<table>
<thead>
<tr>
<th>Title</th>
<th>Dispersive estimates for Schrödinger equations in dimension one (Spectral and Scattering Theory and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MIZUTANI, HARUYA</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2010), B16: 141-151</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2010-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/176840">http://hdl.handle.net/2433/176840</a></td>
</tr>
<tr>
<td>Departmental Bulletin Paper</td>
<td></td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Dispersive estimates for Schrödinger equations in dimension one

By

HARUYA MIZUTANI*

Abstract

We study the time decay of scattering solutions to one dimensional Schrödinger equations and prove a weighted dispersive estimate with stronger time decay than the case of unweighted estimates. Furthermore an asymptotic expansion in time of scattering solutions is given.

§ 1. Introduction

This report is concerned with the time decay of scattering solutions $e^{-itH}P_{ac}u$ to Schrödinger equations

$$i\partial_{t}u = Hu,$$

where

$$H = -\frac{d^2}{dx^2} + V(x), \ x \in \mathbb{R}$$

is a one dimensional Schrödinger operator and $P_{ac}$ is the projection onto the absolutely continuous subspace for $H$. We assume that $V(x)$ is real valued and $V \in L^{1}_{1}$ at least, where $L^{p}_{\gamma}$ is the weighted $L^{p}$ space:

$$L^{p}_{\gamma} := \{ f \mid \langle x \rangle^{\gamma}f \in L^{p}(\mathbb{R}) \}, \quad ||f||_{L^{p}_{\gamma}} := \||\langle x \rangle^{\gamma}f||_{L^{p}},$$

$$\langle x \rangle := \sqrt{1+|x|^2}, \ 1 \leq p \leq \infty, \ \gamma \in \mathbb{R}.$$ 

Under the above conditions, $H$ is self-adjoint on $L^{2}(\mathbb{R})$ with form domain $H^{1}(\mathbb{R})$ and the absolutely continuous spectrum of $H$ is the half line $[0, \infty)$, the singular continuous spectrum of $H$ is absent, and the eigenvalues of $H$ are strictly negative.
Let $H_0 = -\frac{d^2}{dx^2}$. It is well known that the propagator $e^{-itH_0}$ has the following asymptotic expansion in $\mathcal{B}(L^2_s, L^2_{-s})$ for sufficiently large $s > 0$:

$$e^{-itH_0} = t^{-\frac{1}{2}}C_{-1} + t^{-\frac{3}{2}}C_0 + \cdots, \ t \to \infty,$$

where $C_{j-1}$ are given by

$$C_{j-1}u(x) := \frac{1}{\sqrt{4\pi i}j!(4i)^j} \int_{\mathbb{R}}(i|x-y|)^{2j}u(y)dy,$$

and $\mathcal{B}(X,Y)$ denotes the Banach spaces of bounded operators form $X$ to $Y$. The asymptotic expansion of $e^{-itH}$ as $t \to \infty$ in $\mathcal{B}(L^2_s, L^2_{-s})$ was proved by Murata [14], under the assumption that $|V(x)| \leq C\langle x\rangle^{-\sigma}$ for sufficiently large $\sigma > 7$. In higher dimension, such expansions were proved by [11, 10, 14]. In this paper we prove an asymptotic expansion of $e^{-itH}P_{ac}$ in $\mathcal{B}(L^1_s, L^\infty_{-s})$ as $t \to \infty$. 

In order to state our results, we introduce a few notations. The Jost functions $f_{\pm}(\lambda, x)$ are the solutions to the equation

$$-f''(\lambda, x) + V(x)f(\lambda, x) = \lambda^2 f(\lambda, x), \ \lambda, x \in \mathbb{R}$$

satisfying following asymptotic conditions

$$|f_{\pm}(\lambda, x) - e^{\pm i\lambda x}| \to 0 \as x \to \pm\infty.$$ 

It is well known (see [3]) that if $V \in L^1_1$, then the Jost functions are uniquely defined for all $\lambda, x \in \mathbb{R}$. We denote by $W(\lambda)$ their Wronskian

$$W(\lambda) := f_+(\lambda, x) \cdot \partial_x f_-(\lambda, x) - \partial_x f_+(\lambda, x) \cdot f_-(\lambda, x).$$

$W(\lambda)$ is independent of $x$ and does not vanish for $\lambda \neq 0$.

**Definition 1.1.** We say that the potential $V$ is of generic type if $W(0) \neq 0$ and is of exceptional type if $W(0) = 0$. We also say that zero is a resonance of $H$ if the potential $V$ is of exceptional type.

