The Periodic Box-ball System and Tropical Curves (Expansion of Integrable Systems)

Title

Author(s)

Citation

Issue Date

URL

Right

Type

Departmental Bulletin Paper

Textversion

publisher
The Periodic Box-ball System and Tropical Curves

By

Shinsuke IWAO*

Abstract

In this article, we study the box-ball system with finitely many kinds of balls. The box-ball system is obtained from the hungry discrete Toda equation (hpd Toda eq.) by ultradiscretization. We study the applications of the algebraic geometry and the tropical geometry to the ultradiscrete integrable system(s).

§ 1. Introduction

The periodic box-ball system (pBBS) is a discrete dynamical system which consists of an array of finitely many boxes in a line and a finite number of balls (see the figure below). We regard that the rightmost box is connected to the leftmost one. In this article, we consider pBBSs with finitely many kinds of balls. We distinguish these kinds of balls by indices of positive integers written on them.

\[
\begin{array}{cccccc}
\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 2 & 1 \\
\mathbf{4} & \mathbf{2} & 3 & 1 & 3 &\mathbf{1} \\
\end{array}
\]

The \textit{time evolution rule of pBBS} is defined as follows.

i) For each filled box, create a copy of the ball with index 1.

ii) Move each copy respectively to the nearest empty box on the right of it.

iii) Delete the original balls with index 1.

iv) Do the same procedure i)--iii) for the balls with index 2.
v) Repeat these procedures for the balls with 3, 4, \ldots until all the balls have moved.

We show an example of the time evolution of pBBS below. The letter “m” means a ball with index \( m \), and the sign “.” means an empty box. \(( t : \text{time})\).

\[
\begin{align*}
t=0 & \quad .1135.24. \\
1 & \quad .1135.24. \\
2 & \quad .1351.24. \\
3 & \quad .35.1124. \\
4 & \quad .35.1124. \\
\end{align*}
\]

We call a sequence of nondecreasing positive integers (for example “1135”, “24”) soliton. The number of solitons does not change under the time evolution [8]. An \( N \)-soliton system is a pBBS with \( N \) solitons. (The pBBS in the figure above is a 2-soliton system.)

Let \( M \in \mathbb{Z}_{>0} \). Let us consider \( N \)-soliton systems with \( M \) kinds of balls. Choose one soliton arbitrarily and regard it as the “leftmost” soliton formally. (Of course actual leftmost soliton cannot exist by periodicity). At time \( t \in \mathbb{Z} \), let \( Q_{n}^{t+\frac{m-1}{M}} \) be the number of balls with index \( m \) in the \( n \)-th soliton from the left, and \( W_{n}^{t} \) be the number of empty boxes between the \( n \)-th soliton and the \((n+1)\)-th soliton. Note that \( Q_{n+N}^{t} \equiv Q_{n}^{t} \) and \( W_{n+N}^{t} \equiv W_{n}^{t} \). For \( n \in \mathbb{Z} \) and \( t \in (1/M) \cdot \mathbb{Z} \), the time evolution rule of pBBS is expressed as follows [8]:

\[
\begin{align*}
Q_{n}^{t+1} &= \min\{W_{n}^{t}, Q_{n}^{t} + X_{n}^{t}\}, \quad W_{n}^{t+\frac{1}{M}} = Q_{n+1}^{t} + W_{n}^{t} - Q_{n}^{t+1}, \\
& \quad (X_{n}^{t} := \max_{k=0}^{N-1} [\sum_{l=1}^{k} (Q_{n-l}^{t} - W_{n-l}^{t})].) \\
\sum_{n=1}^{N} Q_{n}^{t} &< \sum_{n=1}^{N} W_{n}^{t}.
\end{align*}
\]

The inequality (1.2) reflects the fact that the number of empty boxes must be larger than the number of moving balls. The integer \( W_{n}^{t+\frac{k}{M}} \) \((k = 1, 2, \ldots, M - 1)\) is the number of empty boxes between two solitons when the balls with index \( k \) having finished moving.

Let \( \{Q_{n}^{t}, Q_{n}^{1/M}, \ldots, Q_{n}^{(M-1)/M}, W_{n}^{t}\}\) be the initial condition of pBBS. Let us consider the general solution \( \{Q_{n}^{t}, W_{n}^{t}\}_{n,t} \). Although it seems to be quite difficult to solve the system (1.1)–(1.2) directly, the following proposition gives us another method to obtain the solution.

**Proposition 1.1.** ([5, 8]) Let \( \epsilon > 0 \) be a positive parameter. Assume that there exists a set of functions \( \{I_{n}^{t}(\epsilon), V_{n}^{t}(\epsilon)\}_{n,t} \) such that

\[
\begin{align*}
I_{n}^{t+1} &= I_{n}^{t} + V_{n}^{t} - V_{n-1}^{t+\frac{1}{M}}, \quad V_{n}^{t+\frac{1}{M}} = \frac{I_{n}^{t+1}V_{n}^{t}}{I_{n}^{t+1}}, \quad (I_{n+N}^{t} \equiv I_{n}^{t}, V_{n+N}^{t} \equiv V_{n}^{t}) \\
\prod_{n=1}^{N} I_{n}^{t} &< \prod_{n=1}^{N} V_{n}^{t}.
\end{align*}
\]

\(\square\)
If the limits
\[ Q_n^t := -\lim_{\epsilon \to 0^+} (\epsilon \log I_n^t), \quad W_n^t := -\lim_{\epsilon \to 0^+} (\epsilon \log V_n^t) \]
exist, then the set \( \{Q_n^t, W_n^t\}_{n,t} \) satisfies (1.1) and (1.2).

