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A nondegeneracy result for least energy solutions to a biharmonic problem with nearly critical exponent (Mathematical analysis on the self-organization and self-similarity)

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A nondegeneracy result for least energy solutions to a biharmonic problem with nearly critical exponent

Dedicated to professor Toshitaka Nagai on the occasion of his sixties birthday

By

Tomohiko Sato * and Futoshi Takahashi **

Abstract

Consider the problem $\Delta^2 u = c_0 K(x) u^{p_{\epsilon}}$, $u > 0$ in $\Omega$, $u = \Delta u = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N(N \geq 5)$, $c_0 = (N-4)(N-2)N(N+2)$, $p = (N+4)/(N-4)$, $p_\epsilon = p - \epsilon$ and $K$ is a smooth positive function on $\overline{\Omega}$.

Under some assumptions on the coefficient function $K$, which include the nondegeneracy of its unique maximum point as a critical point of $\text{Hess}K$, we prove that least energy solutions of the problem are nondegenerate for $\epsilon > 0$ small.

§1. Introduction

Consider the problem

$$
\begin{align*}
\Delta^2 u &= c_0 K(x) u^{p_\epsilon} \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &= \Delta u = 0 \quad \text{on } \partial\Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N(N \geq 5)$ is a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $p_\epsilon = p - \epsilon$, $p = (N+4)/(N-4)$ is the critical Sobolev exponent with respect to the Sobolev embedding $H^2 \cap H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\epsilon > 0$ is a small parameter. Here, $K$ is a positive function in $C^2(\overline{\Omega})$. 
We put an assumption on the coefficient function $K$:

(K): $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and $x_0$ is a nondegenerate critical point of $K$.

In the following, as solutions of (1.1) we consider only least energy solutions $u_\varepsilon$ such that

$$\frac{\int_{\Omega} |\Delta u_\varepsilon|^2 dx}{\left(\int_{\Omega} K(x)|u_\varepsilon|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H^{2}\cap H^1_0(\Omega)} \frac{\int_{\Omega} |\Delta u|^2 dx}{\left(\int_{\Omega} K(x)|u|^{p_\varepsilon+1} dx\right)^{\frac{2}{p_\varepsilon+1}}}.$$  

We easily check that least energy solutions blow up in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, and that the maximum point $x_\varepsilon$ of $u_\varepsilon$ converges to a maximum point of $K$ in $\overline{\Omega}$. Therefore we have $x_\varepsilon \rightarrow x_0$ as $\varepsilon \rightarrow 0$, here by the assumption (K), $x_0$ is the unique interior maximum point of $K$.

In this note, we prove the nondegeneracy of least energy solutions to (1.1) when $\varepsilon > 0$ is sufficiently small, under the assumption (K). Here as usual, the nondegeneracy of $u_\varepsilon$ for small $\varepsilon$ means that the problem

$$\begin{cases}
\Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon & \text{in } \Omega, \\
v_\varepsilon = \Delta v_\varepsilon = 0 & \text{on } \partial \Omega
\end{cases}$$

admits no solution except for the trivial one for $\varepsilon > 0$ small enough.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^N (N \geq 5)$ be a smooth bounded domain. Under the assumption (K), least energy solution $u_\varepsilon$ to (1.1) is nondegenerate for $\varepsilon > 0$ small.

The precise asymptotic behavior of least energy solutions as $\varepsilon \rightarrow 0$ when $K \not\equiv 1$ was obtained in [6] under the assumption (K). Using this result, we prove Theorem 1.1 along the line of [7] and [8], the original idea of which comes from [4].

§ 2. Preliminaries

In this section, we recall some facts which are needed in the sequel. Let $G = G(x, z)$ denote the Green function of $\Delta^2$ under the Navier boundary condition:

$$\begin{cases}
\Delta^2 G(\cdot, z) = \delta_z & \text{in } \Omega, \\
G(\cdot, z) = \Delta G(\cdot, z) = 0 & \text{on } \partial \Omega.
\end{cases}$$

We decompose $G$ as $G(x, z) = \Gamma(x, z) - H(x, z)$, where $\Gamma(x, z)$ is the fundamental solution of $\Delta^2$:

$$\Gamma(x, z) = \begin{cases}
\frac{1}{(N-4)(N-2)\sigma_N} |x - z|^{4-N}, & N \geq 5, \\
\frac{1}{\sigma_4} \log |x - z|^{-1}, & N = 4,
\end{cases}$$
where $\sigma_N$ is the volume of the $(N - 1)$ dimensional unit sphere in $\mathbb{R}^N$ and $H(x, z)$ is the regular part of the Green function. Finally, let $R(z) = H(z, z)$ denote the Robin function of the Green function of $\Delta^2$ with the Navier boundary condition. By the maximum principle, we have $R > 0$ on $\Omega$ and $R(z) \to +\infty$ as $z$ tends to the boundary of $\Omega$. In the following, we set $\overline{G} = -\triangle G$. Then $\overline{G}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition, and satisfy

$$
\begin{cases}
-\Delta G = \overline{G}, & \text{in } \Omega, \\
G > 0, & \overline{G} > 0 \\
G = \overline{G} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

