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Dispersive estimates for Schrödinger equations in dimension one

By

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Abstract

We study the time decay of scattering solutions to one dimensional Schrödinger equations and prove a weighted dispersive estimate with stronger time decay than the case of unweighted estimates. Furthermore an asymptotic expansion in time of scattering solutions is given.

§1. Introduction

This report is concerned with the time decay of scattering solutions $e^{-itH}P_{ac}u$ to Schrödinger equations

$$i\partial_{t}u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x), \quad x \in \mathbb{R}$$

is a one dimensional Schrödinger operator and $P_{ac}$ is the projection onto the absolutely continuous subspace for $H$. We assume that $V(x)$ is real valued and $V \in L_{1}^{1}$ at least, where $L_{\gamma}^{p}$ is the weighted $L^{p}$ space:

$$L_{\gamma}^{p} := \{ f | \langle x \rangle^{\gamma}f \in L^{p}(\mathbb{R}) \}, \quad ||f||_{L_{\gamma}^{p}} := ||\langle x \rangle^{\gamma}f||_{L^{p}},$$

$$\langle x \rangle := \sqrt{1+|x|^{2}}, \quad 1 \leq p \leq \infty, \quad \gamma \in \mathbb{R}.$$ 

Under the above conditions, $H$ is self-adjoint on $L^{2}(\mathbb{R})$ with form domain $H^{1}(\mathbb{R})$ and the absolutely continuous spectrum of $H$ is the half line $[0, \infty)$, the singular continuous spectrum of $H$ is absent, and the eigenvalues of $H$ are strictly negative.
Let $H_0 = -\frac{d^2}{dx^2}$. It is well known that the propagator $e^{-itH_0}$ has the following asymptotic expansion in $\mathcal{B}(L^2_s, L^2_{-s})$ for sufficiently large $s > 0$:

$$e^{-itH_0} = t^{-\frac{1}{2}}C_{-1} + t^{-\frac{3}{2}}C_0 + \cdots, \quad t \to \infty,$$

where $C_{j-1}$ are given by

$$C_{j-1}u(x) := \frac{1}{\sqrt{4\pi i}j!(4i)^{j}} \int_{\mathbb{R}} (i|x-y|)^{2j}u(y)dy,$$

and $\mathcal{B}(X, Y)$ denotes the Banach spaces of bounded operators from $X$ to $Y$. The asymptotic expansion of $e^{-itH}$ as $t \to \infty$ in $\mathcal{B}(L^2_s, L^2_{-s})$ was proved by Murata [14], under the assumption that $|V(x)| \leq C\langle x \rangle^{-\sigma}$ for sufficiently large $\sigma > 7$. In higher dimension, such expansions were proved by [11, 10, 14]. In this paper we prove an asymptotic expansion of $e^{-itH}P_{ac}$ in $\mathcal{B}(L^1_s, L^\infty_{-s})$ as $t \to \infty$.

In order to state our results, we introduce a few notations. The Jost functions $f_{\pm}(\lambda, x)$ are the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \quad \lambda, x \in \mathbb{R}$$

satisfying following asymptotic conditions

$$|f_{\pm}(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm\infty.$$ 

It is well known (see [3]) that if $V \in L^1_1$, then the Jost functions are uniquely defined for all $\lambda, x \in \mathbb{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x) \cdot \partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of $x$ and does not vanish for $\lambda \neq 0$.

**Definition 1.1.** We say that the potential $V$ is of generic type if $W(0) \neq 0$ and is of exceptional type if $W(0) = 0$. We also say that zero is a resonance of $H$ if the potential $V$ is of exceptional type.

