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Dispersive estimates for Schrödinger equations in dimension one

By

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Abstract

We study the time decay of scattering solutions to one dimensional Schrödinger equations and prove a weighted dispersive estimate with stronger time decay than the case of unweighted estimates. Furthermore an asymptotic expansion in time of scattering solutions is given.

§ 1. Introduction

This report is concerned with the time decay of scattering solutions $e^{-itH}P_{ac}u$ to Schrödinger equations

$$i\partial_t u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x),\ x \in \mathbb{R}$$

is a one dimensional Schrödinger operator and P_{ac} is the projection onto the absolutely continuous subspace for H. We assume that $V(x)$ is real valued and $V \in L^1_1$ at least, where L^p_γ is the weighted L^p space:

$$L^p_\gamma := \{f \mid \langle x\rangle^\gamma f \in L^p(\mathbb{R})\},\ ||f||_{L^p_\gamma} := ||\langle x\rangle^\gamma f||_{L^p},$$

$$\langle x\rangle := \sqrt{1+|x|^2},\ 1 \le p \le \infty,\ \gamma \in \mathbb{R}.$$

Under the above conditions, H is self-adjoint on $L^2(\mathbb{R})$ with form domain $H^1(\mathbb{R})$ and the absolutely continuous spectrum of H is the half line $[0,\infty)$, the singular continuous spectrum of H is absent, and the eigenvalues of H are strictly negative.

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Let $H_0 = -\frac{d^2}{dx^2}$. It is well known that the propagator e^{-itH_0} has the following asymptotic expansion in $\mathcal{B}(L^2_s, L^2_{-s})$ for sufficiently large $s > 0$:

$$e^{-itH_0} = t^{-\frac{1}{2}}C_{-1} + t^{-\frac{3}{2}}C_0 + \cdots,\ t \to \infty,$$

where C_{j-1} are given by

$$C_{j-1}u(x) := \frac{1}{\sqrt{4\pi i}j!(4i)^j}\int_{\mathbb{R}}(i|x-y|)^{2j}u(y)dy,$$

and $\mathcal{B}(X, Y)$ denotes the Banach spaces of bounded operators form X to Y. The asymptotic expansion of e^{-itH} as $t \to \infty$ in $\mathcal{B}(L^2_s, L^2_{-s})$ was proved by Murata [14], under the assumption that $|V(x)| \le C\langle x\rangle^{-\sigma}$ for sufficiently large $\sigma > 7$. In higher dimension, such expansions were proved by [11, 10, 14]. In this paper we prove an asymptotic expansion of $e^{-itH}P_{ac}$ in $\mathcal{B}(L^1_s, L^\infty_{-s})$ as $t \to \infty$.

In order to state our results, we introduce a few notations. *The Jost functions* $f_\pm(\lambda, x)$ are the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \quad \lambda, x \in \mathbb{R}$$

satisfying following asymptotic conditions

$$|f_\pm(\lambda, x) - e^{\pm i\lambda x}| \to 0 \text{ as } x \to \pm\infty.$$

It is well known (see [3]) that if $V \in L^1_1$, then the Jost functions are uniquely defined for all $\lambda, x \in \mathbb{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x)\cdot\partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x)\cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of x and does not vanish for $\lambda \ne 0$.

Definition 1.1. We say that the potential V is of *generic type* if $W(0) \ne 0$ and is of *exceptional type* if $W(0) = 0$. We also say that *zero is a resonance of H* if the potential V is of exceptional type.

Theorem 1.2. *Let m be a positive integer. Suppose that $V \in L^1_{2m}$ and V is of generic type, or $V \in L^1_{2m+2}$ and V is of exceptional type. Let*

$$s = \begin{cases} 2m-1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then

$$||\langle x\rangle^{-s}(e^{-itH}P_{ac} - P_{m-1})u||_{L^\infty} \le Ct^{-\frac{1}{2}-m}||\langle x\rangle^s u||_{L^1} \tag{1.1}$$

for all $t > 0$*, where* P_{m-1} *is given by*

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.$$

Furthermore, the coefficients C_{j-1} *satisfy the following:*
(1) *If* V *is of generic type, then* $C_{-1} \equiv 0$, $\operatorname{rank} C_{j-1} \leq 2j$ *and*

$$||\langle x\rangle^{-2j+1} C_{j-1} u||_{L^\infty} \leq C||\langle x\rangle^{2j-1} u||_{L^1},\ j = 1, 2, ...m-1.$$