The discrete system (1.3)-(1.4) is called hungry periodic d-Toda equation (hpd Toda eq.), and the operation ‘\( -\lim_{\epsilon \to 0^+} \epsilon \log \cdot \)’ is usually called ultradiscretization. According to the proposition above, the general solution of pBBS is the ultradiscretization of the solution of hpd Toda equation.

In this article, we research the method to calculate the general solutions of pBBSs. This article is organized as follows: In section 2, we introduce the method to solve the hpd Toda equation according to [3]. In section 3, we review the integration theory over tropical curves according to [4]. The results introduced in these two sections are applied to pBBS in section 4. We study the relative cycle of pBBS by using the theory of tropical curves.

§2. Solutions of the hpd Toda equation

We briefly introduce the theory of hpd Toda equation in this section. See [3], for details.

§2.1. Linearization

Let \( y \) be a complex parameter. Define the matrices \( L_t(y) \) and \( R_t(y) \) as follows.

\[
L_t(y) = \begin{pmatrix}
1 & V_1^t \cdot 1/y \\
V_1^t & 1 \\
\vdots & \vdots \\
V_{N-1}^t & 1
\end{pmatrix},
\quad R_t(y) = \begin{pmatrix}
I_1^t & 1 \\
I_2^t & \ddots \\
y & \ddots \\
& \ddots & 1
\end{pmatrix}.
\]

The hpd Toda equation (1.3)-(1.4) is equivalent to the following equation:

\[
L_{t+\frac{1}{M}}(y)R_{t+1}(y) = R_t(y)L_t(y), \quad t \in (1/M) \cdot \mathbb{Z}.
\]

Define the new matrix \( X_t(y) := L_t(y)R_{t+\frac{1}{M}-1}(y) \cdots R_{t+\frac{1}{M}}(y)R_t(y) \). Then equation (2.1) becomes

\[
X_{t+\frac{1}{M}}(y)R_t(y) = R_t(y)X_t(y), \quad t \in (1/M) \cdot \mathbb{Z}.
\]

By equation (2.2), the characteristic polynomial of \( X_t(y) \) does not depend on time \( t \). Therefore the algebraic curve defined by \( \det(X_t(y) - xE) = 0 \) does not depend on \( t \)
either. \((E\) is the unit matrix). Let

\[
\Phi(x, y) := \det(X_{t}(y) - xE)
\]

\[
= f_{0}(x) y^{M} + f_{1}(x) y^{M-1} + \cdots + f_{M}(x) + h y^{-1},
\]

where \(f_{i}(x) (i = 0, 1, \ldots, M)\) is a polynomial in \(x\) of degree \(\leq iN/M\), and \(h\) is a constant \([3, 7]\). The plane complex curve defined by \(\Phi = 0\) is called spectral curve. The spectral curve is determined by the initial data \(\{I_{n}^{0}, I_{n}^{\frac{1}{M}}, \ldots, I_{n}^{\frac{M-1}{M}}, V_{n}^{0}\}_{n}\) of hpd Toda equation. Let \(C\) be the plane curve defined by \(\Phi = 0\). The isospectral set \(T_{C}\) is a set of matrices \(X(y)\) of which the spectral curve is \(C\).

Assume \(C\) is smooth hereafter. If \((x, y) \in C\) and \(X(y) \in T_{C}\), then the number \(x\) is an eigenvalue of \(X(y)\). Let \(v(x, y)\) be an eigenvector of \(X(y)\) which belongs to \(x\). Naturally \(v(x, y)\) is regarded as an element of \(\mathbb{P}^{N-1}\). Moreover, when \(x\) is a simple root of \(\Phi(x, y)\) (for fixed \(y\)), the element \(v(x, y) \in \mathbb{P}^{N-1}\) is determined uniquely. As \(C\) is smooth, the map \((x, y) \mapsto v(x, y)\) can be extended to the proper map \(\psi_{X(y)} : C \to \mathbb{P}^{N-1}\) uniquely. It is known that the image \(\psi_{X(y)}(C) \subset \mathbb{P}^{N-1}\) is a projective curve of degree \(d := g + N - 1\) \([3, \S 2.2]\). \((g\) is the genus of \(C)\).

Let us consider the twisting sheaf \(\mathcal{O}_{\mathbb{P}^{N-1}}(1)\) over \(\mathbb{P}^{N-1}\) and its pullback

\[
\psi_{X(y)}^{*}\mathcal{O}_{\mathbb{P}^{N-1}}(1).
\]

The sheaf \(\psi_{X(y)}^{*}\mathcal{O}_{\mathbb{P}^{N-1}}(1)\) can be identified to the element of \(\text{Pic}^{d}C\) which is defined by the pullback of \(C \cap H \subset \mathbb{P}^{N-1}\). \((H\) is a hyperplane). The map \(\varphi_{C} : T_{C} \ni X(y) \mapsto \psi_{X(y)}^{*}\mathcal{O}_{\mathbb{P}^{N-1}}(1) \in \text{Pic}^{d}C\) is called eigenvector map.