**Theorem 1.2.** Let $m$ be a positive integer. Suppose that $V \in L^1_{2m}$ and $V$ is of generic type, or $V \in L^1_{2m+2}$ and $V$ is of exceptional type. Let

$$s = \begin{cases} 
2m - 1 & \text{if } V \text{ is of generic type,} \\
2m & \text{if } V \text{ is of exceptional type.}
\end{cases}$$

Then

$$||\langle x \rangle^{-s}(e^{-itH}P_{ac} - P_{m-1})u||_{L^\infty} \leq Ct^{-\frac{1}{2}-m}||\langle x \rangle^{s}u||_{L^1}.$$
for all \( t > 0 \), where \( P_{m-1} \) is given by
\[
P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.
\]

Furthermore, the coefficients \( C_{j-1} \) satisfy the following:
(1) If \( V \) is of generic type, then \( C_{-1} \equiv 0 \), rank \( C_{j-1} \leq 2j \) and
\[
||\langle x\rangle^{-2j+1}C_{j-1}u||_{L^{\infty}} \leq C||\langle x\rangle^{2j-1}u||_{L^{1}}, \quad j = 1, 2, ... m - 1.
\]
(2) If \( V \) is of exceptional type, then rank \( C_{j-1} \leq 2j + 1 \) and
\[
||\langle x\rangle^{-2j}C_{j-1}u||_{L^{\infty}} \leq C||\langle x\rangle^{2j}u||_{L^{1}}, \quad j = 0, 1, ... m - 1.
\]

Remark. In exceptional case, we can compute \( C_{-1} \) explicitly:
\[
C_{-1}u = \frac{1}{\sqrt{4\pi i}}\langle u, f_{0}\rangle f_{0},
\]
where \( f_{0} \) is a non trivial bounded solution to the equation \( Hf = 0 \) normalized as
\[
\lim_{x\to\pm\infty} \frac{1}{2}(|f_{0}(x)|^{2} + |f_{0}(-x)|^{2}) = 1.
\]

Dispersive estimates for Schrödinger equations have been studied by many authors. 
\( L^{1} - L^{\infty} \) estimates:
(1.2)
\[
||e^{-itH}P_{ac}u||_{L^{\infty}} \leq C|t|^{-\frac{d}{2}}||u||_{L^{1}},
\]
was proved by [12] under the suitable decay and regularity assumptions for \( V \). Later (1.2) has been proved by [6, 7, 9, 15, 17, 18, 19, 21, 22] under various assumptions on the potential \( V \) and the assumption that zero is neither an eigenvalue nor a resonance of \( H \). When zero is either an eigenvalue or a resonance of \( H \), similar estimates were studied by [4, 23]. Such estimates are very important since (1.2) implies Strichartz estimates which can be applied to prove well-posedness for nonlinear Schrödinger equations. The time decay \( t^{-\frac{1}{2}} \) in \( d = 1 \) is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Goldberg [8] also proved (1.1) with \( m = 1 \) under the assumptions that \( V \in L_{3}^{1} \) and \( V \) is of generic type, or \( V \in L_{4}^{1} \) and \( V \) is of exceptional type. Compared to his results, our assumptions on the potential \( V(x) \), which are used in Theorem 1.2, are weaker.

\section{Sketch of the proof}

To prove Theorem 1.2, we use the representation
\[
\langle e^{-itH}P_{ac}u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it\lambda^{2}} \lambda\langle R(\lambda^{2} + i0)u, v \rangle d\lambda,
\]
where we denote an extended resolvent \( \lim_{\epsilon \to +0} (H - (\lambda + i\epsilon)^2)^{-1} \) by \( R(\lambda^2 + i0) \). We split the propagator into high and low energy parts. For the high energy part, the following proposition holds.

**Proposition 2.1.** Suppose \( V \in L^1_N, N \in \mathbb{N} \) and set \( \lambda_0 := \|V\|_{L^1_N} \). Let \( \chi \) be an even smooth cut-off function such that \( \chi(\lambda) = 1 \) for \( |\lambda| \leq \lambda_0 \) and \( \chi(\lambda) = 0 \) for \( |\lambda| \geq 2\lambda_0 \). Then

\[
\|\langle x \rangle^{-N} e^{-itH}(1 - \chi(\sqrt{H}))u\|_{L^\infty} \leq Ct^{-\frac{1}{2}-N}\|\langle x \rangle^N u\|_{L^1}, \quad u \in L^2 \cap L^1_N
\]

for all \( t > 0 \).