**Lemma 2.1.** For any $z \in \Omega$, there holds

\begin{align}
(2.1) & \int_{\partial \Omega} ((x - z) \cdot \nu_x) \left( \frac{\partial G}{\partial \nu_x} \right) \left( \frac{\partial \overline{G}}{\partial \nu_x} \right) (x, z) ds_x = (N - 4)R(z), \\
(2.2) & \int_{\partial \Omega} \frac{\partial G}{\partial \nu_x}(x, z) \frac{\partial \overline{G}}{\partial \nu_x}(x, z) \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z), & (i = 1, \cdots, N), \\
& \int_{\partial \Omega} \left( \frac{\partial \overline{G}}{\partial x_i} \right) \frac{\partial}{\partial z_j} \left( \frac{\partial G}{\partial \nu_x} \right) (x, z) ds_x + \int_{\partial \Omega} \left( \frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial z_j} \left( \frac{\partial \overline{G}}{\partial \nu_x} \right) (x, z) ds_x \\
(2.3) & = \frac{\partial^2 R}{\partial z_i \partial z_j}(z), & (i, j = 1, \cdots, N).
\end{align}

Here $\nu_x$ is the outer unit normal at $x \in \partial \Omega$.

**Proof.** See [3]:Lemma 3.1 and Lemma 3.3. Note that our sign convention is different from that of [3]. By differentiating (2.2) with respect to $z_j$, noting that $(\frac{\partial G}{\partial \nu_x}(x, z)) \nu_i(x) = \frac{\partial G}{\partial x_i}(x, z)$, $(\frac{\partial \overline{G}}{\partial \nu_x}(x, z)) \nu_i(x) = \frac{\partial \overline{G}}{\partial x_i}(x, z)$ on $\partial \Omega$, we see that (2.3) holds. \qed

**Lemma 2.2.** Let $u_\varepsilon$ be a solution to (1.1) and $v_\varepsilon$ be a solution to (1.2). Denote $\overline{u}_\varepsilon = -\Delta u_\varepsilon$ and $\overline{v}_\varepsilon = -\Delta v_\varepsilon$. Then the following identities hold true:

\begin{align}
(2.4) & \int_{\partial \Omega} ((x - z) \cdot \nu_x) \left( \frac{\partial u_\varepsilon}{\partial \nu_x} \right) \left( \frac{\partial \overline{v}_\varepsilon}{\partial \nu_x} \right) + \left( \frac{\partial \overline{u}_\varepsilon}{\partial \nu_x} \right) \left( \frac{\partial v_\varepsilon}{\partial \nu_x} \right) \right) ds_x \\
& = c_0 \int_{\Omega} u_\varepsilon v_\varepsilon (x - z) \cdot \nabla K(x) dx
\end{align}

for any $z \in \mathbb{R}^N$ and

\begin{align}
(2.5) & \int_{\partial \Omega} \left( \frac{\partial \overline{u}_\varepsilon}{\partial x_i} \right) \left( \frac{\partial v_\varepsilon}{\partial \nu_x} \right) + \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \overline{v}_\varepsilon}{\partial \nu_x} \right) \right) ds_x = c_0 \int_{\Omega} \left( \frac{\partial K}{\partial x_i} \right) u_\varepsilon v_\varepsilon dx
\end{align}

for $i = 1, 2, \cdots, N$. 
Proof. For smooth \( f, g \), we have the formula

\[
\int_{\Omega} \left( (\Delta^2 f)g - (\Delta^2 g)f \right) dx
\]

(2.6)

\[
= \int_{\partial\Omega} \left( \frac{\partial \Delta f}{\partial \nu_x} \right) g - \left( \frac{\partial \Delta g}{\partial \nu_x} \right) f ds_x + \int_{\partial\Omega} \left( \frac{\partial f}{\partial \nu_x} \right) \Delta g - \left( \frac{\partial g}{\partial \nu_x} \right) \Delta f ds_x.
\]

Set \( w_\varepsilon(x) = (x - z) \cdot \nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x) \) where \( \alpha_\varepsilon = \frac{4}{p_\varepsilon - 1} \). Direct computation yields that

\[
\Delta^2 w_\varepsilon = (\alpha_\varepsilon + 4)c_0 K(x) u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} (x - z) \cdot \nabla u_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x - z) \cdot \nabla K(x).
\]

