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Long time asymptotics of heat kernels for one dimensional elliptic operators with periodic coefficients

By

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§ 1. Results

This is a summary of the paper [11]. Let $L$ be an elliptic differential operator on $\mathbb{R}$:

$$L = -\frac{d}{dx}(a(x)\frac{d}{dx}) + c(x),$$

where $a(x)$ and $c(x)$ are real-valued periodic functions with period 1. We assume that $a(x)$ is absolutely continuous, and $a(x) \geq \alpha$ for some positive constant $\alpha$, and that $c \in L_{loc}^{1}(\mathbb{R})$. For each $\gamma \in \mathbb{R}$, let $L_{\gamma}$ be the operator on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by

$$L_{\gamma} = e^{\gamma x}Le^{-\gamma x} = -\left(\frac{d}{dx} - \gamma\right)a(x)\left(\frac{d}{dx} - \gamma\right) + c(x).$$

Regard $L_{\gamma}$ as a closed operator on $L^{2}(\mathbb{T})$ with the domain $D(L_{\gamma}) = \{u \in H^{1}(\mathbb{T}); L_{\gamma}u \in L^{2}(\mathbb{T})\}$. \{ $L_{\gamma}$ \} $\in \mathbb{R}$ is a holomorphic family of type (B). In [1], [5], and [10], the following results were studied. $L_{\gamma}$ has an eigenvalue $E(\gamma) \in \mathbb{R}$ of multiplicity one such that the corresponding eigenspace is generated by a positive function $u_{\gamma}(x)$ on $\mathbb{T}$. Since $(L_{\gamma})^{*} = L_{-\gamma}$, we have $E(\gamma) = E(-\gamma)$. We call $E(\gamma)$ the principal eigenvalue of $L_{\gamma}$.

**Facts.** The function $E(\gamma)$ is real analytic. $E''(\gamma) < 0$ for any $\gamma \in \mathbb{R}$.

$$E''(\gamma) = -2/d_{1}^{2} + O(|\gamma|^{-1}) \text{ as } |\gamma| \to \infty.$$  

$$\sup_{\gamma \in \mathbb{R}} E(\gamma) = E(0) = \inf \sigma(L).$$

For each $k \in \mathbb{R}$, we consider the equation with respect to $\gamma$

$$E'(\gamma) = -k.$$
From the facts above we see that the equation has a unique solution $\gamma = \gamma_k$. For $\gamma \in \mathbb{R}$, let $p_\gamma(x, x')$ be the positive continuous function on $\mathbb{R} \times \mathbb{R}$ defined by
\[ p_\gamma(x, x') = p_\gamma(x', x) = \frac{u_\gamma(x)u_{-\gamma}(x')}{\int_0^1 u_\gamma(x)u_{-\gamma}(x)dx}, \quad x' \leq x. \]
Put
\[ d(x, x') := \int_a^x a(s)^{-1/2}ds \quad \text{and} \quad d_1 := d(1, 0). \]
Let $e^{-tL}(x, x')$ be the integral kernel of the semigroup $e^{-tL}$ of $L$, where $L$ is regarded as the selfadjoint operator on $L^2(\mathbb{R})$ with the domain $D(L) = \{ u \in H^1(\mathbb{R}); Lu \in L^2(\mathbb{R}) \}$.

Our main theorem is the following.

**Theorem 1.1.** $e^{-tL}(x, x')$ admits the following asymptotics as $t \to \infty$:
\[ e^{-tL}(x, x') = \exp[-tE(\gamma_k) - (x - x')\gamma_k] \frac{p_{\gamma_k}(x, x')}{(-2\pi^2 tE''(\gamma_k))^{1/2}} (1 + O(t^{-1})), \]
where $k = d(x, x')/d_1t$. Here the term $O(t^{-1})$ satisfies the estimate $|O(t^{-1})| \leq C/t$ for some constant $C > 0$ independent of $t > 1$, and $x, x' \in \mathbb{R}$.

