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Spectral theory and inverse problems on asymptotically hyperbolic manifolds

By

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Abstract

We shall study spectral properties of Laplace-Beltrami operators on non-compact manifolds having asymptotically hyperbolic ends. We introduce a space of solutions of the associated Helmholtz equation and the S-matrix by observing the asymptotic behavior of solutions at infinity. We then show that this S-matrix determines the Riemannian metric.

§ 1. Spectral and scattering theory on hyperbolic manifolds

Spectral theory for continuous spectrum of Laplace-Beltrami operators on asymptotically hyperbolic manifolds has a long history. Apart from the classical works of Selberg [Se56], Roelcke [Roe66] and Faddeev [Fa67], new issues have been presented on the basis of the development of spectral and scattering theory for Schrödinger operators. Colin de Verdière [Col81] discussed the analytic continuation of Eisenstein series by that of resolvent on hyperbolic spaces. Agmon [Ag86] used modern spectral theories for this problem. Hislop [His94] used Mourre's commutator theory to prove the resolvent estimates for the Laplacian on hyperbolic spaces. The scattering metric proposed by Melrose [Me95] aims at constructing a general calculus on non-compact manifolds on which the scattering theory is developed. Melrose' theory includes the following model. Let $\mathcal{M}$ be a compact n-dimensional Riemannian manifold with boundary. Assume that near the boundary, $\mathcal{M}$ is diffeomorphic to $M \times (0,1)$, M being a compact $n-1$-dimensional manifold, and introduce the following metric

$$ds^2 = \frac{(dy)^2 + A(x,y,dx,dy)}{y^2}, \quad 0 < y < 1, \quad x \in M,$$

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where $A(x, y, dx, dy)$ is a symmetirc covariant tensor such that as $y \to 0$

$$A(x,y,dx,dy) \sim A_0(x,dx) + yA_1(x,dx,dy) + y^2A_2(x,dx,dy) + \cdots, \tag{1.1}$$

A_0 being the Riemannian metric on M. This generalizes the upper half-space model of the hyperbolic space. Spectral structures of the associated Laplace-Beltarmi operator were studied by Mazzeo [Ma88] and Mazzeo-Melrose [MaMe87]. Related inverse problem was studied by Joshi-Sa Barreto [JoSaBa00]. In particular, Sa Barreto [SaBa05] proved that the coincidence of the scattering operators gives rise to an isometry of associated metrics. Here the essential role is played by the boundary control method initiated by Belishev and developed by Belishev-Kurylev [Be87], [Be97], [BeKu92], which makes it possible to reconstruct a Riemannian manifold from the boundary spectral data of the associated Laplace-Beltrami operator.

A feature of Melrose' theory is that it proves the analytic continuation of the resolvent of Laplace-Beltrami operator for a broad class of metric so that it enables us to study the resonance, another important subject in spectral and scattering theory. Let us mention the recent article of Borthwick [Bo07] which studies the inverse problem related to the resonance based on Melrose's theory.

In the case of the Schrödinger operator $-\Delta + V(x)$ on $\mathbf{R}^n$, the behavior of solutions to the Schrödinger equation has a clear difference depending on the decay order of the potential at infinity. If we assume that $V(x) = O(|x|^{-\rho})$, $|x| \to \infty$, the border line is the case $\rho = 1$. This is also true on hyperbolic spaces. The difference occurs in the case $\rho = 1$ of the decay order $d_h^{-\rho}$, where d_h denotes the hyperbolic distance. In (1.1), y corresponds to e^{-d_h}. Hence from the view point of perturbation theory, the theory of scattering metric deals with the case in which the perturbation term is expanded as the power of e^{-d_h}.

This paper is a résumé of the lecture notes [IK09], in which we are aiming at developing the spectral theory and inverse problems on the asymptotically hyperbolic manifolds. We shall deal with the general short-range peturbation of the Riemannian metric, i.e. the one which converges to the standard hyperblic metric in the order $O(d_h^{-1-\epsilon})$ at infinity. As the starting point, we prove the limiting absorption principle, or the existence of the boundary values of the resovent $(-\Delta_g - \lambda \mp i0)^{-1}$ for $\lambda \in \sigma_{cont}(-\Delta_g)$. For the proof, we employ the classical method of integration by parts due to Eidus [Ei69]. Although it is elementary, it enables us to obtain better results as far as the resovent estimates are concerned. We then constrcuct the generalized Fourier transform and characterize the solution space of the Helmholtz equation, by using which we introduce the S-matrix. Our ultimate goal is the inverse problem, i.e. reconstruction of the Riemannian metric from the S-matrix. For this purpose, we adopt the boundary control method (BC-method) of Belishev-Kurylev.

We tried to make our notes as elementary as possible so that one can approach this problem without any deep preliminary knowledge, which made the notes more than 200 pages long. So, we explain here the outline of the theory by giving precise statements of Lemmas and Theorems, leaving all the details in [IK09].

§ 2. Summary of results

We shall study an n-dimensional connected Riemannian manifold $\mathcal{M}$ written as a union of open sets:

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N.$$

The basic assumptions are as follows:

(A-1) *$\overline{\mathcal{K}}$ is compact.*

(A-2) $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset, \quad i \neq j.$

(A-3) *Each $\mathcal{M}_i$, $i = 1, \cdots, N$, is diffeomorphic either to $\mathcal{M}_0 = M \times (0,1)$ or to $\mathcal{M}_\infty = M \times (1,\infty)$,* M being a compact Riemannian manifold of dimension $n-1$. *Here the manifold M is allowed to be different for each i.*

(A-4) *On each $\mathcal{M}_i$, the Riemannian metric ds^2 has the following form*

$$ds^2 = y^{-2}\left((dy)^2 + h(x,dx) + A(x,y,dx,dy)\right),$$

$$A(x,y,dx,dy) = \sum_{i,j=1}^{n-1} a_{ij}(x,y)dx^i dx^j + 2\sum_{i=1}^{n-1} a_{in}(x,y)dx^i dy + a_{nn}(x,y)(dy)^2,$$

where $h(x,dx) = \sum_{i,j=1}^{n-1} h_{ij}(x)dx^i dx^j$ is a positive definite metric on M, and $a_{ij}(x,y), 1 \leq i,j \leq n$, satisfies the following condition

$$|\widetilde{D}_x^\alpha D_y^m\, a(x,y)| \leq C_{\alpha m}(1 + |\log y|)^{-m-1-\epsilon_0}, \quad \forall \alpha, m \tag{2.1}$$

for some $\epsilon_0 > 0$. Here $D_y = y\partial_y$, and $\widetilde{D}_x = \tilde{y}(y)\partial_x$, $\tilde{y}(y) \in C^\infty((0,\infty))$ such that $\tilde{y}(y) = y$ for $y > 2$ and $\tilde{y}(y) = 1$ for $0 < y < 1$.

Letting Δ_g be the Laplace-Beltrami operator of $\mathcal{M}$, we consider the following wave equation

$$\begin{cases} \partial_t^2 u = \Delta_g u \quad \text{on} \quad \mathcal{M}, \\ u\big|_{t=0} = f, \quad \partial_t u\big|_{t=0} = -i\sqrt{-\Delta_g} f, \end{cases}$$

where f is orthogonal to the point spectral subspace for $-\Delta_g$. Then the wave disappears from any compact set in $\mathcal{M}$, and on each end $\mathcal{M}_j$, it will behave like

$$\|u(t) - u_j^{(\pm)}(t)\| \to 0, \quad \text{as} \quad t \to \pm\infty,$$

where $u_j^{(\pm)}(t)$ is the solution to the free wave equation

$$\begin{cases} \partial_t^2 u_j^{(\pm)} = \Delta_{g_j^0} u_j^{(\pm)}, \quad \text{on} \quad \mathcal{M}_j, \\ u_j^{(\pm)}\big|_{t=0} = f_j^{(\pm)}, \quad \partial_t u_j^{(\pm)}\big|_{t=0} = -i\sqrt{-\Delta_{g_j^0}} f_j^{(\pm)}, \end{cases}$$

$\Delta_{g_j^0}$ being the Laplace-Beltrami operator associated with the metric $y^{-2}\big((dy)^2+h_j(x,dx)\big)$. The scattering operator $\mathcal{S}$ assigns the asymptotic data in the remote future to that in the remote past that:

$$\mathcal{S} : \big(f_1^{(-)}, \cdots, f_N^{(-)}\big) \to \big(f_1^{(+)}, \cdots, f_N^{(+)}\big).$$

The inverse scattering is an attempt to recover the metric of $\mathcal{M}$ from the scattering operator $\mathcal{S}$. To study this problem, we first investigate the spectral properties of $-\Delta_g$. Namely

- Location of the essential spectrum.
- Absence of eigenvalues embedded in the continuous spectrum when one of the ends is *regular*, i.e. one $\mathcal{M}_i$ is diffeomorphic to $M \times (0,1)$.
- Discreteness of embedded eigenvalues in the continuous spectrum when all the ends are *cusps*, i.e. $\mathcal{M}_i$ is diffeomorphic to $M \times (1,\infty)$.
- Limiting absorption principle for the resolvent and the absolute continuity of the continuous spectrum.

Oue next issue is the forward problem. Namely

- Construction of the generalized Fourier transform associated with $-\Delta_g$.
- Asymptotic completeness of time-dependent wave operators.
- Characterization of the space of scattering solutions to the Helmhotz equation in terms of the generalized Fourier transform.
- Asymptotic expansion of scattering solutions to the Helmholtz equation and the S-matrix.

As a byproduct, we also study

- Representation of the fundamental solution to the wave equation.
- Radon transform and the propagation of singularities for the wave equation.

Finally, we shall discuss the inverse problem. Namely

- Identification of the Riemannian metric from the scattering matrix.

We show that two asymptotically hyperbolic manifolds satisfying the above assumptions are isometric, if the metrics coincide on one regular end, and also the S-matrices coincide on that end.

The main part of our results is proved under weaker decay assumption on the metric. By examining the proof, we see that the forward and inverse problem of scattering can be solved under the assumption

$$|\widetilde{D}_x^\alpha D_y^m\, a(x,y)| \leq C_{\alpha m}(1+|\log y|)^{-1-\epsilon_0}, \quad \forall \alpha, m \tag{2.2}$$

instead of (2.1).

§ 3. Besov type spaces

The Besov type space introduced by Agmon-Hörmander [AgHo76] furnishes a natural framework to characterize solutions to the Helmholtz equation. We define this space for the hyperbolic space $\mathbf{H}^n$.

We introduce an auxiliary Hilbert space $\mathbf{H}$ endowed with norm $\| \ \|$. We decompose $(0,\infty)$ into $(0,\infty) = \cup_{k\in\mathbf{Z}} I_k$, where

$$I_k = \begin{cases} \big(\exp(e^{k-1}), \exp(e^k)\big], & k \geq 1 \\ \big(e^{-1}, e\big], & k = 0 \\ \big(\exp(-e^{|k|}), \exp(-e^{|k|-1})\big], & k \leq -1. \end{cases}$$

We fix a natural number $n \geq 2$ and put

$$d\mu(y) = \frac{dy}{y^n}.$$

Let $\mathcal{B}$ be the space of $\mathbf{H}$-valued function on $(0,\infty)$ satisfying

$$\|f\|_{\mathcal{B}} = \sum_{k\in\mathbf{Z}} e^{|k|/2}\Big(\int_{I_k}\|f(y)\|^2 d\mu(y)\Big)^{1/2} < \infty.$$

The dual space $\mathcal{B}^*$ is identified with the space equipped with norm

$$\|u\|_{\mathcal{B}}^* = \left(\sup_{R>e} \frac{1}{\log R}\int_{\frac{1}{R}<y<R} \|u(y)\|_{\mathbf{H}}^2 d\mu\right)^{1/2} < \infty,$$

and the following inequality holds :

$$|(f,v)| = \left|\int_0^\infty (f(y),v(y))_{\mathbf{H}} d\mu\right| \leq C\|f\|_{\mathcal{B}}\|v\|_{\mathcal{B}^*}.$$

The following space is also useful:

$$u \in L^{2,s} \Longleftrightarrow \|u\|_s^2 = \int_0^\infty (1+|\log y|)^{2s}\|u(y)\|_{\mathbf{H}}^2\, d\mu(y) < \infty.$$

We have the following inclusion relations :

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}. \tag{3.1}$$

For the hyperbolic space $\mathbf{H}^n$, we employ the upper half-space model $\mathbf{R}^n_+$, and we represent a point of $\mathbf{R}^n_+$ as $(x, y), x \in \mathbf{R}^{n-1}, y > 0$. We then put $\mathbf{H} = L^2(\mathbf{R}^{n-1})$.

§ 4. 1-dimensional problem

We study the Laplace-Beltarmi operator in the upper-half space model by passing to the partial Fourier transformation with respect to $x \in \mathbf{R}^{n-1}$ and reducing it to the 1-dimensional case. Let $n \geq 2$ be an integer, and a parameter $\zeta \in \mathbf{C}$ satisfy $\mathrm{Re}\,\zeta \geq 0$. We consider the differential operator

$$L_0(\zeta) = y^2(-\partial_y^2 + \zeta^2) + (n-2)y\partial_y - \frac{(n-1)^2}{4} \tag{4.1}$$

on the interval $(0, \infty)$. The Green function of $L_0(\zeta) + \nu^2$ for $\mathrm{Re}\,\zeta > 0$ is then written as follows:

$$G_0(\zeta, \nu)f(y) = \int_0^\infty G_0(y, y'; \zeta, \nu) f(y') \frac{dy'}{(y')^n},$$

where the Green kernel is

$$G_0(y, y'; \zeta, \nu) = \begin{cases} (yy')^{(n-1)/2} K_\nu(\zeta y) I_\nu(\zeta y'), & y > y' > 0, \\ (yy')^{(n-1)/2} I_\nu(\zeta y) K_\nu(\zeta y'), & y' > y > 0, \end{cases}$$

I_ν, K_ν being modified Bessel functions. We define $\mathcal{B}, \mathcal{B}^*$ by putting $\mathbf{H} = \mathbf{C}$ in §3. Then we have

$$\|G_0(\zeta, \nu)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}},$$

where the constant C depends on ν, but is independent of ζ when $\mathrm{Re}\,\zeta > 0$.