**Theorem 1.2.** Let $m$ be a positive integer. Suppose that $V \in L^1_{2m}$ and $V$ is of generic type, or $V \in L^1_{2m+2}$ and $V$ is of exceptional type. Let

$$s = \begin{cases} 
2m - 1 & \text{if } V \text{ is of generic type,} \\
2m & \text{if } V \text{ is of exceptional type.} 
\end{cases}$$

Then

$$||\langle x \rangle^{-s}(e^{-itH}P_{ac} - P_{m-1})u||_{L^\infty} \leq Ct^{-\frac{1}{2}-m}||\langle x \rangle^s u||_{L^1}.$$
for all $t > 0$, where $P_{m-1}$ is given by

$$P_{m-1} = \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.$$ 

Furthermore, the coefficients $C_{j-1}$ satisfy the following:

1. If $V$ is of generic type, then $C_{-1} \equiv 0$, rank $C_{j-1} \leq 2j$ and
   $$\| \langle x \rangle^{-2j+1} C_{j-1} u \|_{L^\infty} \leq C \| \langle x \rangle^{2j-1} u \|_{L^1}, \quad j = 1, 2, \ldots, m-1.$$ 
2. If $V$ is of exceptional type, then rank $C_{j-1} \leq 2j+1$ and
   $$\| \langle x \rangle^{-2j} C_{j-1} u \|_{L^\infty} \leq C \| \langle x \rangle^{2j} u \|_{L^1}, \quad j = 0, 1, \ldots, m-1.$$ 

Remark. In exceptional case, we can compute $C_{-1}$ explicitly:

$$C_{-1} u = \frac{1}{\sqrt{4 \pi i}} \langle u, f_0 \rangle f_0,$$

where $f_0$ is a non-trivial bounded solution to the equation $H f = 0$ normalized as

$$\lim_{x \to +\infty} \frac{1}{2} (|f_0(x)|^2 + |f_0(-x)|^2) = 1.$$ 

Dispersive estimates for Schrödinger equations have been studied by many authors. $L^1 - L^\infty$ estimates:

$$\| e^{-itH} P_{ac} u \|_{L^\infty} \leq C |t|^{-\frac{d}{2}} \| u \|_{L^1},$$
was proved by [12] under the suitable decay and regularity assumptions for $V$. Later (1.2) has been proved by [6, 7, 9, 15, 17, 18, 19, 21, 22] under various assumptions on the potential $V$ and the assumption that zero is neither an eigenvalue nor a resonance of $H$. When zero is either an eigenvalue or a resonance of $H$, similar estimates were studied by [4, 23]. Such estimates are very important since (1.2) implies Strichartz estimates which can be applied to prove well-posedness for nonlinear Schrödinger equations. The time decay $t^{-\frac{1}{2}}$ in $d = 1$ is not integrable at infinity and is unsuitable for applying to NLS. We hence are interested in a dispersive estimate whose time decay is integrable at infinity. Goldberg [8] also proved (1.1) with $m = 1$ under the assumptions that $V \in L^\frac{d}{2}_3$ and $V$ is of generic type, or $V \in L^\frac{d}{4}_1$ and $V$ is of exceptional type. Compared to his results, our assumptions on the potential $V(x)$, which are used in Theorem 1.2, are weaker.

§ 2. Sketch of the proof

To prove Theorem 1.2, we use the representation

$$\langle e^{-itH} P_{ac} u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it \lambda^2} \lambda \langle R(\lambda^2 + i0) u, v \rangle d\lambda,$$
where we denote an extended resolvent \( \lim_{\epsilon \to +0} (H - (\lambda + i\epsilon)^2)^{-1} \) by \( R(\lambda^2 + i0) \). We split the propagator into high and low energy parts. For the high energy part, the following proposition holds.

**Proposition 2.1.** Suppose \( V \in L^1_N, N \in \mathbb{N} \) and set \( \lambda_0 := ||V||_{L^1_N} \). Let \( \chi \) be an even smooth cut-off function such that \( \chi(\lambda) = 1 \) for \( |\lambda| \leq \lambda_0 \) and \( \chi(\lambda) = 0 \) for \( |\lambda| \geq 2\lambda_0 \). Then

\[
||\langle x \rangle^{-N} e^{-itH} (1 - \chi(\sqrt{H})) u ||_{L^\infty} \leq Ct^{-\frac{1}{2}-N} ||\langle x \rangle^N u||_{L^1}, \ u \in L^2 \cap L^1_N
\]

for all \( t > 0 \).

