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# Riemann zeta function and the best constants of five series of Sobolev inequalities

By

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## Abstract

We clarified the variational meaning of the special values $\zeta(2M)$ $(M = 1, 2, 3, \cdots)$ of Riemann zeta function $\zeta(z)$. These are essentially the best constants of five series of Sobolev inequalities. In the background, we consider five kinds of boundary value problem (periodic, antiperiodic, Dirichlet, Neumann, Dirichlet-Neumann) for a differential operator $(-1)^M(d/dx)^{2M}$. Green functions for these boundary value problems are given by Bernoulli polynomials. Green functions are simultaneously reproducing kernels for certain Hilbert spaces. Applying Schwarz inequality to the reproducing relation, we have found the best constants of Sobolev inequalities.

## § 1. Conclusion

We consider the following five cases:

P (Periodic), AP (Anti Periodic),

D (Dirichlet), N (Neumann), DN (Dirichlet-Neumann).

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We introduce the following five Sobolev spaces.

$$H = H(\mathrm{X}, M) = \left\{ u(x) \,\middle|\, u(x),\ u^{(M)}(x) \in L^2(0,1), \quad u(x) \text{ satisfies } A(\mathrm{X}) \right\}$$

$$\begin{aligned}
&A(\mathrm{P}) \quad : u^{(i)}(1) - u^{(i)}(0) = 0 \quad (0 \le i \le M-1), \qquad \int_0^1 u(x)dx = 0 \\
&A(\mathrm{AP}) : u^{(i)}(1) + u^{(i)}(0) = 0 \quad (0 \le i \le M-1) \\
&A(\mathrm{D}) \quad : u^{(2i)}(0) = u^{(2i)}(1) = 0 \quad (0 \le i \le [\,(M-1)/2\,]) \\
&A(\mathrm{N}) \quad : u^{(2i+1)}(0) = u^{(2i+1)}(1) = 0 \ (0 \le i \le [\,(M-2)/2\,]), \qquad \int_0^1 u(x)dx = 0 \\
&A(\mathrm{DN}) : u^{(2i)}(0) \ = \ 0 \quad (0 \le i \le [\,(M-1)/2\,]), \\
&\qquad\qquad u^{(2i+1)}(1) \ = \ 0 \quad (0 \le i \le [\,(M-2)/2\,])
\end{aligned}$$

It should be noted that if $M = 1$ the boundary conditions for $u(x)$ in $A(\mathrm{N})$ and for $u(x)$ on $x = 1$ in $A(\mathrm{DN})$ are not required. Furthermore, we introduce Sobolev inner product defined by

$$(u, v)_M \ = \int_0^1 u^{(M)}(x)\,\overline{v}^{(M)}(x)\,dx.$$

Sesquilinear form $(\cdot,\cdot)_M$ is proved to be an inner product of $H$ afterwards. That is, $H$ is Hilbert space with an inner product $(\cdot,\cdot)_M$.

Let $G(\mathrm{X}; x, y) = G(\mathrm{X}, M; x, y)$ $(0 < x, y < 1)$ be Green functions defined by

$$G(\mathrm{P}; x, y) \ = \ (-1)^{M+1} b_{2M}(\,|x-y|\,), \tag{1.1}$$

$$G(\mathrm{AP}; x, y) = (-1)^{M+1} 2^{2M-1} \left[\ b_{2M}\left(\frac{|x-y|}{2}\right) - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{2}\right)\ \right], \tag{1.2}$$

$$G(\mathrm{D}; x, y) = (-1)^{M+1} 2^{2M-1} \left[\ b_{2M}\left(\frac{|x-y|}{2}\right) - b_{2M}\left(\frac{x+y}{2}\right)\ \right], \tag{1.3}$$

$$G(\mathrm{N}; x, y) = (-1)^{M+1} 2^{2M-1} \left[\ b_{2M}\left(\frac{|x-y|}{2}\right) + b_{2M}\left(\frac{x+y}{2}\right)\ \right], \tag{1.4}$$

$$G(\mathrm{DN}; x, y) = (-1)^{M+1} 4^{2M-1} \Bigg[ \\
b_{2M}\left(\frac{|x-y|}{4}\right) - b_{2M}\left(\frac{x+y}{4}\right) + b_{2M}\left(\frac{1}{2} - \frac{x+y}{4}\right) - b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{4}\right)\ \Bigg]. \tag{1.5}$$

$b_{2M}(x)$ is Bernoulli polynomial defined in section 2.

Our conclusion is the following theorem:

**Theorem 1.1.** *For any function $u(x) \in H$, there exists a positive constant $C$*

*which is independent of $u(x)$ such that Sobolev inequality*

$$\left(\sup_{0\le y\le 1}|u(y)|\right)^2 \le C\int_0^1 \left|u^{(M)}(x)\right|^2 dx \tag{1.6}$$

*holds. Among such $C$ the best constant $C(\mathrm{X},M)$ is as follows:*

$$\begin{aligned} C(\mathrm{X},M) &= \sup_{0\le y\le 1} G(y,y) = G(y_0,y_0), \\ C(\mathrm{P},M) &= 2^{-(2M-1)}\pi^{-2M}\zeta(2M), \\ C(\mathrm{D},M) &= C(\mathrm{AP},M) = 2^{-(2M-1)}\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M), \\ C(\mathrm{N},M) &= 2\pi^{-2M}\zeta(2M), \\ C(\mathrm{DN},M) &= 2\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M), \end{aligned} \tag{1.7}$$

*where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ $(\mathrm{Re}\ z > 1)$ is well-known Riemann-zeta function. If we replace $C$ by $C(\mathrm{X},M)$ in (1.6), then the equality holds for $u(x) = c\,G(x,y_0)$ $(0 < x < 1)$ where $c$ is any complex number.*

The engineering meaning of Sobolev inequality is that the square of the maximal bending of a string $(M = 1)$[1] or a beam $(M = 2)$ is estimated from above by the constant multiple of the potential energy.