We put for $f \in C_0^\infty((0, \infty))$ and $k > 0$

$$\left(\mathcal{F}_\zeta f\right)(k) = \frac{\left(2k \sinh(k\pi)\right)^{1/2}}{\pi} \int_0^\infty y^{(n-1)/2} K_{ik}(\zeta y) f(y) \frac{dy}{y^n}. \tag{4.2}$$

Theorem 4.1. *(1) $\mathcal{F}_\zeta$ is uniquely extended to a unitary : $L^2((0, \infty); dy/y^n) \to L^2((0, \infty); dk)$.*
(2) For $f \in D(L_0(\zeta))$, $\left(\mathcal{F}_\zeta L_0(\zeta) f\right)(k) = k^2 \left(\mathcal{F}_\zeta f\right)(k)$.
(3) For $f \in L^2((0, \infty); dy/y^n)$, the inversion formula holds:

$$f = \mathcal{F}_\zeta^* \mathcal{F}_\zeta f = y^{(n-1)/2} \int_0^\infty \frac{\left(2k \sinh(k\pi)\right)^{1/2}}{\pi} K_{ik}(\zeta y)(\mathcal{F}_\zeta f)(k) dk.$$

By $\mathcal{F}_\zeta^* \mathcal{F}_\zeta = 1$ we have,

$$f(y) = \int_0^\infty \int_0^\infty \frac{2\sigma \sinh(\sigma\pi)}{\pi^2} (yy')^{-1/2} K_{i\sigma}(y) K_{i\sigma}(y') f(y') dy' d\sigma,$$

and from $\mathcal{F}_\zeta \mathcal{F}_\zeta^* = 1$,

$$g(\sigma) = \int_0^\infty \int_0^\infty \frac{2\tau\left(\sinh(\sigma\pi)\sinh(\tau\pi)\right)^{1/2}}{\pi^2} \frac{K_{i\sigma}(y)K_{i\tau}(y)}{y} g(\tau)d\tau dy,$$

which are called *Kontrovich-Lebedev*'s inversion formulae.

We often use the following type of notation. Given an operator $\mathcal{F}$ from a Hilbert space $\mathcal{H}$ to another Hilbert space $L^2((0,\infty);\mathbf{h};dk)$, $\mathbf{h}$ being an auxiliary Hilbert space, for $k > 0$ we define an operator $\mathcal{F}(k)$ from a suitable subspace S of $\mathcal{H}$ to $\mathbf{h}$ by

$$\mathcal{F}(k)f = (\mathcal{F}f)(k), \quad f \in S.$$

Conversely if we are given a family of operators $\{\mathcal{F}(k)\}_{k>0}$, with range in $\mathbf{h}$, we define an operator $\mathcal{F}$ with range in $L^2((0,\infty);\mathbf{h};dk)$ by the above formula.

§ 5. The upper-half space model

§ 5.1. Laplace-Beltrami operator

We turn to the upper-half space model. The volume element is $dxdy/y^n$. Therefore

$$L^2(\mathbf{H}^n) = L^2(\mathbf{R}^n_+; \frac{dxdy}{y^n}).$$

The Laplace-Beltrami operator is given by

$$-\Delta_g = y^2(-\partial_y^2 - \Delta_x) + (n-2)y\partial_y, \quad \Delta_x = \sum_{i=1}^{n-1}(\partial/\partial x_i)^2.$$

We put

$$H_0 = -\Delta_g - \frac{(n-1)^2}{4}.$$

The partial Fourier transform $\hat{f}(\xi,y)$ of $f(x,y)$ is defined by

$$\left(F_0 f\right)(\xi,y) = \hat{f}(\xi,y) = (2\pi)^{-(n-1)/2}\int_{\mathbf{R}^{n-1}} e^{-ix\cdot\xi} f(x,y)dx.$$

Letting $L_0(\zeta)$ be as in (4.1), we have

$$\widehat{(H_0 f)}(\xi,y) = \Big(L_0(|\xi|)\hat{f}(\xi,\cdot)\Big)(y).$$

We put

$$R_0(z) = (H_0 - z)^{-1}, \quad z \in \mathbf{C}\setminus\mathbf{R},$$

and define the spaces $\mathcal{B}, \mathcal{B}^*$ by taking $\mathbf{H} = L^2(\mathbf{R}^{n-1};dx)$ in §3.

Theorem 5.1. *(1)* $\sigma(H_0) = [0,\infty)$.
(2) $\sigma_p(H_0) = \emptyset$.
(3) For $\lambda > 0$ *and* $f \in \mathcal{B}$, *the following limit exists in* $\mathcal{B}^*$ *in the weak* $*$*-sense*

$$\lim_{\epsilon\to 0} R_0(\lambda \pm i\epsilon)f =: R_0(\lambda \pm i0)f,$$

and the following inequality holds

$$\|R_0(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \tag{5.1}$$

where the constant C *does not depend on* λ *if it varies over a compact set in* $(0,\infty)$.
(4) We put for $k > 0$, $f \in C_0^\infty(\mathbf{R}^n_+)$,

$$\begin{aligned}\left(\mathcal{F}_0^{(\pm)}(k)f\right)(x) &= \frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}(2\pi)^{-(n-1)/2} \\ &\times \iint_{\mathbf{R}^{n-1}\times(0,\infty)} e^{ix\cdot\xi}\Big(\frac{|\xi|}{2}\Big)^{\mp ik} y^{(n-1)/2}K_{ik}(|\xi|y)\widehat{f}(\xi,y)\frac{d\xi dy}{y^n}.\end{aligned} \tag{5.2}$$

Then we have

$$\frac{k}{\pi i}\big([R_0(k^2+i0) - R_0(k^2-i0)]f, f\big) = \|\mathcal{F}_0^{(\pm)}(k)f\|^2_{L^2(\mathbf{R}^{n-1})}, \tag{5.3}$$

and

$$\|\mathcal{F}_0^{(\pm)}(k)f\|_{L^2(\mathbf{R}^{n-1})} \leq C\|f\|_{\mathcal{B}}, \tag{5.4}$$

where the constant C *is independent of* λ *if it varies over a compact set in* $(0,\infty)$.
(5) We put $(\mathcal{F}_0^{(\pm)}f)(k) = \mathcal{F}_0^{(\pm)}(k)f$. *Then* $\mathcal{F}_0^{(\pm)}$ *is uniquely extended to a unitary operator from* $L^2(\mathbf{H}^n)$ *to* $L^2((0,\infty);L^2(\mathbf{R}^{n-1});dk)$. *For* $f \in D(H_0)$, *we have*

$$(\mathcal{F}_0^{(\pm)}H_0f)(k) = k^2(\mathcal{F}_0^{(\pm)}f)(k). \tag{5.5}$$

§ 5.2. Helmholtz equation

Theorem 5.1 implies

$$\mathcal{F}_0^{(\pm)}(k)^* \in \mathbf{B}(L^2(\mathbf{R}^{n-1});\mathcal{B}^*), \tag{5.6}$$

$$\left(\mathcal{F}_0^{(\pm)}(k)^*\varphi\right)(x,y) = \frac{\big(2k\sinh(k\pi)\big)^{1/2}}{\pi}F_0^*\left[\left(\frac{|\xi|}{2}\right)^{\pm ik} y^{(n-1)/2}K_{ik}(|\xi|y)\widehat{\varphi}(\xi)\right], \tag{5.7}$$

$$F_0^*\psi = (2\pi)^{-(n-1)/2}\int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi}\psi(\xi)d\xi \tag{5.8}$$

and by (5.5)

$$(H_0 - k^2)\mathcal{F}_0^{(\pm)}(k)^*\varphi = 0, \quad \forall\varphi \in L^2(\mathbf{R}^{n-1}).$$

One can then prove the following theorem.

Theorem 5.2. *For $k > 0$*

$$\{u \in \mathcal{B}^*; (H_0 - k^2)u = 0\} = \mathcal{F}_0^{(\pm)}(k)^* \left(L^2(\mathbf{R}^{n-1})\right).$$

Namely any solution in $\mathcal{B}^*$ to the Helmholtz equation is written as a Poisson integral of some L^2-function on the boundary at infinity. In §10, we extend Theorem 5.2 to the manifold $\mathcal{M}$. In the case of $\mathbf{H}^n$, the largest solution space for the Helmholtz was characterized by Helgason [Hel70], who proved that *all* solutions of the equation $(H_0 - \lambda)u = 0$ is written by a Poisson integral of Sato's hyperfunction on the boundary. This result was extended to general symmetric spaces by [Mine75], [KKMOOT78]. This was also extended to the Euclidean space using more general analytic functionals by [HKMO72].

The space $\mathcal{B}^*$ is the smallest (in the following sense) space for the solutions to the Helmholtz equation. Recall the inclusion relations (3.1) in §3. One can show that if $u \in L^{2,-1/2}$ satisfies the Helmholtz equation $(H_0 - k^2)u = 0$ for $k > 0$, then $u = 0$. Therefore all the solutions to the Helmholtz equation decays at most like or slower than the functions in $\mathcal{B}^*$. In spite of that, it contains a sufficiently large number of solutions, since, as will be shown later, the knowledge of this solution space determines the whole manifold $\mathcal{M}$.

§ 6. Modified Radon transform

The Radon transform is usually defined as an integral over some submanifolds. In this section, we define the Radon transform in terms of the Fourier transform. For this purpose it is convenient to change its definition slightly. Let F_0 be the Fourier transformation on $\mathbf{R}^{n-1}$.

Definition 6.1. For $k \in \mathbf{R} \setminus \{0\}$ we define operators $\mathcal{F}^0(k)$ and $\mathcal{F}_0(k)$ by

$$\mathcal{F}^0(k)f(x) = \sqrt{\frac{2}{\pi}}\, k \sqrt{\frac{\sinh(k\pi)}{k\pi}}\, F_0^* \left(\left(\frac{|\xi|}{2}\right)^{-ik} \int_0^\infty y^{\frac{n-1}{2}} K_{ik}(|\xi|y) \widehat{f}(\xi, y) \frac{dy}{y^n} \right),$$

$$\mathcal{F}_0(k) = \frac{\Omega(k)}{\sqrt{2}} \mathcal{F}^0(k), \quad \Omega(k) = \frac{-i}{\Gamma(1-ik)} \sqrt{\frac{k\pi}{\sinh(k\pi)}}.$$

Here $g(k) := (k\pi/\sinh(k\pi))^{1/2}$ is defined on $\mathbf{C} \setminus \{i\tau\,;\, \tau \in (-\infty, 1] \cup [1, \infty)\}$ as a single-valued analytic function. In particular, $g(k) = g(-k)$ for $k > 0$.

Note that, $\mathcal{F}^0(k) = \mathcal{F}_0^{(+)}(k)$ for $k > 0$, and $|\Omega(k)| = 1$.

Lemma 6.2. *(1) $\mathcal{F}_0$ is uniquely extended to an isometry from $L^2(\mathbf{H}^n)$ to $\widehat{\mathcal{H}} := L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}); dk)$, and it diagonalizes H_0 :*

$$\left(\mathcal{F}_0 H_0 f\right)(k, x) = k^2 \left(\mathcal{F}_0 f\right)(k, x).$$

(2) Let r_+ be the projection onto the subspace $\widehat{\mathcal{H}}_+ := L^2((0,\infty); L^2(\mathbf{R}^{n-1}); dk)$. Then the range of $r_+\mathcal{F}_0$ is $\widehat{\mathcal{H}}_+$.
(3) $g \in \widehat{\mathcal{H}}$ belongs to the range of $\mathcal{F}_0$ if and only if

$$\widehat{g}(-k,\xi) = \frac{\Gamma(1-ik)}{\Gamma(1+ik)}\left(\frac{|\xi|}{2}\right)^{2ik}\widehat{g}(k,\xi), \quad \forall k > 0.$$

We then define the modified Radon transform associated with H_0 by

Definition 6.3. For $s \in \mathbf{R}$, we define

$$\big(\mathcal{R}_0 f\big)(s,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iks}\big(\mathcal{F}_0 f\big)(k,x)dk.$$

Recall that $\mathcal{F}_0$ is written explicitly as

$$\mathcal{F}_0(k)f(x) = \frac{-ik}{\sqrt{\pi}\Gamma(1-ik)}F_0^*\Big(\big(\frac{|\xi|}{2}\big)^{-ik}\int_0^\infty y^{\frac{n-1}{2}}K_{ik}(|\xi|y)\widehat{f}(\xi,y)\frac{dy}{y^n}\Big).$$

Theorem 6.4. *$\mathcal{R}_0$ is an isometry from $L^2(\mathbf{H}^n)$ to $\widehat{\mathcal{H}}$. Moreover we have*

$$\mathcal{R}_0 H_0 = -\partial_s^2\mathcal{R}_0.$$

Recall that the solution to the wave equation

$$\begin{cases} \partial_t^2 u + H_0 u = 0, \\ u\big|_{t=0} = f, \ \partial_t u\big|_{t=0} = g \end{cases}$$

is written as

$$u(t) = \cos(t\sqrt{H_0})f + \sin(t\sqrt{H_0})\sqrt{H_0}^{-1}g.$$

Theorem 6.5. *For any $f \in L^2(\mathbf{H}^n)$, we have as $t \to \infty$*

$$\Big\|\cos(t\sqrt{H_0})f - \frac{y^{(n-1)/2}}{\sqrt{2}}(\mathcal{R}_0 f)(-\log y - t, x)\Big\|_{L^2(\mathbf{H}^n)} \to 0,$$

$$\Big\|\sin(t\sqrt{H_0})f - \frac{iy^{(n-1)/2}}{\sqrt{2}}(\mathcal{R}_0 h(-i\partial_s)f)(-\log y - t, x)\Big\|_{L^2(\mathbf{H}^n)} \to 0,$$

where

$$h(-i\partial_s)\phi(s) = \frac{1}{2\pi}\iint_{\mathbf{R}^1\times\mathbf{R}^1} e^{ik(s-s')}h(k)\phi(s')ds'dk,$$

and where $h(k) = 1$ $(k>0)$, $h(k) = -1$ $(k<0)$.

