*RIMS Kôkyûroku Bessatsu*
**B15** (2009), 73–85

# A nondegeneracy result for least energy solutions to a biharmonic problem with nearly critical exponent

*Dedicated to professor Toshitaka Nagai on the occasion of his sixties birthday*

By

Tomohiko SATO * and Futoshi TAKAHASHI **

**Abstract**

Consider the problem $\Delta^2 u = c_0 K(x) u^{p_\varepsilon},\ u > 0$ in $\Omega$, $u = \Delta u = 0$ on $\partial\Omega$, where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N (N \geq 5)$, $c_0 = (N-4)(N-2)N(N+2), p = (N+4)/(N-4), p_\varepsilon = p - \varepsilon$ and $K$ is a smooth positive function on $\overline{\Omega}$.

Under some assumptions on the coefficient function $K$, which include the nondegeneracy of its unique maximum point as a critical point of Hess$K$, we prove that least energy solutions of the problem are nondegenerate for $\varepsilon > 0$ small.

## § 1. Introduction

Consider the problem

$$(1.1) \qquad \begin{cases} \Delta^2 u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 5)$ is a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $p_\varepsilon = p - \varepsilon, p = (N+4)/(N-4)$ is the critical Sobolev exponent with respect to the Sobolev embedding $H^2 \cap H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)$, and $\varepsilon > 0$ is a small parameter. Here, $K$ is a positive function in $C^2(\overline{\Omega})$.

Received February 2, 2009. Revised August 28, 2009. Accepted September 6, 2009.
2000 Mathematics Subject Classification(s): 35B33, 35J35, 35J40.
*Key Words*: nondegeneracy, critical exponent, least energy solution
*Computer Center, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo, 171-8585, Japan.
e-mail: tsato@gakushuin.ac.jp
**Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan.
e-mail: futoshi@sci.osaka-cu.ac.jp
© 2009 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

We put an assumption on the coefficient function $K$:

**(K)**: $K \in C^2(\overline{\Omega}), 0 < K(x) \leq 1, K^{-1}(\max_{\overline{\Omega}} K) = \{x_0\} \subset \Omega$ with $K(x_0) = 1$, and $x_0$ is a nondegenerate critical point of $K$.

In the following, as solutions of (1.1) we consider only least energy solutions $u_\varepsilon$ such that

$$\frac{\int_\Omega |\Delta u_\varepsilon|^2 dx}{\left(\int_\Omega K(x)|u_\varepsilon|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}} = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega K(x)|u|^{p_\varepsilon+1}dx\right)^{\frac{2}{p_\varepsilon+1}}}.$$

We easily check that least energy solutions blow up in the sense that $\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, and that the maximum point $x_\varepsilon$ of $u_\varepsilon$ converges to a maximum point of $K$ in $\overline{\Omega}$. Therefore we have $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, here by the assumption (K), $x_0$ is the unique interior maximum point of $K$.

In this note, we prove the nondegeneracy of least energy solutions to (1.1) when $\varepsilon > 0$ is sufficiently small, under the assumption (K). Here as usual, the nondegeneracy of $u_\varepsilon$ for small $\varepsilon$ means that the problem

$$\begin{cases} \Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon & \text{in } \Omega, \\ v_\varepsilon = \Delta v_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

admits no solution except for the trivial one for $\varepsilon > 0$ small enough.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^N (N \geq 5)$ be a smooth bounded domain. Under the assumption (K), least energy solution $u_\varepsilon$ to (1.1) is nondegenerate for $\varepsilon > 0$ small.*

The precise asymptotic behavior of least energy solutions as $\varepsilon \to 0$ when $K \not\equiv 1$ was obtained in [6] under the assumption (K). Using this result, we prove Theorem 1.1 along the line of [7] and [8], the original idea of which comes from [4].

