition 7.4) or (c) holds by [Sch2], Theorem 3 (or by Proposition 3.2 below).

**Proof of Theorem 3.1.** We may assume $S = S_{\min}$ in the proof. Let $K$ run through the finite extensions of $k$ in $k_S(p)$ and put $X_K = \text{Spec}(\mathcal{O}_K)$. Applying the topological Nakayama-Lemma ([NSW], 5.2.18) to the compact $\mathbb{Z}_p$-module $\lim \mathcal{O}_K^\times \otimes \mathbb{Z}_p$, we see that condition (b) is equivalent to

$$\lim_{K \subset k_S(p)} \mathcal{O}_K^\times /p = 0.$$

Furthermore, by [Sch2], Proposition 2.1, condition (a) is equivalent to

$$\lim_{K \subset k_S(p)} H^i_{et}(X \backslash S) \otimes \mathbb{Z}/p\mathbb{Z} = 0 \text{ for } i \geq 1.$$

Condition (a)$'$ always holds for $i = 1$, $i \geq 4$, and it holds for $i = 3$ provided that $G_S(p)$ is infinite or $S$ is nonempty or $\zeta_p \notin k$ (see [Sch2], Lemma 3.7). The flat Kummer sequence

$$0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$$

induces exact sequences

$$0 \longrightarrow \mathcal{O}_K^\times /p \longrightarrow H^1_{et}(X_K, \mu_p) \longrightarrow _p \text{Pic}(X_K) \rightarrow 0$$
for all $K$. As the field $k_S(p)$ does not have nontrivial unramified $p$-extensions, class field theory implies
\[
\lim_{K \subset k_S(p)} p\text{Pic}(X_K) \subset \lim_{K \subset k_S(p)} \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0.
\]
As we assumed $k$ to be totally imaginary if $p = 2$, the flat duality theorem of Artin-Mazur ([Mil], III Corollary 3.2) induces natural isomorphisms
\[
H^2_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) = H^2_f(X_K, \mathbb{Z}/p\mathbb{Z}) \cong H^1_f(X_K, \mu_p)^\vee.
\]
We conclude that
\[
(*) \quad \lim_{K \subset k_S(p)} H^2_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \left( \lim_{K \subset k_S(p)} \mathcal{O}_K^\times /p \right)^\vee.
\]
We first show the equivalence of (a) and (b) in the case $S = \emptyset$. If (a)' holds, then (*) shows (b)'. If (b) holds, then $\zeta_p \notin k$ or $G_S(p)$ is infinite. Hence we obtain (a)' for $i = 3$. Furthermore, (b)' implies (a)' for $i = 2$ by (*). This finishes the proof of the case $S = \emptyset$.

Now we assume that $S \neq \emptyset$. For $p \in S(K)$, a standard calculation of local cohomology shows that
\[
H^i_p(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \leq 1, \\ H^1(K_p, \mathbb{Z}/p\mathbb{Z})/H^1_{ns}(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 2, \\ H^2(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 3, \\ 0 & \text{for } i \geq 4. \end{cases}
\]
For $p \in S = S_{\min}$, every proper Galois subextension of $k_p(p)|k_p$ admits ramified $p$-extensions. Hence condition (c) is equivalent to
\[
(c)' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} H^i_p(X_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all } i,
\]
and to
\[
(c)'' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} H^2_p(X_K, \mathbb{Z}/p\mathbb{Z}) = 0.
\]
Consider the direct limit over all $K$ of the excision sequences
\[
\cdots \rightarrow \bigoplus_{p \in S(K)} H^i_p(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^i_{et}(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^i_{et}((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow \cdots.
\]
Assume that (a)' holds, i.e. the right hand terms vanish in the limit for $i \geq 1$. Then (*) shows that (b)' is equivalent to (c)''.

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for $i = 2$ in the limit. Condition (c)' then shows (a)''.

We have proven that any two of the conditions (a)–(c) imply the third.
Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW], 10.3.12, respectively)

$$0 \rightarrow \mathcal{O}_K^\times \rightarrow \mathcal{O}_{K,S}^\times \rightarrow \bigoplus_{\mathfrak{p}\in S(K)} (K_{\mathfrak{p}}^\times / U_{\mathfrak{p}}) \rightarrow \text{Pic}(X_K) \rightarrow \text{Pic}((X\setminus S)_{K}) \rightarrow 0$$

by (the flat $\mathbb{Z}$-algebra) $\mathbb{Z}_p$, we obtain exact sequences of finitely generated, hence compact, $\mathbb{Z}_p$-modules. Passing to the projective limit over the finite extensions $K$ of $k$ inside $k_S(p)$ and using $\lim\limits_{\longleftarrow} \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0$, we obtain the exact sequence

$$0 \rightarrow \lim_{K\subset k\in \mathbb{Z}_p} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \rightarrow \lim_{K\subset k\in \mathbb{Z}_p} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p \rightarrow \lim_{K\subset k\in \mathbb{Z}_p} \bigoplus_{\mathfrak{p}\in S(K)} (K_{\mathfrak{p}}^\times / U_{\mathfrak{p}}) \otimes \mathbb{Z}_p \rightarrow 0.$$

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle. \square

If $G_S(p) \neq 1$ and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing $p$. This follows from the next

**Proposition 3.2.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite set of nonarchimedean primes of $k$. If $\text{Spec}(\mathcal{O}_k)\setminus S$ is a $K(\pi, 1)$ for $p$ and $G_S(p) \neq 1$, then every prime $\mathfrak{p} \in S$ with $\zeta_p \in k_{\mathfrak{p}}$ has an infinite inertia group in $G_S(p)$. Moreover, we have

$$k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$$

for all $\mathfrak{p} \in S_{\min} \setminus S_p$.