Corollary 6.6. *For any $f \in L^2(\mathbf{H}^n)$, we have as $t \to \infty$*

$$\sqrt{2}e^{(n-1)(s+t)/2}\Big(\cos(t\sqrt{H_0})f\Big)(x, e^{-s-t}) \to \big(\mathcal{R}_0 f\big)(s,x) \quad \text{in} \quad L^2(\mathbf{R}^n).$$

§ 7. Radon transform and the wave equation

§ 7.1. Radon transform and horosphere

The fundamental solution for the wave equation on $\mathbf{H}^n$ is written explicitly in terms of spherical mean. For $n = 3$, it has the following form:

$$\cos(t\sqrt{H_0})f(z) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi\sinh(t)}\int_{S(z,t)} f(z')dS\right), \tag{7.1}$$

where $S(z;t) = \{z'; d_h(z',z) = t\}$, and $d_h(z',z)$ is the hyperbolic distance, that is,

$$S(z,t) = \left\{(x',y'); |x'-x|^2 + |y' - \cosh(t)y|^2 = \sinh^2(t)y^2\right\}.$$

Therefore $dS = \sinh^2(t)y^2 d\omega$, $d\omega$ being the Euclidean surface element on S^2, and

$$\cos(t\sqrt{H_0})f(z) = \frac{\partial}{\partial t}\left(\frac{\sinh(t)y^2}{4\pi}\int_{S^2} f((x,\cosh(t)y) + \sinh(t)y\omega)d\omega\right).$$

Let $t \to \infty$ and $y \to 0$ keeping $t + \log y = -s$. Then

$$(x,\cosh(t)y) + \sinh(t)y\omega \to (x, \frac{e^{-s}}{2}) + \frac{e^{-s}}{2}\omega,$$

Therefore the sphere $S(z,t)$ converges

$$\Sigma(s,x) = \left\{(x',y'); \left|x'-x\right|^2 + \left|y' - \frac{e^{-s}}{2}\right|^2 = \frac{e^{-2s}}{4}\right\},$$

which is the horosphere tangent to $\{y' = 0\}$. We then have

$$\cos(t\sqrt{H_0})f(z) \sim \frac{-y}{8\pi}\frac{\partial}{\partial s}\left(e^{-s}\int_{\Sigma(s,x)} f d\omega\right),$$

which, compared with Theorem 5.5, implies that

$$\mathcal{R}_0 f(s,x) = \frac{-\sqrt{2}}{8\pi}\frac{\partial}{\partial s}\left(e^{-s}\int_{\Sigma(s,x)} f d\omega\right).$$

From this formula, one can easily see that if f is supported in the region $y > \delta > 0$, $\mathcal{R}_0 f(s,x) = 0$ for $e^{-s} < \delta$. The converse is also true. Namely, if $\mathcal{R}_0 f(s,x) = 0$ for $e^{-s} < \delta$, $f(x,y)$ vanishes for $y < \delta$. This is the *support theorem* for the Radon transform.

§ 7.2. 1-dimensional wave equation

In the Euclidean space, there are 3 ways of constructing fundamental solutions to the wave equation : (1) the method of spherical means, (2) the method of plane waves and (3) the method of Fourier transforms. In the hyperbolic space, the first method is usually adopted. For example, in the work of Helgason [Hel84], a generalization of Asgeirsson's mean value theorem on two-point homogeneous space is used to derive the formula (7.1). In the following we shall apply the Fourier analysis to the fundamental solution. Let us start with the 1-dimensional case. The basic formula is the following one ([DiFe33], p. 302).

Lemma 7.1. *For $x > 0, y > 0$, $|\mathrm{Re}\,\nu| < 1/4$, we have*

$$K_\nu(x)K_\nu(y) = \frac{\pi}{2\sin(\nu\pi)}\int_{\log(y/x)}^\infty J_0(\sqrt{2xy\cosh t - x^2 - y^2})\sinh(\nu t)dt.$$

We put

$$\rho(k) = \frac{2k\sinh(\pi k)}{\pi^2},$$

and define for $\zeta > 0$

$$U_{adv}(t,y,y';\zeta) = \frac{(yy')^{\frac{n-1}{2}}}{2\pi}\int_{\mathbf{R}^2}\frac{K_{ik}(\zeta y)K_{ik}(\zeta y')}{k^2-(\omega+i0)^2}\rho(k)e^{-it\omega}dkd\omega,$$

$$U_{ret}(t,y,y';\zeta) = \frac{(yy')^{\frac{n-1}{2}}}{2\pi}\int_{\mathbf{R}^2}\frac{K_{ik}(\zeta y)K_{ik}(\zeta y')}{k^2-(\omega-i0)^2}\rho(k)e^{-it\omega}dkd\omega.$$

Lemma 7.2. *(1) For $t > 0$ and $y, y' > 0$, we have*

$$U_{adv}(t,y,y';\zeta) = (yy')^{\frac{n-1}{2}}\theta\big(t - \big|\log\frac{y}{y'}\big|\big)J_0\big(\zeta\sqrt{2yy'\cosh t - y^2 - (y')^2}\big),$$

and for $t < 0$,

$$U_{adv}(t,y,y';\zeta) = 0.$$

(2) For $t \in \mathbf{R}$,

$$U_{ret}(t,y,y';\zeta) = U_{adv}(-t,y,y';\zeta).$$

Lemma 7.3. *(1) For $f \in C_0^\infty((0,\infty))$, we put*

$$u_+(t,y,\zeta) = \int_0^\infty U_{adv}(t,y,y';\zeta)f(y')\frac{dy'}{(y')^n}.$$

Then the following formulas hold:

$$(L_0(\zeta) - \partial_t^2)u_+(t,y,\zeta) = f(y)\delta(t), \tag{7.2}$$

$$u(t,y,\zeta) = 0 \quad \text{for} \quad t < 0, \tag{7.3}$$

$$(\partial_t u)(+0,y,\zeta) = f(y). \tag{7.4}$$

We now define

$$U(t,y,y';\zeta) = \frac{1}{2}\left(U_{adv}(t,y,y';\zeta) - U_{ret}(t,y,y';\zeta)\right).$$

Lemma 7.4. *For $f \in C_0^\infty((0,\infty))$, we put*

$$u(t,y,\zeta) = \int_0^\infty U(t,y,y';\zeta)f(y')\frac{dy'}{(y')^n}.$$

Then we have

$$(\partial_t^2 - L_0(\zeta))u(t,y,\zeta) = 0,$$
$$u(0,y,\zeta) = 0,$$
$$\partial_t u(0,y,\zeta) = f(y).$$

§ 7.3. Wave equation in $\mathbf{H}^n$

We define an operator $P(t,y,y')$ by

$$P(t,y,y')f(x) = (2\pi)^{-\frac{n-1}{2}}\int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi}p(\xi;t,y,y')\widehat{f}(\xi)d\xi, \tag{7.5}$$

$$p(\xi;t,y,y') = J_0(|\xi|\sqrt{2yy'\cosh(t) - y^2 - (y')^2}),$$

which is a Fourier multiplier acting on functions of $x \in \mathbf{R}^{n-1}$, depending on parameters t, y, y'. Since $J_0(z)$ is an even function of z, $p(\xi;t,y,y')$ is smooth with respect to ξ and all the other parameters y, y' and t. The solution of the Cauchy problem

$$\begin{cases} \partial_t^2 u + H_0 u = 0, \\ u(0) = 0, \quad \partial_t u(0) = f \end{cases}$$

is written as

$$u(t,x,y) = \frac{1}{2}\int_0^\infty (yy')^{\frac{n-1}{2}}\left(\theta(t - |\log\frac{y}{y'}|) - \theta(-t - |\log\frac{y}{y'}|)\right)\left(P(t,y,y')f(\cdot,y')\right)(x)\frac{dy'}{(y')^n}.$$

Differentiating this formula with respect to t, we get the fundamental solution.

Theorem 7.5. *Let P be defined by (7.5). Then we have the following formula:*

$$\begin{aligned}
&\cos(t\sqrt{H_0})f(x,y) \\
&= \frac{1}{2}\int_0^\infty (yy')^{\frac{n-1}{2}}\left(\delta\big(t - |\log\frac{y}{y'}|\big) + \delta\big(t + |\log\frac{y}{y'}|\big)\right)P(t,y,y')f(\cdot,y')(x)\frac{dy'}{(y')^n} \\
&+ \frac{1}{2}\int_0^\infty (yy')^{\frac{n-1}{2}}\left(\theta\big(t - |\log\frac{y}{y'}|\big) - \theta\big(-t - |\log\frac{y}{y'}|\big)\right)\partial_t P(t,y,y')f(\cdot,y')(x)\frac{dy'}{(y')^n}.
\end{aligned}$$

In view of Corollary 6.6, we can derive an explicit form of the modified Radon transform $\mathcal{R}_0 f$ by letting $t \to \infty$ and $y \to 0$ keeping $-t - \log y = s$.

Theorem 7.6. *For $f \in C_0^\infty(\mathbf{H}^n)$ and $s \in \mathbf{R}$, we have*

$$\mathcal{R}_0 f(s,x) = \frac{e^{(n-1)s/2}}{\sqrt{2}} f(x, e^{-s}) - \frac{e^{-s}}{\sqrt{2}} \int_0^{e^{-s}} y^{-\frac{n-1}{2}} A(s,y) f(\cdot, y) \frac{dy}{y},$$

where $A(s,y)f(\cdot,y)$ is defined by

$$A(s,y)f(\cdot,y) = (2\pi)^{-(n-1)/2} \int_{\mathbf{R}^{n-1}} e^{ix\cdot\xi} A(\xi; s, y) \widehat{f}(\xi, y) d\xi,$$

$$A(\xi; s, y) = \frac{|\xi|^2}{2} B(|\xi| \sqrt{e^{-s}y - y^2}), \quad B(z) = \frac{J_1(z)}{z}.$$

Passing to the Fourier transform we get the following formula.

Lemma 7.7. *For $y > 0$*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} e^{iks} \frac{-ik}{\Gamma(1-ik)} 2^{ik} K_{ik}(y) dk = 2e^{-s}\delta(e^{-s} - y) - e^{-s}\theta(e^{-s} - y) B\big(\sqrt{e^{-s}y - y^2}\big),$$

where θ is the Heaviside function and $B(z) = J_1(z)/z$.

§ 8. Classification of 2-dimensional hyperbolic manifolds

The hyperbolic manifold is, by definition, a complete Riemannian manifold with all sectional curvatures equal to -1. General hyperbolic manifolds are constructed by the action of discrete groups on the upper-half space. The resulting quotient manifold is either compact, or non-compact but finte volume, or non-compact with infinite volume. In the latter two cases, the manifold can be split into bounded part and unbounded part, this latter being called the end. To study the general structure of ends is beyond our scope. We briefly look at the 2-dimensional case.

Recall that $\mathbf{C}_+ = \{z = x + iy\,; y > 0\}$ is a 2-dimensional hyperbolic space equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}. \tag{8.1}$$

Let $\partial\mathbf{C}_+ = \partial\mathbf{H}^2 = \{(x,0)\,; x \in \mathbf{R}\} \cup \infty = \mathbf{R} \cup \infty$. For a matrix

$$\gamma = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in SL(2, \mathbf{R})$$

the Möbius transformation is defined by

$$\mathbf{C}_+ \ni z \to \gamma z := \frac{az+b}{cz+d}, \tag{8.2}$$

which is an isometry on $\mathbf{H}^2$. This transformation γ is classified into 3 categories :

$$\begin{aligned} \textit{elliptic} &\Longleftrightarrow \text{there is only one fixed point in } \mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| < 2, \\ \textit{parabolic} &\Longleftrightarrow \text{there is only one degenerate fixed point on } \partial\mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| = 2, \\ \textit{hyperbolic} &\Longleftrightarrow \text{there are two fixed points on } \partial\mathbf{C}_+ \\ &\Longleftrightarrow |\mathrm{tr}\,\gamma| > 2. \end{aligned}$$

Let Γ be a discrete subgroup of $SL(2,\mathbf{R})$, which is usually called a *Fuchsian* group. Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be the fundamental domain for the action (8.2). Γ is said to be *geometrically finite* if $\mathcal{M}$ is chosen to be a finite-sided convex polygon. The sides are then geodesics of $\mathbf{H}^2$. The geometric finiteness is equivalent to that Γ is finitely generated.

As a simple example, consider the cyclic group Γ generated by the action $z \to z+1$. This is parabolic with fixed point ∞. The associated fundamental domain is $\mathcal{M} = (-1/2, 1/2] \times (0,\infty)$, which is a hyperbolic manifold with metric (8.1). It has two infinities : $(-1/2,1/2] \times \{0\}$ and ∞. The part $(-1/2,1/2] \times (0,1)$ has an infinite volume, which we call *regular infinity* in this paper. The part $(-1/2,1/2] \times (1,\infty)$ has a finite volume, and is called the *cusp*. The sides $x = \pm 1/2$ are geodesics.

Another simple example is the cyclic group generated by the hyperbolic action $z \to \lambda z$, $\lambda > 1$. The sides of the fundamental domain $\mathcal{M} = \{1 \leq |z| \leq \lambda\}$ are semi-circles orthogonal to $\{y = 0\}$, which are geodesics. The quotient manifold is diffeomorphic to $S^1 \times (-\infty,\infty)$. It is parametrized by (t,r), where $t \in \mathbf{R}/\log\lambda\mathbf{Z}$ and r is the signed distance from the segment $\{(0,t)\,;1 \leq t \leq \lambda\}$. The metric is then written as

$$ds^2 = (dr)^2 + \cosh^2 r\,(dt)^2. \tag{8.3}$$

The part $x > 0$ (or $x < 0$) is called the *funnel*. Letting $y = 2e^{-r}$, one can rewrite (8.3) as

$$ds^2 = \Big(\frac{dy}{y}\Big)^2 + \Big(\frac{1}{y} + \frac{y}{4}\Big)^2 (dt)^2.$$

This means that the funnel is a small perturbation of the regular infinity.

Let $\Lambda(\Gamma)$ be the set of all limit points of the orbit $\{\gamma z\,;\gamma \in \Gamma\}$, i.e. $w \in \Lambda(\Gamma)$ if there exist $z_0 \in \mathbf{C}_+$ and $\gamma_n \in \Gamma$ such that $\gamma_n \cdot z_0 \to w$. Since Γ acts discontinuously on $\mathbf{C}_+$, $\Lambda(\Gamma) \subset \partial\mathbf{H}^2$. If $\Lambda(\Gamma)$ is a finite set, Γ is said to be *elementary*. In this case, $\mathcal{M}$ is either $\mathbf{H}^2$, or the quotient manifold by hyperbolic, or parabolic cyclic groups. For non-elementary case, we have the following theorem.