## § 2. Preliminaries

In this section, we recall some facts which are needed in the sequel. Let $G = G(x, z)$ denote the Green function of $\Delta^2$ under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = \Delta G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

We decompose $G$ as $G(x, z) = \Gamma(x, z) - H(x, z)$, where $\Gamma(x, z)$ is the fundamental solution of $\Delta^2$:

$$\Gamma(x, z) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N} |x - z|^{4-N}, & N \geq 5, \\ \frac{1}{\sigma_4} \log |x - z|^{-1}, & N = 4, \end{cases}$$

where $\sigma_N$ is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$ and $H(x,z)$ is the regular part of the Green function. Finally, let $R(z) = H(z,z)$ denote the Robin function of the Green function of $\Delta^2$ with the Navier boundary condition. By the maximum principle, we have $R > 0$ on $\Omega$ and $R(z) \to +\infty$ as $z$ tends to the boundary of $\Omega$. In the following, we set $\overline{G} = -\Delta G$. Then $\overline{G}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition, and satisfy

$$\begin{cases} -\Delta G = \overline{G}, \ -\Delta \overline{G} = \delta_z & \text{in } \Omega, \\ G > 0, \ \overline{G} > 0 & \text{in } \Omega, \\ G = \overline{G} = 0 & \text{on } \partial\Omega. \end{cases}$$

**Lemma 2.1.** *For any $z \in \Omega$, there holds*

$$\int_{\partial\Omega} ((x-z)\cdot\nu_x) \left(\frac{\partial G}{\partial \nu_x}\right) \left(\frac{\partial \overline{G}}{\partial \nu_x}\right)(x,z) ds_x = (N-4)R(z), \tag{2.1}$$

$$\int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x,z) \frac{\partial \overline{G}}{\partial \nu_x}(x,z)\nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z), \quad (i = 1, \cdots, N), \tag{2.2}$$

$$\begin{aligned} &\int_{\partial\Omega} \left(\frac{\partial \overline{G}}{\partial x_i}\right) \frac{\partial}{\partial z_j}\left(\frac{\partial G}{\partial \nu_x}\right)(x,z) ds_x + \int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i}\right) \frac{\partial}{\partial z_j}\left(\frac{\partial \overline{G}}{\partial \nu_x}\right)(x,z) ds_x \\ &= \frac{\partial^2 R}{\partial z_i \partial z_j}(z), \quad (i,j = 1, \cdots, N). \end{aligned} \tag{2.3}$$

*Here $\nu_x$ is the outer unit normal at $x \in \partial\Omega$.*

*Proof.* See [3]:Lemma 3.1 and Lemma 3.3. Note that our sign convention is different from that of [3]. By differentiating (2.2) with respect to $z_j$, noting that $\left(\frac{\partial G}{\partial \nu_x}(x,z)\right)\nu_i(x) = \frac{\partial G}{\partial x_i}(x,z)$, $\left(\frac{\partial \overline{G}}{\partial \nu_x}(x,z)\right)\nu_i(x) = \frac{\partial \overline{G}}{\partial x_i}(x,z)$ on $\partial\Omega$, we see that (2.3) holds. □

**Lemma 2.2.** *Let $u_\varepsilon$ be a solution to (1.1) and $v_\varepsilon$ be a solution to (1.2). Denote $\overline{u}_\varepsilon = -\Delta u_\varepsilon$ and $\overline{v}_\varepsilon = -\Delta v_\varepsilon$. Then the following identities hold true:*

$$\begin{aligned} &\int_{\partial\Omega} ((x-z)\cdot\nu_x) \left\{ \left(\frac{\partial u_\varepsilon}{\partial \nu_x}\right)\left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x}\right) + \left(\frac{\partial \overline{u}_\varepsilon}{\partial \nu_x}\right)\left(\frac{\partial v_\varepsilon}{\partial \nu_x}\right) \right\} ds_x \\ &= c_0 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon (x-z)\cdot\nabla K(x) dx \end{aligned} \tag{2.4}$$

*for any $z \in \mathbb{R}^N$ and*

$$\int_{\partial\Omega} \left\{ \left(\frac{\partial \overline{u}_\varepsilon}{\partial x_i}\right)\left(\frac{\partial v_\varepsilon}{\partial \nu_x}\right) + \left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\left(\frac{\partial \overline{v}_\varepsilon}{\partial \nu_x}\right) \right\} ds_x = c_0 \int_\Omega \left(\frac{\partial K}{\partial x_i}\right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \tag{2.5}$$