Since \( v_\varepsilon \) satisfies \( \Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon \), we have

\[
(\Delta^2 w_\varepsilon)v_\varepsilon - (\Delta^2 v_\varepsilon)w_\varepsilon = (\alpha_\varepsilon + 4 - p_\varepsilon \alpha_\varepsilon)c_0 K(x) u_\varepsilon^{p_\varepsilon} v_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon} (x - z) \cdot \nabla K(x) v_\varepsilon = 0.
\]

Integrating this identity on \( \Omega \) with the formula (2.6), and noting that

\[
w_\varepsilon(x) = (x - z) \cdot v_\varepsilon \left( \frac{\partial u_\varepsilon}{\partial \nu_x} \right), \quad \Delta w_\varepsilon(x) = (x - z) \cdot v_\varepsilon \left( \frac{\partial \Delta u_\varepsilon}{\partial \nu_x} \right)
\]

for \( x \in \partial \Omega \), we have (2.4).

On the other hand, differentiating the equation in (1.1) with respect to \( x_i \), we have

\[
\Delta^2 \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) + c_0 \left( \frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} \quad \text{in } \Omega.
\]

Multiplying this by \( v_\varepsilon \), and the equation of \( v_\varepsilon \) by \( \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) \) and subtracting, we obtain

\[
(\Delta^2 v_\varepsilon) \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) - \left( \Delta^2 \left( \frac{\partial u_\varepsilon}{\partial x_i} \right) \right) v_\varepsilon = 0.
\]

Finally, integration by parts formula (2.6) yields (2.5). \( \square \)

Next is the asymptotic result by [6]. In what follows, we use a symbol \( \| \cdot \| \) to denote the \( L^\infty \) norm of functions.

**Theorem 2.3.** Let \( \Omega \subset \mathbb{R}^N, N \geq 5 \) be a smooth bounded domain. Let \( u_\varepsilon \) be a least energy solution to (1.1) for \( \varepsilon > 0 \) and let \( x_\varepsilon \in \Omega \) be a point such that \( u_\varepsilon(x_\varepsilon) = \| u_\varepsilon \| \). Assume (K). Then after passing to a subsequence, the following estimate holds true:

There exists a constant \( C > 0 \) independent of \( \varepsilon \) such that for any \( R_\varepsilon \to \infty \) with \( r_\varepsilon = R_\varepsilon \| u_\varepsilon \|^{-\frac{p_\varepsilon - 1}{4}} \to 0 \),

\[
\left\{ \begin{array}{l}
\frac{u_\varepsilon(x)}{\| u_\varepsilon \|} \leq C \frac{\| u_\varepsilon \|}{\left( 1 + \| u_\varepsilon \|^{\frac{N-4}{2}} |x - x_\varepsilon| \right)^{N-4}}, \quad \text{for } |x - x_\varepsilon| \leq r_\varepsilon, \\
\frac{u_\varepsilon(x)}{\| u_\varepsilon \|} \leq \frac{C}{\| u_\varepsilon \|} \frac{1}{|x - x_\varepsilon|^{N-4}}, \quad \text{for } \{ |x - x_\varepsilon| > r_\varepsilon \} \cap \Omega.
\end{array} \right.
\]

(2.7)
Furthermore, as $\epsilon \to 0$,

\begin{equation}
|x_{\epsilon} - x_0| = O(\|u_{\epsilon}\|^{-2}) \quad N = 5,
\end{equation}

\begin{equation}
|x_{\epsilon} - x_0| = o(\|u_{\epsilon}\|^{-\frac{2}{N-4}}) \quad N \geq 6,
\end{equation}

\begin{equation}
\|u_{\epsilon}\|^{\epsilon} \to 1,
\end{equation}

\begin{equation}
\|u_{\epsilon}\|u_{\epsilon}(x) \to 2(N-4)(N-2)\sigma_N G(x, x_0) \quad \text{in } C_{loc}^3(\overline{\Omega} \setminus \{x_0\}),
\end{equation}

\begin{equation}
\epsilon \|u_{\epsilon}\|^2 \to \frac{2^{15}}{21}\pi R(x_0) \quad N = 5,
\end{equation}

\begin{equation}
\epsilon \|u_{\epsilon}\|^\frac{4}{N-4} \to -\frac{2}{(N-2)(N-4)}\triangle K(x_0) \quad N \geq 7.
\end{equation}

Now, consider the scaled function

\begin{equation}
\tilde{u}_{\epsilon}(y) := \frac{1}{\|u_{\epsilon}\|}u_{\epsilon}\left(\frac{y}{\|u_{\epsilon}\|^\frac{p_{\epsilon}-1}{4}} + x_{\epsilon}\right), \quad y \in \Omega_{\epsilon} := \|u_{\epsilon}\|^\frac{p_{\epsilon}-1}{4}(\Omega - \varepsilon).
\end{equation}