We here list explicit forms of $C(\mathrm{X},M)$ $(M = 1,2,3,4,5)$.

| $M$ | $C(\mathrm{P},M)$ | $C(\mathrm{D},M), C(\mathrm{AP},M)$ | $C(\mathrm{N},M)$ | $C(\mathrm{DN},M)$ |
|---|---|---|---|---|
| 1 | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | 1 |
| 2 | $\frac{1}{720}$ | $\frac{1}{48}$ | $\frac{1}{45}$ | $\frac{1}{3}$ |
| 3 | $\frac{1}{30240}$ | $\frac{1}{480}$ | $\frac{2}{945}$ | $\frac{2}{15}$ |
| 4 | $\frac{1}{1209600}$ | $\frac{17}{80640}$ | $\frac{1}{4725}$ | $\frac{17}{315}$ |
| 5 | $\frac{1}{47900160}$ | $\frac{31}{1451520}$ | $\frac{2}{93555}$ | $\frac{62}{2835}$ |

The present paper is composed of eight sections. In section 2, we explain about Bernoulli polynomial [2, 3], which plays an important role in this paper. In section 3, we present five boundary value problems for $(-1)^M(d/dx)^{2M}$. In section 4, we show that Green function $G(x,y)$ is expressed in terms of Bernoulli polynomial by "reflection method". In section 5, it is clarified that Green function $G(x,y)$ is a reproducing kernel for $H$ and $(\cdot,\cdot)_M$. Section 6 is devoted to the proof of Theorem 1.1. In section 7, we investigate the diagonal value of Green function. Finally, in section 8, we present a discrete version of Theorem 1.1 $(\mathrm{DN}, M=1)$.

## § 2. Bernoulli polynomial

As a preparation, we introduce Bernoulli polynomials $b_j(x)$ defined by the following recurrence relation:

$$\begin{cases} b_0(x) \;=\; 1 \\ b_j'(x) \;=\; b_{j-1}(x), \quad \displaystyle\int_0^1 b_j(x)\,dx \;=\; 0 \qquad (j=1,2,3,\cdots). \end{cases}$$

Here we list explicit forms of $b_j(x)$ $(j=0,1,\cdots,8)$.

$$\begin{aligned}
&b_0(x) = 1, \qquad b_1(x) = x-\frac{1}{2}, \qquad b_2(x) = \frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{12},\\
&b_3(x) = \frac{1}{6}x^3-\frac{1}{4}x^2+\frac{1}{12}x, \qquad b_4(x) = \frac{1}{24}x^4-\frac{1}{12}x^3+\frac{1}{24}x^2-\frac{1}{720},\\
&b_5(x) = \frac{1}{120}x^5-\frac{1}{48}x^4+\frac{1}{72}x^3-\frac{1}{720}x,\\
&b_6(x) = \frac{1}{720}x^6-\frac{1}{240}x^5+\frac{1}{288}x^4-\frac{1}{1440}x^2+\frac{1}{30240},\\
&b_7(x) = \frac{1}{5040}x^7-\frac{1}{1440}x^6+\frac{1}{1440}x^5-\frac{1}{4320}x^3+\frac{1}{30240}x,\\
&b_8(x) = \frac{1}{40320}x^8-\frac{1}{10080}x^7+\frac{1}{8640}x^6-\frac{1}{17280}x^4+\frac{1}{60480}x^2-\frac{1}{1209600}.
\end{aligned}$$

They are also defined by the generating function

$$\frac{e^{xt}}{t^{-1}(e^t-1)} = \sum_{j=0}^{\infty} b_j(x)\,t^j \qquad (\,|\,t\,|<2\pi\,).$$

Bernoulli polynomial $b_j(x)$ is $j$-th polynomial with respect to $x$. We list up the properties of Bernoulli polynomial $b_j(x)$ which are required in this paper.

$$b_j(1-x) \;=\; (-1)^j\, b_j(x) \qquad (j=0,1,2,\cdots). \tag{2.1}$$

$$b_j(1)-b_j(0) \;=\; \begin{cases} 1 & (j=1) \\ 0 & (j\neq 1). \end{cases} \tag{2.2}$$

$$b_{2j+1}(0) \;=\; \begin{cases} -1/2 & (j=0) \\ 0 & (j=1,2,3,\cdots). \end{cases} \tag{2.3}$$

$$b_{2j+1}(1/2) \;=\; 0 \qquad (j=0,1,2,\cdots). \tag{2.4}$$

$$(-1)^{j+1}b_{2j}(0) \;=\; B_j\big/(2j)! \qquad (j=0,1,2,\cdots). \tag{2.5}$$

In (2.5), $B_j$ is Bernoulli number defined by

$$\sum_{j=0}^{n-1} (-1)^j \binom{2n}{2j} B_j \;=\; -\,n \quad (n=1,2,3,\cdots), \qquad B_0 \;=\; -1.$$