Let $\lambda_0 = \inf \sigma(L)$.

**Corollary 1.2.** There exists a positive constant $C$ such that
\[ \sup_{x, x' \in \mathbb{R}} |e^{-t(L - \lambda_0)}(x, x') - \frac{p_0(x, x')}{(4\pi m_a t)^{1/2}} \exp\left(-\frac{(x - x')^2}{4m_at}\right)| \leq \frac{C}{t}, \]
where
\[ m_a := -\frac{E''(0)}{2} = \left( \int_0^1 u_0^2(x)dx \int_0^1 u_0^{-2}(x)a^{-1}(x)dx \right)^{-1}. \]
In particular, if $c(x) \equiv 0$, then
\[ \sup_{x, x' \in \mathbb{R}} |e^{-tL}(x, x') - \frac{1}{(4\pi \overline{a} t)^{1/2}} \exp\left(-\frac{(x - x')^2}{4\overline{a}t}\right)| \leq \frac{C}{t}, \]
where $\overline{a} := (\int_0^1 a^{-1}(x)dx)^{-1}$.

If $c(x) \equiv 0$, the corollary is known in [2].

**Corollary 1.3.** There exist positive constants $C$ and $T$ such that for any $t \geq T$ and $x, x' \in \mathbb{R}$,
\[ \frac{e^{-C(x-x')^2/t}}{C\sqrt{t}} \leq e^{-t(L - \lambda_0)}(x, x') \leq C\frac{e^{-(x-x')^2/Ct}}{\sqrt{t}}. \]
There are some other results for long-time behaviors of the heat kernel for operators with periodic coefficients (e.g. [9], [8], [4]).

§ 2. Outline of the proof of Theorem

Corresponding to the equation \((L - \lambda)\phi = 0\), we consider the equation

\[
\frac{d}{dx} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} 0 & a(x)^{-1} \\ (c(x) - \lambda) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}
\]

for \(\lambda \in \mathbb{C}\). By the standard iteration method of ordinary differential equations, we can find unique solutions \((\varphi_1(x, \lambda), \varphi_2(x, \lambda))\) and \((\psi_1(x, \lambda), \psi_2(x, \lambda))\) to (2.1) with the initial conditions

\[
\begin{pmatrix} \varphi_1(0, \lambda) \\ \varphi_2(0, \lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_1(0, \lambda) \\ \psi_2(0, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

respectively, in the space of \(\mathbb{C}^2\)-valued absolutely continuous functions. We can also see that \(\varphi_j(x, \lambda)\) and \(\psi_j(x, \lambda)\) are \(C([-R, R])\)-valued entire functions of \(\lambda\) for any \(R\).

Since \(L\) is of limit-point type at \(\infty\) and at \(-\infty\), for \(\lambda\) non-real, the equation

\[
(L - \lambda)x = 0
\]

has unique solutions in \(L^2(0, \infty)\) and in \(L^2(-\infty, 0)\) up to a constant multiple, and these solutions can be written as

\[
\begin{align*}
\chi_+(x, \lambda) &= \varphi_1(x, \lambda) + m_+(\lambda)\psi_1(x, \lambda) \in L^2(0, \infty), \\
\chi_-(x, \lambda) &= \varphi_1(x, \lambda) + m_-(\lambda)\psi_1(x, \lambda) \in L^2(-\infty, 0),
\end{align*}
\]

where \(m_{\pm}(\lambda)\) are analytic functions on \(\mathbb{C}_+ \cup \mathbb{C}_-\) such that \(\pm \text{Im} \lambda \text{Im} m_{\pm}(\lambda) > 0\). The functions \(\chi_{\pm}(x, \lambda)\) and \(m_{\pm}(\lambda)\) depend on the coefficients \(a(x)\) and \(c(x)\), and we sometimes denote these by \(\chi_{\pm}(x, \lambda; a, c)\) and \(m_{\pm}(\lambda; a, c)\). Let \(w(\lambda)\) be the analytic function on \(\mathbb{C}_+ \cup \mathbb{C}_-\) defined by