$\tilde{u}_{\epsilon}$ satisfies $0 < \tilde{u}_{\epsilon} \leq 1$, $\tilde{u}_{\epsilon}(0) = 1$, and

\begin{equation}
\begin{cases}
\Delta^2 \tilde{u}_{\epsilon} = c_0 K\left(\frac{y}{\|u_{\epsilon}\|^\frac{p_{\epsilon}-1}{4}} + x_{\epsilon}\right)\tilde{u}_{\epsilon}^{p_{\epsilon}} & \text{in } \Omega_{\epsilon}, \\
\tilde{u}_{\epsilon} = \Delta \tilde{u}_{\epsilon} = 0 & \text{on } \partial \Omega_{\epsilon}.
\end{cases}
\end{equation}

Since $\|u_{\epsilon}\| \to \infty$ as $\epsilon \to 0$ and $x_{\epsilon}$ does not approach to $\partial \Omega$, we see $\Omega_{\epsilon} \to \mathbb{R}^N$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_{\epsilon}$ that

\begin{equation}
\tilde{u}_{\epsilon} \to U \quad \text{compact uniformly in } \mathbb{R}^N
\end{equation}

as $\epsilon \to 0$ for some function $U$. Passing to the limit, we obtain that $U$ is a solution of

\begin{equation}
\begin{cases}
\Delta^2 U = c_0 U^p & \text{in } \mathbb{R}^N, \\
0 < U \leq 1, \ U(0) = 1, \\
\lim_{|y| \to \infty} U(y) = 0.
\end{cases}
\end{equation}

According to the uniqueness theorem by Chang Shou Lin [5], we obtain

\begin{equation}
U(y) = \left(\frac{1}{1 + |y|^2}\right)\frac{N-4}{2}.
\end{equation}

In terms of $\tilde{u}_{\epsilon}$ in (2.12), the estimate (2.7) reads

\begin{equation}
\tilde{u}_{\epsilon}(y) \leq \begin{cases}
CU(y) & \text{for } |y| \leq R_{\epsilon}, \\
C\frac{1}{|y|^{N-4}} & \text{for } |\{y| > R_{\epsilon}\} \cap \Omega_{\epsilon},
\end{cases}
\end{equation}

where $R_{\epsilon} \to \infty$ is any sequence as in the above.

Here, we recall a theorem by Bartsch, Weth and Willem [1].
Lemma 2.4. Let \( v_0 \) be a solution to
\[
\begin{align*}
\Delta^2 v_0 &= c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |\Delta v_0|^2 dy < \infty.
\end{align*}
\]
Then there exist \( a_j (j = 1, 2, \cdots, N), b \in \mathbb{R} \) such that \( v_0 \) can be written as
\[
v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1 + |y|^2)^{(N-2)/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}}.
\]

§ 3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.
We argue by contradiction. We assume there exists a non-trivial solution \( v_\epsilon \) to (1.2) satisfying \( \|v_\epsilon\| = \|u_\epsilon\| \) for any \( \epsilon > 0 \).
Consider the scaled function
\[
\tilde{v}_\epsilon(y) = \frac{1}{\|u_\epsilon\|} v_\epsilon \left( \frac{y}{\|u_\epsilon\|^{p_\epsilon - 1}} + x_\epsilon \right), \quad y \in \Omega_\epsilon = \|u_\epsilon\|^{p_\epsilon - 1} (\Omega - x_\epsilon).
\]
We see \( 0 < \tilde{v}_\epsilon \leq 1 \) and \( \tilde{v}_\epsilon \) satisfies
\[
\begin{align*}
\Delta^2 \tilde{v}_\epsilon &= c_0 p_\epsilon K \left( \frac{y}{\|u_\epsilon\|^{p_\epsilon - 1}} + x_\epsilon \right) \tilde{u}_\epsilon^{p_\epsilon - 1} \tilde{v}_\epsilon \quad \text{in } \Omega_\epsilon, \\
\tilde{v}_\epsilon &= 0 \quad \text{on } \partial \Omega_\epsilon, \\
\|\tilde{v}_\epsilon\|_{L^\infty(\Omega_\epsilon)} &= 1.
\end{align*}
\]
By \( \|\tilde{v}_\epsilon\|_{L^\infty(\Omega_\epsilon)} = 1 \), elliptic estimate implies that
\[
\tilde{v}_\epsilon \to v_0 \quad \text{compact uniformly in } \mathbb{R}^N
\]
for some \( v_0 \) and \( v_0 \) satisfies
\[
\Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.
\]
Also by arguing as in [7], we have
\[
\int_{\Omega_\epsilon} |\Delta \tilde{v}_\epsilon|^2 dy \leq C
\]
for some \( C > 0 \) independent of \( \epsilon > 0 \) small. By (3.4) and Fatou’s lemma, we also have
\[
\int_{\mathbb{R}^N} |\Delta v_0|^2 dy \leq C.
\]
Thus by Lemma 2.4, we have

\begin{equation}
(3.5) \quad v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1 + |y|^2)^{(N-2)/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}}.
\end{equation}

In the following, we divide the proof into three steps.