(2) *If* V *is of exceptional type, then* $\operatorname{rank} C_{j-1} \leq 2j+1$ *and*

$$||\langle x\rangle^{-2j} C_{j-1} u||_{L^\infty} \leq C||\langle x\rangle^{2j} u||_{L^1},\ j = 0, 1, ...m-1.$$

Remark. In exceptional case, we can compute C_{-1} explicitly:

$$C_{-1}u = \frac{1}{\sqrt{4\pi i}} \langle u, f_0\rangle f_0,$$

where f_0 is a non trivial bounded solution to the equation $Hf = 0$ normalized as

$$\lim_{x\to+\infty} \frac{1}{2}(|f_0(x)|^2 + |f_0(-x)|^2) = 1.$$

Dispersive estimates for Schrödinger equations have been studied by many authors. $L^1 - L^\infty$ estimates:

$$||e^{-itH} P_{ac} u||_{L^\infty} \leq C|t|^{-\frac{d}{2}} ||u||_{L^1}, \tag{1.2}$$

was proved by [12] under the suitable decay and regularity assumptions for V. Later (1.2) has been proved by [6, 7, 9, 15, 17, 18, 19, 21, 22] under various assumptions on the potential V and the assumption that zero is neither an eigenvalue nor a resonance of H. When zero is either an eigenvalue or a resonance of H, similar estimates were studied by [4, 23]. Such estimates are very important since (1.2) implies Strichartz estimates which can be applied to prove well-posedness for nonlinear Schrödinger equations. The time decay $t^{-\frac{1}{2}}$ in $d = 1$ is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Goldberg [8] also proved (1.1) with $m = 1$ under the assumptions that $V \in L^1_3$ and V is of generic type, or $V \in L^1_4$ and V is of exceptional type. Compared to his results, our assumptions on the potential $V(x)$, which are used in Theorem 1.2, are weaker.

§ 2. Sketch of the proof

To prove Theorem 1.2, we use the representation

$$\langle e^{-itH} P_{ac} u, v\rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \langle R(\lambda^2 + i0)u, v\rangle d\lambda,$$

where we denote an extended resolvent $\lim_{\varepsilon\to+0}(H-(\lambda+i\varepsilon)^2)^{-1}$ by $R(\lambda^2+i0)$. We split the propagator into high and low energy parts. For the high energy part, the following proposition holds.

Proposition 2.1. *Suppose $V\in L^1_N, N\in\mathbb{N}$ and set $\lambda_0:=||V||_{L^1_N}$. Let χ be an even smooth cut-off function such that $\chi(\lambda)=1$ for $|\lambda|\le\lambda_0$ and $\chi(\lambda)=0$ for $|\lambda|\ge 2\lambda_0$. Then*

$$||\langle x\rangle^{-N}e^{-itH}(1-\chi(\sqrt{H}))u||_{L^\infty}\le Ct^{-\frac{1}{2}-N}||\langle x\rangle^N u||_{L^1},\ u\in L^2\cap L^1_N$$

for all $t>0$.

Proof. Set $\tilde\chi(\lambda):=1-\chi(\lambda)$. Let η be an even smooth function on $\mathbb{R}$ such that $\eta(\lambda)=1$ if $|\lambda|\le 1$, $\eta(\lambda)=0$ if $|\lambda|\ge 2$ and let $\tilde\chi_L(\lambda):=\eta(\lambda/L)\tilde\chi(\lambda)$ for $L\ge 1$. Using the Born series expansion of $R(\lambda^2+i0)$, we have

$$\begin{aligned}\langle e^{itH}\tilde\chi_L(\sqrt{H})u,v\rangle=\frac{1}{\pi i}\sum_{n=0}^{\infty}\frac{1}{(-2i)^{n+1}}\int_{\mathbb{R}^{n+3}}e^{-it\lambda^2+i\lambda\sum_{j=0}^{n}|x_{j+1}-x_j|}\frac{\tilde\chi_L(\lambda)}{\lambda^n}\\ \times u(x_0)\prod_{j=1}^{n}V(x_j)v(x_{n+1})d\lambda dx_0...dx_{n+1}.\end{aligned}$$