**Proof.** Set \( \tilde{\chi}(\lambda) := 1 - \chi(\lambda) \). Let \( \eta \) be an even smooth function on \( \mathbb{R} \) such that \( \eta(\lambda) = 1 \) if \( |\lambda| \leq 1 \), \( \eta(\lambda) = 0 \) if \( |\lambda| \geq 2 \) and let \( \tilde{\chi}_L(\lambda) := \eta(\lambda/L)\tilde{\chi}(\lambda) \) for \( L \geq 1 \). Using the Born series expansion of \( R(\lambda^2 + i0) \), we have

\[
\langle e^{itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle = \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{1}{(-2i)^{n+1}} \int_{\mathbb{R}^{n+3}} e^{-it\lambda^2 + i\lambda} \sum_{j=0}^n |x_{j+1} - x_j| \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} u(x_0) \prod_{j=1}^n V(x_j)v(x_{n+1}) d\lambda dx_0...dx_{n+1}.
\]

Let us consider the oscillatory integral

\[
\Phi(t, a) = \int_{\mathbb{R}} e^{-it\lambda^2 + ia\lambda} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} d\lambda, \quad a \in \mathbb{R}.
\]

Using integration by parts and Fourier inversion formula, we have

\[
|\Phi(t, a)| \leq Ct^{-\frac{1}{2}-N}\||\mathcal{F}P_{\lambda} (e^{ia\lambda} \tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^1} \leq Ct^{-\frac{1}{2}-N} \sum_{k=0}^{N} |a|^{N-k} \||\mathcal{F}\partial_{\lambda}^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1},
\]

where \( P_{\lambda} = \frac{\partial}{\partial_{\lambda}} \frac{1}{\lambda} \) and \( \mathcal{F} \) is the Fourier transform with respect to \( \lambda \). A direct computation yields

\[
\sup_{L \geq 10} \sup_{0 \leq k \leq N} \||\mathcal{F}\partial_{\lambda}^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1} \leq C(n + 2N)^N \lambda_0^{-n-N}
\]

for \( n \geq 0 \). Since

\[
\sum_{j=0}^{n} |x_{j+1} - x_j| \leq \prod_{j=1}^{n} (1 + |x_j|),
\]

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we conclude that
\[
\sup_{L \geq 1} |\langle e^{itH} \tilde{\chi}_{L}(\sqrt{H})u, v \rangle|
\leq Ct^{-\frac{1}{2}-N} \sum_{n=0}^{\infty} 2^{-n} n^{N} \lambda_{0}^{-n-N} \| (1+|x|)^{N} V \|_{L^{1}} \| (1+|x|)^{N} u \|_{L^{1}} \| (1+|x|)^{N} v \|_{L^{1}}
\leq Ct^{-\frac{1}{2}-N} \| \langle x \rangle^{N} u \|_{L^{1}} \| \langle x \rangle^{N} v \|_{L^{1}}
\]
for all \( t > 0 \).

\section{2.1. Jost Functions}

Using the Jost functions and their Wronskian, the integral kernel of the resolvent \( R(\lambda^{2} \pm i0) \) is given by
\[
f_{+}(\pm \lambda, y)f_{-}(\pm \lambda, x)/W(\pm \lambda) \chi_{\{x<y\}} + f_{+}(\pm \lambda, x)f_{-}(\pm \lambda, y)/W(\pm \lambda) \chi_{\{x>y\}}.
\]
In this subsection, we collect results on the Jost functions \( f_{\pm}(\lambda, x) \) needed later. \( f_{+}(\lambda, x) \) and \( f_{+}(-\lambda, x) \) are independent for \( \lambda \neq 0 \) since their Wronskian
\[
W[f_{+}(\lambda, \cdot), f_{+}(-\lambda, \cdot)] := f_{+}(\lambda, x) \cdot \partial_{x}f_{+}(-\lambda, x) - \partial_{x}f_{+}(\lambda, x) \cdot f_{+}(-\lambda, x)
= \lim_{x \rightarrow \pm \infty} [e^{i\lambda x}(-i\lambda)e^{-i\lambda x} - i\lambda e^{i\lambda x}e^{-i\lambda x}]
= -2i\lambda \neq 0.
\]
Similarly \( W[f_{-}(\lambda, \cdot), f_{-}(-\lambda, \cdot)] = 2i\lambda \). These imply the relations
\[
T(\lambda)f_{-}(\lambda, x) = R_{1}(\lambda)f_{+}(\lambda, x) + f_{+}(-\lambda, x),
T(\lambda)f_{+}(\lambda, x) = R_{2}(\lambda)f_{-}(\lambda, x) + f_{-}(-\lambda, x),
\]
where \( T(\lambda), R_{1}(\lambda) \) and \( R_{2}(\lambda) \) are called the transmission and reflection coefficients, respectively. It is well known that
\[
|T(\lambda)|^{2} + |R_{j}(\lambda)|^{2} = 1, \quad j = 1, 2.
\]
Furthermore the following holds (see [3], [13] and [1]).