*for $i = 1, 2, \cdots, N$.*

*Proof.* For smooth $f, g$, we have the formula

$$\begin{aligned}&\int_\Omega \left((\Delta^2 f)g - (\Delta^2 g)f\right) dx \\ &= \int_{\partial\Omega} \left(\frac{\partial \Delta f}{\partial \nu_x}\right) g - \left(\frac{\partial \Delta g}{\partial \nu_x}\right) f ds_x + \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu_x}\right) \Delta g - \left(\frac{\partial g}{\partial \nu_x}\right) \Delta f ds_x. \end{aligned} \tag{2.6}$$

Set $w_\varepsilon(x) = (x-z)\cdot \nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x)$ where $\alpha_\varepsilon = \frac{4}{p_\varepsilon - 1}$. Direct computation yields that

$$\Delta^2 w_\varepsilon = (\alpha_\varepsilon + 4) c_0 K(x) u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}(x-z)\cdot\nabla u_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon}(x-z)\cdot \nabla K(x).$$

Since $v_\varepsilon$ satisfies $\Delta^2 v_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon-1} v_\varepsilon$, we have

$$(\Delta^2 w_\varepsilon) v_\varepsilon - (\Delta^2 v_\varepsilon) w_\varepsilon = (\alpha_\varepsilon + 4 - p_\varepsilon \alpha_\varepsilon) c_0 K(x) u_\varepsilon^{p_\varepsilon} v_\varepsilon + c_0 u_\varepsilon^{p_\varepsilon}(x-z)\cdot\nabla K(x) v_\varepsilon = 0.$$

Integrating this identity on $\Omega$ with the formula (2.6), and noting that

$$w_\varepsilon(x) = (x-z)\cdot\nu_x(\frac{\partial u_\varepsilon}{\partial \nu_x}), \quad \Delta w_\varepsilon(x) = (x-z)\cdot \nu_x(\frac{\partial \Delta u_\varepsilon}{\partial \nu_x})$$

for $x \in \partial\Omega$, we have (2.4).

On the other hand, differentiating the equation in (1.1) with respect to $x_i$, we have

$$\Delta^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1}\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) + c_0 \left(\frac{\partial K}{\partial x_i}\right) u_\varepsilon^{p_\varepsilon} \quad \text{in } \Omega.$$

Multiplying this by $v_\varepsilon$, and the equation of $v_\varepsilon$ by $\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)$ and subtracting, we obtain

$$(\Delta^2 v_\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial x_i}\right) - \left(\Delta^2\left(\frac{\partial u_\varepsilon}{\partial x_i}\right)\right) v_\varepsilon = 0.$$

Finally, integration by parts formula (2.6) yields (2.5). □

Next is the asymptotic result by [6]. In what follows, we use a symbol $\|\cdot\|$ to denote the $L^\infty$ norm of functions.

**Theorem 2.3.** *Let $\Omega \subset \mathbb{R}^N, N \geq 5$ be a smooth bounded domain. Let $u_\varepsilon$ be a least energy solution to* (1.1) *for $\varepsilon > 0$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|$. Assume $(K)$. Then after passing to a subsequence, the following estimate holds true:*

*There exists a constant $C > 0$ independent of $\varepsilon$ such that for any $R_\varepsilon \to \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}} \to 0$,*

$$\begin{cases} u_\varepsilon(x) \leq C \dfrac{\|u_\varepsilon\|}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-4}}|x - x_\varepsilon|^2\right)^{\frac{N-4}{2}}}, & \textit{for } |x - x_\varepsilon| \leq r_\varepsilon, \\ u_\varepsilon(x) \leq \frac{C}{\|u_\varepsilon\|}\frac{1}{|x-x_\varepsilon|^{N-4}}, & \textit{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \end{cases} \tag{2.7}$$