Let us consider the oscillatory integral

$$\Phi(t,a)=\int_{\mathbb{R}}e^{-it\lambda^2+ia\lambda}\frac{\tilde\chi_L(\lambda)}{\lambda^n}d\lambda,\ a\in\mathbb{R}.$$

Using integration by parts and Fourier inversion formula, we have

$$\begin{aligned}|\Phi(t,a)|&\le Ct^{-\frac{1}{2}-N}||\mathcal{F}P_\lambda^N(e^{ia\lambda}\tilde\chi_L(\lambda)\lambda^{-n})||_{L^1}\\&\le Ct^{-\frac{1}{2}-N}\sum_{k=0}^{N}|a|^{N-k}||\mathcal{F}\partial_\lambda^k(\tilde\chi_L(\lambda)\lambda^{-n-N})||_{L^1},\end{aligned}$$

where $P_\lambda=\frac{\partial}{\partial_\lambda}\frac{1}{\lambda}$ and $\mathcal{F}$ is the Fourier transform with respect to λ. A direct computation yields

$$\sup_{L\ge1}\sup_{0\le k\le N}||\mathcal{F}\partial_\lambda^k(\tilde\chi_L(\lambda)\lambda^{-n-N})||_{L^1}\le C(n+2N)^N\lambda_0^{-n-N}$$

for $n\ge 0$. Since

$$\sum_{j=0}^{n}|x_{j+1}-x_j|\le\prod_{j=1}^{n}(1+|x_j|),$$

we conclude that

$$\begin{aligned}
&\sup_{L\geq 1} |\langle e^{itH}\tilde{\chi}_L(\sqrt{H})u, v\rangle| \\
&\leq Ct^{-\frac{1}{2}-N}\sum_{n=0}^{\infty} 2^{-n}n^N\lambda_0^{-n-N}||(1+|x|)^N V||_{L^1}^n ||(1+|x|)^N u||_{L^1}||(1+|x|)^N v||_{L^1} \\
&\leq Ct^{-\frac{1}{2}-N}||\langle x\rangle^N u||_{L^1}||\langle x\rangle^N v||_{L^1}
\end{aligned}$$

for all $t > 0$. □

§ 2.1. Jost Functions

Using the Jost functions and their Wronskian, the integral kernel of the resolvent $R(\lambda^2 \pm i0)$ is given by

$$\frac{f_+(\pm\lambda, y)f_-(\pm\lambda, x)}{W(\pm\lambda)}\chi_{\{x<y\}} + \frac{f_+(\pm\lambda, x)f_-(\pm\lambda, y)}{W(\pm\lambda)}\chi_{\{x>y\}}.$$

In this subsection, we collect results on the Jost functions $f_\pm(\lambda, x)$ needed later. $f_+(\lambda, x)$ and $f_+(-\lambda, x)$ are independent for $\lambda \neq 0$ since their Wronskian

$$\begin{aligned}
W[f_+(\lambda,\cdot), f_+(-\lambda,\cdot)] &:= f_+(\lambda, x)\cdot\partial_x f_+(-\lambda, x) - \partial_x f_+(\lambda, x)\cdot f_+(-\lambda, x) \\
&= \lim_{x\to+\infty}[e^{i\lambda x}(-i\lambda)e^{-i\lambda x} - i\lambda e^{i\lambda x}e^{-i\lambda x}] \\
&= -2i\lambda \neq 0.
\end{aligned}$$

Similarly $W[f_-(\lambda,\cdot), f_-(-\lambda,\cdot)] = 2i\lambda$. These imply the relations

$$\begin{aligned}
T(\lambda)f_-(\lambda, x) &= R_1(\lambda)f_+(\lambda, x) + f_+(-\lambda, x), \\
T(\lambda)f_+(\lambda, x) &= R_2(\lambda)f_-(\lambda, x) + f_-(-\lambda, x),
\end{aligned} \tag{2.1}$$

where $T(\lambda)$, $R_1(\lambda)$ and $R_2(\lambda)$ are called the *transmission* and *reflection* coefficients, respectively. It is well known that

$$|T(\lambda)|^2 + |R_j(\lambda)|^2 = 1, \quad j = 1, 2.$$

Furthermore the following holds (see [3], [13] and [1]).