Theorem 8.1. *Let $\mathcal{M} = \Gamma\backslash\mathbf{H}^2$ be a non-elementary geometrically finite hyperbolic manifold. Then there exists a compact subset $\mathcal{K}$ such that $\mathcal{M}\setminus\mathcal{K}$ is a finite disjoint union of cusps and funnels.*

We need to add a remark about Theorem 8.1. Let Γ be a Fuchsian group. For a point $z_0 \in \overline{\mathbf{R}^2_+}$, we put

$$\Gamma_{z_0} = \{\gamma \in \Gamma\,;\gamma \cdot z_0 = z_0\}.$$

If $\Gamma_{z_0} \neq \{1\}$, z_0 is called a fixed point of Γ. A fixed point in $\mathbf{R}^2_+$ is called an elliptic fixed point. Let $\mathcal{M}_{sing}$ be the set of elliptic fixed points of Γ. By a suitable choice of local coordinates, $\mathcal{M} = \Gamma\backslash\mathbf{H^2}$ becomes a Riemann surface, moreover by introducing the metric $y^{-2}\left((dx)^2 + (dy)^2\right)$, $\mathcal{M} \setminus \mathcal{M}_{sing}$ is a hyperbolic manifold. However, this metric is singular around the points from $\mathcal{M}_{sing}$. Around the elliptic fixed point $z_0 \in \mathcal{M}$, $\mathcal{M}_{sing}$ admits a covering space compatible with the local coorodiante system, and $\mathcal{M}$ is called *orbifold.* Theorem 8.1 also holds for the orbifold case. In this note, we do not enter into the orbifold structure.

§ 9. Model space

By the above classification, it is natural to consider the manifolds whose ends are close to a part of $\Omega = M \times (0,\infty)$, M being a compact manifold, and the metric of Ω is given by

$$ds^2 = \frac{(dy)^2 + h(x,dx)}{y^2}, \tag{9.1}$$

where $h(x,dx) = \sum_{i,j=1}^{n-1} h_{ij}(x)dx^i dx^j$ is the metric on M, x being local coordinates on M. Let Δ_M be the Laplace-Beltrami operator on M, $0 = \lambda_1 < \lambda_2 \leq \cdots$ be the eigenvalues, and $\varphi_m(x)$ the associated complete orthonormal system of eigenvectors of Δ_M. The Laplace-Beltrami operator on Ω is given by

$$H_0 = -y^2(\partial_y^2 + \Delta_M) + (n-2)y\partial_y - \frac{(n-1)^2}{4}.$$

Spectral properties of H_0 can be studied in essentially the same way as in §5. We have only to replace the space $L^2(\mathbf{R}^{n-1})$ by $L^2(M)$ and the Fourier transform by the Fourier series. Here the Fourier coefficient of $f(x,y)$ is denoted by

$$\widehat{f}_m(y) = \int_M f(x,y)\overline{\varphi_m(x)}\sqrt{g_M}dx, \tag{9.2}$$

with $g_M = \det\left(h_{ij}(x)\right)$. For $f \in C_0^\infty(\Omega)$, we have

$$\widehat{(H_0 f)}_m(y) = \left(L_0(\sqrt{\lambda_m})\widehat{f}_m(\cdot)\right)(y),$$

where $L_0(\zeta)$ is defined by (4.1). For $\lambda_m \neq 0$, the Green operator of $L_0(\sqrt{\lambda_m}) - \lambda \mp i\epsilon$ is

$$\left(L_0(\sqrt{\lambda_m}) - \lambda \mp i\epsilon)\right)^{-1} = G_0(\sqrt{\lambda_m}, \mp i\sqrt{\lambda \pm i\epsilon}).$$

The Fourier transformation associated with $L_0(\sqrt{\lambda_m})$ is given in (4.2):

$$\left(F_{0m}f\right)(k) = \frac{\left(2k\sinh(k\pi)\right)^{1/2}}{\pi}\int_0^\infty y^{(n-1)/2}K_{ik}(\sqrt{\lambda_m}y)f(y)\frac{dy}{y^n}.$$

Then we obtain the following theorem.

Theorem 9.1. *Let $\lambda_m \neq 0$.*
(1) F_{0m} is a unitary operator from $L^2((0,\infty);dy/y^n)$ onto $L^2((0,\infty),dk)$.
(2) For $f \in D(L_0(\sqrt{\lambda_m}))$

$$(F_{0m}L_0(\sqrt{\lambda_m})f)(k) = k^2(F_{0m}f)(k).$$

(3) For $f \in L^2((0,\infty);dy/y^n)$ the inversion formula holds :

$$\begin{aligned} f &= \left(F_{0m}\right)^* F_{0m}f \\ &= y^{(n-1)/2}\int_0^\infty \frac{(2k\sinh(k\pi))^{1/2}}{\pi}K_{ik}(\sqrt{\lambda_m}y)(F_{0m}f)(k)dk. \end{aligned}$$

We consider the case $\lambda_m = 0$, i.e. $m = 0$:

$$L_0(0) = -y^2\partial_y^2 + (n-2)y\partial_y - \frac{(n-1)^2}{4}.$$

Since this is Euler's operator, we have

$$(L_0(0) - \lambda \mp i\epsilon))^{-1} = G_0(\mp i\sqrt{\lambda \pm i\epsilon}),$$

$$G_0(\nu)f(y) = \int_0^\infty G_0(y,y';\nu)f(y')\frac{dy'}{(y')^n}, \tag{9.3}$$

$$G_0(y,y',\nu) = \frac{1}{2\nu}\begin{cases} y^{\frac{n-1}{2}+\nu}(y')^{\frac{n-1}{2}-\nu}, & 0 < y < y', \\ y^{\frac{n-1}{2}-\nu}(y')^{\frac{n-1}{2}+\nu}, & 0 < y' < y. \end{cases} \tag{9.4}$$

Then we can prove

$$\|G_0(\nu)f\|_{\mathcal{B}^*} \leq \frac{C}{|\nu|}\|f\|_{\mathcal{B}},$$

where the constant C is independent of ν. The Fourier transformation associated with $L_0(0)$ is

$$\begin{aligned} \left(F_0 f\right)(k) &= \left(F_0^{(+)}(k)f, F_0^{(-)}(k)f\right), \\ F_0^{(\pm)}(k)f &= \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{\frac{n-1}{2}\pm ik}f(y)\frac{dy}{y^n}. \end{aligned}$$

Theorem 9.2. *(1) F_0 is unitary from $L^2((0,\infty);dy/y^n)$ to $(L^2((0,\infty);dk))^2$.*
(2) For $f \in D(L_0(0))$,

$$(F_0L_0(0)f)(k) = k^2(F_0f)(k).$$

(3) For $f \in L^2((0,\infty);dy/y^n)$, the inversion formula holds:

$$\begin{aligned} f &= F_0^*F_0f \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty y^{(n-1)/2}\left(y^{-ik}F_0^{(+)}(k)f + y^{ik}F_0^{(-)}(k)f\right)dk. \end{aligned}$$

We now return to the operator H_0. Recall that the generalized Fourier transformation is derived from the asymptotic behavior of the resolvent at infinity. For $\Omega = M \times (0,\infty)$, there are two infinities ; $y = 0$ and $y = \infty$, the former corresponding to the funnel or the regular infinity of the parabolic cylinder, the latter to the cusp. We put the suffix *reg* or c for the Fourier transforms associated with regular infinity or cusp.

Definition 9.3. We define

$$\mathbf{h} = L^2(M) \oplus \mathbf{C}, \quad \widehat{\mathcal{H}} = L^2((0,\infty);\mathbf{h};dk),$$

$$\mathcal{F}_0^{(\pm)} = \left(\mathcal{F}_{0\,reg}^{(\pm)}, \mathcal{F}_{0\,c}^{(\pm)}\right),$$

$$\left(\mathcal{F}_{0\,reg}^{(\pm)}f\right)(k,x) = \sum_{j=0}^\infty C_m^{(\pm)}(k)\varphi_m(x)\left(F_{0m}^{(\pm)}\widehat{f}_m(\cdot)\right)(k), \tag{9.5}$$

$$F_{0m}^{(\pm)} = \begin{cases} F_{0m} & (\lambda_m \neq 0) \\ F_0^{(\pm)} & (\lambda_m = 0), \end{cases} \tag{9.6}$$

$$C_m^{(\pm)}(k) = \begin{cases} \left(\dfrac{\sqrt{\lambda_m}}{2}\right)^{\mp ik} & (\lambda_m \neq 0) \\ \dfrac{\pm i}{k\omega_\pm(k)}\sqrt{\dfrac{\pi}{2}} & (\lambda_m = 0), \end{cases} \tag{9.7}$$

$$\left(\mathcal{F}_{0\,c}^{(\pm)}f\right)(k) = \frac{1}{\sqrt{|M|}}\left(F_0^{(\mp)}\widehat{f}_0(\cdot)\right)(k), \tag{9.8}$$

where $|M|$ is the volume of M.

We define $\mathcal{B},\mathcal{B}^*$ by putting $\mathcal{H} = L^2(M)$ in §3, and let $R_0(z) = (H_0 - z)^{-1}$.

Theorem 9.4. *(1)* $\sigma(H_0) = [0,\infty)$.
(2) $\sigma_p(H_0) = \emptyset$.
(3) For $\lambda > 0, f \in \mathcal{B}$, *the following limit exists in the* $*$*-weak sense*

$$\lim_{\epsilon\to 0} R_0(\lambda \pm i\epsilon)f =: R_0(\lambda \pm i0)f.$$

Moreover

$$\|R_0(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}},$$

where the constant C *does not depend on* λ *if* λ *varies over a comapct set in* $(0,\infty)$.
(4) Letting $\mathcal{F}_0^{(\pm)}(k)f = (\mathcal{F}_0^{(\pm)}f)(k)$, *we have*

$$\|\mathcal{F}_0^{(\pm)}(k)f\|_{\mathbf{h}} \leq C\|f\|_{\mathcal{B}},$$

where the constant C *does not depend on* λ *if* λ *varies over a comapct set in* $(0,\infty)$.
(5) $\mathcal{F}_0^{(\pm)}$ *is uniquely extended to a unitary operator from* $L^2(\Omega)$ *to* $\widehat{\mathcal{H}}$. *Moreover if* $f \in D(H_0)$

$$(\mathcal{F}_0^{(\pm)}H_0 f)(k) = k^2(\mathcal{F}_0^{(\pm)}f)(k).$$

The relation of $\mathcal{F}_0^{(\pm)}$ and the asymptotic behavior of the resolvent is as follows.

Theorem 9.5. *For* $k > 0$, $f \in \mathcal{B}$, *we have*

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1/R<y<1}\|R_0(k^2 \pm i0)f - v_{reg}^{(\pm)}\|^2_{L^2(M)}\frac{dy}{y^n} = 0, \tag{9.9}$$

$$v_{reg}^{(\pm)} = \omega_{\pm}(k)\,y^{(n-1)/2\mp ik}\mathcal{F}_{0\,reg}^{(\pm)}(k)f,$$

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1<y<R}\|R_0(k^2 \pm i0)f - v_{c}^{(\pm)}\|^2_{L^2(M)}\frac{dy}{y^n} = 0, \tag{9.10}$$

$$v_{c}^{(\pm)} = \omega_{\pm}^{(c)}(k)\,y^{(n-1)/2\pm ik}\mathcal{F}_{0\,c}^{(\pm)}(k)f, \quad \omega_{\pm}^{(c)}(k) = \pm\frac{i}{k}\sqrt{\frac{\pi}{2}}.$$

§ 10. Manifolds with hyperbolic ends

§ 10.1. Resolvent estimates

We study the manifold $\mathcal{M}$ satisfying the assumptions (A-1) $\sim$ (A-4) in §2. Let us note that the upper half-space $\mathbf{H}^n$ satisfies these assumptions. In fact, we take $\mathcal{M}$ to be $\mathcal{K}\cup\mathcal{M}_1$, where$\mathcal{M}_1$ is diffeomorphic to $S^{n-1}\times(1,\infty)$ equipped with the metric $(dr)^2+\sinh^2 r(d\theta)^2$, the hyperbolic mertic written by geodesic polar coordinates. Taking $e^r = 2/y$, we arrive at at the above model.

If $\mathcal{M}_i$ is diffeomorphic to $M \times (0,1)$, one can transfom the above metric into the form

$$ds^2 = y^{-2}\Big((dy)^2 + h(x,dx) + \sum_{i,j=1}^{n-1} a_{ij}(x,y)dx^i dx^j\Big) \tag{10.1}$$

where each $a_{ij}(x,y)$ satisfies the condition (2.1).

Theorem 10.1. *(1) $H\big|_{C_0^\infty(\mathcal{M})}$ is essentially self-adjoint.*
(2) $\sigma_e(H) = [0,\infty)$.

We shall prove the limiting absorption by using the method of integration by parts (see e.g. [Ei69]). We explain the main idea by adopting $H_0 = -\Delta_g$ in the upper-half space model $\mathbf{R}^n_+$. Let us note that $u_\pm = R_0(\lambda \pm i0)f$ behaves like

$$\hat{u}_\pm(\xi,y) \sim C_m(\xi)y^{(n-1)/2\mp i\sqrt{\lambda}} \quad (y \to 0).$$

Therefore we infer

$$\Big(y\partial_y - (\frac{n-1}{2} \mp i\sqrt{\lambda})\Big)u_\pm = o(y^{(n-1)/2}) \quad (y \to 0).$$

This suggests the importance of the term $\Big(y\partial_y - (\frac{n-1}{2} \mp i\sqrt{\lambda})\Big)u_\pm$ to derive the estimates for $u_\pm$. We put

$$\sigma_\pm = \frac{n-1}{2} \mp i\sqrt{z}.$$

Here for $z = re^{i\theta}, r > 0, -\pi < \theta < \pi$, we take the branch of $\sqrt{z}$ as $\sqrt{r}e^{i\theta/2}$. In the following, we estimate $u = R_0(\lambda + i0)$. The first step is to rewrite the equation $(H_0 - z)u = f$ as follows:

$$D_y(D_y - \sigma_\pm)u = \sigma_\mp\big(D_y - \sigma_\pm\big)u - D_x^2 u - f. \tag{10.2}$$

Let $(\ ,\ )$, $\|\cdot\|$ denote the inner product and norm of $L^2(\mathbf{R}^{n-1})$, respectively. The proof of the limiting absorption principle is reduced to the following 3 a-priori estimates.