*Furthermore, as* $\varepsilon \to 0$,

$$(2.8) \qquad (1) \begin{cases} |x_\varepsilon - x_0| = O(\|u_\varepsilon\|^{-2}) & N = 5, \\ |x_\varepsilon - x_0| = o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) & N \geq 6, \end{cases}$$

$$(2.9) \qquad (2) \|u_\varepsilon\|^\varepsilon \to 1,$$

$$(2.10) \qquad (3) \|u_\varepsilon\| u_\varepsilon(x) \to 2(N-4)(N-2)\sigma_N G(x, x_0) \quad \text{in } C^3_{loc}(\overline{\Omega} \setminus \{x_0\}),$$

$$(2.11) \qquad (4) \begin{cases} \varepsilon \|u_\varepsilon\|^2 \to \frac{2^{15}}{21}\pi R(x_0) & N = 5, \\ \varepsilon \|u_\varepsilon\|^2 \to -\frac{1}{4}\Delta K(x_0) + 480\pi^3 R(x_0) & N = 6, \\ \varepsilon \|u_\varepsilon\|^{\frac{4}{N-4}} \to -\frac{2}{(N-2)(N-4)}\Delta K(x_0) & N \geq 7. \end{cases}$$

Now, consider the scaled function

$$(2.12) \qquad \tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}(\Omega - \varepsilon).$$

$\tilde{u}_\varepsilon$ satisfies $0 < \tilde{u}_\varepsilon \leq 1, \tilde{u}_\varepsilon(0) = 1$, and

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon)\tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since $\|u_\varepsilon\| \to \infty$ as $\varepsilon \to 0$ and $x_\varepsilon$ does not approach to $\partial\Omega$, we see $\Omega_\varepsilon \to \mathbb{R}^N$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$(2.13) \qquad \tilde{u}_\varepsilon \to U \quad \text{compact uniformly in } \mathbb{R}^N$$

as $\varepsilon \to 0$ for some function $U$. Passing to the limit, we obtain that $U$ is a solution of

$$\begin{cases} \Delta^2 U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \ U(0) = 1, \\ \lim_{|y| \to \infty} U(y) = 0. \end{cases}$$

According to the uniqueness theorem by Chang Shou Lin [5], we obtain

$$(2.14) \qquad U(y) = \left(\frac{1}{1 + |y|^2}\right)^{\frac{N-4}{2}}.$$

In terms of $\tilde{u}_\varepsilon$ in (2.12), the estimate (2.7) reads

$$(2.15) \qquad \tilde{u}_\varepsilon(y) \leq \begin{cases} CU(y) & \text{for } |y| \leq R_\varepsilon, \\ C\frac{1}{|y|^{N-4}} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon, \end{cases}$$

where $R_\varepsilon \to \infty$ is any sequence as in the above.

Here, we recall a theorem by Bartsch, Weth and Willem [1].

**Lemma 2.4.** *Let $v_0$ be a solution to*

$$\begin{cases} \Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad in\, \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\Delta v_0|^2 dy < \infty. \end{cases}$$

*Then there exist $a_j$ $(j = 1, 2, \cdots, N), b \in \mathbb{R}$ such that $v_0$ can be written as*

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}.$$

## § 3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1.

We argue by contradiction. We assume there exists a non-trivial solution $v_\varepsilon$ to (1.2) satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$.

Consider the scaled function

$$\tilde{v}_\varepsilon(y) = \frac{1}{\|u_\varepsilon\|} v_\varepsilon\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon\right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}(\Omega - x_\varepsilon). \tag{3.1}$$

We see $0 < \tilde{v}_\varepsilon \le 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{v}_\varepsilon = c_0 p_\varepsilon K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon)\tilde{u}_\varepsilon^{p_\varepsilon-1}\tilde{v}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = \Delta\tilde{v}_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1. \end{cases} \tag{3.2}$$

By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, elliptic estimate implies that

$$\tilde{v}_\varepsilon \to v_0 \quad \text{compact uniformly in } \mathbb{R}^N \tag{3.3}$$

for some $v_0$ and $v_0$ satisfies

$$\Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Also by arguing as in [7], we have

$$\int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy \le C \tag{3.4}$$

for some $C > 0$ independent of $\varepsilon > 0$ small. By (3.4) and Fatou's lemma, we also have

$$\int_{\mathbb{R}^N} |\Delta v_0|^2 dy \le C.$$

Thus by Lemma 2.4, we have

$$v_0 = \sum_{j=1}^{N} a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b\frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}. \tag{3.5}$$

In the following, we divide the proof into three steps.