Lemma 2.2. (1) *Suppose that V is of generic type and $V \in L^1_N$, $N \geq 1$. Then T, R_1 and $R_2 \in C^{N-1}(\mathbb{R})$ and for $1 \leq k \leq N-1$,*

$$|\partial_\lambda^k T(\lambda)| + |\partial_\lambda^k R_1(\lambda)| + |\partial_\lambda^k R_2(\lambda)| \leq C\langle\lambda\rangle^{-1}, \ \lambda \in \mathbb{R}. \tag{2.2}$$

Furthermore, we have

$$\begin{aligned}
&T(\lambda) = \alpha\lambda + o(\lambda), \quad \alpha \neq 0, \quad \lambda \to 0, \\
&R_1(0) = R_2(0) = -1.
\end{aligned}$$

(2) *Suppose that V is of exceptional type and $V \in L^1_N$, $N \geq 2$. Then T, R_1 and $R_2 \in C^{N-2}(\mathbb{R})$ and (2.2) holds for $1 \leq k \leq N-2$. Furthermore, as $\lambda \to 0$, we have*

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$
$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1),$$
$$R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0,x) \neq 0$.

The following inequality was proved by Artbazar and Yajima [1]:

$$|\partial_\lambda^k f_\pm(\lambda,x)| \leq C\langle x\rangle^k (1+\max(\mp x, 0)),\ (\lambda,x) \in \mathbb{R}^2. \tag{2.3}$$

for $0 \leq k \leq N-1$. Using Lemma 2.2, we can improve the above estimates.

Lemma 2.3. (1) *Suppose that V is of generic type and $V \in L^1_N$, $N \geq 1$. Then*

$$|\partial_\lambda^k (T(\lambda) f_\pm(\lambda,x))| \leq C\langle x\rangle^k, \quad \lambda \neq 0,\ x \in \mathbb{R},$$

for $0 \leq k \leq N-1$. If in addition $N \geq 2$, then

$$|\partial_\lambda^k f_\pm(0,x)| \leq C\langle x\rangle^k, \quad x \in \mathbb{R},$$

for $1 \leq k \leq N-1$ and k odd.
(2) *Suppose that V is of exceptional type and $V \in L^1_N$, $N \geq 2$, then*

$$|\partial_\lambda^k f_\pm(\lambda,x)| \leq C\langle x\rangle^k, \quad (\lambda,x) \in \mathbb{R}^2.$$

for $0 \leq k \leq N-2$.

Lemma 2.3 follows from (2.1), (2.3) and Lemma 2.2, and we omit the proof.
We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi,x) := \int_{\mathbb{R}} e^{2i\lambda\xi}(m_\pm(\lambda,x)-1)d\lambda,$$

where $m_\pm(\lambda,x) := e^{\mp i\lambda x} f_\pm(\lambda,x)$ are the modified Jost functions. Then the function $B_+(\xi,x)$ satisfies the Marchenko equation:

$$B_+(\xi,x) = \int_{x+\xi}^\infty V(\sigma)d\sigma + \int_0^\xi d\zeta \int_{x+\xi-\zeta}^\infty V(\sigma)B_+(\zeta,\sigma)d\sigma \tag{2.4}$$

and $B_-(\xi,x)$ also satisfies a corresponding equation. It is well known (see [3]) that if

$V \in L^1_1$, then the function $B_+(\xi, x)$ is well defined for $\xi \geq 0,\ x \in \mathbb{R}$ and satisfies the following estimates

$$|B_+(\xi, x)| \leq e^{\gamma(x)}\eta(x+\xi), \quad \xi \geq 0,\ x \in \mathbb{R}, \tag{2.5}$$

where $\eta(x) = \int_x^\infty |V(\sigma)|d\sigma$, $\gamma(x) = \int_x^\infty (\sigma - x)|V(\sigma)|d\sigma$. $B_-(\xi, x)$ also satisfies a similar inequalities. Since $\gamma(x)$ is dominated by $||V||_{L^1_1}$ for all $x \geq 0$, (2.5) implies $||B_+(\cdot, x)||_{L^1}$ is bounded for $x \geq 0$ with the bound depending on $||V||_{L^1_1}$. Similarly $||B_-(\cdot, x)||_{L^1}$ is bounded for $x \leq 0$. Iterating Marchenko equations, we can prove the following (see [2]).