\[
w(\lambda) := \int_0^1 \frac{m_+(\lambda; a_s, c_s)}{a(s)} ds,
\]

where \(a_s(x) = a(x + s)\) and \(c_s(x) = c(x + s)\). Let \(D(\lambda)\) be the discriminant of (2.2):

\[
D(\lambda) := \varphi_1(1, \lambda) + \psi_2(1, \lambda).
\]

It is known that there exists a sequence of real numbers

\[-\infty < \lambda_0 < \mu_0 < \mu_1 < \lambda_1 < \lambda_2 < \cdots\]

such that it tends to infinity, \(D(\lambda_n) = 2\), \(D(\mu_n) = -2\), and the spectrum of \(L\) is written as

\[
\sigma(L) = \{\lambda \in \mathbb{R}; |D(\lambda)| \leq 2\} = \cup_{n=0}^\infty ([\lambda_{2n}, \mu_{2n}] \cup [\mu_{2n+1}, \lambda_{2n+1}]).
\]

We summarize some basic facts. The proofs are based on the theories of Johnson and Morser [3] and Marchenko [6].
Lemma 2.1. (i) \( \chi_\pm(x, \lambda) \) are expressed as
\[
\chi_\pm(x, \lambda) = \exp \int_0^x \frac{m_\pm(\lambda; a_s, c_s)}{a(s)} ds = e^{\pm w(\lambda)x} u_\pm(x, \lambda) = e^{\pm w(\lambda)d(x,0)/d_1} v_\pm(x, \lambda),
\]
where \( u_\pm(x, \lambda) \) and \( v_\pm(x, \lambda) \) are some periodic functions of \( x \) with period 1.

(ii) \( w(\lambda), \; \chi_\pm(x, \lambda), \; u_\pm(x, \lambda) \) and \( v_\pm(x, \lambda) \) extend analytically from \( \mathbf{C}_+ \) to \( \mathbf{C}_- \) through the interval \((-\infty, \lambda_0)\).

(iii) \( \text{Im} \; w(\lambda) > 0 \) for \( \lambda \) non-real, and \( \text{Re} \; w(\lambda) < 0 \) for \( \lambda \in \mathbf{C} \setminus [\lambda_0, \infty) \).

(iv) \( w(\lambda) < 0 \) for \( \lambda < \lambda_0 \), and \( \lim_{\lambda \uparrow \lambda_0} w(\lambda) = 0 \).

(v) For any \( \varepsilon > 0 \) and \( \delta > 0 \) there exists a positive constant \( T \) such that for any \( \kappa = \sigma + i\tau \in \mathbf{C}_+ \) with \( \delta|\sigma| \leq \tau \) and \( \tau \geq T \),
\[
\sup_{x \in \mathbf{R}} |v_\pm(x, \kappa^2) - (a(0)/a(x))^{1/4}| \leq \varepsilon.
\]

(vi) The function \( w(\lambda) \) admits the following expressions:
\[
w(\lambda) = \alpha + \frac{1}{\pi} \int_{\lambda_0}^{\infty} \left( \frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) n(t) dt, \quad \lambda \in \mathbf{C} \setminus [\lambda_0, \infty),
\]
where \( \alpha \in \mathbf{R} \) and \( n(t) := \text{Im} \; w(t+i0) \) is a continuous nondecreasing function such that \( \text{supp} \; dn = \sigma(L) \).