Let \(v(x, y) = (g_{1}, g_{2}, \ldots, g_{N-1}, 1)^{T}\), where \(g_{1}, \ldots, g_{N-1}\) are appropriate meromorphic functions of \(x\) and \(y\). Then the element \(\varphi_{C}(X(y)) \in \text{Pic}^{d}C\) is the positive divisor \(E\) of minimal degree such that \((g_{k}) + E \geq 0\) \((\forall k)\). In fact, \(E\) is the pullback of the hyperplane \(\{X_{N} = 0\}\), where \(X_{1} : X_{2} : \cdots : X_{N}\) is a set of homogeneous coordinates for \(\mathbb{P}^{N-1}\).

**Remark 2.1.** By concrete calculations, we can obtain the relation : \((g_{1})_{\infty} = E\) \([3, \text{Remark 2.4}], [7]\). This relation is quite convenient to calculate the image of given matrix \(X(y)\) by the eigenvector map.

Let \(P : (x, y) = (\infty, \infty)\) and \(Q : (x, y) = (\infty, 0)\) be two points on \(C\). It follows that:

\begin{itemize}
  \item 1 There exists a general effective divisor \(D\) of degree \(g\) s.t. \(\varphi_{C}(X(y)) = (g_{1})_{\infty} = D + (N - 1)Q\).
  \item 2 \(\varphi_{C} : T_{C} \to \text{Pic}^{d}C\) is injective.
\end{itemize}

\(\bullet\) See [7].

By the statement \(\bullet 2\), to analyze \(X(y) \in T_{C}\) and to analyze \(\varphi_{C}(X(y)) \in \text{Pic}^{d}C\) is equivalent.
Remark 2.2. There uniquely exist two points \( P(\infty, \infty) \), \( Q(\infty, 0) \) on \( C \). This is a special property of the spectral curves of hpd Toda equation. In [7], they researched the spectral curves on which \( \{\text{g.c.d.}(N, M)\} \) points \( P_1, P_2, \ldots, P_G : (x, y) = (\infty, \infty) \) exist. \( (G = \{\text{g.c.d.}(N, M)\}) \).

Let us proceed to the hpd Toda equation. Let \( X_t(y) \in \mathcal{T}_C \) be the matrix defined by the equation below (2.1). Let \( \sigma \) (resp. \( \tau \)) be the action on \( \mathcal{T}_C \) induced by the shift of the indices of \( Q^n_t \) and \( W^n_t \) as \( n \mapsto n + 1 \) (resp. \( t \mapsto t + 1 \)). These actions naturally induce the action on \( \Image(\varphi_C) \).

It is possible to calculate these actions concretely. Define the complex numbers \( y_0, y_1, \ldots, y_{M-1} \) by \( y_k := (-1)^N \cdot \prod_{n=1}^{N} I_n^{k/M} \). Hence it can be checked easily that there exist \( M \) points \( A_0(0, y_0), A_1(0, y_1), \ldots, A_{M-1}(0, y_{M-1}) \) on \( C \).

Theorem 2.1.

1. \( \varphi_C(\sigma(X(y))) = \varphi_C(X(y)) + P - Q \),
2. \( \varphi_C(\tau(X(y))) = \varphi_C(X(y)) + A_0 + A_1 + \cdots + A_{M-1} - M \cdot P \).

Proof. [3], Prop 2.6 and Prop 2.16. \( \square \)

The points \( P, Q, A_0, \ldots, A_{M-1} \) do not depend on \( t \) nor \( n \). And the actions of \( \sigma \) and \( \tau \) are commutative. Therefore, the arbitrary composite of \( \sigma \) and \( \tau \) can be expressed as:

Corollary 2.2. Let \( D_n := P - Q, D_t := A_0 + A_1 + \cdots + A_{M-1} - M \cdot P \). Then
\[
\varphi_C(\sigma^r \circ \tau^s(X(y))) = \varphi_C(X(y)) + rD_n + sD_t.
\]
\( \square \)

In a word, the actions of \( \sigma \) and \( \tau \) is linearized on \( \text{Pic}^d C \).

§ 3. The lattice integral over tropical curve

We introduce the theory of tropical curves in this section. The lattice integral over tropical curve is an analogy of the holomorphic integral over complex curve [1, 6]. We study the relation between these two “integrals”, according to [4].

§ 3.1. The tropical curve

A Tropical curve is an algebraic curve defined over the tropical semifield \( \mathbb{T} := \mathbb{R} \cup \{+\infty\} \) equipped with the min-plus operation: “\( x + y^* = \min\{x, y\} \)” and “\( xy^* = x + y \).” Here we consider tropical plane curves only.
Definition 3.1. Let $\phi_1, \phi_2, \ldots, \phi_n$ be polynomials of the form: $\phi_i(X, Y) = a_iX + b_iY + c_i$ ($a_i, b_i \in \mathbb{Z}, c_i \in \mathbb{R}$). A tropical plane curve $\mathcal{C}$ is a subset of $\mathbb{R}^2$ defined by

$$(3.1) \quad \mathcal{C} := \{(X, Y) \in \mathbb{R}^2 \mid \text{the function } \min_{i=1}^{n} [\phi_i(X, Y)] \text{ is not smooth.}\}$$

Hereafter we call these curves “tropical curves” simply. A tropical curve consists of finitely many segments and half-lines, that are called edges of the tropical curve. A vertex of a tropical curve is a point which is contained by more than two segments or half-lines.