**Step 1.** $b = 0$.

**Step 2.** $a_j = 0, j = 1, \cdots, N$.

**Step 3.** $v_0 = 0$ leads to a contradiction.

First, by using the Kelvin transformation and a local supremum estimate for weak solutions to a linear biharmonic equation by Caristi and Mitidieri [2], we can obtain the pointwise estimate for the scaled function $\tilde{v}_\varepsilon$, just as in [7] Lemma 3.1.

**Lemma 3.1.** *Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate*

$$|\tilde{v}_\varepsilon(y)| \le CU(y), \quad \forall y \in \Omega_\varepsilon \tag{3.6}$$

*for some $C > 0$.*

Also by Lemma 3.1 and Theorem 2.3 (2.7), we have the following convergence result. For a proof, see Lemma 3.2 in [7].

**Lemma 3.2.** *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing $x_0$. Then we have*

$$\|u_\varepsilon\| v_\varepsilon \to -2(N-2)(N-4)\sigma_N b G(\cdot, x_0) \quad \text{in } C^3(\omega). \tag{3.7}$$

**Proof of Step 1.** Here, we prove only the case $N \ge 7$. Proof of the cases $N = 5$ and $N = 6$ will be done by a similar argument; see [8] for the second order $-\Delta$ case.

Putting $z = x_0$ in (2.4) and multiplying $\|u_\varepsilon\|^{4/(N-4)}$, we have

$$\begin{aligned}
&\|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x-x_0)\cdot\nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x}\right) \left(\frac{\partial \|u_\varepsilon\| \overline{v}_\varepsilon}{\partial \nu_x}\right) ds_x \\
&+ \|u_\varepsilon\|^{\frac{4}{N-4}-2} \int_{\partial\Omega} ((x-x_0)\cdot\nu_x) \left(\frac{\partial \|u_\varepsilon\| \overline{u}_\varepsilon}{\partial \nu_x}\right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x}\right) ds_x \\
&= \|u_\varepsilon\|^{\frac{4}{N-4}} c_0 \int_\Omega u_\varepsilon^{p_\varepsilon} v_\varepsilon (x-x_0)\cdot \nabla K(x) dx.
\end{aligned} \tag{3.8}$$

As $\frac{4}{N-4} < 2$ if $N \ge 7$, LHS of (3.8) converges to 0 as $\varepsilon \to 0$. On the other hand, by Taylor's formula and the change of variables, we write

$$\text{(RHS) of (3.8)} =: C_1 + C_2 + C_3 + C_4$$

where, putting $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$,

$$C_1 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{2})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy,$$

$$C_2 = 2c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i (x_{\varepsilon j} - x_{0j}) dy,$$

$$C_3 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} (x_{\varepsilon i} - x_{0i})(x_{\varepsilon j} - x_{0j}) dy,$$

$$C_4 = c_0 \|u_\varepsilon\|^{\frac{4}{N-4}+p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left( O \left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon - x_0 \right|^3 \right) dy.$$

By (2.15), (3.6), (2.9), (2.8) and the dominated convergence theorem, we see

$$C_2 = O(\|u_\varepsilon\|^{\frac{2}{N-4}+\frac{N-3}{4}\varepsilon}) \times O\left( \int_{\mathbb{R}^N} U^p v_0(y)|y| dy + o(1) \right) \times o(\|u_\varepsilon\|^{-\frac{2}{N-4}}) = o(1),$$

$$C_3 = O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \times O\left( \int_{\mathbb{R}^N} U^p v_0(y) dy + o(1) \right) \times o(\|u_\varepsilon\|^{-\frac{4}{N-4}}) = o(1),$$

$$\begin{aligned} C_4 &= O(\|u_\varepsilon\|^{\frac{4}{N-4}+\frac{N-4}{4}\varepsilon}) \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \left( O\left( \left| \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{2}}} \right|^3 \right) + O(|x_\varepsilon - x_0|^3) \right) \\ &= O(\|u_\varepsilon\|^{\frac{4}{N-4}}) \times O(\|u_\varepsilon\|^{-\frac{6}{N-4}}) \times O\left( \int_{\mathbb{R}^N} U^p v_0(y)(|y|^3 + 1) dy + o(1) \right) \\ &= O(\|u_\varepsilon\|^{-\frac{2}{N-4}}) \end{aligned}$$