Lemma 2.2. (1) Suppose that \( V \) is of generic type and \( V \in L_{N}^{1}, \ N \geq 1 \). Then \( T, R_{1} \) and \( R_{2} \in C^{N-1}(\mathbb{R}) \) and for \( 1 \leq k \leq N-1 \),
\[
|\partial_{\lambda}^{k}T(\lambda)| + |\partial_{\lambda}^{k}R_{1}(\lambda)| + |\partial_{\lambda}^{k}R_{2}(\lambda)| \leq C(\lambda)^{-1}, \ \lambda \in \mathbb{R}.
\]
Furthermore, we have
\[
T(\lambda) = \alpha \lambda + o(\lambda), \quad \alpha \neq 0, \quad \lambda \rightarrow 0,
R_{1}(0) = R_{2}(0) = -1.
\]
(2) Suppose that $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$. Then $T$, $R_1$ and $R_2 \in C^{N-2}(\mathbb{R})$ and (2.2) holds for $1 \leq k \leq N-2$. Furthermore, as $\lambda \to 0$, we have

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$

$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1),$$

$$R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0, x) \neq 0$.

The following inequality was proved by Artbazar and Yajima [1]:

$$|\partial^k_{\lambda} f_\pm(\lambda, x)| \leq C\langle x\rangle^k (1+\max(\mp x, 0)), \quad (\lambda, x) \in \mathbb{R}^2,$$

for $0 \leq k \leq N-1$. Using Lemma 2.2, we can improve the above estimates.

**Lemma 2.3.** (1) Suppose that $V$ is of generic type and $V \in L^1_N$, $N \geq 1$. Then

$$|\partial^k_{\lambda} (T(\lambda) f_\pm(\lambda, x))| \leq C\langle x\rangle^k, \quad \lambda \neq 0, \quad x \in \mathbb{R},$$

for $0 \leq k \leq N-1$. If in addition $N \geq 2$, then

$$|\partial^k_{\lambda} f_\pm(0, x)| \leq C\langle x\rangle^k, \quad x \in \mathbb{R},$$

for $1 \leq k \leq N-1$ and $k$ odd.

(2) Suppose that $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

$$|\partial^k_{\lambda} f_\pm(\lambda, x)| \leq C\langle x\rangle^k, \quad (\lambda, x) \in \mathbb{R}^2,$$

for $0 \leq k \leq N-2$.

Lemma 2.3 follows from (2.1), (2.3) and Lemma 2.2, and we omit the proof.

We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi, x) := \int_{\mathbb{R}} e^{2i\lambda \xi} (m_\pm(\lambda, x) - 1) d\lambda,$$

where $m_\pm(\lambda, x) := e^{\mp i\lambda x} f_\pm(\lambda, x)$ are the modified Jost functions. Then the function $B_+(\xi, x)$ satisfies the Marchenko equation:

$$B_+(\xi, x) = \int_{x+\xi}^{\infty} V(\sigma) d\sigma + \int_{0}^{\xi} d\zeta \int_{x+\xi-\zeta}^{\infty} V(\sigma) B_+(\zeta, \sigma) d\sigma,$$

and $B_-(\xi, x)$ also satisfies a corresponding equation. It is well known (see [3]) that if
$V \in L^1_1$, then the function $B_+(\xi, x)$ is well defined for $\xi \geq 0$, $x \in \mathbb{R}$ and satisfies the following estimates

\begin{equation}
|B_+(\xi, x)| \leq e^{\gamma(x)} \eta(x + \xi), \quad \xi \geq 0, \ x \in \mathbb{R},
\end{equation}

where $\eta(x) = \int_x^\infty |V(\sigma)|d\sigma$, $\gamma(x) = \int_x^\infty (\sigma - x)|V(\sigma)|d\sigma$. $B_-(\xi, x)$ also satisfies a similar inequalities. Since $\gamma(x)$ is dominated by $||V||_{L^1_1}$ for all $x \geq 0$, (2.5) implies $||B_+(\cdot, x)||_{L^1_1}$ is bounded for $x \geq 0$ with the bound depending on $||V||_{L^1_1}$. Similarly $||B_-(\cdot, x)||_{L^1_1}$ is bounded for $x \leq 0$. Iterating Marchenko equations, we can prove the following (see [2]).