*Proof.* Set \( \tilde{\chi}(\lambda) := 1 - \chi(\lambda) \). Let \( \eta \) be an even smooth function on \( \mathbb{R} \) such that \( \eta(\lambda) = 1 \) if \( |\lambda| \leq 1 \), \( \eta(\lambda) = 0 \) if \( |\lambda| \geq 2 \) and let \( \tilde{\chi}_L(\lambda) := \eta(\lambda/L) \tilde{\chi}(\lambda) \) for \( L \geq 1 \). Using the Born series expansion of \( R(\lambda^2 + i0) \), we have

\[
\langle e^{itH} \tilde{\chi}_L(\sqrt{H}) u, v \rangle = \frac{1}{\pi i} \sum_{n=0}^\infty \frac{1}{(-2i)^{n+1}} \int_{\mathbb{R}^{n+3}} e^{-it\lambda^2 + i\lambda \sum_{j=0}^{n} |x_{j+1} - x_j|} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} 
\times u(x_0) \prod_{j=1}^{n} V(x_j) v(x_{n+1}) d\lambda dx_0 ... dx_{n+1}.
\]

Let us consider the oscillatory integral

\[
\Phi(t,a) = \int_{\mathbb{R}} e^{-it\lambda^2 + ia\lambda} \frac{\tilde{\chi}_L(\lambda)}{\lambda^n} d\lambda, \ a \in \mathbb{R}.
\]

Using integration by parts and Fourier inversion formula, we have

\[
|\Phi(t,a)| \leq Ct^{-\frac{1}{2}-N} ||\mathcal{F}P_\lambda^N(e^{ia\lambda} \tilde{\chi}_L(\lambda)\lambda^{-n})||_{L^1} 
\leq Ct^{-\frac{1}{2}-N} \sum_{k=0}^{N} |a|^{N-k} ||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1},
\]

where \( P_\lambda = \frac{\partial}{\partial_\lambda} \frac{1}{\lambda} \) and \( \mathcal{F} \) is the Fourier transform with respect to \( \lambda \). A direct computation yields

\[
\sup_{L \geq 1} \sup_{0 \leq k \leq N} ||\mathcal{F}\partial_\lambda^k(\tilde{\chi}_L(\lambda)\lambda^{-n-N})||_{L^1} \leq C(n + 2N)^N \lambda_0^{n-N}
\]

for \( n \geq 0 \). Since

\[
\sum_{j=0}^{n} |x_{j+1} - x_j| \leq \prod_{j=1}^{n} (1 + |x_j|),
\]

and
we conclude that

\[
\sup_{L \geq 1} |\langle e^{itH} \tilde{\chi}_L(\sqrt{H}) u, v \rangle| \\
\leq Ct^{-\frac{1}{2}-N} \sum_{n=0}^{\infty} 2^{-n} n^{-N} \lambda_0^{-n-N} ||(1+|x|)^N V||_{L^1} ||(1+|x|)^N u||_{L^1} ||(1+|x|)^N v||_{L^1} \\
\leq Ct^{-\frac{1}{2}-N} ||(x)^N u||_{L^1} ||(x)^N v||_{L^1}
\]

for all \( t > 0 \).

\[\square\]

\section{2.1. Jost Functions}

Using the Jost functions and their Wronskian, the integral kernel of the resolvent \( R(\lambda^2 \pm i0) \) is given by

\[
\frac{f_+(\lambda, y) f_-(\lambda, x)}{W(\pm \lambda)} \chi_{\{x<y\}} + \frac{f_+(\lambda, x) f_-(\lambda, y)}{W(\pm \lambda)} \chi_{\{x>y\}}.
\]

In this subsection, we collect results on the Jost functions \( f_\pm(\lambda, x) \) needed later. \( f_+(\lambda, x) \) and \( f_+(\lambda, x) \) are independent for \( \lambda \neq 0 \) since their Wronskian

\[
W[f_+(\lambda, \cdot), f_-(\lambda, \cdot)] = 2i\lambda.
\]

These imply the relations

\[
T(\lambda) f_-(\lambda, x) = R_1(\lambda) f_+(\lambda, x) + f_+(\lambda, x), \\
(2.1) \\
T(\lambda) f_+(\lambda, x) = R_2(\lambda) f_-(\lambda, x) + f_-(\lambda, x),
\]

where \( T(\lambda), R_1(\lambda) \) and \( R_2(\lambda) \) are called the \textit{transmission} and \textit{reflection} coefficients, respectively. It is well known that

\[
|T(\lambda)|^2 + |R_j(\lambda)|^2 = 1, \quad j = 1, 2.
\]