as $\varepsilon \to 0$. As for $C_1$, we see

$$\begin{aligned} C_1 &= c_0 \|u_\varepsilon\|^{(\frac{N-2}{4})\varepsilon} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy \\ &\to c_0 \int_{\mathbb{R}^N} U^p(y) v_0(y) \sum_{i,j=1}^N b_{ij} y_i y_j dy = \frac{c_0}{N} b \Delta K(x_0) \int_{\mathbb{R}^N} U^p(y) \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}} |y|^2 dy. \end{aligned}$$

Thus letting $\varepsilon \to 0$ in (3.8), we have

$$0 = \Delta K(x_0) \times b.$$

Hence we obtain $b = 0$, because our nondegeneracy assumption of $x_0$ assures that $\Delta K(x_0) < 0$ strictly. □

**Proof of Step 2.**

In this step, we prove $a_j = 0, j = 1, 2, \cdots, N$ in (3.5) by using the next lemma.

**Lemma 3.3.** *Assume $b = 0$ and $a = (a_1, \cdots, a_N) \neq 0$ in (3.5). Then we have*

$$\|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon \to 2(N-2)\sigma_N \sum_{j=1}^{N} a_j \left( \frac{\partial G}{\partial z_j}(x,z) \right) \Big|_{z=x_0}$$

*in $C^3_{loc}(\overline{\Omega} \setminus \{x_0\})$.*

**Proof.** Since $-\Delta \overline{v}_\varepsilon = c_0 p_\varepsilon K(x) u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$ in $\Omega$, $\overline{v}_\varepsilon = 0$ on $\partial\Omega$, the Green representation formula implies that

$$\overline{v}_\varepsilon(x) = c_0 p_\varepsilon \int_\Omega \overline{G}(x,z) K(z) u_\varepsilon^{p_\varepsilon - 1}(z) v_\varepsilon(z) dz \tag{3.9}$$

for any $x \in \overline{\Omega} \setminus \{x_0\}$, here $\overline{G}(x,z) = -\Delta_x G(x,z)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By a change of variables, we see

$$\begin{aligned} & c_0 p_\varepsilon \int_\Omega \overline{G}(x,z) K(z) u_\varepsilon^{p_\varepsilon - 1}(z) v_\varepsilon(z) dz \\ &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) dy \end{aligned}$$

where $\overline{G}_\varepsilon(x,y) = \overline{G}(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon)$ and $K_\varepsilon(y) = K(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon)$. By (2.13) and (3.3), we obtain

$$K_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) \to \sum_{j=1}^{N} a_j \left( \frac{\partial}{\partial y_j} \frac{-1}{(N+4)} U^p(y) \right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^{N} a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{(N+4)} U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here, the right hand side is assumed to be 0 outside of $\Omega_\varepsilon$. By the unique solvability, we have the solution $w_\varepsilon$ of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+3)})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon - 1} \tilde{v}_\varepsilon(y) = O(|y|^{-(N+4)})$ by (2.15) and (3.6). Also we have

$$w_\varepsilon \to \frac{-1}{(N+4)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \to \frac{-1}{(N+4)} \int_{\mathbb{R}^N} U^p dy = \left( \frac{-1}{N+4} \right) \left( \frac{2\sigma_N}{N(N+2)} \right)$$

by the dominated convergence theorem. Using integration by parts, we have

$$\overline{v}_\varepsilon(x) = c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \int_{\Omega_\varepsilon} \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy$$

$$= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \overline{G}_\varepsilon(x,y) K_\varepsilon(y) \right\} w_\varepsilon(y) dy$$

$$= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial z_j} \left\{ \overline{G}(x,z) K(z) \right\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon} w_\varepsilon(y) dy.$$