**Lemma 2.4.** Let $N \in \mathbb{N}$, $N \geq 1$ and suppose $V \in L^1_N$. Then the functions $B_{\pm}(\xi, x)$ satisfy the estimates

\begin{equation}
||B_{\pm}(\cdot, x)||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \ x \in \mathbb{R},
\end{equation}

where $C$ depends on $||V||_{L^1_N}$.

The following Lemma follows from Lemma 2.4 and the representation

$$W(\lambda) = -2i\lambda + \int_\mathbb{R} V(\sigma) m_+(\lambda, \sigma)d\sigma.$$

**Lemma 2.5.** Let $\chi \in C^\infty_0(\mathbb{R})$.

1. Let $N \in \mathbb{N}$, $N \geq 1$ and assume that $V \in L^1_N$ and $V$ is of generic type, then

$$\mathcal{F}(\frac{\chi}{W}) \in L^1_{N-1}.$$

2. Let $N \in \mathbb{N}$, $N \geq 2$ and assume that $V \in L^1_N$ and $V$ is of exceptional type, then

$$\mathcal{F}(\frac{\lambda \chi}{W}) \in L^1_{N-2}.$$

Here $W(\lambda)$ is the Wronskian of the Jost functions.

Lemma 2.4, Lemma 2.5 and (2.1) imply the following.

**Lemma 2.6.** Let $\chi \in C^\infty(\mathbb{R})$. Suppose that $V \in L^1_N$, $N \geq 1$. Then

\begin{equation}
||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(x)^{N-1}(1 + \max(\mp x, 0)), \ x \in \mathbb{R}.
\end{equation}

Furthermore,

1. If $V$ is of generic type, then

$$||\mathcal{F}(\chi(\cdot)T(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(x)^{N-1}, \ x \in \mathbb{R}.$$  

2. If $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

$$||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-2}} \leq C(x)^{N-2}, \ x \in \mathbb{R}.$$
§ 2.2. The Low Energy Estimates

In this subsection, we prove the following Proposition to complete the proof of Theorem 1.2.

**Proposition 2.7.** Let \( m \in \mathbb{N} \) and let \( \chi \) be an even smooth cut-off function such that \( \chi(\lambda) = 1 \) close to zero. Suppose that \( V \in L_{2m}^{1} \) and \( V \) is of generic type, or \( V \in L_{2m+2}^{1} \) and \( V \) is of exceptional type. Let \( P_{m-1} \) as in Theorem 1.2. Let

\[
s = \begin{cases} 
2m - 1 & \text{if } V \text{ is of generic type}, \\
2m & \text{if } V \text{ is of exceptional type}.
\end{cases}
\]

Then

\[
\| \langle x \rangle^{-s} (e^{-i t H} \chi(\sqrt{H}) P_{ac} - P_{m-1}) u \|_{L^\infty} \leq C t^{-\frac{1}{2} - m} \| \langle x \rangle^{s} u \|_{L^{1}}
\]

for all \( t > 0 \).

**Proof.** We consider the generic case. The proof of the exceptional case is similar and we omit the proof. Set

\[
K(\lambda, x, y) := T(\lambda) f_+(\lambda, y) f_-(\lambda, x), \quad G(\lambda, x, y) := \frac{K(\lambda, x, y)}{\lambda}.
\]

We start from the representation

\[
\langle e^{-i t H} \chi(\sqrt{H}) P_{ac} u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-i t \lambda^2} \lambda \chi(\lambda) \langle R(\lambda^2 + i0) u, v \rangle d\lambda
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} e^{-i t \lambda^2} \lambda \chi(\lambda) \tilde{G}(\lambda, x, y) d\lambda \right) u(y) \overline{v(x)} dy dx,
\]

where \( \tilde{G}(\lambda, x, y) \) denotes the kernel of \(-2iR(\lambda^2 + i0)\) and is given by

\[
(2.8) \quad \tilde{G}(\lambda, x, y) = \begin{cases} 
G(\lambda, x, y) & \text{for } x < y, \\
G(\lambda, y, x) & \text{for } x > y.
\end{cases}
\]