The condition appeared in (3.1) is rewritten as:

$$(3.2) \quad \#\{i \mid \phi_i(X, Y) = \min_{i=1}^{n} [\phi_i(X, Y)]\} \geq 2.$$ 

Hence, for $A = (X, Y) \in \mathcal{C}$, we define the non-negative integer $I(A)$ by

$$I(A) := \#\{i \mid \phi_i(X, Y) = \min_{i=1}^{n} [\phi_i(X, Y)]\} - 2.$$ 

Definition 3.2. The tropical curve $\mathcal{C}$ is regular if

$I(A) = 0 \iff A$ is a inner point of an edge, \quad $I(A) = 1 \iff A$ is a vertex.$$

Remark 3.1. $\mathcal{C}$ is regular $\Rightarrow$ $\mathcal{C}$ is 3-valent.

For the purpose of arguing about the ultradiscretization of algebraic curves, it is convenient to consider the algebraic curves over the Puiseux field. Let $e := e^{-1/\varepsilon}$ and $K := \bigcup_{d=1}^{\infty} \mathbb{C}((e^{1/d}))$. The Puiseux field $K$ has the valuation map $\text{val} : K \rightarrow \mathbb{Q} \cup \{+\infty\}$ ($\text{val}(e) = 1, \text{val}(0) = +\infty$). When the ultradiscrete limit $X := -\lim_{\varepsilon \to 0} \varepsilon \log x (x \in K)$ exists, it follows that $\text{val}(x) = X$.

Define the subring $R := \{x \in K \mid \text{val}(x) \geq 0\} \subset K$ and the multiplicative group $R^* := \{x \in K \mid \text{val}(x) = 0\}$.

Let $\Phi(x, y) = \sum_{i,j} r_{i,j} x^i y^j \in K[x, y]$ be a polynomial over $K$. The tropicalization of the algebraic curve $C : \Phi(x, y) = 0$ is a set

$$(3.2) \quad \text{TropC} := \{(X, Y) \in \mathbb{R}^2 \mid \min_{i,j} [\text{val}(r_{i,j}) + iX + jY] \text{ is not smooth.}\}.$$ 

§ 3.2. Examples

3.2.1. Example I

$$(3.3) \quad \Phi(x, y) = y^2 + (x^2 + ex + e^5)y + e^{11}$$
Let $C$ be the algebraic curve defined by $\Phi$. Then it follows that:

$$\text{Trop } C = \{ (X, Y) \mid \min\{2Y, 2X + Y, X + Y + 1, Y + 5, 20\} \text{ is not smooth}\}$$

(figure 1).

Trop $C$ has four vertices. The vertex $(X, Y) = (1, 2)$ is associated with the points in the curve $C$ that are expressed by $(x, y) = (re^1, se^2)$ $(r, s \in R^*)$. Substituting $x = re^1$ and $y = se^2$ to $\Phi(x, y) = 0$, we obtain $0 = s^2e^4 + (r^2e^2 + re^2 + e^5)se^2 + e^{11} = (s^2 + r^2s + rs)e^4 + se^7 + e^{11}$. When we take the limit $\epsilon \to 0^+$ $(e \to 0)$, the factor $s^2 + r^2s + rs$ must be dominant. Now we consider the Riemann surface defined by $s^2 + r^2s + rs = 0$. First note that $r, s \in R^*$, which implies $r, s \neq 0$. The surface $\{s + r^2 + r = 0\}$ is a Riemann sphere with genus 0.

Similarly, vertices $(1, 9), (4, 5), (4, 6)$ are also associated with the following “small” Riemann surfaces respectively:

$$r^2s + rs + 1 = 0, \quad s^2 + rs + s = 0, \quad rs + s + 1 = 0.$$  

Each curve is of genus 0.

Figure 1 shows us the relation between $C$ and Trop $C$ schematically. Each Riemann sphere is associated with a vertex. Let us call these “small” Riemann surfaces subsurfaces.

### 3.2.2. Example II

$$C : \Phi(x, y) = y^2 + (x^3 + ex^2 + e^3x + e^6)y + e^{10} = 0$$
The subsurfaces associated with \((1, 3), (1, 7)\) and \((2, 5)\) are expressed by:

\[
s + r^3 + r^2 = 0, \quad r^3 s + r^2 s + 1 = 0, \quad s^2 + r^2 s + rs + 1 = 0
\]

respectively. The first two are of genus 0, and the third one is of genus 1.

The figure above shows us the relation between \(C\) and Trop\(C\). Although the genus of \(C\) is 2, the genus of Trop\(C\) is 1.

**Definition 3.3.** The algebraic curve \(C\) over \(K\) is non-degenerate if its subsurfaces are irreducible and of genus 0.

**§ 3.3. Lattice integral**

The *lattice integral* is a linear form over a tropical curve, which is defined by the metric over rational graph [6].