Note that $p_\varepsilon - (\frac{p_\varepsilon - 1}{4})N - (\frac{p_\varepsilon - 1}{4}) = -(\frac{N-2}{N-4}) + \varepsilon(\frac{N-3}{4})$. Now, we see

$$\begin{aligned} &\frac{\partial}{\partial z_j} \left\{ \overline{G}(x,z) K(z) \right\} \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon} \\ &\to \left( \frac{\partial \overline{G}}{\partial z_j}(x, x_0) \right) K(x_0) + \overline{G}(x, x_0) \left( \frac{\partial K}{\partial z_j}(x_0) \right) \\ &= \frac{\partial \overline{G}}{\partial z_j}(x, x_0) \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^N$ as $\varepsilon \to 0$, since $x_0$ is a critical point of $K$ with $K(x_0) = 1$. Therefore, we have the convergence

$$\begin{aligned} \|u_\varepsilon\|^{\frac{N-2}{N-4}} \overline{v}_\varepsilon(x) \to &- c_0 p \left( \frac{-1}{N+4} \right) \left( \frac{2\sigma_N}{N(N+2)} \right) \sum_{j=1}^N a_j \left( \frac{\partial \overline{G}}{\partial z_j}(x,z) \right) \Big|_{z=x_0} \\ &= 2(N-2)\sigma_N \sum_{j=1}^N a_j \left( \frac{\partial \overline{G}}{\partial z_j}(x,z) \right) \Big|_{z=x_0} \end{aligned}$$

for any $x \in \overline{\Omega} \setminus \{x_0\}$. Elliptic estimates implies this convergence holds true in $C^1_{loc}(\overline{\Omega} \setminus \{x_0\})$. This proves Lemma. □

Now, assume the contrary that $a = (a_1, \cdots, a_N) \neq 0$. We multiply both sides of (2.5) in Lemma 2.2 by $\|u_\varepsilon\|^{(N-2)/(N-4)} \times \|u_\varepsilon\|^{-1}$ to get

$$\|u_\varepsilon\|^{-2} \left[ \int_{\partial\Omega} \left( \frac{\partial \|u_\varepsilon\| \overline{u}_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon}{\partial \nu_x} \right) ds_x + \left( \frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} \overline{v}_\varepsilon}{\partial \nu_x} \right) ds_x \right]$$

(3.10)

$$= \|u_\varepsilon\|^{-1 + \frac{N-2}{N-4}} c_0 \int_\Omega \left( \frac{\partial K}{\partial x_i} \right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx$$

As $\varepsilon \to 0$, we see that

$$\int_{\partial\Omega} \left( \frac{\partial \|u_\varepsilon\| \overline{u}_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon}{\partial \nu_x} \right) ds_x + \left( \frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left( \frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} \overline{v}_\varepsilon}{\partial \nu_x} \right) ds_x$$

tends to

$$\begin{aligned} &4(N-4)(N-2)^2 \sigma_N^2 \sum_{j=1}^N a_j \times \\ &\int_{\partial\Omega} \left\{ \left( \frac{\partial \overline{G}}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left( \frac{\partial G}{\partial z_j} \right) (x, x_0) + \left( \frac{\partial G}{\partial x_i} \right) \frac{\partial}{\partial \nu_x} \left( \frac{\partial \overline{G}}{\partial z_j} \right) (x, x_0) \right\} ds_x \\ &= 4(N-4)(N-2)^2 \sigma_N^2 \sum_{j=1}^N a_j \frac{\partial^2 R}{\partial z_i \partial z_j}(z) \Big|_{z=x_0}, \end{aligned}$$

here we have used Theorem 2.3 (2.10), Lemma 3.3 and Lemma 2.1 (2.3). Thus we have (LHS) of (3.10) tends to 0 as $\varepsilon \to 0$.