Next we derive Fourier expansion formula of $b_j(\{x\})$, where

$$\{x\} = x - [x], \qquad [x] = \sup\{n \in \mathbf{Z} \,|\, n \le x\},$$

denotes a decimal part of a real number $x$. $\{x\}$ is a periodic function of $x$ with period 1. For $j=1,2,3,\cdots$, we have

$$b_j(\{x\}) \;=\; -\sum_{k\neq 0} \left(\sqrt{-1}\,2\pi k\right)^{-j} \exp\!\left(\sqrt{-1}\,2\pi k x\right),$$

that is to say

$$b_{2j}(\{x\}) \;=\; (-1)^{j+1}\, 2 \sum_{k=1}^{\infty} (2\pi k)^{-2j}\, \cos(2\pi k x), \tag{2.6}$$

$$b_{2j+1}(\{x\}) \;=\; (-1)^{j+1}\, 2 \sum_{k=1}^{\infty} (2\pi k)^{-(2j+1)}\, \sin(2\pi k x). \tag{2.7}$$

For $j=0,1,2,\cdots$, the relation

$$(-1)^{j+1}\, b_{2j}(0) \;=\; 2 \sum_{k=1}^{\infty} (2\pi k)^{-2j} \;=\; \frac{2}{(2\pi)^{2j}}\, \zeta(2j), \tag{2.8}$$

$$b_{2j}(1/2) \;=\; -\left(1 - 2^{-(2j-1)}\right) b_{2j}(0) \tag{2.9}$$

follows from the above Fourier expansion of $b_j(\{x\})$. The following lemma concerning Bernoulli polynomials play important roles hereafter.

**Lemma 2.1.** $u(x) = (-1)^{j+1} b_{2j}(x)$ $(j=1,2,3,\cdots)$ *satisfy the following properties:*

$$\max_{0\le x\le 1} u(x) \;=\; u(0) \;=\; u(1) \;>\; 0, \qquad \min_{0\le x\le 1} u(x) \;=\; u(1/2) \;<\; 0,$$

$$\max_{0\le x\le 1} |\, u(x)\,| \;=\; u(0) \;=\; u(1),$$

$$u'(x) \;<\; 0 \qquad (0<x<1/2), \qquad u'(x) \;>\; 0 \qquad (1/2<x<1).$$

## § 3. Boundary value problem

Let $f(x)$ be a bounded continuous function on an interval $0<x<1$ satisfying the following solvability condition $S(\mathrm{X})$:

$$S(\mathrm{P}),\ S(\mathrm{N}) \;:\; \int_0^1 f(y)\,dy \;=\; 0, \qquad S(\mathrm{AP}),\ S(\mathrm{D}),\ S(\mathrm{DN}) \;:\; \text{none}.$$

We consider the boundary value problem:

BVP (X, $M$)

$$\begin{cases} (-1)^M u^{(2M)} \;=\; f(x) \qquad (0<x<1) & (3.1)\\ B(\mathrm{X}) & (3.2)\\ O(\mathrm{X}) & (3.3)\end{cases}$$

where the boundary condition $B(\mathrm{X})$ and orthogonality condition $O(\mathrm{X})$ are given as follows:

$$\begin{array}{lll} B(\mathrm{P}) & : \; u^{(i)}(1)-u^{(i)}(0)=0 & \big(\,0\le i\le 2M-1\,\big)\\ B(\mathrm{AP}) & : \; u^{(i)}(1)+u^{(i)}(0)=0 & \big(\,0\le i\le 2M-1\,\big)\\ B(\mathrm{D}) & : \; u^{(2i)}(0)=u^{(2i)}(1)=0 & \big(\,0\le i\le M-1\,\big)\\ B(\mathrm{N}) & : \; u^{(2i+1)}(0)=u^{(2i+1)}(1)=0 & \big(\,0\le i\le M-1\,\big)\\ B(\mathrm{DN}) & : \; u^{(2i)}(0)=u^{(2i+1)}(1)=0 & \big(\,0\le i\le M-1\,\big)\end{array}$$

$$O(\mathrm{P}),\ O(\mathrm{N}) \;:\; \int_0^1 u(x)\,dx \;=\; 0, \qquad O(\mathrm{AP}),\ O(\mathrm{D}),\ O(\mathrm{DN}) \;:\; \text{none}.$$

Concerning the uniqueness and existence of the solution to BVP$(\mathrm{X}, M)$, we have the following theorem:

**Theorem 3.1.** *For any bounded continuous function $f(x)$ on an interval $0<x<1$ which satisfies the solvability condition $S(\mathrm{X})$, BVP$(\mathrm{X}, M)$ has a unique classical solution $u(x)$ expressed as*

$$u(x) \;=\; \int_0^1 G(\mathrm{X}, M; x, y)\, f(y)\, dy \qquad (0<x<1). \tag{3.4}$$

*Green function $G(\mathrm{X}; x, y) = G(\mathrm{X}, M; x, y)$ $(0<x,y<1)$ is given by* (1.1) $\sim$ (1.5).