**Proof.** We may assume $S = S_{\min}$. Suppose $\mathfrak{p} \in S$ with $\zeta_p \in k_{\mathfrak{p}}$ does not ramify in $k_S(p)|k$. Setting $S' = S \setminus \{\mathfrak{p}\}$, we have $k_{S'}(p) = k_S(p)$, in particular,

$$H^1_{et}(X\setminus S', \mathbb{Z}/p\mathbb{Z}) \sim H^1_{et}(X\setminus S, \mathbb{Z}/p\mathbb{Z}).$$

In the following, we omit the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation. Using the vanishing of $H^3_{et}(X\setminus S)$, the étale excision sequence yields a commutative exact diagram

$$H^2(G_{S'}(p)) \sim H^2(G_S(p)) \downarrow \leftarrow \downarrow \leftarrow \leftarrow \leftarrow \leftarrow$$

$$H^2_p(X) \rightarrow H^2_{et}(X\setminus S') \rightarrow H^2_{et}(X\setminus S) \rightarrow H^2_p(X) \rightarrow H^2_{et}(X\setminus S') \rightarrow H^3_{et}(X\setminus S').$$

Hence $\alpha$ is split-surjective and $\mathbb{Z}/p\mathbb{Z} \cong H^3_p(X) \sim H^3_{et}(X\setminus S')$. This implies $S' = \emptyset$, hence $S = \{\mathfrak{p}\}$, and $\zeta_p \in k$. The same applies to every finite extension of $k$ in $k_S(p)$, hence $\mathfrak{p}$ is inert in $k_S(p) = k_{\emptyset}(p)$. This implies that the natural homomorphism

$$\text{Gal}(k_p^{nr}(p)|k_{\mathfrak{p}}) \rightarrow G_{\emptyset}(k)(p)$$
is surjective. Therefore \(G_S(p) = G_{\emptyset}(p)\) is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the \(K(\pi, 1)\)-property, it is trivial, in contradiction to our assumptions.

This shows that all \(p \in S\) with \(\zeta_p \in k_p\) ramify in \(k_S(p)\). As this applies to every finite extension of \(k\) inside \(k_S(p)\), the inertia groups must be infinite. For \(p \in S_{\min} \setminus S_p\) this implies \(k_S(p)_p = k_p(p)\).

**Theorem 3.3.** Let \(k\) be a number field and let \(p\) be a prime number. Assume that \(k\) is totally imaginary if \(p = 2\). Let \(S\) be a finite nonempty set of nonarchimedean primes of \(k\). Assume that conditions (a)–(c) of Theorem 3.1 hold and that \(\zeta_p \in k_p\) for all \(p \in S\). Then \(G_S(p)\) is a pro-
\(p\) duality group of dimension 2.

**Proof.** Condition (a) implies \(H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \sim H^3_{et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0\). Hence \(cd\ G_S(p) \leq 2\). On the other hand, by (c), the group \(G_S(p)\) contains \(Gal(k_p(p)|k_p)\) as a subgroup for all \(p \in S\). As \(\zeta_p \in k_p\) for \(p \in S\), these local groups have cohomological dimension 2, hence so does \(G_S(p)\).

In order to show that \(G_S(p)\) is a duality group, we have to show that

\[
D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \lim_{\rightarrow} H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee \quad \text{vanish for} \quad i = 0, 1,
\]

where \(U\) runs through the open subgroups of \(G_S(p)\) and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of \(D_0\) is obvious, as \(G_S(p)\) is infinite. We therefore have to show that

\[
\lim_{K \subset k_S(p)} H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.
\]

We put \(X = Spec(\mathcal{O}_k)\) and denote the embedding by \(j : (X \setminus S)_K \to X_K\). By the flat duality theorem of Artin-Mazur, we have natural isomorphisms

\[
H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H^2_{fl, c}((X \setminus S)_K, \mu_p) = H^2_{fl}(X_K, j_!\mu_p).
\]

The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

\[
(*) \quad \bigoplus_{p \in S(K)} K_p^X / K_p^X p \to H^2_{fl}(X_K, j_!\mu_p) \to H^2_{fl}((X \setminus S)_K, \mu_p).
\]

As \(\zeta_p \in k_p\) and \(k_S(p)_p = k_p(p)\) for \(p \in S\) by assumption, the left hand term of (\(\ast\)) vanishes when passing to the limit over all \(K\). We use the Kummer sequence to obtain an exact sequence

\[
(**) \quad Pic((X \setminus S)_K)/p \longrightarrow H^2_{fl}((X \setminus S)_K, \mu_p) \longrightarrow pBr((X \setminus S)_K).
\]
The left hand term of (**) vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection

\[ p\text{Br}((X \setminus S)_K) \hookrightarrow \bigoplus_{p \in S(K)} p\text{Br}(K_p). \]

As \( k_S(p) \) realizes the maximal unramified \( p \)-extension of \( k_p \) for \( p \in S \), the limit of the middle term in (**) and hence also the limit of the middle term in (*) vanishes. This shows that \( G_S(p) \) is a duality group of dimension 2. \( \square \)

**Remark:** The dualizing module can be calculated to

\[ D \cong \text{tor}_p(C_S(k_S(p))), \]

i.e. \( D \) is isomorphic to the \( p \)-torsion subgroup in the \( S \)-idèle class group of \( k_S(p) \). The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

**References**