Lemma 10.2. *Let $\varphi(y) \in C^1((0,\infty);\mathbf{R})$ and $0 < a < b < \infty$.*
(1) For any $w \in C_0^\infty(\mathbf{H}^n)$, we have

$$\mathrm{Re}\int_a^b \varphi(D_y w, w)\frac{dy}{y^n} = -\frac{1}{2}\int_a^b (D_y\varphi)\|w\|^2\frac{dy}{y^n} + \Big[\frac{\varphi\|w\|^2}{2y^{n-1}}\Big]_{y=a}^{y=b} + \frac{n-1}{2}\int_a^b \varphi\|w\|^2\frac{dy}{y^n}.$$

(2) For any $u \in C_0^\infty(\mathbf{H}^n)$, we have

$$\begin{aligned}&\mathrm{Re}\int_a^b \varphi\big((D_y - \sigma_\pm)u, -D_x^2 u\big)\frac{dy}{y^n}\\ &= \Big[\frac{\varphi\|D_x u\|^2}{2y^{n-1}}\Big]_{y=a}^{y=b} - \frac{1}{2}\int_a^b (D_y\varphi)\|D_x u\|^2\frac{dy}{y^n} + \Big(\frac{n-3}{2} - \mathrm{Re}\,\sigma_\pm\Big)\int_a^b \varphi\|D_x u\|^2\frac{dy}{y^n}.\end{aligned}$$

Lemma 10.3. *Let $u = R_0(z)f$ with $\mathrm{Im}\sqrt{z} \geq 0$. Put $w_\pm = (D_y - \sigma_\pm)u$. Then for any $C^1 \ni \varphi \geq 0$ and constants $0 < a < b$, we have*

$$
\begin{aligned}
& -\int_a^b (D_y\varphi)\|w_+\|^2 \frac{dy}{y^n} + \left[\frac{\varphi(\|w_+\|^2 - \|D_x u\|^2)}{y^{n-1}}\right]_{y=a}^{y=b} \\
& \leq -\int_a^b \big(D_y\varphi + 2\varphi\big)\|D_x u\|^2 \frac{dy}{y^n} - 2\mathrm{Re}\int_a^b \varphi(f, w_+)\frac{dy}{y^n},
\end{aligned}
\tag{10.3}
$$

$$
\begin{aligned}
& \int_a^b (D_y\varphi)\|w_-\|^2 \frac{dy}{y^n} - \left[\frac{\varphi(\|w_-\|^2 - \|D_x u\|^2)}{y^{n-1}}\right]_{y=a}^{y=b} \\
& \leq \int_a^b \big(D_y\varphi + 2\varphi\big)\|D_x u\|^2 \frac{dy}{y^n} + 2\mathrm{Re}\int_a^b \varphi(f, w_-)\frac{dy}{y^n}.
\end{aligned}
\tag{10.4}
$$

Lemma 10.4. *We put $u = R_0(z)f$ for $f \in \mathcal{B}$ and let z vary over the region*

$$J = \{z \in \mathbf{C}\,;\, a \leq \mathrm{Re}\, z \leq b,\ 0 < \mathrm{Im}\, z < 1\},$$

where $0 < a < b$ are arbitrarily chosen constants. Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\int_0^\infty \|D_x u\|^2 \frac{dy}{y^n} \leq \epsilon\|u\|^2_{\mathcal{B}^*} + C_\epsilon\|f\|^2_{\mathcal{B}}.$$

As in the case of [Ei69], an important role is played by the radiation condition. We put

$$\sigma_\pm(\lambda) = \frac{n-1}{2} \mp i\sqrt{\lambda}.$$

We say that a solution $u \in \mathcal{B}^*$ of the equation

$$(H - \lambda)u = f \in \mathcal{B}$$

satisfies the outgoing radiation condition, when $\mathcal{M}_i$ has a regular infinity

$$\lim_{R\to\infty} \frac{1}{\log R}\int_{1/R}^{1/2} \|(D_y - \sigma_+(\lambda))u(\cdot, y)\|^2_{L^2(\mathbf{E}_i)}\frac{dy}{y^n} = 0, \tag{10.5}$$

and when $\mathcal{M}_i$ has a cusp

$$\lim_{R\to\infty} \frac{1}{\log R}\int_2^R \|(D_y - \sigma_-(\lambda))u(\cdot, y)\|^2_{L^2(\mathbf{E}_i)}\frac{dy}{y^n} = 0. \tag{10.6}$$

The incoming radiation condition is defined similarly with $\sigma_+(\lambda)$ replaced by $\sigma_-(\lambda)$.

Theorem 10.5. *Let $\lambda > 0$ and suppose $u \in \mathcal{B}^*$ satisfies $(H - \lambda)u = 0$ and the radiation condition. Then:*
(1) If one of $\mathcal{M}_i$'s has a regular infinity, then $u = 0$.
(2) If all $\mathcal{M}_i$ have a cusp, $u \in L^{2,s}$, $\forall s > 0$.

These preparations are sufficient to prove the limiting absorption principle for H.

Theorem 10.6. *For $\lambda \in \sigma_e(H) \setminus \sigma_p(H)$, there exists a limit*

$$\lim_{\epsilon\to 0} R(\lambda \pm i\epsilon) \equiv R(\lambda \pm i0) \in \mathbf{B}(\mathcal{B};\mathcal{B}^*)$$

in the weak $*$-sense. Moreover for any compact interval $I \subset \sigma_e(H) \setminus \sigma_p(H)$ there exists a constant $C > 0$ such that

$$\|R(\lambda \pm i0)f\|_{\mathcal{B}^*} \leq C\|f\|_{\mathcal{B}}, \quad \lambda \in I.$$

For $f \in \mathcal{B}$, we put $u = R(\lambda \pm i0)f$. Then u is a unique solution to the equation $(H - \lambda)u = f$ satisfying the outgoing (for the case $+$), incoming (for the case $-$) radiation condition. For $f, g \in \mathcal{B}$, $(R(\lambda \pm i0)f, g)$ is continuous with respect to $\lambda > 0$.

§ 10.2. Fourier transforms associated with H

We use the following partition of unity. Fix $x_0 \in \mathcal{K}$ arbitrarily, and pick $\chi_0 \in C_0^\infty(\mathcal{M})$ such that

$$\chi_0(x) = \begin{cases} 1, & \mathrm{dis}\,(x, x_0) < R, \\ 0, & \mathrm{dis}\,(x, x_0) > R + 1. \end{cases}$$

Taking R large enough, we define $\chi_j \in C^\infty(\mathcal{M})$ such that

$$\chi_j(x) = \begin{cases} 1 - \chi_0(x), & x \in \mathcal{M}_j, \\ 0, & x \notin \mathcal{M}_j. \end{cases}$$

Then we have

$$\begin{cases} \sum_{j=0}^N \chi_j = 1, \\ \mathrm{supp}\,\chi_j \subset \mathcal{M}_j, \quad 1 \leq j \leq N, \\ \chi_0 = 1 \quad \text{on} \quad \mathcal{K}. \end{cases} \tag{10.7}$$

For $1 \leq j \leq N$, we construct $\widetilde{\chi}_j \in C^\infty(\mathcal{M})$ such that

$$\mathrm{supp}\,\widetilde{\chi}_j \subset \mathcal{M}_j, \quad \widetilde{\chi}_j = 1 \quad \text{on} \quad \mathrm{supp}\,\chi_j.$$

Let H_{0j} be the Laplace-Beltrami operator on $M_j \times (0,\infty)$ and χ_j as in (10.7). Since

$$(H_{0j} - \lambda)\chi_j R(\lambda \pm i0) = \chi_j + ([H_{0j}, \chi_j] - \chi_j V)\, R(\lambda \pm i0),$$

letting

$$R_{0j}(z) = (H_{0j} - z)^{-1},$$

we have

$$\chi_j R(\lambda \pm i0) = R_{0j}(\lambda \pm i0)\chi_j + R_{0j}(\lambda \pm i0)\,([H_{0j}, \chi_j] - \chi_j V)\, R(\lambda \pm i0).$$

This resolvent equation enables us to construct the Fourier transformation for H by the perturbation argument.

Definition of $\mathcal{F}_{0\,j}^{(\pm)}(k)$. We define $\mathcal{F}_{0j}^{(\pm)}(k)$ as follows. Let $\lambda_{j,1}, \lambda_{j,2}, \cdots$ be the eigenvalues of the Laplace-Beltrami operator on M_j and $\varphi_{j,1}, \varphi_{j,2}, \cdots$ be the associated eigenvectors.

(i) For $1 \leq j \leq M$ (the case of regular infinity)

$$\big(\mathcal{F}_{0\,j}^{(\pm)}(k)f\big)(x) = \sum_{m\geq 0} C_m^{(\pm)}(k)\varphi_{j,m}(x)F_{0,m}^{(\pm)}(k)\widehat{f}_m(\cdot),$$

where the right-hand side is defined by (9.5), (9.6) with M replaced by M_j, and $C_m^{(\pm)}(k)$ is from (9.7).

(ii) For $M+1 \leq j \leq N$ (the case of cusp)

$$\mathcal{F}_{0\,j}^{(\pm)}(k)f = \frac{1}{\sqrt{|\mathbf{E_j}|}}F_{0,0}^{(\mp)}(k)\widehat{f}(0,\cdot).$$

We put

$$\mathcal{F}_{j,m}^{(\pm)}(k) = F_{0,m}^{(\pm)}(k)\left(\chi_j + ([H_{0j},\chi_j] - \chi_j V)R(k^2 \pm i0)\right). \tag{10.8}$$

Definition of $\mathcal{F}^{(\pm)}(k)$. The Fourier transformation associated with H is defined by

$$\mathcal{F}^{(\pm)}(k) = \big(\mathcal{F}_1^{(\pm)}(k), \cdots, \mathcal{F}_N^{(\pm)}(k)\big),$$

where for $1 \leq j \leq M$

$$\begin{aligned}\mathcal{F}_j^{(\pm)}(k) &= \mathcal{F}_{0\,j}^{(\pm)}(k)\big[\chi_j + ([H_{0j},\chi_j] - \chi_j V)\,R(k^2 \pm i0)\big]\\ &= \sum_{m\geq 0} C_m^{(\pm)}(k)\varphi_{j,m}(x)\mathcal{F}_{j,m}^{(\pm)}(k),\end{aligned}$$

and for $M+1 \leq j \leq N$

$$\begin{aligned}\mathcal{F}_j^{(\pm)}(k) &= \mathcal{F}_{0\,j}^{(\pm)}(k)\big[\chi_j + ([H_{0j},\chi_j] - \chi_j V)\,R(k^2 \pm i0)\big]\\ &= \frac{1}{\sqrt{|M_j|}}\mathcal{F}_{j,0}^{(\pm)}(k).\end{aligned} \tag{10.9}$$

For functions $f, g \in \mathcal{B}^*$ on $\mathcal{M}$, by $f \simeq g$ we mean that on each end

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{1<y<R}\|f(y) - g(y)\|_{L^2(M_i)}^2\frac{dy}{y^n} = 0,$$

$$\lim_{R\to\infty}\frac{1}{\log R}\int_{\log(1/R)<y<1}\|f(y) - g(y)\|_{L^2(M_i)}^2\frac{dy}{y^n} = 0.$$

Theorem 10.7. *Let $f \in \mathcal{B}$, $k^2 \in \sigma_e(H) \setminus \sigma_p(H)$, and χ_j the partition of unity from (10.7). Then we have*

$$R(k^2 \pm i0)f \simeq \omega_\pm(k) \sum_{j=1}^{M} \chi_j y^{(n-1)/2 \mp ik} \mathcal{F}_j^{(\pm)}(k) f$$
$$+ \omega_\pm^{(c)}(k) \sum_{j=M+1}^{N} \chi_j y^{(n-1)/2 \pm ik} \mathcal{F}_j^{(\pm)}(k) f.$$

We put

$$\mathbf{h}_\infty = \left(\oplus_{i=1}^M L^2(M_i)\right) \oplus \left(\oplus_{i=M+1}^N \mathbf{C}\right), \tag{10.10}$$

and for $\varphi, \psi \in \mathbf{h}_\infty$ we define the inner product by

$$(\varphi, \psi)_{\mathbf{h}_\infty} = \sum_{i=1}^{M} (\varphi_i, \psi_i)_{L^2(M_i)} + \sum_{j=M+1}^{N} \varphi_j \overline{\psi_j} |M_j|.$$

We also put

$$\mathcal{F}^{(\pm)}(k) f = \left(\mathcal{F}_1^{(\pm)}(k) f, \cdots, \mathcal{F}_N^{(\pm)}(k) f\right).$$

$$\widehat{\mathcal{H}} = L^2((0,\infty); \mathbf{h}_\infty; dk).$$

Theorem 10.8. *We define $\left(\mathcal{F}^{(\pm)} f\right)(k) = \mathcal{F}^{(\pm)}(k) f$ for $f \in \mathcal{B}$. Then $\mathcal{F}^{(\pm)}$ is uniquely extended to a bounded operator from $L^2(\mathcal{M})$ to $\widehat{\mathcal{H}}$ with the following properties.*

(1) Ran $\mathcal{F}^{(\pm)} = \widehat{\mathcal{H}}$.
(2) $\|f\| = \|\mathcal{F}^{(\pm)} f\|$ *for* $f \in \mathcal{H}_{ac}(H)$.
(3) $\mathcal{F}^{(\pm)} f = 0$ *for* $f \in \mathcal{H}_p(H)$.
(4) $\left(\mathcal{F}^{(\pm)} H f\right)(k) = k^2 \left(\mathcal{F}^{(\pm)} f\right)(k)$ *for* $f \in \mathrm{Dom}\, H$.
(5) $\mathcal{F}^{(\pm)}(k)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$ *and* $(H - k^2)\mathcal{F}^{(\pm)}(k)^* = 0$ *for* $k^2 \in (0,\infty) \setminus \sigma_p(H)$.
(6) For $f \in \mathcal{H}_{ac}(H)$, the inversion formula holds:

$$f = \left(\mathcal{F}^{(\pm)}\right)^* \mathcal{F}^{(\pm)} f = \sum_{j=1}^{N} \int_0^\infty \mathcal{F}_j^{(\pm)}(k)^* \left(\mathcal{F}_j^{(\pm)} f\right)(k) dk.$$

§ 10.3. S matrix

Theorem 5.2 is extended to $\mathcal{M}$.