Let \(E\) be an edge of Trop\(C\). The point \(x \in E\) satisfies \(E : x = v_0 + tv, (0 \leq t \leq \ell)\). The vector \(v_0\) is the starting point of \(E\) and \(v\) is the primitive tangent vector for \(E\). Now we define the *tropical length* of \(E\) by \(\ell(E) := \ell\).

Let \(\beta_1, \ldots, \beta_g\) be a basis of \(H_1(\text{Trop}C, \mathbb{Z})\). We can define a natural bi-linear form \((\cdot, \cdot)\) over the space of paths on \(H_1(\text{Trop}C, \mathbb{Z})\). For this, we define \((r, r) := \ell(r)\) for non-self-intersecting path, and extend it to any pairs of paths bilinearly.

This bilinear form is called *lattice integral* because \((\beta_i, \cdot)\) can be regarded as an element \(H_1(\mathbb{Z})^\vee = H^0(\Omega^1)\). (\(\Omega^1\) is the cotangent sheaf of Trop\(C\)).

A *period matrix* of Trop\(C\) is a \(g \times g\) matrix \(B_{\text{Trop}C} := (\beta_i, \beta_j)_{i,j}\).
The Periodic Box-ball System and Tropical Curves

Figure 2. An example of metric graph. It follows that $(\beta_1, \beta_1) = \ell_1 + \ell_3 + \ell_5 + \ell_6$, $(\beta_2, \beta_2) = \ell_2 + \ell_4 + \ell_6 + \ell_7$, $(\beta_1, \beta_2) = (\beta_2, \beta_1) = -\ell_6$.

§ 3.4. Complex integrals

Now we study the relation between the lattice integrals and the usual complex integrals. For this, we consider the simplest case here.

Definition 3.4. Let $C$ be a curve given by the zeros of $\Phi \in K[x, y]$. When $C$ satisfies the following conditions, we said that $C$ has a good tropicalization.

1) $C$ is non-degenerate, 2) $\text{Trop} C$ is regular, 3) generic condition.

Generic condition is an assumption for the coefficients of $\Phi \in K[x, y]$. It may happens that $C$ becomes singular or has some strange property when these coefficients take special values. We assume the generic condition to avoid these situations. See [4],§3, for details.

Next we proceed to the complex integrals. Let $C$ be a Riemann surface of genus $g$, and $\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g \in H_1(C, \mathbb{Z})$ be a canonical basis. When choosing $\beta_i$, we make them have the same position and orientation as $\beta_1, \ldots, \beta_g \in H_1(\text{Trop}, \mathbb{Z})$ on $\text{Trop} C$ (see the figure below). Define the holomorphic differentials $\omega_1, \ldots, \omega_g$ on $C$ s.t. $\int_{\alpha_i} \omega_j = \delta_{i,j}$, and the $g \times g$ matrix $B_C := (\int_{\beta_i} \omega_j)_{i,j}$.

The following is the main result of [4].

Theorem 3.1. When $C$ has a good tropicalization, it follows that

$$B_C \sim \frac{-1}{2\pi i \varepsilon} B_{\text{Trop} C} \quad (\varepsilon \to 0^+).$$

Corollary 3.2. Let $\gamma$ be an oriented path on $C$. When we define a path $\gamma$ (abuse of symbol) on $\text{Trop} C$ such that it has the same position and orientation as $\gamma_{\text{on} C}$, then:

$$\int_{\gamma} \omega_i \sim \frac{-1}{2\pi i \varepsilon} (\gamma, \beta_i) \quad (\varepsilon \to 0^+).$$
§ 4. Application for pBBSs

In this section, we study the method to solve the problems about pBBSs by applying the results of theorem 2.1 and theorem 3.1.

§ 4.1. Abel-Jacobi mapping

Let $Q_{n}^{0}, Q_{n}^{\frac{1}{M}}$, $Q_{n}^{\frac{2}{M}}, \ldots, Q_{n}^{\frac{M-1}{M}}, W_{n}^{0}$ be an initial condition of pBBS. Using this, we define the initial condition of hpd Toda equation by $I_{n}^{\frac{k}{M}} := \kappa_{n,k} e^{Q_{n}^{k/M}}, V_{n}^{0} := \lambda_{n} e^{W_{n}^{0}} (k = 0, 1, \ldots, M - 1)$, where $\kappa_{n,k}$ and $\lambda_{n}$ are arbitrary elements of $R^\ast$.

Let $C$ be the spectral curve defined by the initial data. According to section 2, the eigenvector mapping $\varphi_{C}$ sends the matrix $X(y)$ (2.2) into $\text{Pic}^{d} C$. We can have the general effective divisor $D$ such that $\varphi_{C}(X(y)) = D + (N - 1)Q$.

Fix a point $P_{0} \in C$. Let $\text{Div}_{+}^{g}(C)$ be the set of effective divisors of degree $g$ on $C$. Define the Abel-Jacobi mapping $A : \text{Div}_{+}^{g}(C) \to \mathbb{C}^{g}/(\mathbb{Z}^{g} + B_{C}\mathbb{Z}^{g})$ by

$$A(D) := \sum_{i=1}^{g} \left( \int_{P_{0}}^{P_{i}} \omega_{1}, \int_{P_{0}}^{P_{i}} \omega_{2}, \ldots, \int_{P_{0}}^{P_{i}} \omega_{g} \right).$$

The image of $A$ does not depend on the choice of integral paths. By the classical result, if $D$ is a general divisor, it is possible to determine the original divisor $D$ uniquely from the vector $A(D)$. Therefore, to analyze the divisor $D + (N - 1)Q$ and to analyze the vector $A(D)$ is equivalent.