**Proof of Theorem 3.1** The uniqueness of the solution to BVP$(\mathrm{X}, M)$ is shown in section 4. Differentiating $u(x)$ $(0<x<1)$ in (3.4) $i$ $(0\le i\le 2M)$ times and using Theorem 3.2 (2), (3), (4) and (6), we can show that the existence of the solution to BVP$(\mathrm{X}, M)$. We have Theorem 3.1. ■

**Theorem 3.2.** *Green function $G(\mathrm{X}; x, y) = G(\mathrm{X}, M; x, y)$ satisfies the following properties:*

(1) $G(\mathrm{X}; x, y) = G(\mathrm{X}; y, x) \qquad (\mathrm{X}=\mathrm{P},\mathrm{AP},\mathrm{D},\mathrm{N},\mathrm{DN}, \quad 0<x,y<1).$

(2) $(-1)^M \partial_x^{2M} G(\mathrm{X}; x, y) = \begin{cases} 0 & (\mathrm{X}=\mathrm{AP},\mathrm{D},\mathrm{DN})\\ -1 & (\mathrm{X}=\mathrm{P},\mathrm{N}) \qquad (0<x,y<1, \quad x\ne y).\end{cases}$

(3) *For* $0 \le i \le 2M-1$, *we have*

$$\partial_x^i G(\mathrm{P};x,y)\Big|_{x=1} - \partial_x^i G(\mathrm{P};x,y)\Big|_{x=0} = 0,$$

$$\partial_x^i G(\mathrm{AP};x,y)\Big|_{x=1} + \partial_x^i G(\mathrm{AP};x,y)\Big|_{x=0} = 0 \qquad (0 < y < 1).$$

*For* $0 \le i \le M-1$, *we have*

$$\partial_x^{2i} G(\mathrm{D};x,y)\Big|_{x=0,1} = 0, \qquad \partial_x^{2i+1} G(\mathrm{N};x,y)\Big|_{x=0,1} = 0,$$

$$\partial_x^{2i} G(\mathrm{DN};x,y)\Big|_{x=0} = \partial_x^{2i+1} G(\mathrm{DN};x,y)\Big|_{x=1} = 0 \qquad (0 < y < 1).$$

(4) $\partial_x^i G(\mathrm{X};x,y)\Big|_{y=x-0} - \partial_x^i G(\mathrm{X};x,y)\Big|_{y=x+0} =$

$$\begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases} \quad (\mathrm{X} = \mathrm{P}, \mathrm{AP}, \mathrm{D}, \mathrm{N}, \mathrm{DN}, \quad 0 < x < 1).$$

(5) $\partial_x^i G(\mathrm{X};x,y)\Big|_{x=y+0} - \partial_x^i G(\mathrm{X};x,y)\Big|_{x=y-0} =$

$$\begin{cases} 0 & (0 \le i \le 2M-2) \\ (-1)^M & (i = 2M-1) \end{cases} \quad (\mathrm{X} = \mathrm{P}, \mathrm{AP}, \mathrm{D}, \mathrm{N}, \mathrm{DN}, \quad 0 < y < 1).$$

(6) $\displaystyle\int_0^1 G(\mathrm{X};x,y)dx \ = \ 0 \quad (\mathrm{X} = \mathrm{P}, \mathrm{N}, \quad 0 < y < 1).$

(7) $G(\mathrm{X};x,y) \ > \ 0 \quad (\mathrm{X} = \mathrm{D}, \mathrm{DN}, \quad 0 < x, y < 1).$

**Proof of Theorem 3.2** Employing properties of Bernoulli polynomial, one can easily show that $G(x,y)$ given by (1.1) $\sim$ (1.5) satisfy properties (1) $\sim$ (6). We only prove (7) by induction with respect to $M$. If $M = 1$, we have

$$G(\mathrm{D},1;x,y) \ = \ (x \wedge y)\,(1 - x \vee y) \ > \ 0 \qquad (0 < x, y < 1).$$

For every fixed $y$ $(0 < y < 1)$, $u(x) = G(\mathrm{D},M;x,y)$ $(M = 2,3,4,\cdots)$ satisfies

$$\begin{cases} -\,u'' \ = \ G(\mathrm{D},M-1;x,y) & (0 < x < 1) \\ u(0) \ = \ u(1) \ = \ 0. \end{cases}$$

Thus, we can show

$$u(x) = G(\mathrm{D},M;x,y) = \int_0^1 G(\mathrm{D},1;x,z)\,G(\mathrm{D},M-1;z,y)\,dz \ > \ 0$$

$$(M = 2,3,4,\cdots, \quad 0 < x < 1).$$

Moreover, using $G(\mathrm{D},M;x,y) > 0$ $(M = 1,2,3,\cdots,\ 0 < x, y < 1)$, (1.5) is rewritten equivalently as

$$G(\mathrm{DN},M;x,y) \ = \ 2^{2M-1}\left[\ G\Big(\mathrm{D},M;\frac{x}{2},\frac{y}{2}\Big) \ + \ G\Big(\mathrm{D},M;\frac{x}{2},1-\frac{y}{2}\Big)\ \right] \ > \ 0$$

$$(M = 1,2,3,\cdots, \quad 0 < x, y < 1).$$

This shows (7). ■

Concerning the uniqueness of Green function, we have the following theorem:

**Theorem 3.3.** *The smooth function $G(x,y) = G(\mathrm{X}, M; x, y)$ on an open set $0 < x, y < 1,\ x \neq y$ satisfying properties in Theorem* 3.2 (2), (3), (4) *and* (6) *is unique.*

**Proof of Theorem 3.3** Suppose that we have another function $\widetilde{G}(x,y)$ satisfying the same properties in Theorem 3.2 (2), (3), (4) and (6). For any function $f(x)$ satisfying $S(\mathrm{X})$,

$$u(x) \;=\; \int_0^1 \widetilde{G}(x,y)\, f(y)\, dy \qquad (0 < x < 1)$$

satisfies BVP(X,$M$). From Theorem 3.1, we have

$$\int_0^1 \widetilde{G}(x,y)\, f(y)\, dy \;=\; \int_0^1 G(x,y)\, f(y)\, dy \qquad (0 < x < 1).$$

This shows $\widetilde{G}(x,y) = G(x,y)$ $(0 < x, y < 1)$. ■

## § 4. The method of reflection

In this section, we derive the expression (1.2), (1.3), (1.4) and (1.5) by the so-called method of reflection.