**Step 1.** $b = 0$.

**Step 2.** $a_j = 0, j = 1, \ldots, N$.

**Step 3.** $v_0 = 0$ leads to a contradiction.

First, by using the Kelvin transformation and a local supremum estimate for weak solutions to a linear biharmonic equation by Caristi and Mitidieri [2], we can obtain the pointwise estimate for the scaled function $\tilde{v}_\varepsilon$, just as in [7] Lemma 3.1.

**Lemma 3.1.** Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate

\begin{equation}
(3.6) \quad |\tilde{v}_\varepsilon(y)| \leq C U(y), \quad \forall y \in \Omega_\varepsilon
\end{equation}

for some $C > 0$.

Also by Lemma 3.1 and Theorem 2.3 (2.7), we have the following convergence result. For a proof, see Lemma 3.2 in [7].

**Lemma 3.2.** Let $\omega \subset \Omega$ be any neighborhood of $\partial \Omega$ not containing $x_0$. Then we have

\begin{equation}
(3.7) \quad \|u_\varepsilon\|v_\varepsilon \rightarrow -2(N - 2)(N - 4)\sigma_N b G(\cdot, x_0) \quad \text{in } C^3(\omega).
\end{equation}

**Proof of Step 1.** Here, we prove only the case $N \geq 7$. Proof of the cases $N = 5$ and $N = 6$ will be done by a similar argument; see [8] for the second order $-\Delta$ case.

Putting $z = x_0$ in (2.4) and multiplying $\|u_\varepsilon\|^4/(N-4)$, we have

\begin{align}
&\|u_\varepsilon\|^4/(N-4) \int_{\partial \Omega} \left( (x - x_0) \cdot \nu_x \right) \left( \frac{\partial \|u_\varepsilon\|u_\varepsilon}{\partial \nu_x} \right) \left( \frac{\partial \|u_\varepsilon\|\tilde{v}_\varepsilon}{\partial \nu_x} \right) ds_x \\
&+ \|u_\varepsilon\|^4/(N-4) \int_{\partial \Omega} \left( (x - x_0) \cdot \nu_x \right) \left( \frac{\partial \|u_\varepsilon\|\tilde{u}_\varepsilon}{\partial \nu_x} \right) \left( \frac{\partial \|u_\varepsilon\|v_\varepsilon}{\partial \nu_x} \right) ds_x \\
&= \|u_\varepsilon\|^4/(N-4) \int_{\partial \Omega} w^\varepsilon u_\varepsilon (x - x_0) \cdot \nabla K(x) dx.
\end{align}

As $\frac{4}{N-4} < 2$ if $N \geq 7$, LHS of (3.8) converges to 0 as $\varepsilon \to 0$. On the other hand, by Taylor’s formula and the change of variables, we write

(RHS) of (3.8) $= C_1 + C_2 + C_3 + C_4$
where, putting $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$,


c_1 = c_0 \| u_\epsilon \|^{\frac{N-2}{4} + p_\epsilon + 1 - (\frac{p_\epsilon - 1}{N-4})N - (\frac{p_\epsilon - 1}{2})} \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \sum_{i,j=1}^{N} b_{ij} y_i y_j dy,

c_2 = 2c_0 \| u_\epsilon \|^{\frac{N-2}{4} + \frac{N-4}{4} + 1 - (\frac{p_\epsilon - 1}{4})N - (\frac{p_\epsilon - 1}{4})} \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \sum_{i,j=1}^{N} b_{ij} (x_\epsilon y_i - x_0 y_i)(x_\epsilon y_j - x_0 y_j) dy,

c_3 = c_0 \| u_\epsilon \|^{\frac{N-2}{4} + p_\epsilon + 1 - (\frac{p_\epsilon - 1}{N-4})N - (\frac{p_\epsilon - 1}{2})} \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \sum_{i,j=1}^{N} b_{ij} (x_\epsilon - x_0)(x_\epsilon i - x_0 i)(x_\epsilon j - x_0 j) dy,

c_4 = c_0 \| u_\epsilon \|^{\frac{N-2}{4} + p_\epsilon + 1 - (\frac{p_\epsilon - 1}{N-4})N - (\frac{p_\epsilon - 1}{2})} \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \left( O \left( \frac{y}{\| u_\epsilon \|^\frac{p_\epsilon - 1}{2}} + x_\epsilon - x_0 \right)^3 \right) dy.