(vii) Let \( \theta(\zeta) \) be the analytic function on \( \mathbf{C}_+ \) defined by
\[
\theta(\zeta) := -iw(\zeta^2 + \lambda_0), \quad \zeta \in \mathbf{C}_+.
\]
Then \( \theta(\zeta) \) admits the expression:
\[
\theta(\zeta) = d_1 \zeta + \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{t-\zeta} - \frac{t}{1+t^2} \right) \tilde{\gamma}(t) dt, \quad \zeta \in \mathbf{C}_+,
\]
where \( \tilde{\gamma}(t) = \text{Im} \; \theta(t+i0) \) is a continuous, bounded, even, and non-negative function such that \( \text{supp} \; \tilde{\gamma}(t) = \tilde{\rho}, \) where \( \tilde{\rho} := \cup_{n=0}^{\infty} [(\nu_{2n}, \nu_{2n+1}) \cup (\nu'_{2n+1}, \nu'_{2n+2}) \cup (-\nu_{2n+1}, -\nu_{2n}) \cup (-\nu'_{2n+2}, -\nu'_{2n+1})] \) with \( \nu_n := \sqrt{\mu_n - \lambda_0} \) and \( \nu'_n := \sqrt{\lambda_n - \lambda_0} \). Furthermore, \( \theta(\zeta) \) extends analytically through the interval \((-\nu_0, \nu_0)\) from \( \mathbf{C}_+ \) to a domain in \( \mathbf{C}_- \).

We write
\[
\theta(\zeta) = \theta_1(\xi, \eta) + i\theta_2(\xi, \eta), \quad \zeta = \xi + i\eta.
\]
By Lemma 2.1(vii) we can see that \( \theta_1(\xi, \eta) \) and \( \theta_2(\xi, \eta) \) are odd and even in \( \xi \), respectively. We have \( w(\lambda_0 - \eta^2) = -\theta_2(0, \eta) \) for \( \eta \geq 0 \). By Lemma 2.1(i), \( (L - \chi_\pm(x, \lambda) = e^{\pm w(\lambda)x} u_\pm(x, \lambda) = e^{\pm w(\lambda)d(x,0)/d_1} v_\pm(x, \lambda),
\]
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\[ \lambda e^{\pm w(\lambda)x}u_\pm(x, \lambda) = 0, \]
and so \((L - \lambda_0 + \eta^2) \exp(\mp \theta_2(0, \eta)x)u_\pm(x, \lambda_0 - \eta^2) = 0\). Since \(m_\pm(\lambda_0 - \eta^2)\) are real, \(u_\pm(x, \lambda_0 - \eta^2)\) are positive. Thus \(\lambda_0 - \eta^2\) is the principal eigenvalue of \(L(\pm i \theta_2(0, \eta))\), i.e.,

\[
(2.3) \quad E(\theta_2(0, \eta)) = E(-\theta_2(0, \eta)) = \lambda_0 - \eta^2,
\]
and the corresponding eigenspace is generated by \(u_\pm(x, \lambda_0 - \eta^2)\).

By Lemma 2.1(iv), for \(\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)\), \(\chi_+(x, \lambda) \in L^2(0, \infty)\) and \(\chi_-(x, \lambda) \in L^2(-\infty, 0)\). Note that for \(\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)\), the relation

\[
[\chi_+(x, \lambda), \chi_-(x, \lambda)]w'(\lambda) = \int_0^1 u_+(x, \lambda)u_-(x, \lambda)dx
\]
holds (see [7]), and the left-hand side does not vanish. Therefore, the integral kernel \(R_\lambda(x, y)\) of the resolvent \((L - \lambda)^{-1}\) is expressed as

\[
R_\lambda(x, x') = R_\lambda(x', x) = \frac{\chi_+(x, \lambda)\chi_-(x', \lambda)}{[\chi_+(x, \lambda), \chi_-(x, \lambda)]} = w'(\lambda)e^{w(\lambda)(x-x')}
\]

\[
= \frac{\chi_+(x, \lambda)\chi_-(x', \lambda)}{\int_0^1 u_+(x, \lambda)u_-(x, \lambda)dx}
\]

\[
(2.4) \quad = \frac{\chi_+(x, \lambda)\chi_-(x', \lambda)}{\int_0^1 u_+(x, \lambda)u_-(x, \lambda)dx}, \quad x' \leq x,
\]
for \(\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)\).