Furthermore the following holds (see [3], [13] and [1]).

\begin{lemma}
(1) Suppose that \( V \) is of generic type and \( V \in L^1_N \), \( N \geq 1 \). Then \( T, R_1 \) and \( R_2 \in C^{N-1}(\mathbb{R}) \) and for \( 1 \leq k \leq N - 1 \),

\[
|\partial^{k}T(\lambda)| + |\partial^{k}R_1(\lambda)| + |\partial^{k}R_2(\lambda)| \leq C(\lambda)^{-1}, \quad \lambda \in \mathbb{R}.
\]

Furthermore, we have

\[
T(\lambda) = \alpha \lambda + o(\lambda), \quad \alpha \neq 0, \quad \lambda \to 0, \\
R_1(0) = R_2(0) = -1.
\]

\end{lemma}
(2) Suppose that $V$ is of exceptional type and $V \in L_N^1$, $N \geq 2$. Then $T$, $R_1$ and $R_2 \in C^{N-2}(\mathbb{R})$ and (2.2) holds for $1 \leq k \leq N-2$. Furthermore, as $\lambda \to 0$, we have

$$T(\lambda) = \frac{2a}{1+a^2} + o(1),$$

$$R_1(\lambda) = \frac{1-a^2}{1+a^2} + o(1),$$

$$R_2(\lambda) = \frac{a^2-1}{1+a^2} + o(1),$$

with $a := \lim_{x \to -\infty} f_+(0, x) \neq 0$.

The following inequality was proved by Artbazar and Yajima [1]:

$$(2.3) \quad |\partial^k_\lambda f_\pm(\lambda, x)| \leq C\langle x\rangle^k (1 + \max(\mp x, 0)), \quad (\lambda, x) \in \mathbb{R}^2,$$

for $0 \leq k \leq N - 1$. Using Lemma 2.2, we can improve the above estimates.

**Lemma 2.3.**

(1) Suppose that $V$ is of generic type and $V \in L_N^1$, $N \geq 1$. Then

$$|\partial^k_\lambda (T(\lambda) f_\pm(\lambda, x))| \leq C\langle x\rangle^k, \quad \lambda \neq 0, \ x \in \mathbb{R},$$

for $0 \leq k \leq N - 1$. If in addition $N \geq 2$, then

$$|\partial^k_\lambda f_\pm(0, x)| \leq C\langle x\rangle^k, \ x \in \mathbb{R},$$

for $1 \leq k \leq N - 1$ and $k$ odd.

(2) Suppose that $V$ is of exceptional type and $V \in L_N^1$, $N \geq 2$, then

$$|\partial^k_\lambda f_\pm(\lambda, x)| \leq C\langle x\rangle^k, \ (\lambda, x) \in \mathbb{R}^2,$$

for $0 \leq k \leq N - 2$.

Lemma 2.3 follows from (2.1), (2.3) and Lemma 2.2, and we omit the proof.

We next study Fourier properties of the Jost functions. Set

$$B_\pm(\xi, x) := \int_\mathbb{R} e^{2i\lambda \xi} (m_\pm(\lambda, x) - 1) d\lambda,$$

where $m_\pm(\lambda, x) := e^{\mp i\lambda x} f_\pm(\lambda, x)$ are the modified Jost functions. Then the function $B_+(\xi, x)$ satisfies the Marchenko equation:

$$(2.4) \quad B_+(\xi, x) = \int_{x+\xi}^\infty V(\sigma) d\sigma + \int_0^\xi d\zeta \int_{x+\xi-\zeta}^\infty V(\sigma) B_+(\zeta, \sigma) d\sigma$$

and $B_-(\xi, x)$ also satisfies a corresponding equation. It is well known (see [3]) that if
$V \in L^1_1$, then the function $B_+(\xi, x)$ is well defined for $\xi \geq 0$, $x \in \mathbb{R}$ and satisfies the following estimates

\begin{equation}
|B_+(\xi, x)| \leq e^{\gamma(x)} \eta(x + \xi), \quad \xi \geq 0, \quad x \in \mathbb{R},
\end{equation}

where $\eta(x) = \int_x^{\infty} |V(\sigma)| d\sigma$, $\gamma(x) = \int_x^{\infty} (\sigma - x)|V(\sigma)| d\sigma$. $B_-(\xi, x)$ also satisfies a similar inequalities. Since $\gamma(x)$ is dominated by $||V||_{L^1}$ for all $x \geq 0$, (2.5) implies $||B_+(\cdot, x)||_{L^1}$ is bounded for $x \geq 0$ with the bound depending on $||V||_{L^1}$. Similarly $||B_-(\cdot, x)||_{L^1}$ is bounded for $x \leq 0$. Iterating Marchenko equations, we can prove the following (see [2]).