Let $\mathcal{D}(X(y))$ be the general effective divisor expressed by $\varphi_{C}(X(y)) - (N - 1)Q$. By the statements of theorem 2.1 and corollary 2.2, the actions of $\sigma$ and $\tau$ are expressed by the following equation:

$$(4.1) \quad A(\mathcal{D}(\sigma^{r} \circ \tau^{s}(X(y)))) = A(\mathcal{D}(X(y))) + rv_{n} + sv_{t} \in \mathbb{C}^{g}/(\mathbb{Z}^{g} + B_{C}\mathbb{Z}^{g}).$$
where \( v_n = \left( \int_Q^P \omega_1, \ldots, \int_Q^P \omega_g \right) \),
\( v_t = \sum_{i=0}^{M-1} \left( \int_P^{A_i} \omega_1, \ldots, \int_P^{A_i} \omega_g \right) \).

§ 4.2. Fundamental and relative period of pBBS

**Definition 4.1.** The fundamental period of pBBS is the number of minimal time steps until the state of pBBS returns to the same state as the initial condition.

Although it is possible to calculate the fundamental periods of pBBSs, we consider another “period” here for simplicity.

**Definition 4.2.** The relative period of pBBS is the minimal positive integer \( T > 0 \) such that \( \exists q \in \mathbb{Z} \Rightarrow Q_{n+q}^{t+T} \equiv Q_n^t \) and \( W_{n+q}^{t+T} \equiv W_n^t \).

**Remark 4.1.** The fundamental period can be calculated from the relative period by combinatorial methods [9].

Equation (4.1) shows us
\[
\varphi_C(\sigma^q \circ \tau^T(X(y))) = \varphi_C(X(y)) \Leftrightarrow qv_n + Tv_t \equiv 0 \pmod{(\mathbb{Z}^g + B_C\mathbb{Z}^g)}.
\]

§ 4.3. How to calculate the relative period

In this section, we study the method to calculate the relative periods. As a concrete example, consider the state of pBBS below:

\( t=0 \)
11.\ldots1112.\ldots111\ldots11111\ldots
1 .\ldots1112.\ldots111\ldots1111.
2 11.\ldots11111\ldots1112.\ldots1111\ldots1
3 .\ldots111\ldots11111\ldots112.\ldots1111.\ldots

Then we have
\[
(Q_1^0, Q_1^{1/2}, Q_2^0, Q_2^{1/2}, Q_3^0, Q_3^{1/2}, Q_4^0, Q_4^{1/2}) = (2, 0, 3, 1, 3, 0, 5, 0),
\]
\[
(W_1^0, W_2^0, W_3^0, W_4^0) = (3, 4, 3, 6).
\]

(Note that we can define the “leftmost” soliton arbitrarily). Define \( I_1^0 = \kappa_{1,0} e^2, I_1^{1/2} = \kappa_{1,1} e^0; I_2^0 = \kappa_{2,0} e^3, \ldots; V_1^0 = \lambda_1 e^3, V_1^0 = \lambda_2 e^4, \ldots \) etc. \((\kappa_{n,0}, \kappa_{n,1}, \lambda_n \in \mathbb{R}^+))\).

Therefore, the determinant \( \Phi(x,y) = \det(X(y) - xE) \) can be expanded:
\[
\Phi = y^2 - 2(x^2 + r_{2,1} e^0 x + r_{2,0} e^1)y
+ (x^4 + r_{1,3} e^2 x^3 + r_{1,2} e^5 x^2 + r_{1,1} e^9 x + r_{1,0} e^{14}) - r e^{30} y^{-1}
\]
Because the field $K$ is algebraically closed, we can rewrite $\Phi$ into:

$$\Phi = y^2 - 2(x + u_{2,0}e^0)(x + u_{2,1}e^1)y + (x + u_{1,0}e^2)(x + u_{1,1}e^3)(x + u_{1,2}e^4)(x + u_{1,3}e^5) - re^{30}y^{-1}$$

($u_{i,j} \in R^*$). Denote the algebraic curve $\{\Phi = 0\}$ by $C$. Hereafter we consider the tropical curve $\text{Trop} C$. See the figure in the previous page.

The point $A_0 : (x, y) = \left(0, \prod_{n=1}^{4} I_n^0\right) \in C$ is associated with the point $(X, Y) = (\text{val}(0), \text{val}(\prod_{n=1}^{4} I_n^0)) = (+\infty, 13)$ on $\text{Trop} C$. We will abuse the symbol $A_0$ to mean the point on the tropical curve. Similarly, we use (abuse) the symbol $A_1$, $P$ and $Q$ as $A_1 : (X, Y) = (+\infty, 1)$, $P : (X, Y) = (-\infty, -\infty)$, $Q : (X, Y) = (-\infty, +\infty) \in \text{Trop} C$. ($\text{val}(\infty) = -\text{val}(0) = -\infty$).
Fix oriented paths $\gamma_{Q \to P}, \gamma_{P \to A_0}, \gamma_{P \to A_1} \subset \text{Trop } C$ which connect two points $Q \to P, P \to A_0, P \to A_1$ respectively.