Theorem 10.9. *If $k^2 \notin \sigma_p(H)$, we have*

$$\mathcal{F}^{(\pm)}(k)\mathcal{B} = \mathbf{h}_\infty,$$

$$\{u \in \mathcal{B}^*\,;\, (H - k^2)u = 0\} = \mathcal{F}^{(\pm)}(k)^* \mathbf{h}_\infty.$$

We derive an asymptotic expansion of solutions to the Helmholtz equation. Let V_q be the differential operator defined by

$$V_q = -[H_{0q}, \chi_q] + V\chi_q \quad (1 \leq q \leq N).$$

For $1 \leq p \leq M,\ 1 \leq q \leq N$, we define

$$\widehat{S}_{pq}(k) = \delta_{pq} J_p(k) - \frac{\pi i}{k} \mathcal{F}_p^{(+)}(k) V_q^* \left(\mathcal{F}_{0\,q}^{(-)}(k)\right)^*,$$

$$J_p(k)\psi = \sum_{m\geq 1} \left(\frac{\sqrt{\lambda_{p,m}}}{2}\right)^{-2ik} \varphi_{p,m}(x)\widehat{\psi}_m \quad (1 \leq p \leq M).$$

For $M+1 \leq p \leq N, 1 \leq q \leq N$, we define

$$\widehat{S}_{pq}(k) = -\frac{\pi i}{k} \mathcal{F}_p^{(+)}(k) V_q^* \left(\mathcal{F}_{0\,q}^{(-)}(k)\right)^*.$$

Theorem 10.10. *For $\psi = (\psi_1, \cdots, \psi_N) \in \mathbf{h}_\infty$, we have*

$$\begin{aligned}
\left(\mathcal{F}^{(-)}(k)\right)^* \psi &= \sum_{p=1}^N \left(\mathcal{F}_p^{(-)}(k)\right)^* \psi_p \\
&\simeq \frac{ik}{\pi}\omega_-(k) \sum_{p=1}^M \chi_p y^{(n-1)/2+ik} \psi_p \\
&+ \frac{ik}{\pi}\omega_-(k) \sum_{p=M+1}^N \chi_p y^{(n-1)/2-ik} \widehat{\psi}_{p0} \\
&- \frac{ik}{\pi}\omega_+(k) \sum_{p=1}^M \sum_{q=1}^N \chi_p y^{(n-1)/2-ik} \widehat{S}_{pq}(k)\psi_q \\
&- \frac{ik}{\pi}\omega_+^{(c)}(k) \sum_{p=M+1}^N \sum_{q=1}^N \chi_p y^{(n-1)/2+ik} \widehat{S}_{pq}(k)\psi_q.
\end{aligned}$$

We define an operator-valued $N \times N$ matrix $\widehat{S}(k)$ by

$$\widehat{S}(k) = \left(\widehat{S}_{pq}(k)\right).$$

Theorem 10.11. *(1) For any $u \in \mathcal{B}^*$ satisfying $(H - k^2)u = 0$, there exists a unique $\psi^{(\pm)} \in \mathbf{h}_\infty$ such that*

$$\begin{aligned}
u \simeq \omega_-(k) \sum_{p=1}^M \chi_p y^{(n-1)/2+ik} \psi_p^{(-)} + \omega_-(k) \sum_{p=M+1}^N \chi_p y^{(n-1)/2-ik} \widehat{\psi}_{p0}^{(-)} \\
- \omega_+(k) \sum_{p=1}^M \chi_p y^{(n-1)/2-ik} \psi_p^{(+)} - \omega_+^{(c)}(k) \sum_{p=M+1}^N \chi_p y^{(n-1)/2+ik} \psi_p^{(+)}.
\end{aligned}$$

(2) For any $\psi^{(-)} \in \mathbf{h}_\infty$*, there exists a unique* $\psi^{(+)} \in \mathbf{h}_\infty$ *and* $u \in \mathcal{B}^*$ *satisfying* $(H - k^2)u = 0$*, for which the expansion (1) holds. Moreover*

$$\psi^{(+)} = \widehat{S}(k)\psi^{(-)}.$$

Theorem 10.12. $\widehat{S}(k)$ *is unitary on* $\mathbf{h}_\infty$ *and*

$$\mathcal{F}^{(+)}(k) = \widehat{S}(k)\mathcal{F}^{(-)}(k).$$

§ 11. Singularity expansion of the Radon transform

We look at the upper-half space model $\mathbf{R}^n_+$ from a different view point. We shall assume that the perturbed metric has the following properties:

$$ds^2 = y^{-2}\Big((dx)^2 + (dy)^2 + A(x,y,dx,dy)\Big),$$

where $A(x,y,dx,dy)$ is a symmetric covariant tensor of the form

$$A(x,y,dx,dy) = \sum_{i,j=1}^{n-1} a_{ij}(x,y)dx^i dx^j + 2\sum_{i=1}^{n-1} a_{in}(x,y)dx^i dy + a_{nn}(x,y)(dy)^2,$$

and each $a_{ij}(x,y)$ $(1 \leq i,j \leq n)$ is assumed to satisfy the condition

$$|\widetilde{D}_x^\alpha D_y^\beta a(x,y)| \leq C_{\alpha\beta}(1 + d_h(x,y))^{-\beta-1-\epsilon_0}, \quad \epsilon_0 > 0, \tag{11.1}$$

where $d_h(x,y)$ is the hyperbolic distance between (x,y) and $(0,1)$.

Recall that the homogeneous distribution $\big(s\big)_\pm^\alpha$ is defined for $\mathrm{Re}\,\alpha > -1$ by

$$\big(s\big)_\pm^\alpha = \begin{cases} |s|^\alpha/\Gamma(\alpha+1), & \pm s > 0, \\ 0, & \pm s < 0, \end{cases}$$

and for $n = 1,2,3,\cdots$ and $\mathrm{Re}\,\alpha > -1$

$$\big(s\big)_\pm^{\alpha-n} = \left(\pm\frac{d}{ds}\right)^n \big(s\big)_\pm^\alpha.$$

Then one can constuct the generalized Fourier tansform by using the method in §10 and define the modified Radon transform associated with this metric in the same way as in §6.

Theorem 11.1. *Let* $s_0 > -\log y_0/4$*, and take sufficienlly large* $N > 0$*. Then*

$$\mathcal{R}_+ = \sum_{j=0}^{N} \mathcal{R}_+^{(j)} + R_N,$$

where R_N is a local regularizer of order N with respect to $s > s_0$, and

$$\left(\mathcal{R}_+^{(j)} f\right)(s,x) = \int_0^{\infty} (s+\log y)_-^{j-1} y^{-\frac{n-1}{2}} P_j(y) f(x,y)\chi(y)\frac{dy}{y},$$

$$P_j(y) = \frac{(-i)^j}{\sqrt{2}} a_j(x,y,-i\partial_x)^*.$$

Here $a_j(x,y,\xi)$ is is a polynomial in ξ of order $2j$. Hence $a_j(x,y,-i\partial_x)$ is a differential operator of order $2j$. The above theorem in particular yields the following expression

$$\left(\mathcal{R}_+^{(j)} f\right)(s,x) = \begin{cases} \dfrac{e^{(n-1)s/2}}{\sqrt{2}}\chi(e^{-s})f(x,e^{-s}), \quad (j=0), \\ \displaystyle\int_0^{e^{-s}} \frac{(s+\log y)^{j-1}}{(j-1)!} y^{-\frac{n-1}{2}} P_j(y) f(x,y)\chi(y)\frac{dy}{y}, \quad (j \geq 1), \end{cases} \tag{11.2}$$

where $\chi(y) \in C^{\infty}(\mathbf{R})$ such that $\chi(y) = 1$ $(y < y_0/4)$, $\chi(y) = 0$ $(y > y_0/3)$. This is a generalization of Theorem 7.6 in the sense of singularity expansion.

§ 12. Inverse problems for hyperbolic ends

§ 12.1. Inverse scattering at regular ends

Let $\mathcal{M}$ be a manifold satisfying the assumptions (A.1) $\sim$ (A.4) in §2 with ends of number $N \geq 1$. We assume that at least *one of the ends has a regular infinity.* Let $\mathcal{M}_1$ be such an end. Nemely, in the notation §2, $\mathcal{M}_1$ is diffeomorphic to $M \times (0,1)$. Let $\Gamma \subset \mathcal{M}$ be a compact submanifold of codimension 1 such that $\mathcal{M}$ is split into 2 parts $\mathcal{L}_1$ and $\mathcal{K}_1$ in the following way :

$$\mathcal{M} = \mathcal{L}_1 \cup \mathcal{K}_1, \quad \mathcal{L}_1 \cap \mathcal{K}_1 = \Gamma,$$

where $\mathcal{L}_1$ and $\mathcal{K}_1$ are assumed to be submanifolds of $\mathcal{M}$ with boundary Γ inheriting the Riemannian structure of $\mathcal{M}$. Assume also that $\mathcal{L}_1$ is non-compact and has infinity common to $\mathcal{M}_1$.

Let Δ_g be the Laplace-Beltrami operator on $\mathcal{M}$, and $H(\mathcal{L}_1)$ and $H(\mathcal{K}_1)$ be $-\Delta_g - (n-1)^2/4$ on $\mathcal{L}_1$ and $\mathcal{K}_1$ with Neumann boundary condition on the boundary Γ. Then one can solve the Neumann problem on $\mathcal{K}_1$:

$$\begin{cases} (H(\mathcal{K}_1) - \lambda)u = 0 \quad \text{in} \quad \mathcal{K}_1, \\ \partial_\nu u = f \quad \text{on} \quad \Gamma, \end{cases}$$

where we assume the outgoing radiation condition if $\mathcal{K}_1$ is non-compact. Using the solution u of this equation, we define the Neumann to Dirichlet map (N-D map) by

$\Lambda^{(+)}(\lambda)f = u\big|_{\partial\Omega}$. Note that $\Lambda^{(+)}(\lambda)$ is also defined for $z \in \sigma(H(\mathcal{K}_1))$, and $\Lambda^{(+)}(\lambda)$ is the boundary value of $\Lambda(z)$ as $z \to \lambda + i0$. Therefore $\Lambda^{(+)}(\lambda)$ defined for $\lambda > 0$ has a unique analytic continuation to $\mathbf{C} \setminus \sigma(H(\mathcal{K}_1))$.

Since $\mathcal{M}$ has N-ends, the S-matrix for $\mathcal{M}$ is an $N \times N$-matrix:

$$\widehat{S}(k) = \left(\widehat{S}_{ij}(k)\right)_{1 \le i,j \le N}.$$

Theorem 12.1. *Suppose $k^2 \neq 0$ is not a Neumann eigenvalue for $H^{(1)}$ and $H^{(2)}$ on $\mathcal{K}_1$. Let $\Lambda_j^{(+)}(k)$ be the N-D map for $H^{(j)}$, $j = 1, 2$, on $\mathcal{K}_1$. Suppose $G^{(1)} = G^{(2)}$ on $\mathcal{L}_1$. Then $\widehat{S}_{11}^{(1)}(k) = \widehat{S}_{11}^{(2)}(k)$ if and only if $\Lambda_1^{(+)}(k^2) = \Lambda_2^{(+)}(k^2)$.*

Recall that $\mathcal{K}_1$ is a non-compact manifold with compact boundary Γ. The operator $H(\mathcal{K}_1) = -\Delta_g$ has two parts of spectral representations: the generalized Fourier transform, which we denote by $\mathcal{F}_c^{(+)}$ here, corresponding to the absolutely continuous spectrum for $H(\mathcal{K}_1)$, and the discrete Fourier transform, denoted by $\mathcal{F}_p$, corresponding to the point specrum for $H(\mathcal{K}_1)$.

We put $\Omega_{ex} = \mathcal{K}_1$ and $\partial\Omega_{ex} = \Gamma$. Let $r_\Gamma \in \mathbf{B}(H^1(\Omega_{ex}); H^{1/2}(\Gamma))$ be the trace operator to Γ. Define $\delta_\Gamma \in \mathbf{B}(H^{-1/2}(\Gamma); H^{-1}(\Omega_{ex}))$ as its adjoint:

$$(\delta_\Gamma f, w)_{L^2(\Omega_{ex})} = (f, r_\Gamma w)_{L^2(\Gamma)}, \quad f \in H^{-1/2}(\Gamma), \quad w \in H^1(\Omega_{ex}).$$

Accordingly, we write as

$$r_\Gamma = \delta_\Gamma^*.$$

Lemma 12.2. *The N-D map $\Lambda(z)$ defined for $z \in \mathbf{C} \setminus \mathbf{R}$ is split ito two parts*

$$\Lambda(z) = \int_0^\infty \frac{\delta_\Gamma^* \mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k)\delta_\Gamma}{k^2 - z} dk + \sum_i \frac{\delta_\Gamma^* P_i \delta_\Gamma}{\lambda_i - z}. \tag{12.1}$$

Let us call the set

$$\left\{\delta_\Gamma^* \mathcal{F}_c^{(+)}(k)^* \mathcal{F}_c^{(+)}(k)\delta_\Gamma; k > 0\right\} \cup \left\{\left(\lambda_i, \delta_\Gamma^* P_i \delta_\Gamma\right)\right\}_{i=1}^\infty \tag{12.2}$$

the boundary spectral projection (**BSP**) for $H(\mathcal{K}_1)$. By (12.1), we have

$$\Lambda(z) = \delta_\Gamma^*(H(\mathcal{K}_1) - z)^{-1}\delta_\Gamma. \tag{12.3}$$

Lemma 12.3. *Knowing the N-D map $\Lambda^{(+)}(k)$ for all k such that $k^2 \notin \sigma_p H(\mathcal{K}_1)$ is equivalent to knowing BSP for $H(\mathcal{K}_1)$.*

We now pass to the boundary control method (BC-method) to reconstruct the manifold from BSP. The BC-method does not rely on the special manifold structure, and works if we know the N-D map for the asoociated Laplace-Beltrami operator. The

BC-method was first applied to compact manifolds ([BeKu92]), and was extended to non-compact manifolds (see e.g. [Be97], [KK98]).

Let us formulate the inverse problem on non-compact Riemannian manifolds. Let $M^{(1)}$ and $M^{(2)}$ be Riemannian manifolds (not necessarily compact) with boundary, on which is inhelited the Riemannian metric induced from $M^{(r)}$. We say that $M^{(1)}$ and $M^{(2)}$ have a common part $\Gamma^{(1)} = \Gamma^{(2)}$ on the boundary if there exists an open set $\Gamma^{(r)} \subset \partial M^{(r)}$ and a diffeomorphism $\phi : \Gamma^{(1)} \to \Gamma^{(2)}$. Let $\Lambda^{(r)}(z)$ be the N-D map for the Laplace-Beltrami operator on $M^{(r)}$. Then we define

$$\Lambda^{(1)}(z)\Big|_{\Gamma^{(1)}} = \Lambda^{(2)}(z)\Big|_{\Gamma^{(2)}} \Longleftrightarrow \phi \circ \Lambda^{(1)}(z)\Big|_{\Gamma^{(1)}} = \Lambda^{(2)}(z)\Big|_{\Gamma^{(2)}} \circ \phi. \tag{12.4}$$

One can then show that (with some additional assumptions) if $M^{(1)}$ and $M^{(2)}$ have Γ in common and the same N-D map on Γ (in thse sense that the above (12.4) holds for all $z \notin \mathbf{R}$), then $M^{(1)}$ and $M^{(2)}$ are isometric. Assuming this for the moment, we have proven the following theorem.