Consider the integral

\[
(2.9) \quad I(t, G) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i t \lambda^2} \lambda \chi(\lambda) G(\lambda, x, y) d\lambda,
\]

for \( x < y \). The proof for the case \( x > y \) is analogous. Integrating by parts \( (2.9) \), we have

\[
(2.10) \quad I(t, G) = \frac{1}{4\pi i t} \int_{\mathbb{R}} e^{-i t \lambda^2} \partial_\lambda (\chi(\lambda) G(\lambda, x, y)) d\lambda.
\]
The case $m = 1$: it suffice to prove that
\[ |I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \ x < y. \]

Using the Fourier inversion formula, we obtain
\[ |I(t, G)| \leq Ct^{-\frac{3}{2}} \| (\mathcal{F} \partial_{\lambda} \chi(\cdot) G(\cdot, x, y)) \|_{L^1} \]
for all $t > 0$ and $x < y$, where $\mathcal{F}$ is the Fourier transform with respect to $\lambda$. By Young’s inequality, Lemma 2.5 (1) and Lemma 2.6, we have
\[ \| (\mathcal{F} \partial_{\lambda} \chi G)(\cdot, x, y) \|_{L^1} \leq C \langle x \rangle \langle y \rangle, \ x < y \]
and this implies
\[ |I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle \]
for $x < y$.

The case $m \geq 2$: Applying the stationary phase theorem to the integral (2.10), we have
\[ I(t, G) = \frac{1}{\sqrt{\pi i}} \sum_{j=1}^{m-1} \frac{t^{-\frac{1}{2}-j}}{(j-1)!(4i)^j} (\partial_{\lambda}^{2j-1} G)(0, x, y) + t^{-\frac{1}{2}-m} S_{m-1}(t, G) \]
with
\[ |S_{m-1}(t, G)| \leq C \| (\mathcal{F} \partial_{\lambda}^{2m-1} \chi G)(\cdot, x, y) \|_{L^1} \]
\[ \leq C \langle x \rangle^{2m-1} \langle y \rangle^{2m-1}, \ x < y. \]
For the last inequality, we used Lemma 2.5 (1) and Lemma 2.6. We now define the coefficients $C_{j-1}$: since $T(0) = 0$, we have
\[ (\partial_{\lambda}^{2j-1} G)(0, x, y) = \frac{1}{2j} (\partial_{\lambda}^{2j} K)(0, x, y). \]
Considering the fact that
\[ (\partial_{\lambda}^{2j} K)(0, x, y) = (\partial_{\lambda}^{2j} K)(0, y, x), \ x, y \in \mathbb{R}, \ j = 1, 2, ..., m, \]
we define $C_{j-1}$ and $P_{m-1}$ by
\[ C_{j-1} u(x) := \frac{1}{\sqrt{4\pi ij!(4i)^j}} \int_{\mathbb{R}} (\partial_{\lambda}^{2j} K)(0, x, y) u(y) dy, \ x \in \mathbb{R}, \]
(2.11)
\[ P_{m-1} := \sum_{j=0}^{m-1} t^{-\frac{3}{2}-j} C_{j-1}. \]
Then we have
\[ \| \langle x \rangle^{-2m+1}(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u \|_{L^\infty} \leq Ct^{-\frac{1}{2}-m}\| \langle x \rangle^{2m-1}u \|_{L^1}. \]

By the definition of $C_{j-1}$ and Corollary 2.3 (1), we can see that
\[ \text{rank } C_{j-1} \leq 2j, \]
and there exists $C > 0$ such that
\[ \| \langle x \rangle^{-2j+1}C_{j-1}u \|_{L^\infty} \leq C\| \langle x \rangle^{2j-1}u \|_{L^1}. \]

In particular,
\[ C_{-1} \equiv 0. \]

These complete the proof. \hfill \square

References

Dispersive estimates for Schrödinger equations in dimension one