By (2.15), (3.6), (2.9), (2.8) and the dominated convergence theorem, we see

\[ C_2 = O(\| u_\epsilon \|^{\frac{N-2}{4} + \frac{N-4}{4}}) \times O \left( \int_{\mathbb{R}^N} U^p v_0(y) |y| dy + o(1) \right) \times o(\| u_\epsilon \|^{-\frac{2}{N-4}}) = o(1), \]

\[ C_3 = O(\| u_\epsilon \|^{\frac{N-2}{4} + \frac{N-4}{4}}) \times O \left( \int_{\mathbb{R}^N} U^p v_0(y) dy + o(1) \right) \times o(\| u_\epsilon \|^{-\frac{2}{N-4}}) = o(1), \]

\[ C_4 = O(\| u_\epsilon \|^{\frac{N-2}{4} + \frac{N-4}{4}}) \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \left( O \left( \frac{y}{\| u_\epsilon \|^{\frac{p_\epsilon - 1}{2}}} \right)^3 + O(|x_\epsilon - x_0|^3) \right) \]

\[ = O(\| u_\epsilon \|^{\frac{N-2}{4}}) \times O(\| u_\epsilon \|^{-\frac{N-2}{4}}) \times O \left( \int_{\mathbb{R}^N} U^p v_0(y) |y|^3 + 1 dy + o(1) \right) \]

\[ = O(\| u_\epsilon \|^{-\frac{2}{N-4}}) \]

as $\epsilon \to 0$. As for $C_1$, we see

\[ C_1 = c_0 \| u_\epsilon \|^{\frac{N-2}{4}} \int_{\Omega_\epsilon} \tilde{u}_\epsilon^{p_\epsilon} \tilde{v}_\epsilon(y) \sum_{i,j=1}^{N} b_{ij} y_i y_j dy \]

\[ \to c_0 \int_{\mathbb{R}^N} U^p(y) v_0(y) \sum_{i,j=1}^{N} b_{ij} y_i y_j dy = \frac{c_0}{N} b \Delta K(x_0) \int_{\mathbb{R}^N} U^p(y) \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}} |y|^2 dy. \]

Thus letting $\epsilon \to 0$ in (3.8), we have

\[ 0 = \Delta K(x_0) \times b. \]

Hence we obtain $b = 0$, because our nondegeneracy assumption of $x_0$ assures that $\Delta K(x_0) < 0$ strictly. \qed

**Proof of Step 2.**

In this step, we prove $a_j = 0, j = 1, 2, \cdots, N$ in (3.5) by using the next lemma.
Lemma 3.3. Assume $b = 0$ and $a = (a_1, \cdots, a_N) \neq 0$ in (3.5). Then we have
\[
\|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon \rightarrow 2(N-2)\sigma_N \sum_{j=1}^{N} a_j \left( \frac{\partial G}{\partial z_j}(x, z) \right) \bigg|_{z=x_0}
\]
in $C^3_{loc}(\Omega \setminus \{x_0\})$.