By the inverse Laplace transformation we have

\[
e^{-t(L-\lambda_0)} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(L - \lambda_0 - \lambda)^{-1}d\lambda,
\]
where \(C = C_1 \cup C_2 \cup C_3\) and

\[
C_1 = \{re^{-i\theta_0}; r : \infty \rightarrow r_0\}, \quad C_2 = \{r_0e^{i\theta}; \theta : 2\pi - \theta_0 \rightarrow \theta_0\}, \quad C_3 = \{re^{i\theta_0}; r : r_0 \rightarrow \infty\},
\]
for \(0 < \theta_0 < \pi/2\) and \(r_0 > 0\). Changing the integral variable so that \(\lambda = \zeta^2\) and taking account of the estimate \(\|L - \lambda_0 \lambda^{-1}\| \leq C/|\lambda|\) for \(\lambda \in \mathbb{C}\) with \(0 < \theta_0 \leq \arg \lambda \leq 2\pi - \theta_0\), we have by Cauchy’s integral theorem

\[
e^{-t(L-\lambda_0)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\zeta^2}(L - \lambda_0 - \zeta^2)^{-1}2\zeta d\zeta,
\]
where \(\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\) and

\[
\Gamma_1 = \{re^{i(\pi-\theta_1)} - a_1 + ia_2; r : \infty \rightarrow 0\}, \quad \Gamma_2 = \{\xi + ia_2; \xi : -a_1 \rightarrow a_1\},
\]
\[
\Gamma_3 = \{re^{i\theta_1} + a_1 + ia_2; r : 0 \rightarrow \infty\},
\]
for any $a_1, a_2 > 0$ and $0 < \theta_1 < \pi/4$. We consider the function

$$e(t, x, x') := \frac{1}{2\pi i} \int_{\Gamma} e^{-t\zeta^2} R_{\zeta^2+\lambda_0}(x, x') 2\zeta d\zeta$$

in the case $x' \leq x$. Let $q_\zeta(x, x')$ be the $C(T \times T)$-valued analytic function of $\zeta \in \mathbb{C}_+$ defined by

$$q_\zeta(x, x') = q_\zeta(x', x) = \frac{v_+(x, \zeta^2 + \lambda_0)v_-(x', \zeta^2 + \lambda_0)}{\int_0^1 v_+(x, \zeta^2 + \lambda_0)v_-(x, \zeta^2 + \lambda_0)dx}, \quad x' \leq x.$$  

By (2.4) and (2.5) we have

$$e(t, x, x') = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\phi_k(\zeta)} q_\zeta(x, x') \frac{\theta'(\zeta)}{d(x, x')} d\zeta,$$

where

$$\phi_k(\zeta) := \zeta^2 - ik\theta(\zeta), \quad k := d(x, x')/d_1 t \geq 0.$$ 

In the following we regard $k$ as a nonnegative parameter.

We can write

$$\phi_k(\zeta) = \text{Re}\phi_k(\xi, \eta) + i\text{Im}\phi_k(\xi, \eta) = (\xi^2 - \eta^2 + k\theta_2(\xi, \eta)) + i(2\xi\eta - k\theta_1(\xi, \eta)).$$

Put $l_0 := v_0/2$. In the region $\{ (\xi, \eta); |\xi| \leq l_0, \eta \geq 0 \}$, we shall find a critical point of $\phi_k(\zeta)$, that is a solution to

$$2\xi + k\partial_\xi \theta_2(\xi, \eta) = 0, \quad -2\eta + k\partial_\eta \theta_2(\xi, \eta) = 0.$$  