Lemma 2.4. *Let $N \in \mathbb{N}$, $N \geq 1$ and suppose $V \in L^1_N$. Then the functions $B_\pm(\xi, x)$ satisfy the estimates*

$$||B_\pm(\cdot, x)||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R}, \tag{2.6}$$

where C depends on $||V||_{L^1_N}$.

The following Lemma follows from Lemma 2.4 and the representation

$$W(\lambda) = -2i\lambda + \int_{\mathbb{R}} V(\sigma)m_+(\lambda, \sigma)d\sigma.$$

Lemma 2.5. *Let $\chi \in C_0^\infty(\mathbb{R})$.*
(1) Let $N \in \mathbb{N}$, $N \geq 1$ and assume that $V \in L^1_N$ and V is of generic type, then

$$\mathcal{F}(\frac{\chi}{W}) \in L^1_{N-1}.$$

(2) Let $N \in \mathbb{N}$, $N \geq 2$ and assume that $V \in L^1_N$ and V is of exceptional type, then

$$\mathcal{F}(\frac{\lambda\chi}{W}) \in L^1_{N-2}.$$

Here $W(\lambda)$ is the Wronskian of the Jost functions.

Lemma 2.4, Lemma 2.5 and (2.1) imply the following.

Lemma 2.6. *Let $\chi \in C_0^\infty(\mathbb{R})$. Suppose that $V \in L^1_N$, $N \geq 1$. Then*

$$||\mathcal{F}(\chi(\cdot)f_\pm(\cdot, x))||_{L^1_{N-1}} \leq C\langle x\rangle^{N-1}(1 + \max(\mp x, 0)),\ x \in \mathbb{R}. \tag{2.7}$$

Furthermore,
(1) If V is of generic type, then

$$||\mathcal{F}(\chi(\cdot)T(\cdot)f_\pm(\cdot, x))||_{L^1_{N-1}} \leq C\langle x\rangle^{N-1},\ x \in \mathbb{R}.$$

(2) If V is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

$$||\mathcal{F}(\chi(\cdot)f_\pm(\cdot, x))||_{L^1_{N-2}} \leq C\langle x\rangle^{N-2},\ x \in \mathbb{R}.$$

§ 2.2. The Low Energy Estimates

In this subsection, we prove the following Proposition to complete the proof of Theorem 1.2.

Proposition 2.7. *Let $m \in \mathbb{N}$ and let χ be an even smooth cut-off function such that $\chi(\lambda) = 1$ close to zero. Suppose that $V \in L^1_{2m}$ and V is of generic type, or $V \in L^1_{2m+2}$ and V is of exceptional type. Let P_{m-1} as in Theorem 1.2. Let*

$$s = \begin{cases} 2m-1 & \text{if } V \text{ is of generic type,} \\ 2m & \text{if } V \text{ is of exceptional type.} \end{cases}$$

Then

$$||\langle x\rangle^{-s}(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u||_{L^\infty} \le Ct^{-\frac{1}{2}-m}||\langle x\rangle^s u||_{L^1}$$

for all $t > 0$.