Proof. Since $-\Delta \overline{v}_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon$ in $\Omega$, $\overline{v}_\varepsilon = 0$ on $\partial \Omega$, the Green representation formula implies that
\[
(3.9) \quad \overline{v}_\varepsilon(x) = c_0 p_\varepsilon \int_{\Omega} \overline{G}(x, z) K(z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz
\]
for any $x \in \overline{\Omega} \setminus \{x_0\}$, here $\overline{G}(x, z) = -\triangle_x G(x, z)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By a change of variables, we see
\[
c_0 p_\varepsilon \int_{\Omega} \overline{G}(x, z) K(z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz
\]
\[
= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon-1} \frac{(p_\varepsilon-1)^N}{\|u_\varepsilon\|^{p_\varepsilon-1}} \int_{\Omega_\varepsilon} \overline{G}_\varepsilon(x, y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) dy
\]
where $\overline{G}_\varepsilon(x, y) = \overline{G}(x, \frac{y}{\|u_\varepsilon\|^{p_\varepsilon-1}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{p_\varepsilon-1}} + x_\varepsilon)$. By (2.13) and (3.3), we obtain
\[
K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) \rightarrow \sum_{j=1}^{N} a_j \left( \frac{\partial}{\partial y_j} \frac{-1}{(N+4)} U^p(y) \right)
\]
uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE
\[
\sum_{j=1}^{N} a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N
\]
with the initial condition $w_\varepsilon|_{\Gamma_\alpha} = \frac{-1}{(N+4)} U^p(y)$, where $\Gamma_\alpha = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here, the right hand side is assumed to be 0 outside of $\Omega_\varepsilon$. By the unique solvability, we have the solution $w_\varepsilon$ of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+3)})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) = O(|y|^{-(N+4)})$ by (2.15) and (3.6). Also we have
\[
w_\varepsilon \rightarrow \frac{-1}{(N+4)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N
\]
and
\[
\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{(N+4)} \int_{\mathbb{R}^N} U^p dy = \left( \frac{-1}{(N+4)} \left( \frac{2\sigma_N}{N(N+2)} \right) \right)
\]
by the dominated convergence theorem. Using integration by parts, we have
\[
\overline{u}_\varepsilon(x) = c_0 p_{\varepsilon}\|u_{\varepsilon}\|^{p_{\varepsilon}-(p_{\varepsilon}-1)N} \int_{\Omega_{\varepsilon}} \overline{G}_\varepsilon(x, y) K_\varepsilon(y) \sum_{j=1}^{N} a_j \frac{\partial w_{\varepsilon}}{\partial y_j} dy
\]
\[
= -c_0 p_{\varepsilon}\|u_{\varepsilon}\|^{p_{\varepsilon}-(p_{\varepsilon}-1)N} \sum_{j=1}^{N} a_j \int_{\Omega_{\varepsilon}} \frac{\partial}{\partial y_j} \{\overline{G}_\varepsilon(x, y) K_\varepsilon(y)\} w_{\varepsilon}(y) dy
\]
\[
= -c_0 p_{\varepsilon}\|u_{\varepsilon}\|^{p_{\varepsilon}-(p_{\varepsilon}-1)N-(p_{\varepsilon}-1)} \sum_{j=1}^{N} a_j \int_{\Omega_{\varepsilon}} \frac{\partial}{\partial z_j} \{\overline{G}(x, z) K(z)\} \left|_{z=\frac{y}{\|u_{\varepsilon}\|^\frac{p_{\varepsilon}-1}{4}}+x_\varepsilon}\right. w_{\varepsilon}(y) dy.
\]
Note that \( p_{\varepsilon}-(p_{\varepsilon}-1)N-(p_{\varepsilon}-1) = -(\frac{N-2}{N-4}) + \varepsilon(\frac{N-3}{4}) \). Now, we see
\[
\frac{\partial}{\partial z_j} \{\overline{G}(x, z) K(z)\} \left|_{z=\frac{y}{\|u_{\varepsilon}\|^\frac{p_{\varepsilon}-1}{4}}+x_\varepsilon}\right. \rightarrow \left(\frac{\partial \overline{G}}{\partial z_j}(x, x_0)\right) K(x_0) + \overline{G}(x, x_0) \left(\frac{\partial K}{\partial z_j}(x_0)\right)
\]
\[
= \frac{\partial \overline{G}}{\partial z_j}(x, x_0)
\]
uniformly on compact subsets of \( \mathbb{R}^N \) as \( \varepsilon \to 0 \), since \( x_0 \) is a critical point of \( K \) with \( K(x_0) = 1 \). Therefore, we have the convergence
\[
\|u_{\varepsilon}\|^{\frac{N-2}{N-4}} \overline{u}_\varepsilon(x) \to -c_0 \left(\frac{-1}{N+4}\right) \left(\frac{2\sigma_N}{N(N+2)}\right) \sum_{j=1}^{N} a_j \left(\frac{\partial \overline{G}}{\partial z_j}(x, z)\right) \left|_{z=x_0}\right.
\]
\[
= 2(N-2)\sigma_N \sum_{j=1}^{N} a_j \left(\frac{\partial \overline{G}}{\partial z_j}(x, z)\right) \left|_{z=x_0}\right.
\]
for any \( x \in \overline{\Omega} \setminus \{x_0\} \). Elliptic estimates implies this convergence holds true in \( C_{loc}^{1}(\overline{\Omega} \setminus \{x_0\}) \). This proves Lemma. \( \square \)

Now, assume the contrary that \( a = (a_1, \cdots, a_N) \neq 0 \). We multiply both sides of (2.5) in Lemma 2.2 by \( \|u_{\varepsilon}\|^{(N-2)/(N-4)} \times \|u_{\varepsilon}\|^{-1} \) to get
\[
\|u_{\varepsilon}\|^{-2} \left[ \int_{\partial \Omega} \left(\frac{\partial u_{\varepsilon}}{\partial x_i} \right) \left(\frac{\partial u_{\varepsilon}}{\partial v_x} \right) ds_x + \left(\frac{\partial \|u_{\varepsilon}\|}{\partial x_i} \right) \left(\frac{\partial \|u_{\varepsilon}\|^\frac{N-2}{N-4} \overline{u}_\varepsilon}{\partial v_x} \right) ds_x \right]
\]
\[
= \|u_{\varepsilon}\|^{-1+\frac{N-3}{N-4}} c_0 \int_{\Omega} \left(\frac{\partial K}{\partial x_i} \right) u_{\varepsilon} v_{\varepsilon} dx \tag{3.10}
\]
As $\varepsilon \to 0$, we see that