Lemma 2.2. There exists a constant $C > 0$ such that for $|\xi| \leq l_0$ and $\eta \geq 0$,

$$C^{-1} \leq \partial_\eta \theta_2(\xi, \eta) \leq C, \quad C^{-1} \leq \partial_\xi[\eta/\partial_\eta \theta_2(\xi, \eta)] \leq C.$$ 

Let $k \geq 0$. Since $\theta_2(\xi, \eta)$ is an even function in $\xi$, the first equation of (2.7) is satisfied for $\xi = 0, \eta \geq 0$. By Lemma 2.2 there exists a unique solution $\eta = \eta(\xi, k)$ to the second equation of (2.7), i.e.

$$\frac{\eta}{\partial_\eta \theta_2(\xi, \eta)} = \frac{k}{2}.$$ 

Thus $(0, \eta(0, k))$ is a critical point of $\phi_k$.

We replace $\Gamma$ with $\Gamma'_1 \cup \Gamma'_2$, where

$$\Gamma'_1 := \{ \xi + i\eta(\xi, k); \xi : -l_0 \rightarrow l_0 \},$$

$$\Gamma'_2 := \{ \xi + i[-(\xi + l_0)/2 + \eta(-l_0, k)]; \xi : -\infty \rightarrow -l_0 \}$$

$$\cup \{ \xi + i[(\xi - l_0)/2 + \eta(l_0, k)]; \xi : l_0 \rightarrow \infty \}.$$
We can see that the amplitude function \( q_{\zeta}(x, x') \theta'(\zeta) \) in the integrand in (2.6) satisfies
\[
|q_{\zeta}(x, x') \theta'(\zeta)| \leq C \text{ for } \zeta \in \Gamma_1' \cup \Gamma_2', \text{ for some } C > 0 \text{ independent of } k \geq 0 \text{ and } x, x' \in \mathbb{R}.
\]
For the integral on \( \Gamma_1' \) we have
\[
(2.8) \quad \frac{1}{2\pi} \int_{\Gamma_1'} e^{-t \phi_k(\zeta)} q_{\zeta}(x, x') \theta'(\zeta) d\zeta
\]
\[
= \frac{1}{2\pi} \int_{-l_0}^{l_0} \exp[-t\phi_k(\xi + i\eta(\xi, k))]q_{\xi+i\eta(\xi, k)}(x, x')\theta'(\xi + i\eta(\xi, k))(1 + i\partial_\xi \eta(\xi, k)) d\xi.
\]
The following holds.

**Lemma 2.3.**

(i) There exists a constant \( C > 0 \) independent of \( k \geq 0 \) such that
\[
C^{-1} \leq \partial_\xi^2[\phi_k(\xi + i\eta(\xi, k))]|_{\xi=0} = \partial_\xi^2[\text{Re} \phi_k(\xi, \eta(\xi, k))]|_{\xi=0} \leq C.
\]

(ii) For any integer \( n \geq 1 \) there exists a constant \( C_n > 0 \) such that for any \( \xi, |\xi| \leq l_0, \text{ and } k \geq 0, \)
\[
|\partial_\xi^n[\phi_k(\xi + i\eta(\xi, k))]| \leq C_n.
\]

(iii) For any integer \( n \geq 0 \) there exists a constant \( C_n > 0 \) such that for any \( \xi, |\xi| \leq l_0, \text{ and } k \geq 0 \text{ and } x, x' \in \mathbb{R}, \)
\[
|\partial_\xi^n[q_{\xi+i\eta(\xi, k)}(x, x') \theta'(\xi + i\eta(\xi, k))(1 + i\partial_\xi \eta(\xi, k))]| \leq C_n.
\]