Theorem 12.4. *Let $\mathcal{M}$ be a manifold satisfying the assumptions (A.1) $\sim$ (A.4) in §2. We assume that one of the ends has a regular infinity, and denote it by $\mathcal{M}_1$. Suppose we are given two metrics $G^{(j)}$, $j = 1, 2$, on $\mathcal{M}$ satisfying (A-3). Assume that $G^{(1)} = G^{(2)}$ on $\mathcal{M}_1$. If $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$ for all $k > 0$, then $G^{(1)}$ and $G^{(2)}$ are isometric on $\mathcal{M}$.*

We can actually prove a stronger version of Theorem 12.4, i.e. it is valid for two manifolds whose number of of ends are not known a-priori.

Theorem 12.5. *Let $\mathcal{M}^{(p)}$, $p = 1, 2$, be manifolds satisfying the assumptions (A.1)$\sim$ (A.4) in §2 endowed with metric $G^{(p)}$, $p = 1, 2$. We assume that for both of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ one of the ends has a regular infinity, and denote it by $\mathcal{M}_1^{(p)}$, $p = 1, 2$. Assume that $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_1^{(2)}$ are isometric, and $\widehat{S}_{11}(k) = \widehat{S}_{11}(k)$ for all $k > 0$. Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.*

§ 13. Brief introduction to the boundary control method

§ 13.1. Wave equation and Gel'fand inverse problem

In the remaining sections, we give a brief explanation of the BC method. Let M be an n-dimensional connected Riemannian manifold with boundary ∂M. We shall consider the IBVP (initial-boundary value problem) for the wave equation

$$\partial_t^2 u = \Delta_g u \quad \text{on} \quad M \times (0, \infty).$$

We impose the initial condition

$$u\big|_{t=0} = \partial_t u\big|_{t=0} = 0,$$

and the boundary condition

$$\partial_\nu u\big|_{\partial M\times(0,\infty)} = f \in C_0^\infty(\partial M \times (0,\infty)).$$

Here ν is the outer unit normal to ∂M. Let $u^f(x,t)$ be the solution to the above IBVP. We measure u^f on $\partial M \times (0,\infty)$, and call

$$\Lambda^h : f \to u^f\big|_{\partial M\times(0,\infty)} \tag{13.1}$$

a *hyperbolic Neumann-Dirichlet map.* The basic question we address is the following one.

Question Assume we know Λ^h. Can we determine (M,g), i.e. the manifold M and the metric g?

This is the *Gel'fand inverse problem* (stated in a slightly different form). Note that Λ^h is an opeartor defined on $\partial M\times(0,\infty)$. Starting from the knowledge on $\partial M\times(0,\infty)$, the first issue is the topology of M, and the second issue is the Riemannian structure.

The answer to the above question is affirmative when M is compact, and also for non-compact M with some additional geometric assumptions. To fix the idea, in the following, M means either any compact connected Riemannian manifold with boundary, or when dealing with the non-compact case, the manifold $\mathcal{K}_1$ discussed in the previous section. However, the arguments given below also work for non-compact manifolds possesing the spectral representation as in the case of $\mathcal{K}_1$. Note that in both cases ∂M is compact.

§ 13.2. Spectral formulation

Let us begin with the compact manifold case. Consider the Neumann Laplacian A_N:

$$A_N u = -\Delta_g u, \quad u \in H^2(M), \quad \partial_\nu u\big|_{\partial M} = 0.$$

The spectrum of A_N consists of real numbers

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty.$$

Let φ_k be the associated eigenvectors

$$-\Delta_g\varphi_k = \lambda_k\varphi_K, \quad \partial_\nu\varphi_k\big|_{\partial M} = 0.$$

Without loss of generality we can assume φ_k to be real-valued. We call $\left\{\left(\lambda_k, \varphi_k\big|_{\partial M}\right)\right\}_{k=1}^\infty$ the *boundary spectral data* (BSD). The original Gel'fand inverse problem is:

Question Given BSD, can we determine (M,g)?

The relation of BSD to the hyperbolic Neumann-Dirichlet map is represented by the following (formal) formula:

$$\left(\Lambda^h f\right)(x,t) = \int_{\partial M}\int_{\mathbf{R}_+} G(x,y,t-s)f(y,s)dS_y ds.$$

$$G(x,y,t) = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\lambda_k}t)}{\sqrt{\lambda_k}} \varphi_k(x)\varphi_k(y)\big|_{\partial M\times\partial M}. \tag{13.2}$$

The *Boundary Control* (BC) method goes back to the famous results by M. G. Krein, in the mid-fifties, on the 1−dimensional inverse scattering theory. Compared with the fundamental methods by Gel'fand-Levitan and Marchenko, the method of Krein is distinguished by the systematic use of the finite propagation speed for the wave equation. However, the ideas based upon the domain of influence, etc. coming from this finite velocity are "disguised" in the work of Krein due to their formulation in the frequency domain (or the stationary equation), where they turn out to be conditions on analyticity of the corresponding Fourier transform of the solution. This principal hyperbolic nature of Krein's method was revealed by Blagovestchenskii who was working in the time-domain (or the time-dependnet equation) using the finite velocity of the wave propagation and ideas of controllability in the filled domain to derive a Volterra-type equation for unknown functions. These ideas have become crucial for the extension of the method to multidimensions pioneered by Belishev [Be87], [BeKu92]. One more important ingredient of the BC-method, namely, the possibility to evaluate the inner product of waves sent into M from ∂M also goes back to the 1-dimensional case to the work of Blagovestchenskii.

The BC method has the following features.

(1) BC method is hyperbolic.

Since the propagation speed of wave motion is finite, and singularities of waves are related with geodesics, this implies the close connection of BC method with geometry.

(2) BC method is not perturbative.

We do not assume that the given metric is close to some standard one. In this sense, the BC method does not have the character of perturbation theory.

§ 13.3. Outline of the procedure

The crucial tool of the BC-method is the space of boundary distance functions $R(M)$ to be defined in §16, and the reconstruction of the manifold M is done by the following 3 steps :

- BSP determines $R(M)$ (§19).
- $R(M)$ is topologically isomorphic to M (§16).
- $R(M)$ determines the Riemannian metric of M (§18).

This is an effective interplay of linear partial differential equations and geometry. The main ingredients of the 1st step are Blagovestchenskii's idenitity, which represents the solution of IBVP of the wave equation by BSD, and Tatar's uniqueness theorem, which guarantees the conrollablity of IBVP. The 2nd step is of character of general topology. The 3rd step is purely from differential geometry, in which the coordinate system of M is constructed by $R(M)$ and the metric tensor is computed.

§ 14. Blagovestchenskii idenitity

Given a solution u^f of the wave equation

$$\begin{cases} \partial_t^2 u = \Delta_g u, \\ \partial_\nu u\big|_{\partial M\times\mathbf{R}_+} = f \in C_0^\infty(\partial M\times(0,\infty)), \\ u\big|_{t=0} = \partial_t u\Big|_{t=0} = 0, \end{cases} \tag{14.1}$$

we have when M is compact

$$u_k^f(t) = \int_0^t ds \int_{\partial M} dS_g \frac{\sin(\sqrt{\lambda_k}(t-s))}{\sqrt{\lambda_k}} f(y,s)\varphi_k(y). \tag{14.2}$$

This formula shows that $u_k^f(t)$ is represented by λ_k and $\varphi_k\big|_{\partial M}$, i.e. BSD.

Lemma 14.1. *The following Blagovestchenskii idenitity holds:*

$$(u^f(t), u^h(s)) = \sum_k u_k^f(t)\overline{u_k^h(s)}, \qquad \forall t,s\in\mathbf{R},\ \forall f,h\in C_0^\infty(\partial M\times\mathbf{R}_+) \tag{14.3}$$

Lemma 14.1 is the first corner stone of BC method. We let

$$S(t,\lambda) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}, \quad \widetilde{S}(t,s,\lambda) = S(t,\lambda)S(s,\lambda), \tag{14.4}$$

and rewrite the right-hand side of (14.3) as

$$\sum_i \int_0^t \int_0^s dt'ds'\widetilde{S}(t-t',s-s',\lambda_i)\left(\delta_\Gamma^* P_i\delta_\Gamma f(t'), h(s')\right). \tag{14.5}$$

This implies the following corollary.

Corollary 14.2. *The inner product $(u^f(t), u^h(s))$ is written only by BSP.*

This is also true when $-\Delta_g$ has the continuous spectrum. The Laplace-Beltrami operator on $\mathcal{K}_1$ admits the spectral representation $\mathcal{F}_c^{(+)}$. In this case, to modify the formula (14.3), we have only to add the integral of $\mathcal{F}_c^{(+)}(k)^*\mathcal{F}_c^{(+)}(k)$ to the right-hand side of (14.5):

$$\begin{aligned} &\int_0^\infty dk \int_0^t\int_0^s dt'ds'\,\widetilde{S}(t-t's-s',k^2)\left(\delta_\Gamma^*\mathcal{F}_c^{(+)}(k)^*\mathcal{F}_c^{(+)}(k)\delta_\Gamma f(t'), h(s')\right) \\ &+\sum_i \int_0^t\int_0^s dt'ds'\widetilde{S}(t-t',s-s',\lambda_i)\left(\delta_\Gamma^* P_i\delta_\Gamma f(t'), h(s')\right). \end{aligned} \tag{14.6}$$

Again $(u^f(t), u^h(s))$ is written only by BSP.

§ 15. Controllabilty and observability

§ 15.1. Controllability

Let u^f be a unique solution to IBVP (14.1). We take a bounded open set S on the boundary, $S \subset \partial M$, and put for $t_0 > 0$

$$M(S, t_0) = \{x \in M; d(x, S) \leq t_0\}.$$

Lemma 15.1. *If* $\operatorname{supp} f \subset S \times (0, \infty)$*, then*

$$\operatorname{supp} u^f(\cdot, t_0) \subset M(S, t_0), \quad \forall t_0 > 0.$$

For a measurabe subset $D \subset M$ and $f \in L^2(D)$, we define $f = 0$ on $M \setminus D$ and regard $L^2(D)$ as a closed subspace of $L^2(M)$. In view of Lemma 15.1, we can define the map W_{t_0} by

$$W_{t_0} : C_0^\infty(S \times (0, t_0)) \ni f \to u^f\big|_{t=t_0} \in L^2(M(S, t_0)).$$

The crucial fact is the following theorem due to Tataru ([Ta95]).

Theorem 15.2. $\overline{\operatorname{Ran}(W_{t_0})} = L^2(M(S, t_0))$.

By this theorem, for any $\epsilon > 0$ and $a \in L^2(M)$ such that $\operatorname{supp} a \subset M(S, t_0)$, there exists $f = f_{\epsilon,a} \in C_0^\infty(S \times (0, t_0))$ satisfying $\|u^f(\cdot, t_0) - a\|_{L^2(M)} < \epsilon$. Therefore the property described in Theorem 15.2 should be called *approximate controllability.*

§ 15.2. Observability

Let us also consider the adjoint problem of (14.1):

$$\begin{cases} \partial_t^2 v = \Delta_g v \quad \text{in} \quad M \times \mathbf{R}, \\ \partial_\nu v\big|_{\partial M \times \mathbf{R}} = 0, \\ v\big|_{t=t_0} = 0, \quad \partial_t v\big|_{t=t_0} = \psi \in L^2(M). \end{cases} \tag{15.1}$$

We define the *observability operator* by

$$\mathcal{O}_{t_0}\psi = v^\psi\big|_{S \times (0, t_0)}, \quad \psi \in L^2(M(S, t_0)),$$

where v^ψ is the weak solution to (15.1). Note that $v^\psi\big|_{\partial M \times \mathbf{R}} \in C(\mathbf{R}, H^{1/2}(\partial M))$, and

$$\|\mathcal{O}_{t_0}\psi\|_{L^2(S \times (0,t_0))} \leq C\|\psi\|_{L^2(M)}, \tag{15.2}$$

where $C = C_{t_0}$ is a constant. We can show that

Lemma 15.3. *For any* $f \in C_0^\infty(S \times (0, t_0))$, $\psi \in L^2(M(S, t_0))$*, we have*

$$(W_{t_0}f, \psi)_{L^2(M(S,t_0))} = -(f, \mathcal{O}_{t_0}\psi)_{L^2(S \times (0,t_0))}.$$

Lemma 15.3 imply that

$$W_{t_0} = -\mathcal{O}_{t_0}^* \in \mathbf{B}(L^2(S \times (0, t_0)); L^2(M)). \tag{15.3}$$

Therefore Theorem 15.2 is equivalent to the statement that

$$\operatorname{Ker} \mathcal{O}_{t_0} = \{0\}. \tag{15.4}$$

This property is called *observability.* The claim (15.4) means the following:

Assume v satisfies

$$\begin{cases} \partial_t^2 v = \Delta_g v \quad \text{in} \quad M \times \mathbf{R}, \\ v\big|_{t=t_0} = 0, \quad \operatorname{supp} \partial_t v\big|_{t=t_0} \subset M(S, t_0), \\ \partial_\nu v\big|_{\partial M \times (0,t_0)} = 0, \quad v\big|_{S \times (0,t_0)} = 0. \end{cases}$$

Then $\partial_t v\big|_{t=t_0} = 0$.

§ 15.3. Uuiqueness theorem

If we put $u(t) = v(t + t_0)$, then $u(t) = -u(-t)$ and the above claim is formulated as follows.

Assume u satisfies

$$\begin{cases} \partial_t^2 u = \Delta_g u \quad \text{in} \quad M \times \mathbf{R}, \\ u\big|_{t=0} = 0, \quad \operatorname{supp} \partial_t u\big|_{t=0} \subset M(S, t_0), \\ \partial_\nu u\big|_{\partial M \times (-t_0,t_0)} = 0, \quad u\big|_{S \times (-t_0,t_0)} = 0. \end{cases}$$

Then $\partial_t u\big|_{t=0} = 0$.