At first, we derive (1.2), (1.3) and (1.4) starting from BVP(P,$M$) on an interval $0 < x < 2$. In the previous work [2], we proved the following theorem:

**Theorem 4.1** ([2]). *For any bounded continuous function $f(x)$ on an interval $0 < x < 2$ which satisfies the solvability condition*

$$\int_0^2 f(y)\, dy \;=\; 0, \tag{4.1}$$

*the periodic boundary value problem*

BVP (P,$M$)

$$\begin{cases} (-1)^M u^{(2M)} \;=\; f(x) & (0 < x < 2) \quad (4.2) \\ u^{(i)}(2) \;-\; u^{(i)}(0) \;=\; 0 & (0 \le i \le 2M-1) \quad (4.3) \\ \displaystyle\int_0^2 u(x)\, dx \;=\; 0 & \quad (4.4) \end{cases}$$

*has a unique classical solution $u(x)$ which is given by*

$$u(x) \;=\; \int_0^2 (-1)^{M+1} 2^{2M-1} b_{2M}\!\left(\frac{|x-y|}{2}\right) f(y)\, dy \qquad (0 < x < 2). \tag{4.5}$$

For any bounded continuous function $f(x)$ on an interval $0 < x < 1$, we extend the domain of definition to $0 < x < 2$ by the symmetry

$$f(x) = \begin{cases} -f(x-1) & \text{(AP)} \\ -f(2-x) & \text{(D)} \\ f(2-x) & \text{(N)} \end{cases} \qquad (1 < x < 2). \tag{4.6}$$

In the case (N), we assume that

$$\int_0^1 f(y)\,dy = 0. \tag{4.7}$$

This extended function $f(x)$ (AP,D,N) satisfies (4.1). For this extended $f(x)$, the solution $u(x)$ of BVP(P,$M$) is given by (4.5). For $0 < x < 1$, we have

$$(-1)^{M+1} 2^{-(2M-1)} u(x) = I_1 + I_2$$

$$I_1 = \int_0^1 b_{2M}\left(\frac{|x-y|}{2}\right) f(y)\,dy, \qquad I_2 = \int_1^2 b_{2M}\left(\frac{y-x}{2}\right) f(y)\,dy.$$

Using symmetry of $f(x)$ (4.6), we have

$$\text{(AP)} \quad I_2 = -\int_1^2 b_{2M}\left(\frac{y-x}{2}\right) f(y-1)\,dy = -\int_0^1 b_{2M}\left(\frac{1}{2} - \frac{x-y}{2}\right) f(y)\,dy =$$
$$-\int_0^x b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{2}\right) f(y)\,dy - \int_x^1 b_{2M}\left(\frac{1}{2} + \frac{|x-y|}{2}\right) f(y)\,dy =$$
$$-\int_0^1 b_{2M}\left(\frac{1}{2} - \frac{|x-y|}{2}\right) f(y)\,dy$$

$$\text{(D)} \quad I_2 = -\int_1^2 b_{2M}\left(\frac{y-x}{2}\right) f(2-y)\,dy = -\int_0^1 b_{2M}\left(1 - \frac{x+y}{2}\right) f(y)\,dy =$$
$$-\int_0^1 b_{2M}\left(\frac{x+y}{2}\right) f(y)\,dy$$

$$\text{(N)} \quad I_2 = \int_1^2 b_{2M}\left(\frac{y-x}{2}\right) f(2-y)\,dy = \int_0^1 b_{2M}\left(1 - \frac{x+y}{2}\right) f(y)\,dy =$$
$$\int_0^1 b_{2M}\left(\frac{x+y}{2}\right) f(y)\,dy$$

where we use (2.1). Thus, we obtain

$$u(x) = \int_0^1 G(x,y)\, f(y)\,dy \qquad (0 < x < 1), \tag{4.8}$$

where $G(x,y)$ is given by (1.2), (1.3) and (1.4).

Secondly, we derive $G(\text{DN}; x, y)$ (1.5) starting from BVP(D,$M$) on an interval $0 < x < 2$. In [4], we have proved the following theorem:

**Theorem 4.2** ([4]). *For any bounded continuous function $f(x)$ on an interval $0 < x < 2$, Dirichlet boundary value problem*

BVP (D,$M$)

$$\begin{cases} (-1)^M u^{(2M)} \;=\; f(x) & (0 < x < 2) \\ u^{(2i)}(0) \;=\; u^{(2i)}(2) \;=\; 0 & (0 \le i \le M-1) \end{cases} \tag{4.9, 4.10}$$

*has a unique classical solution $u(x)$ given by*

$$u(x) = \int_0^2 (-1)^{M+1} 4^{2M-1} \left[ \, b_{2M}\!\left(\frac{|x-y|}{4}\right) - b_{2M}\!\left(\frac{x+y}{4}\right) \right] f(y)\, dy$$
$$(0 < x < 2). \tag{4.11}$$