**Lemma 2.4.** Let $N \in \mathbb{N}$, $N \geq 1$ and suppose $V \in L^1_N$. Then the functions $B_{\pm}(\xi, x)$ satisfy the estimates

\begin{equation}
||B_{\pm}(\cdot, x)||_{L^1_{N-1}} \leq C(1 + \max(\mp x, 0))^N, \quad x \in \mathbb{R},
\end{equation}

where $C$ depends on $||V||_{L^1_N}$.

The following Lemma follows from Lemma 2.4 and the representation

\[ W(\lambda) = -2i\lambda + \int_{\mathbb{R}} V(\sigma)m_+(\lambda, \sigma)d\sigma. \]

**Lemma 2.5.** Let $\chi \in C_0^\infty(\mathbb{R})$.

1. Let $N \in \mathbb{N}$, $N \geq 1$ and assume that $V \in L^1_N$ and $V$ is of generic type, then

\[ \mathcal{F}(\frac{\chi}{W}) \in L^1_{N-1}. \]

2. Let $N \in \mathbb{N}$, $N \geq 2$ and assume that $V \in L^1_N$ and $V$ is of exceptional type, then

\[ \mathcal{F}(\frac{\lambda \chi}{W}) \in L^1_{N-2}. \]

Here $W(\lambda)$ is the Wronskian of the Jost functions.

Lemma 2.4, Lemma 2.5 and (2.1) imply the following.

**Lemma 2.6.** Let $\chi \in C_0^\infty(\mathbb{R})$. Suppose that $V \in L^1_N$, $N \geq 1$. Then

\begin{equation}
||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(x)^{N-1}(1 + \max(\mp x, 0)), \quad x \in \mathbb{R}.
\end{equation}

Furthermore,

1. If $V$ is of generic type, then

\[ ||\mathcal{F}(\chi(\cdot)T(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-1}} \leq C(x)^{N-1}, \quad x \in \mathbb{R}. \]

2. If $V$ is of exceptional type and $V \in L^1_N$, $N \geq 2$, then

\[ ||\mathcal{F}(\chi(\cdot)f_{\pm}(\cdot, x))||_{L^1_{N-2}} \leq C(x)^{N-2}, \quad x \in \mathbb{R}. \]
§ 2.2. The Low Energy Estimates

In this subsection, we prove the following Proposition to complete the proof of Theorem 1.2.

**Proposition 2.7.** Let \( m \in \mathbb{N} \) and let \( \chi \) be an even smooth cut-off function such that \( \chi(\lambda) = 1 \) close to zero. Suppose that \( V \in L_{2m}^{1} \) and \( V \) is of generic type, or \( V \in L_{2m+2}^{1} \) and \( V \) is of exceptional type. Let \( P_{m-1} \) as in Theorem 1.2. Let

\[
 s = \begin{cases} 
 2m - 1 & \text{if } V \text{ is of generic type,} \\
 2m & \text{if } V \text{ is of exceptional type.} 
\end{cases}
\]

Then

\[
 ||\langle x \rangle^{-s}(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u||_{L^\infty} \leq Ct^{-\frac{1}{2}-m}||\langle x \rangle^{s}u||_{L^1}
\]

for all \( t > 0 \).