Put \( \eta_k := \eta(0, k) \). Taking account of this lemma, we apply a saddle point method to the right-hand side of (2.8). Then it is equal to
\[
\exp[-t \phi_k(i\eta_k)] \left( \frac{2\pi t \partial_\xi^2[\text{Re} \phi(\xi, \eta(\xi, k))]|_{\xi=0})^{1/2}}{\partial_\xi^2[\text{Re} \phi_k(\xi, \eta(\xi, k))]|_{\xi=0}} q_{i\eta_k}(x, x') \theta_2(0, \eta_k) (1 + O(t^{-1}))
\]
\[
= \frac{\exp[-t(k \theta_2(0, \eta_k) - \eta_k^2)]}{(2\pi t(2 - k \partial_\eta^2 \theta_2(0, \eta_k)))^{1/2}} q_{i\eta_k}(x, x') \theta_2(0, \eta_k)(1 + O(t^{-1}))
\]
where \( O(t^{-1}) \) satisfies \( |O(t^{-1})| \leq C t^{-1} \) with a constant \( C > 0 \) independent of \( t > 1 \) and \( x, x' \in \mathbb{R} \).

On the other hand, for the integral on \( \Gamma_2' \) we can show that
\[
\left| \int_{\Gamma_2'} e^{-t \phi_k(\zeta)} q_{\zeta}(x, x') \theta'(\zeta) d\zeta \right| \leq C e^{-Ct} \exp[-t(k \theta_2(0, \eta_k) - \eta_k^2)],
\]
for some positive constant \( C \). Thus we have proved
\[
(2.9) \quad e(t, x, x') = \exp[-t(k \theta_2(0, \eta_k) - \eta_k^2)] \frac{\partial_\eta \theta_2(0, \eta_k) q_{i\eta_k}(x, x')}{(2\pi t(2 - k \partial_\eta^2 \theta_2(0, \eta_k)))^{1/2}} (1 + O(t^{-1})).
\]

By (2.3) and Lemma 2.2 we have
\[
E'(\theta_2(0, \eta)) = -2\eta/\partial_\eta \theta_2(0, \eta), \quad \eta \geq 0.
\]
This implies that $\eta(0, k)$ is the unique solution to $E'(\theta_2(0, \eta)) = -k$. By Lemma 2.2, the map $\eta \in [0, \infty) \mapsto \theta_2(0, \eta) \in [0, \infty)$ is a one-to-one correspondence, therefore the equation $E'(\gamma) = -k \leq 0$ has a unique solution $\gamma = \gamma_k$, that is

$$(2.10) \quad \gamma_k = \theta_2(0, \eta(0, k)).$$

This together with (2.3) yields that

$$(2.11) \quad E(\gamma_k) = \lambda_0 - \eta(0, k)^2,$$

$$(2.12) \quad -E''(\gamma_k) = \frac{[2 - k \partial_\eta^2 \theta_2(0, \eta(0, k))]/\partial_\eta \theta_2(0, \eta(0, k))^2}{(2-k \partial_\eta \theta_2(0, \eta(0, k)))^2}. $$

Furthermore, note that

$$u_{\pm \gamma_k}(x) = c_{\pm} u_{\pm}(x, E(\gamma_k)) = c_{\pm} e^{\pm \gamma_k (x - d(x,0)/d_1)} v_{\pm}(x, E(\gamma_k))$$

for some constants $c_{\pm}$. Hence

$$(2.13) \quad p_{\gamma_k}(x, x') = e^{\gamma_k [(x-x')-d(x,x')/d_1]/d_1} q_{\eta_k}(x,x').$$

By (2.9-13) we have the desired asymptotics

$$e^{-tL}(x,x') = \exp[-t(E(\gamma_k) + k \gamma_k)] \frac{q_{\eta_k}(x,x')}{(-2 \pi t E''(\gamma_k))^{1/2}} (1 + O(t^{-1}))$$

$$= \exp[-tE(\gamma_k) - (x - x') \gamma_k] \frac{p_{\gamma_k}(x,x')}{(-2 \pi t E''(\gamma_k))^{1/2}} (1 + O(t^{-1})).$$

References