On the other hand, again we solve the linear PDE

$$\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N \tag{3.11}$$

with the initial condition $w_\varepsilon|_{\Gamma_a} = \frac{-1}{2N} U^{p+1}(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here as before, the RHS of (3.11) is understood as 0 outside of $\Omega_\varepsilon$. The solution $w_\varepsilon$ satisfies the estimate $w_\varepsilon(y) = O(|y|^{-2N+1})$ as $|y| \to \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon} \tilde{v}_\varepsilon(y) = O(U^{p_\varepsilon+1}(y)) = O(|y|^{-2N})$ by (2.15) and (3.6). As before, we have

$$w_\varepsilon \to \frac{-1}{2N} U^{p+1} \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \to \frac{-1}{2N} \int_{\mathbb{R}^N} U^{p+1} dy = \frac{-1}{2N} \sigma_N C_N$$

by the dominated convergence theorem, where $C_N = \int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} dr = \frac{\Gamma(N/2)^2}{2\Gamma(N)}$. Thus, (RHS of (2.5)) $\times \|u_\varepsilon\|^{\frac{N-2}{N-4}-1}$ is

$$
\begin{aligned}
& c_0\|u_\varepsilon\|^{-1+\frac{N-2}{N-4}} \int_\Omega \left(\frac{\partial K}{\partial x_i}\right) u_\varepsilon^{p_\varepsilon} v_\varepsilon dx \\
&= c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i}\right) (\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon)\tilde{u}_\varepsilon^{p_\varepsilon}\tilde{v}_\varepsilon dy \\
&= c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \left(\frac{\partial K}{\partial x_i}\right) (\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon)\sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\
&= -c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j}\left\{ \left(\frac{\partial K}{\partial x_i}\right) (\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon) \right\} w_\varepsilon(y) dy \\
&= -c_0\|u_\varepsilon\|^{\frac{N-2}{N-4}+p_\varepsilon-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x)\Big|_{x=\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}}+x_\varepsilon} w_\varepsilon(y) dy \\
&\to -c_0 \left(\lim_{\varepsilon\to 0}\|u_\varepsilon\|^{(\frac{N-3}{4})\varepsilon}\right) \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) \left(\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} w_\varepsilon(y) dy\right) \\
&= \frac{1}{2N} c_0 \sigma_N C_N \sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0).
\end{aligned}
$$

Thus we have

$$\sum_{j=1}^N a_j \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0) = 0.$$

By our assumption of the nondegeneracy of $x_0$, the matrix $\left(\frac{\partial^2 K}{\partial x_i \partial x_j}\right)(x_0)$ is invertible. Therefore we obtain that $a_j = 0$ for all $j = 1, \cdots, N$. Thus we have proved Step 2. □

**Proof of Step 3.**

By Step 1 and Step 2, we have obtained that the limit function $\lim_{\varepsilon\to 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$ and $|y_\varepsilon| \to \infty$, because the above convergence $\tilde{v}_\varepsilon \to v_0 \equiv 0$ is uniform on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 3.1. This proves Theorem 1.1. □

**Acknowledgements.** We thank the referee for his/her careful reading of our manuscript. Part of this work was supported by JSPS Grant-in-Aid for Scientific Research (C), No. 20540216.

## References

[1] T. Bartsch, and T. Weth, and M. Willem: *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator,* Calc. Var. and PDE. **18** (2003) 253-268.

[2] G. Caristi, and E. Mitidieri: *Harnack inequality and applications to solutions of biharmonic equations,* Operator Theory: Advances and Appl. **168** (2006) 1-26.

[3] K. S. Chou, and D. Geng: *Asymptotics of positive solutions for a biharmonic equation involving critical exponent,* Diff. Int. Eq. **13** (2000) 921-940.

[4] M. Grossi: *A nondegeneracy result for some nonlinear elliptic equation,* NoDEA **12** (2005) 227-241.

[5] C. S. Lin: *A classification of solutions of a conformally invariant fourth order equation in* $\mathbb{R}^n$, Comment. Math. Helv. **73** (1998) 206-231.

[6] F. Takahashi: *Asymptotic behavior of least energy solutions for a biharmonic problem with nearly critical growth,* Asymptotic Anal. **60** (2008) 213-226.

[7] F. Takahashi: *An asymptotic nondegeneracy result for a biharmonic equation with the nearly critical growth,* NoDEA, **15** (2008) 623-637.

[8] F. Takahashi: *Nondegeneracy of least energy solutions for an elliptic problem with nearly critical nonlinearity,* to appear in Proc. Royal Soc. Edinburgh.