For any bounded continuous function $f(x)$ on an interval $0 < x < 1$, we extend the domain of definition to $0 < x < 2$ by the symmetry

$$f(x) \;=\; f(2-x) \qquad (1 < x < 2). \tag{4.12}$$

For this extended $f(x)$, the solution $u(x)$ of BVP(D,$M$) is given by (4.11). For $0 < x < 1$, we have

$$(-1)^{M+1} 4^{-(2M-1)} u(x) \;=\; I_1 \;+\; I_2$$
$$I_1 \;=\; \int_0^1 \left[ \, b_{2M}\!\left(\frac{|x-y|}{4}\right) - b_{2M}\!\left(\frac{x+y}{4}\right) \right] f(y)\, dy$$
$$I_2 \;=\; \int_1^2 \left[ \, b_{2M}\!\left(\frac{y-x}{4}\right) - b_{2M}\!\left(\frac{x+y}{4}\right) \right] f(y)\, dy.$$

Using (4.12), we have

$$I_2 \;=\; \int_1^2 \left[ \, b_{2M}\!\left(\frac{y-x}{4}\right) - b_{2M}\!\left(\frac{x+y}{4}\right) \right] f(2-y)\, dy \;=$$
$$\int_0^1 \left[ \, b_{2M}\!\left(\frac{2-x-y}{4}\right) - b_{2M}\!\left(\frac{2+x-y}{4}\right) \right] f(y)\, dy \;=$$
$$\int_0^x \left[ \, b_{2M}\!\left(\frac{1}{2} - \frac{x+y}{4}\right) - \, b_{2M}\!\left(\frac{1}{2} + \frac{|x-y|}{4}\right) \right] f(y)\, dy +$$
$$\int_x^1 \left[ \, b_{2M}\!\left(\frac{1}{2} - \frac{x+y}{4}\right) - \, b_{2M}\!\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \right] f(y)\, dy \;=$$
$$\int_0^1 \left[ \, b_{2M}\!\left(\frac{1}{2} - \frac{x+y}{4}\right) - \, b_{2M}\!\left(\frac{1}{2} - \frac{|x-y|}{4}\right) \right] f(y)\, dy,$$

where we have used (2.1). Finally we have

$$u(x) \;=\; \int_0^1 G(x,y)\, f(y)\, dy \qquad (0 < x < 1), \tag{4.13}$$

where $G(x,y)$ is given by (1.5).

## § 5. Reproducing kernel

In this section, it is shown that Green function $G(x,y) = G(\mathrm{X}, M; x, y)$ is a reproducing kernel for a set of Hilbert space $H$ and its inner product $(\cdot,\cdot)_M$ introduced in section 1. The next theorem claims that Green function $G(x,y)$ is a reproducing kernel for $H$ and $(\cdot,\cdot)_M$.

**Theorem 5.1.** (1) *For any $u(x) \in H$, we have the following reproducing relation:*

$$u(y) \;=\; (\,u(x), G(x,y)\,)_M \;=\; \int_0^1 u^{(M)}(x)\,\partial_x^M G(x,y)\,dx \quad (0 \le y \le 1). \tag{5.1}$$

(2) $$G(y,y) \;=\; \int_0^1 \left|\partial_x^M G(x,y)\right|^2 dx \qquad (0 \le y \le 1). \tag{5.2}$$

**Proof of Theorem 5.1** For functions $u = u(x)$ and $v = v(x) = G(x,y)$ with $y$ arbitrarily fixed in $0 \le y \le 1$, we have

$$u^{(M)}\,\overline{v}^{(M)} \;-\; u\,(-1)^M \overline{v}^{(2M)} \;=\; \left(\sum_{j=0}^{M-1}(-1)^{M-1-j}\,u^{(j)}\,\overline{v}^{(2M-1-j)}\right)'.$$

Integrating this identity with respect to $x$ on intervals $0 < x < y$ and $y < x < 1$ and using Theorem 3.2 (2), (3), (5) and (6), we have (5.1). (5.2) is obtained by putting $u(x) = G(x,y)$ in (5.1). We have proved Theorem 5.1. ■

## § 6. Diagonal value of Green function

We investigate the diagonal value of Green function $G(\mathrm{X}; y, y) = G(\mathrm{X}, M; y, y)$ $(0 < y < 1)$, which is given by

$$G(\mathrm{P}; y, y) \;=\; (-1)^{M+1} b_{2M}(0),$$

$$G(\mathrm{AP}; y, y) = (-1)^{M+1} 2^{2M-1}\Big[\; b_{2M}(0) - b_{2M}(1/2)\;\Big],$$

$$G(\mathrm{D}; y, y) = (-1)^{M+1} 2^{2M-1}\Big[\; b_{2M}(0) - b_{2M}(y)\;\Big],$$

$$G(\mathrm{N}; y, y) = (-1)^{M+1} 2^{2M-1}\Big[\; b_{2M}(0) + b_{2M}(y)\;\Big],$$

$$G(\mathrm{DN}; y, y) = (-1)^{M+1} 4^{2M-1}\left[\; b_{2M}(0) - b_{2M}\Big(\frac{y}{2}\Big) + b_{2M}\Big(\frac{1}{2} - \frac{y}{2}\Big) - b_{2M}\Big(\frac{1}{2}\Big)\;\right].$$

From Lemma 2.1, it is shown that

$$\max_{0\le y\le 1} G(\mathrm{X};y,y) \;=\; G(\mathrm{X};y_0,y_0)$$

$$\begin{cases} y_0 \text{ is an arbitrarily fixed number satisfying } 0\le y_0\le 1 & (\mathrm{X}=\mathrm{P},\mathrm{AP}) \\ y_0 = 1/2 & (\mathrm{X}=\mathrm{D}) \\ y_0 = 0,1 & (\mathrm{X}=\mathrm{N}) \\ y_0 = 1 & (\mathrm{X}=\mathrm{DN}). \end{cases}$$