$$\int_{\partial \Omega} \left( \frac{\partial \| u_\varepsilon \| \overline{u}_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \| u_\varepsilon \| \overline{v}_\varepsilon}{\partial \nu_x} \right) ds_x + \left( \frac{\partial || u_\varepsilon \| u_\varepsilon}{\partial x_i} \right) \left( \frac{\partial || u_\varepsilon \| \overline{v}_\varepsilon}{\partial \nu_x} \right) ds_x$$

tends to

$$4(N - 4)(N - 2)^2 \sigma_N^2 \sum_{j=1}^{N} a_j \times$$

$$\int_{\partial \Omega} \left\{ \left( \frac{\partial \overline{G}}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left( \frac{\partial \overline{G}}{\partial z_j} \right) (x, x_0) + \left( \frac{\partial \overline{G}}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left( \frac{\partial \overline{G}}{\partial z_j} \right) (x, x_0) \right\} ds_x$$

$$= 4(N - 4)(N - 2)^2 \sigma_N^2 \sum_{j=1}^{N} a_j \frac{\partial^2 R}{\partial z_i \partial z_j} (z) \bigg|_{z=x_0},$$

here we have used Theorem 2.3 (2.10), Lemma 3.3 and Lemma 2.1 (2.3). Thus we have (LHS) of (3.10) tends to 0 as $\varepsilon \to 0$.

On the other hand, again we solve the linear PDE

(3.11)

$$\sum_{j=1}^{N} a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon (y), \quad y \in \mathbb{R}^N$$

with the initial condition $w_\varepsilon |_{\Gamma_a} = -\frac{1}{2N} U^{p+1}(y)$, where $\Gamma_a = \{ x \in \mathbb{R}^N | x \cdot a = 0 \}$. Here as before, the RHS of (3.11) is understood as 0 outside of $\Omega_\varepsilon$. The solution $w_\varepsilon$ satisfies the estimate $w_\varepsilon (y) = O(|y|^{-2N+1})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon (y) = O(U^{p+1}(y)) = O(|y|^{-2N})$ by (2.15) and (3.6). As before, we have

$$w_\varepsilon \to \frac{-1}{2N} U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon (y) dy \to \frac{-1}{2N} \int_{\mathbb{R}^N} U^{p+1} dy = \frac{-1}{2N} \sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2 \Gamma(N)}$. Thus,
(RHS of (2.5)) \times \|u_\varepsilon\|^{\frac{N-2}{N-4}-1} is
\begin{align*}
c_0\|u_\varepsilon\|^{-1+\frac{N-2}{N-4}} \int_\Omega \left( \frac{\partial K}{\partial x_i} \right) u_\varepsilon^p v_\varepsilon dx \\
= c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})} \int_\Omega \left( \frac{\partial K}{\partial x_i} \right) \left( \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \tilde{u}_\varepsilon^p \tilde{v}_\varepsilon dy \\
= c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})} \int_\Omega \left( \frac{\partial K}{\partial x_i} \right) \left( \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
= -c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_\Omega \frac{\partial \tilde{v}_\varepsilon}{\partial y_j} \left( \left( \frac{\partial K}{\partial x_i} \right) \left( \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \right) w_\varepsilon(y) dy \\
= -c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_\Omega \left( \frac{\partial^2 K}{\partial x_i \partial x_j} \right)(x) \left( \lim_{\varepsilon \to 0} \int_\Omega w_\varepsilon(y) dy \right) \\
= -c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) \left( \lim_{\varepsilon \to 0} \int_\Omega w_\varepsilon(y) dy \right) \\
= \frac{1}{2N} c_0 \sigma_N C_N \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0).
\end{align*}

Thus we have
\[ \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) = 0. \]

By our assumption of the nondegeneracy of $x_0$, the matrix \( \left( \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) \right) \) is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \cdots, N$. Thus we have proved Step 2. \( \Box \)

**Proof of Step 3.**

By Step 1 and Step 2, we have obtained that the limit function \( \lim_{\varepsilon \to 0} \tilde{v}_\varepsilon = v_0 \equiv 0 \). Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$ and $|y_\varepsilon| \to \infty$, because the above convergence $\tilde{v}_\varepsilon \to v_0 \equiv 0$ is uniform on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 3.1. This proves Theorem 1.1. \( \Box \)

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**References**
A nondegeneracy result to a biharmonic problem