Then we obtain the following data from the figure of Trop $C$.

$$B_{\text{Trop } C} = \begin{pmatrix} 26 & -9 & 0 & 0 \\ -9 & 16 & -5 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$v_{n,\text{Trop } C} := \left( (\beta_i, \gamma_{Q \to P}) \right)_i \equiv (15, 0, 0, 1) \pmod{B_{\text{Trop } C} \mathbb{Z}^4},$$

$$v_{t,\text{Trop } C} := \left( (\beta_i, \gamma_{P \to A_0} + \gamma_{P \to A_1}) \right)_i \equiv (1, 1, 1, 0) \pmod{B_{\text{Trop } C} \mathbb{Z}^4}.$$ 

The vectors $v_{n,\text{Trop } C}, v_{t,\text{Trop } C} \pmod{B_{\text{Trop } C} \mathbb{Z}^4}$ do not depend on the choice of $\gamma_{Q \to P}, \gamma_{P \to A_i}$.

Here we note the relation:

$$N v_{n,\text{Trop } C} = 4 v_{n,\text{Trop } C} \equiv (60, 0, 0, 4)$$

$$= 3 (26, -9, 0, 0) + 2 (-9, 16, -5, 0) + (0, -5, 10, 0) + (0, 0, 0, 4)$$

$$\equiv (0, 0, 0, 0) \pmod{B_{\text{Trop } C} \mathbb{Z}^4},$$

that reflects the fact that the action $\sigma : n \mapsto n + 1$ satisfies $\sigma^N = \text{id}$.

Assume $C$ has a good tropicalization. Theorem 3.1 and corollary 3.2 imply

$$B_C \sim -\frac{1}{2\pi i \epsilon} B_{\text{Trop } C},$$

$$v_n \sim -\frac{1}{2\pi i \epsilon} v_{n,\text{Trop } C} \pmod{B_{\text{Trop } C} \mathbb{Z}^4},$$

$$v_t \sim -\frac{1}{2\pi i \epsilon} v_{t,\text{Trop } C} \pmod{B_{\text{Trop } C} \mathbb{Z}^4}.$$ 

Substituting this relation into the last equation $qv_n + T v_t \equiv 0 \pmod{(\mathbb{Z}^g + B_C \mathbb{Z}^g)}$, we obtain

$$qv_n + T v_t \equiv 0 \pmod{(\mathbb{Z}^4 + B_C \mathbb{Z}^4)}$$

$$\Leftrightarrow \exists r', r \in \mathbb{Z}^4 \text{ s.t. } qv_n + T v_t = r' + B_C r$$

$$\Rightarrow \exists r \in \mathbb{Z}^4 \text{ s.t. } -\frac{1}{2\pi i \epsilon} (qv_{n,\text{Trop } C} + T v_{t,\text{Trop } C}) = -\frac{1}{2\pi i \epsilon} B_{\text{Trop } C} r$$

(4.3)

$$\Leftrightarrow qv_{n,\text{Trop } C} + T v_{t,\text{Trop } C} \equiv 0 \pmod{B_{\text{Trop } C} \mathbb{Z}^4}.$$ 

(The second arrow is ‘$\Rightarrow$’. See section 5.)

Equation (4.3) is equivalent to:

$$q (B_{\text{Trop } C}^{-1} \cdot v_{n,\text{Trop } C}) + T (B_{\text{Trop } C}^{-1} \cdot v_{t,\text{Trop } C}) \equiv 0 \pmod{\mathbb{Z}^4}.$$
Therefore,
\[ q(3/4, 1/2, 1/4, 1/4) + T(1/10, 8/45, 17/90, 0) \equiv 0 \pmod{\mathbb{Z}^4}. \]

From this, we can conclude that the minimal positive integer \( T > 0 \) which satisfies equation (4.3) is 90. In fact, the relative period of given state of pBBS is 90 (see Appendix A).

§ 5. Concluding remark

We have studied the algebraic geometrical and tropical geometrical method to analyze the pBBSs in the previous section. Through this technique, we can obtain automatically some of important results, for example, the fundamental period formula of the pBBS with one kind of balls [9].

However, we have admitted some assumptions in this article. To justify our theory, we must deal with the following subjects.

1. Regularity of Trop \( C \).
   To apply theorem 3.1, Trop \( C \) must be regular. However, some special initial condition makes Trop \( C \) non-regular. Because regularity is essential for the proof of theorem 3.1, this theorem fails in this situation.

2. Generic condition.
   Theorem 3.1 also requires that the coefficients of \( \Phi(x, y) \) (4.2) must satisfy some generic condition. We do not describe in detail about this condition here, but it is sufficient to prove:

   \textbf{Conjecture 5.1.} The coefficients of \( \Phi(x, y) = \det (X(y) - xE) \) are functionally independent as the functions of \( I_{n}^{0}, \ldots, I_{n}^{M-1}, V_{n}^{0} \).

   This conjecture should be true if we want to claim “pBBS is a completely integrable system”, but the author does not know the proof.