Expilicit forms of $\max_{0\le y\le 1} G(\mathrm{X};y,y)$ are given by

$$\max_{0\le y\le 1} G(\mathrm{P};y,y) \;=\; G(\mathrm{P};y_0,y_0) \;=\; (-1)^{M+1} b_{2M}(0) \;=\; 2^{-(2M-1)}\pi^{-2M}\zeta(2M),$$

$$\max_{0\le y\le 1} G(\mathrm{AP};y,y) \;=\; G(\mathrm{AP};y_0,y_0) \;=\; (-1)^{M+1}2^{2M-1}\Big[\, b_{2M}(0) - b_{2M}(1/2)\,\Big] \;=$$
$$(-1)^{M+1}\left(2^{2M}-1\right) b_{2M}(0) \;=\; 2^{-(2M-1)}\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M),$$

$$\max_{0\le y\le 1} G(\mathrm{D};y,y) \;=\; G(\mathrm{D};1/2,1/2) \;=\; (-1)^{M+1}2^{2M-1}\Big[\, b_{2M}(0) - b_{2M}(1/2)\,\Big] \;=$$
$$(-1)^{M+1}\left(2^{2M}-1\right) b_{2M}(0) \;=\; 2^{-(2M-1)}\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M),$$

$$\max_{0\le y\le 1} G(\mathrm{N};y,y) \;=\; G(\mathrm{N};0,0) \;=\; G(\mathrm{N};1,1) \;=$$
$$(-1)^{M+1}2^{2M} b_{2M}(0) \;=\; 2\,\pi^{-2M}\zeta(2M),$$

$$\max_{0\le y\le 1} G(\mathrm{DN};y,y) \;=\; G(\mathrm{DN};1,1) \;=\; (-1)^{M+1}2^{4M-1}\Big[\, b_{2M}(0) - b_{2M}(1/2)\,\Big] \;=$$
$$(-1)^{M+1}2^{2M}\left(2^{2M}-1\right) b_{2M}(0) \;=\; 2\left(2^{2M}-1\right)\pi^{-2M}\zeta(2M),$$

where we have used (2.8) and (2.9). Hence we have

$$\max_{0\le y\le 1} G(\mathrm{X};y,y) \;=\; G(\mathrm{X};y_0,y_0) \;=\; C(\mathrm{X},M) \tag{6.1}$$

where $C(\mathrm{X},M)$ is given by Theorem 1.1.

## § 7. Sobolev inequality

In this section, we give a proof of Theorem 1.1.

**Proof of Theorem 1.1** Applying Schwarz inequality to (5.1) and using (5.2), we have

$$|\,u(y)\,|^2 \;\le\; \int_0^1 |\,\partial_x^M G(x,y)\,|^2\,dx \int_0^1 \left|\,u^{(M)}(x)\,\right|^2 dx \;=\; G(y,y)\int_0^1 \left|\,u^{(M)}(x)\,\right|^2 dx.$$

Noting that $\max_{0\le y\le 1} G(y,y) = G(y_0,y_0)$, we have following Sobolev inequality:

$$\Big( \sup_{0\le y\le 1} |\, u(y)\,| \Big)^2 \le G(y_0,y_0) \int_0^1 \Big| u^{(M)}(x) \Big|^2 dx. \tag{7.1}$$

This inequality shows that $(\cdot,\cdot)_M$ is positive definite. It should be noted that it requires Schwarz inequality but does not require "positive definiteness" of the inner product for the purpose of proving (7.1).

In the second place, we apply this inequality to $u(x) = G(x,y_0) \in H$ and have

$$\Big( \sup_{0\le y\le 1} |\, G(y,y_0)\,| \Big)^2 \le G(y_0,y_0) \int_0^1 |\, \partial_x^M G(x,y_0)\,|^2 dx = \Big( G(y_0,y_0) \Big)^2 .$$

Together with a trivial inequality

$$\Big( G(y_0,y_0) \Big)^2 \le \Big( \sup_{0\le y\le 1} |\, G(y,y_0)\,| \Big)^2 ,$$

we finally obtain

$$\Big( \sup_{0\le y\le 1} |\, G(y,y_0)\,| \Big)^2 = G(y_0,y_0) \int_0^1 |\, \partial_x^M G(x,y_0)\,|^2 dx, \tag{7.2}$$

which completes the proof of Theorem 1.1. ■

## § 8. Discrete Sobolev inequality (DN, $M=1$)

We finally consider a discrete analogue of Sobolev inequality. In the previous paper [5], we consider a discrete version of Theorem 1.1 (P, $M = 1,2,\cdots$). Here, we focus our attention on Theorem 1.1 (DN, $M=1$).

We assume that $N = 2,3,4,\cdots$. We consider the following set of simultaneous equations:

$$\begin{cases} -u(i+1) + 2\,u(i) - u(i-1) = f(i) & (1\le i\le N) \\ u(0) = 0, \qquad u(N+1) - u(N) = 0, \end{cases}$$

which are regarded as a discrete version of BVP(DN, 1). This is rewritten equivalently as $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$ where

$$\boldsymbol{A} = \Big( a_{ij} \Big) = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{pmatrix},$$

$$\boldsymbol{u} = {}^t(\, u(1), \cdots, u(N)\, ), \qquad \boldsymbol{f} = {}^t(\, f(1), \cdots, f(N)\, ) \in \boldsymbol{C}^N .$$

Let $\boldsymbol{\delta}_j$ be a vector defined by

$$\boldsymbol{\delta}_j \;=\; {}^t(0,\cdots,0,\overset{j}{\overbrace{1}},0,\cdots,0) \qquad (1 \le j \le N).$$