Proof. We consider the generic case. The proof of the exceptional case is similar and we omit the proof. Set

$$K(\lambda, x, y) := T(\lambda)f_+(\lambda, y)f_-(\lambda, x),\ G(\lambda, x, y) := \frac{K(\lambda, x, y)}{\lambda}.$$

We start from the representation

$$\begin{aligned} \langle e^{-itH}\chi(\sqrt{H})P_{ac}u, v\rangle &= \frac{1}{\pi i}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\langle R(\lambda^2 + i0)u, v\rangle d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)\tilde{G}(\lambda, x, y)d\lambda\right)u(y)\overline{v(x)}dydx, \end{aligned}$$

where $\tilde{G}(\lambda, x, y)$ denotes the kernel of $-2iR(\lambda^2 + i0)$ and is given by

$$\tilde{G}(\lambda, x, y) = \begin{cases} G(\lambda, x, y) & \text{for } x < y, \\ G(\lambda, y, x) & \text{for } x > y. \end{cases} \tag{2.8}$$

Consider the integral

$$I(t, G) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\lambda^2}\lambda\chi(\lambda)G(\lambda, x, y)d\lambda, \tag{2.9}$$

for $x < y$. The proof for the case $x > y$ is analogous. Integrating by parts (2.9), we have

$$I(t, G) = \frac{1}{4\pi it}\int_{\mathbb{R}} e^{-it\lambda^2}\partial_\lambda(\chi(\lambda)G(\lambda, x, y))d\lambda. \tag{2.10}$$

The case $m = 1$: it suffice to prove that

$$|I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \ x < y.$$

Using the Fourier inversion formula, we obtain

$$|I(t, G)| \leq Ct^{-\frac{3}{2}} ||(\mathcal{F}\partial_\lambda \chi(\cdot) G(\cdot, x, y)||_{L^1}$$

for all $t > 0$ and $x < y$, where $\mathcal{F}$ is the Fourier transform with respect to λ. By Young's inequality, Lemma 2.5 (1) and Lemma 2.6, we have

$$||(\mathcal{F}\partial_\lambda \chi G)(\cdot, x, y)||_{L^1} \leq C\langle x \rangle \langle y \rangle, \ x < y$$

and this implies

$$|I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle$$

for $x < y$.

The case $m \geq 2$: Applying the stationary phase theorem to the integral (2.10), we have

$$I(t, G) = \frac{1}{\sqrt{\pi i}} \sum_{j=1}^{m-1} \frac{t^{-\frac{1}{2}-j}}{(j-1)!(4i)^j} (\partial_\lambda^{2j-1} G)(0, x, y) + t^{-\frac{1}{2}-m} S_{m-1}(t, G)$$

with

$$\begin{aligned} |S_{m-1}(t, G)| &\leq C||(\mathcal{F}\partial_\lambda^{2m-1} \chi G)(\cdot, x, y)||_{L^1} \\ &\leq C\langle x \rangle^{2m-1} \langle y \rangle^{2m-1}, \ x < y. \end{aligned}$$

For the last inequality, we used Lemma 2.5 (1) and Lemma 2.6. We now define the coefficients C_{j-1}: since $T(0) = 0$, we have

$$(\partial_\lambda^{2j-1} G)(0, x, y) = \frac{1}{2j} (\partial_\lambda^{2j} K)(0, x, y).$$

Considering the fact that

$$(\partial_\lambda^{2j} K)(0, x, y) = (\partial_\lambda^{2j} K)(0, y, x), \ x, y \in \mathbb{R}, \ j = 1, 2, ..., m,$$

we define C_{j-1} and P_{m-1} by

$$\begin{aligned} C_{j-1} u(x) &:= \frac{1}{\sqrt{4\pi i} j!(4i)^j} \int_{\mathbb{R}} (\partial_\lambda^{2j} K)(0, x, y) u(y) dy, \ x \in \mathbb{R}, \\ P_{m-1} &:= \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}. \end{aligned} \tag{2.11}$$

Then we have

$$||\langle x\rangle^{-2m+1}(e^{-itH}\chi(\sqrt{H})P_{ac}-P_{m-1})u||_{L^\infty}\le Ct^{-\frac{1}{2}-m}||\langle x\rangle^{2m-1}u||_{L^1}.$$

By the definition of C_{j-1} and Corollary 2.3 (1), we can see that

$$\operatorname{rank} C_{j-1}\le 2j,$$

and there exists $C>0$ such that

$$||\langle x\rangle^{-2j+1}C_{j-1}u||_{L^\infty}\le C||\langle x\rangle^{2j-1}u||_{L^1}.$$

In particular,

$$C_{-1}\equiv 0.$$

These complete the proof. □

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