3. Does the torus \( \mathbb{R}^g/B_{\text{Trop} C}\mathbb{Z}^g \) have enough information of pBBS?
   Fix a spectral curve \( C \). Let \( T_C \) be the iso-spectral set (section 2.1). Denote the set of initial conditions of pBBSs which give the tropical curve Trop \( C \) by \( \mathfrak{T}_{\text{Trop} C} \).

   Let us consider the composite:

   \[ \mathfrak{T}_{\text{Trop} C} \xrightarrow{*} T_C \xrightarrow{\text{Pic}^d C} \mathbb{C}^d/(\mathbb{Z}^d + B_C \mathbb{Z}^d) \xrightarrow{**} \mathbb{R}^g/B_{\text{Trop} C}\mathbb{Z}^g \]

   \( \xrightarrow{*} \) is ultradiscretization, \( \xrightarrow{**} \) is an operation to define an initial condition of hpd Toda eq. from an initial condition of pBBS (section 4.1), \( \xrightarrow{***} \) is an operation to pick up the coefficient of \(-1/(2\pi i \epsilon)\). Note that \( \xrightarrow{**} \) depends on the choice of elements \( R^* \).

   The questions are:
Q.1. Is $\chi : \mathfrak{T}_{\text{TrOpC}} \to \mathbb{R}^{g}/B_{\text{TrOpC}}\mathbb{Z}^{g}$ well-defined?

Q.2. What is the image of $\chi$? Is $\chi$ injective?

If these are true, all arrow in (4.3) (section 4.3) become “$\Leftrightarrow$”.

**Remark 5.1.** For $M = 1$, R. Inoue and T. Takenawa answered these questions in [2]. In fact, they proved that $\chi$ is an isomorphism.

References


§ Appendix A.

t= 0 11...1112....111...11111...... 11...1112....111...11111. 111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
t=10
1....12.1111...111...1111
.1111...12....11111...111...
111....12....1112....1111....1
....1111....11....1112....111.
11....11...1111....1112....1
....111....1111...1112....1112
112....1111...111....11111....
....112....11....11111....1111.
111....112....11....11111....1
....111....112.1111....1111.
111....111....12....11111....1


t=20
1....111....11....1112....1111
.11111...111....1112.......
....111....11....11111....1112.
1112....111....11....11111....1
....11112....11....1111....111.
111....11112....11....1111....1
....1111....12.1111...11111.
111....11111....12....111....11
....1111....11111..12....111.


t=30
11111....1111....11112....
....11111....11....11111....12.
112....11111....11....11111....1
....11112....11....11111....11.
111....11112....11....1111....1
....1111....11112.11....111.
11....11111....12.11111....1
....1111....1111....12....111.
11....11111....11111....12....1


t=40
.111....1111....11....11112.
112....1111....1111....11....1
....112....11111....111....111.
11....112....11111....11....111
11111....112....1111....111....1
1....111....11112....11....111
The Periodic Box-ball System and Tropical Curves

\[
\begin{array}{ll}
\text{t=50} & \begin{array}{l}
.1111...111.....11112..11.... \\
.....1111...1111.....112.1111 \\
11111...1111...1111.....12... \\
.....1111...1111.....11111...12.
\end{array} \\
& \begin{array}{l}
112.....111...1111.....111...1 \\
.....11112.....111.....1111...11. \\
1......11112.....111.....1111...1 \\
.1111.....112.11111.....11. \\
1......11111.....112.....1111...1 \\
.1111.....11111.....12.....11. \\
1......1111.....1111.....112...1 \\
.111.....1111.....111.....112 \\
12.....11111.....1111.....111 ..... \\
.....12.....1111.....11111.....111 ...
\end{array}
\end{array}
\]

\[
\begin{array}{ll}
\text{t=60} & \begin{array}{l}
11...112.....1111.....111...11 \\
.....11...11112.....1111.....111... \\
1......11112.....1111.....1111...11. \\
.1111...11112.....1111.....1111...11. \\
1......11111.....1112.....1111...11. \\
.1111...11111.....11111.....112. \\
12.....11...11111.....11111.....111 . \\
.....112.11.....11111.....11111. \\
11...12.11111.....1111.....11 \\
.....111....12.....1111.....11111 ... 
\end{array}
\end{array}
\]

\[
\begin{array}{ll}
\text{t=70} & \begin{array}{l}
1.....111....112.....1111.....1111 \\
11111.....111.....1112.....1111..... \\
.....111....1111.....11112.....1111. \\
1111.....111.....11112.....1112...1 \\
.....11111.....1111.....11112.....1112 \\
1112.....11....1111.....11111..... \\
.....1112.11.....111.....11111.....1111. \\
111.....12.11111.....11111.....11 \\
.....111111.....12.....111.....11111. \\
.............11.....1112.....111.....11111 
\end{array}
\end{array}
\]

\[
\begin{array}{ll}
\text{t=80} & \begin{array}{l}
11111.....11.....112.....1111..... \\
.....11111.....11.....112.....1111. \\
1111.....11.....11111.....112...1 \\
.....1111.....11.....11111.....1111. \\
111.....11.....11111.....112...11 \\
.....1111.....11.....11111.....11112 
\end{array}
\end{array}
\]
1112...11111..11......111......
....1112....11..11111....111..
11......1112..11.....1111....11
..11111.....112.111......111..
1......11111....12..1111.....11
.111........111..112...111111.
t=90 1...11111......11...1112....11