We also introduce an ordinary unitary inner product

$$(\boldsymbol{u},\boldsymbol{v}) \;=\; \boldsymbol{v}^*\,\boldsymbol{u} \;=\; {}^t\overline{\boldsymbol{v}}\,\boldsymbol{u} \;=\; \sum_{j=1}^{N} \overline{v}(j)\,u(j),$$

Sobolev inner product

$$(\boldsymbol{u},\boldsymbol{v})_A \;=\; (\boldsymbol{A}\boldsymbol{u},\boldsymbol{v}) \;=\; \boldsymbol{v}^*\,\boldsymbol{A}\,\boldsymbol{u} \;=\; \sum_{i,j=1}^{N} \overline{v}(i)\,a_{ij}\,u(j)$$

and Sobolev energy

$$\|\,\boldsymbol{u}\,\|_A^2 \;=\; (\boldsymbol{u},\boldsymbol{u})_A \;=\; \sum_{i,j=1}^{N} \overline{u}(i)\,a_{ij}\,u(j) \;=\; |\,u(1)\,|^2 + \sum_{j=1}^{N-1} |\,u(j+1)-u(j)\,|^2.$$

We remark that $(\cdot,\cdot)_A$ is positive definite.

The conclusion of this section is as follows:

**Theorem 8.1.** *For any $\boldsymbol{u} \in \boldsymbol{C}^N$, there exists a positive constant $C$ which is independent of $\boldsymbol{u}$ such that the discrete Sobolev inequality*

$$\left(\max_{1\le j\le N} |\,u(j)\,|\right)^2 \;\le\; C\,\|\,\boldsymbol{u}\,\|_A^2 \tag{8.1}$$

*holds. Among such $C$ the best constant is $C_0 = N$. If we replace $C$ by $C_0$, then the equality holds for $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_N$ where $\boldsymbol{G} = \boldsymbol{A}^{-1}$ is given by the following expression:*

$$\boldsymbol{G} \;=\; \Big(\; g_{ij} \;\Big) \;=\; \Big(\; \min\{i,j\} \;\Big) \;=\; \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & N \end{pmatrix}. \tag{8.2}$$

(8.2) is easily proved by using Gauss's sweeping-out method.

**Theorem 8.2.** (1) *For any $\boldsymbol{u} \in \boldsymbol{C}^N$, we have the following reproducing relation:*

$$u(j) \;=\; (\boldsymbol{u},\boldsymbol{G}\boldsymbol{\delta}_j)_A \qquad (1 \le j \le N). \tag{8.3}$$

(2) $$g_{jj} \;=\; (\boldsymbol{G}\boldsymbol{\delta}_j,\boldsymbol{G}\boldsymbol{\delta}_j)_A \qquad (1 \le j \le N). \tag{8.4}$$

**Proof of Theorem 8.2** Noting that $\boldsymbol{G}^* = \boldsymbol{G}$, we have

$$(\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j)_A = (\boldsymbol{A}\boldsymbol{u}, \boldsymbol{G}\boldsymbol{\delta}_j) = \boldsymbol{\delta}_j^* \boldsymbol{G}^* \boldsymbol{A}\boldsymbol{u} = \boldsymbol{\delta}_j^* \boldsymbol{u} = u(j).$$

Applying $\boldsymbol{u} = \boldsymbol{G}\boldsymbol{\delta}_j \in \mathbf{C}^N$ to (8.3), we have

$$(\boldsymbol{G}\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j)_A = (\boldsymbol{A}\boldsymbol{G}\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j) = (\boldsymbol{\delta}_j, \boldsymbol{G}\boldsymbol{\delta}_j) = \boldsymbol{\delta}_j^* \boldsymbol{G}^* \boldsymbol{\delta}_j = \boldsymbol{\delta}_j^* \boldsymbol{G}\boldsymbol{\delta}_j = g_{jj}.$$

This shows Theorem 8.2. ■

**Proof of Theorem 8.1** Applying Schwarz inequality to (8.3) and using (8.4), we have $|\, u(j)\,|^2 \le \|\, \boldsymbol{u}\,\|_A^2 \|\, \boldsymbol{G}\boldsymbol{\delta}_j\,\|_A^2 = g_{jj} \|\, \boldsymbol{u}\,\|_A^2$ . Taking the maximum with respect to $1 \le j \le N$, we have the following discrete Sobolev inequality:

$$\left( \max_{1\le j\le N} |\, u(j)\,| \right)^2 \le C_0 \|\, \boldsymbol{u}\,\|_A^2, \qquad C_0 = \max_{1\le j\le N} g_{jj} = g_{NN} = N. \tag{8.5}$$

In the second place, if we apply this inequality to $\boldsymbol{u} = \boldsymbol{G}\,\boldsymbol{\delta}_N \in \mathbf{C}^N$, then we have

$$\left( \max_{1\le j\le N} |\, g_{jN}\,| \right)^2 \le C_0 \|\, \boldsymbol{G}\boldsymbol{\delta}_N\,\|_A^2 = C_0^2.$$

Combining this and trivial inequality $C_0^2 = g_{NN}^2 \le \left( \max_{1\le j\le N} |\, g_{jN}\,| \right)^2$ , we obtain

$$\left( \max_{1\le j\le N} |\, g_{jN}\,| \right)^2 = C_0 \|\, \boldsymbol{G}\boldsymbol{\delta}_N\,\|_A^2, \tag{8.6}$$

which completes the proof of Theorem 8.1. ■

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