**Proof.** We consider the generic case. The proof of the exceptional case is similar and we omit the proof. Set

\[
 K(\lambda, x, y) := T(\lambda)f_+(\lambda, y)f_-(\lambda, x), \quad G(\lambda, x, y) := \frac{K(\lambda, x, y)}{\lambda}.
\]

We start from the representation

\[
 \langle e^{-itH}\chi(\sqrt{H})P_{ac}u, v \rangle = \frac{1}{\pi i} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) \langle R(\lambda^2 + i0)u, v \rangle d\lambda
\]

\[
 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) \tilde{G}(\lambda, x, y) d\lambda \right) u(y)\overline{v(x)} dy dx,
\]

where \( \tilde{G}(\lambda, x, y) \) denotes the kernel of \(-2iR(\lambda^2 + i0)\) and is given by

\[
 (2.8) \quad \tilde{G}(\lambda, x, y) = \begin{cases} 
 G(\lambda, x, y) & \text{for } x < y, \\
 G(\lambda, y, x) & \text{for } x > y.
\end{cases}
\]

Consider the integral

\[
 (2.9) \quad I(t, G) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\lambda^2} \lambda \chi(\lambda) G(\lambda, x, y) d\lambda,
\]

for \( x < y \). The proof for the case \( x > y \) is analogous. Integrating by parts (2.9), we have

\[
 (2.10) \quad I(t, G) = \frac{1}{4\pi i t} \int_{\mathbb{R}} e^{-it\lambda^2} \partial_\lambda(\chi(\lambda) G(\lambda, x, y)) d\lambda.
\]
The case \( m = 1 \): it suffice to prove that
\[
|I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle, \ x < y.
\]

Using the Fourier inversion formula, we obtain
\[
|I(t, G)| \leq Ct^{-\frac{3}{2}} ||(\mathcal{F}\partial_{\lambda} \chi G)(\cdot, x, y)||_{L^1}
\]
for all \( t > 0 \) and \( x < y \), where \( \mathcal{F} \) is the Fourier transform with respect to \( \lambda \). By Young’s inequality, Lemma 2.5 (1) and Lemma 2.6, we have
\[
||((\mathcal{F}\partial_{\lambda} \chi G)(\cdot, x, y)||_{L^1} \leq C \langle x \rangle \langle y \rangle, \ x < y
\]
and this implies
\[
|I(t, G)| \leq Ct^{-\frac{3}{2}} \langle x \rangle \langle y \rangle
\]
for \( x < y \).

The case \( m \geq 2 \): Applying the stationary phase theorem to the integral (2.10), we have
\[
I(t, G) = \frac{1}{\sqrt{\pi i}} \sum_{j=1}^{m-1} \frac{t^{-\frac{1}{2}-j}}{(j-1)!(4i)^j} (\partial^{2j-1}_{\lambda} G)(0, x, y) + t^{-\frac{1}{2}-m} S_{m-1}(t, G)
\]
with
\[
|S_{m-1}(t, G)| \leq C ||(\mathcal{F}\partial^{2m-1}_{\lambda} \chi G)(\cdot, x, y)||_{L^1}
\]
\[
\leq C \langle x \rangle^{2m-1} \langle y \rangle^{2m-1}, \ x < y.
\]
For the last inequality, we used Lemma 2.5 (1) and Lemma 2.6. We now define the coefficients \( C_{j-1} \): since \( T(0) = 0 \), we have
\[
(\partial^{2j-1}_{\lambda} G)(0, x, y) = \frac{1}{2j} (\partial^{2j}_{\lambda} K)(0, x, y).
\]
Considering the fact that
\[
(\partial^{2j}_{\lambda} K)(0, x, y) = (\partial^{2j}_{\lambda} K)(0, y, x), \ x, y \in \mathbb{R}, \ j = 1, 2, ..., m,
\]
we define \( C_{j-1} \) and \( P_{m-1} \) by
\[
C_{j-1} u(x) := \frac{1}{\sqrt{4\pi i j!(4i)^j}} \int_{\mathbb{R}} (\partial^{2j}_{\lambda} K)(0, x, y) u(y) dy, \ x \in \mathbb{R},
\]
(2.11)
\[
P_{m-1} := \sum_{j=0}^{m-1} t^{-\frac{1}{2}-j} C_{j-1}.
\]
Then we have
\[
||\langle x \rangle^{-2m+1}(e^{-itH}\chi(\sqrt{H})P_{ac} - P_{m-1})u||_{L^\infty} \leq Ct^{-\frac{1}{2}-m}||\langle x \rangle^{2m-1}u||_{L^1}.
\]
By the definition of $C_{j-1}$ and Corollary 2.3 (1), we can see that
\[
\text{rank } C_{j-1} \leq 2j,
\]
and there exists $C > 0$ such that
\[
||\langle x \rangle^{-2j+1}C_{j-1}u||_{L^\infty} \leq C||\langle x \rangle^{2j-1}u||_{L^1}.
\]
In particular,
\[
C_{-1} \equiv 0.
\]
These complete the proof. \qed

References

Dispersive estimates for Schrödinger equations in dimension one