Actually, we can prove a stronger version of this claim. Namely:

Theorem 15.4. *Assume $u \in C^\infty(M \times (-t_0, t_0))$ satisfies*

$$\begin{cases} \partial_t^2 u = \Delta_g u \quad \text{in} \quad M \times (-t_0, t_0), \\ \partial_\nu u\big|_{S \times (-t_0,t_0)} = 0, \quad u\big|_{S \times (-t_0,t_0)} = 0. \end{cases}$$

Then $u\big|_{t=0} = 0$ in the double cone of influence $K(S, t_0)$, i.e.

$$K(S, t_0) = \{(x, t)\, ;\, d(x, S) \le t_0 - |t|\}.$$

This sort of theorem (Holmgren-John type uniqueness theorem) has a long story, starting from the classical result by Holmgren:

Theorem 15.5. *Let u be a classical, i.e. C^2, solution to the partial differential equation $P(x, D_x)u = 0$ with analytic coeffcients. If $u = 0$ in one side of a non-characteristic surface Σ, then* $\operatorname{supp} u \cap \Sigma = \emptyset$*, i.e. $u = 0$ near Σ.*

Recall that for a differential operator $P(x, D_x) = \sum_{|\alpha|\leq m} p_\alpha(x) D_x^\alpha$ defined on an open set U in $\mathbf{R}^n$, its *principal part* is defined by $P_m(x, \xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$. A surface Σ of co-dimension 1 in U is said to be *non-characteristic* to $P(x, D_x)$, if $P_m(x, \nu_x) \neq 0$ for any $x \in \Sigma$ and normal ν_x to Σ at x. Theorem 4.6 was first proved by E. Holmgren in 1901 and extended by F. John in 1949. This theorem has been tried to be extended to the C^∞-coefficient case by Robbiano or Hörmander, and finally Tataru [Ta95] succeeded in obtaining the result in full generality. The importance of *non-analyticity* should be strongly emphasized in applications to inverse problems.

§ 16. Topological reconstruction of M by $R(M)$

§ 16.1. Reconstruction from boundary distance functions

The key idea of the BC-method is to reconstruct the *boundary distance function*, $r_x(z)$, defined as follows: For any $x \in M$, r_x is defined by

$$r_x(z) = d(x, z), \quad z \in \partial M,$$

$d(x, y)$ being the distance of $x, y \in M$. We define the map R by

$$R : M \ni x \to r_x(\cdot) \in C(\partial M).$$

If ∂M is compact, $R(M)$ becomes a metric space by the distance

$$d_\infty(r_1, r_2) = \|r_1(\cdot) - r_2(\cdot)\|_{L^\infty(\partial M)},$$

and the following inclusion relation hold

$$R(M) \subset C^{0,1}(\partial M) \subset L^\infty(\partial M),$$

where $C^{0,1}(\partial M)$ is the space of Lipschitz continuous functions on ∂M. The utility of the boundary distance function is seen in the following lemma.

Lemma 16.1. *If ∂M is compact, $(R(M), d_\infty)$ is homeomorphic to (M, d).*

$R(M)$ is a set of functions indexed by the points $x \in M$. However in the inverse problem we are now considering, we know neither M nor x, since they are the objects we are trying to reconstruct. So, changing the notation, we let $r_1 = r_x, r_2 = r_y$, where $x, y \in M$. Assume we can find new distance $\widehat{d}(r_1, r_2)$ from $d_\infty(r_1, r_2)$ so that $\widehat{d}(r_1, r_2) = d(x, y)$ for x, y such that $r_1 = r_x$, $r_2 = r_y$. Then $(R(M), \widehat{d})$ becomes isometric, as a metric space, to (M, d). By the Myers-Steenrod theorem, this implies that there is a unique Riemannian manifold structure on $R(M)$ such that $R : M \to R(M)$ is isometry. In the following, we give a direct way of reconstructing the Riemannian manifold structure on $R(M)$ to make R a Riemannian isometry from M to $R(M)$, without leaning over the abstract nature of the Myers-Steenrod theorem.

§ 17. Boundary cut locus

To introduce a Riemannian manifold structure on $R(M)$, we use geodesics emanating from the boundary ∂M. We then need to discuss the maximal region on which we can introduce the boundary normal coordinates, and also the structure of the cut locus. The geometrical tools necessary for this step are standard, which we shall explain in this section. For a Riemannian manifold M, let $T_x(M)$ be the tangent space at $x \in M$. Recall that for $\xi, \eta \in T_x(M)$, the inner product and the length are defined by

$$g_x(\xi,\eta) = g_{ij}(x)\xi^i\eta^j = \sum_{i,j=1}^n g_{ij}(x)\xi^i\eta^j, \quad |\xi|_g = \sqrt{g_x(\xi,\xi)}$$

Put $S_x(M) = \{\xi \in T_x(M)\,;\, |\xi|_g = 1\}$. Let $T(M)$ and $T^*(M)$ be the tangent bundle and the cotangent bundle of M, respectively.

§ 17.1. Variation and Jacobi fields

Let $c(t)$ be a curve on M. For a vector field $X(t)$ on M, with components $(X^1(t), \cdots, X^n(t))$ in local coordinates, the *covariant differential* $\frac{D}{dt}X(t)$ along $c(t)$ is defined by

$$\frac{D}{dt}X^k(t) = \dot{X}^k(t) + \Gamma^k_{ij}(c(t))\dot{c}^i(t)X^j(t),$$

where we used the abbreviation $\dot{f}(t) = \frac{df(t)}{dt}$. A vector field $Z(t)$ is said to be *parallel* along $c(t)$ if it satisfies $\frac{D}{dt}Z(t) = 0$. In particular, $c(t)$ is a geodesic if and only if $c(t)$ is parallel along $c(t)$. The energy of the curve $c(t)$ is defined by

$$E(c) = \frac{1}{2}\int_a^b g_{c(t)}(\dot{c}(t),\dot{c}(t))dt.$$

A C^∞-map : $[a,b] \times (-\epsilon,\epsilon) \ni (t,s) \to H(t,s) \in M$ is said to be a *variation* of $c(t)$ if $H(t,0) = c(t)$ $(a \le t \le b)$. The *curvature tensor* R is defined by

$$(R(X,Y)Z)^l = R^l_{ijk}X^iY^jZ^k,$$

$$R^l_{ijk} = \frac{\partial\Gamma^l_{jk}}{\partial x^i} - \frac{\partial\Gamma^l_{ik}}{\partial x^j} + \Gamma^l_{ir}\Gamma^r_{jk} - \Gamma^l_{jr}\Gamma^r_{ik},$$

where X, Y, Z are vector fields on M.

Lemma 17.1. *Let $H(t,s)$ be a variation of $c(t)$, and put $c_s(t) = H(t,s)$. We define the vector field $Y(t)$ along $c(t)$ by*

$$Y(t) = \frac{\partial}{\partial s}H(t,s)\Big|_{s=0}.$$

Then the following formulas hold.
(1) The 1st variation formula:

$$\frac{d}{ds}E(c_s)\Big|_{s=0} = g_{c(b)}(Y(b),\dot{c}(b)) - g_{c(a)}(Y(a),\dot{c}(a)) - \int_a^b g_{c(t)}\Big(Y(t), \frac{D}{dt}\dot{c}(t)\Big)dt,$$

where D/dt is the covariant diffrential along $c(t)$.
(2) The 2nd variation formula:

$$\begin{aligned}\frac{d^2}{ds^2}E(c_s)\Big|_{s=0} &= g_{c(b)}(S(b),\dot{c}(b)) - g_{c(a)}(S(a),\dot{c}(a)) \\ &\quad + \int_a^b \Big\{ g_{c(t)}\Big(\frac{D}{dt}Y(t), \frac{D}{dt}Y(t)\Big) - g_{c(t)}\big(R(Y(t),\dot{c}(t))\dot{c}(t), Y(t)\big) \\ &\quad - g_{c(t)}\Big(S(t), \frac{D}{dt}\dot{c}(t)\Big)\Big\}dt,\end{aligned}$$

where letting D/ds be the covariant differential along the curve $C_t(s) : s \to H(t,s)$,

$$S(t) = \frac{D}{ds}\frac{\partial H(t,s)}{\partial s}\Big|_{s=0}.$$

Note that if $C_t(s)$ is a geodesic, $S(t) = 0$.

If $c(t)$ is a geodesic, $Y(t)$ is called the *Jacobi field* along $c(t)$.

Lemma 17.2. *Let $c(t)$ $(a \le t \le b)$ be a geodesic on M. Then a vector field $Y(t)$ is a Jacobi field along $c(t)$ if and only if $Y(t)$ satisfies*

$$\left(\frac{D}{dt}\right)^2 Y + R(Y,\dot{c})\dot{c} = 0, \quad a \le t \le b, \tag{17.1}$$

where D/dt is the covariant differential along $c(t)$.

§ 17.2. Focal point

In the following, we consider the *boundary normal geodesic*, denoted by $\gamma_z(t)$ or $\exp_{\partial M}(z,t)$, starting from $z \in \partial M$ with initial direction the inner unit normal at z. Fixing t, we define the map $\exp_{\partial M}(t)$ by

$$\exp_{\partial M}(t) : \partial M \ni z \to \gamma_z(t) \in M.$$

Let $d_{\partial M}\exp_{\partial M}(t)\big|_{z=z_0} : T_{z_0}(\partial M) \to T_{\gamma_{z_0}(t)}(M)$ be the differential of $\exp_{\partial M}(t)$ evaluated at z_0.

Definition 17.3. Let $\gamma_{z_0}(t)$ be the boundary normal geodesic starting from $z_0 \in \partial M$. The point $\gamma_{z_0}(t_0) = \exp_{\partial M}(z_0, t_0)$ is called a *focal point* along $\gamma_{z_0}(t)$ if

$$\mathrm{rank}\left(d_{\partial M} \exp_{\partial M}(t_0)\big|_{z=z_0}\right) < \dim M - 1.$$

Lemma 17.4. *Let $\gamma_{z_0}(t) = \exp_{\partial M}(z_0, t)$ be the boundary normal geodesic emanating from $z_0 \in \partial M$. If $\gamma_{z_0}(t_0)$ is a focal point for some $t_0 > 0$, there exists a Jacobi field $Y(t)$ along $\gamma_{z_0}(t)$ such that*

$$Y(t_0) = 0, \quad 0 \neq Y(0) \in T_{z_0}(\partial M), \tag{17.2}$$

$$g_{z_0}\left(\frac{DY}{dt}(0), Y(0)\right) = 0, \tag{17.3}$$

where D/dt is the covariant differential along $\gamma_{z_0}(t)$.

The above geometric preliminaries are sufficient to prove the following lemma.

Lemma 17.5. *Let $c(t) = \gamma_{z_0}(t)$ $(0 \leq t \leq t_0)$ be a boundary normal geodesic starting from $z_0 \in \partial M$. If $\gamma_{z_0}(t_1)$ is a focal point along $c(t)$ for some $0 < t_1 < t_0$, there is a curve with end points z_0 and $\gamma_{z_0}(t_0)$ which is strictly shorter than the geodesic $c(t)$ $(0 \leq t \leq t_0)$.*

Lemma 17.5 implies that the distance from $\gamma_z(t_0)$ to ∂M is shorter than the length of $\gamma_z(t)$ $(0 \leq t \leq t_0)$ and is attained by a boundary normal geodesic $\gamma_w(t)$, where $w \in \partial M$, $w \neq z_0$.

§ 17.3. Boundary cut point

Let $\gamma_z(\cdot)$ be a boundary normal geodesic starting from $z \in \partial M$. A point $\gamma_z(t)$ is said to be *uniquely minimizing* along the geodesic $\gamma_z(\cdot)$ if $t = d(\gamma_z(t), \partial M)$ and $t < d(\gamma_z(t), w)$ for any $w \in \partial M$ such that $w \neq z$, i.e. $\{\gamma_z(s)\,;\, 0 \leq s \leq t\}$ is a unique shortest geodesic from $\gamma_z(t)$ to ∂M. Uniquely minimizing points have the following property: If $\gamma_z(t)$ is uniquely minimizing along γ_z, then so is $\gamma_z(s)$ for any $0 < s < t$.

This property implies that either there is a value $0 < t < \infty$ such that $\gamma_z(s)$ is uniquely minimizing for any $0 < s < t$, and $\gamma_z(\tau)$ is not uniquely minimizing for any $\tau > t$, or $\gamma_z(t)$ is uniquely minimizing for any $t > 0$.

Definition 17.6. Along the boundary normal geodesic γ_z starting from $z \in \partial M$, there is a critical distance, which is denoted by $\tau(z)$, such that $\{\gamma_z(s); 0 \leq s \leq t\}$ is a unique shortest curve from $\gamma_z(t)$ to ∂M when $t < \tau(z)$, but $\{\gamma_z(s); 0 \leq s \leq t\}$ is no more a unique shortest curve when $t > \tau(z)$, i.e. there is $w \in \partial M$ such that $d(w, \gamma_z(t)) < t$. Such a point $\gamma_z(\tau(z))$ is called a *boundary cut point* of z along γ_z. If $\tau(z) = \infty$, we say that there is no boundary cut point along the boundary normal geodesic γ_z.

Lemma 17.7. *For $z_0 \in \partial M$, let $\tau(z_0)$ be as in Definition 17.6. At the boundary cut point, $d(\gamma_{z_0}(\tau(z_0)), z_0) = \tau(z_0)$, and at least one (possibly both) of the following statements holds:*
(a) $\gamma_{z_0}(\tau(z_0))$ is an ordinary boundary cut point, i.e. there is $w \in \partial M$ such that $w \neq z_0$ and $\gamma_{z_0}(\tau(z_0)) = \gamma_w(\tau(z_0))$.
(b) $\gamma_{z_0}(\tau(z_0))$ is the first focal point along γ_{z_0}, i.e.

$$\operatorname{rank}\Big(d_{\partial M} \exp_{\partial M}(t)\big|_{z=z_0}\Big) = \dim M - 1 \quad \text{if} \quad t < \tau(z_0),$$
$$\operatorname{rank}\Big(d_{\partial M} \exp_{\partial M}(t)\big|_{z=z_0}\Big) < \dim M - 1 \quad \text{if} \quad t = \tau(z_0).$$

We introduce a topology in $\mathbf{R}_+ \cup \infty$ by taking intervals (a, b) and $(a, \infty] = (a, \infty) \cup \infty$ as a base for the open sets.

Lemma 17.8. *The function $\tau(z)$ in Definition 17.6 is continuous from ∂M to $\mathbf{R}_+ \cup \infty$.*

§ 17.4. Boundary cut locus

Definition 17.9. The *boundary cut locus* ω is defined by

$$\omega = \{x = \gamma_z(\tau(z))\,;\, z \in \partial M\},$$

where $\gamma_z(\tau(z))$ is the boundary cut point of z along the boundary normal geodesic γ_z in Definition 17.6.

Let us investigate the structure of ω. We put

$$B_n(M) = \mathop{\cup}_{z \in \partial M} \big\{ \exp_{\partial M}(z, t)\,;\, 0 \leq t < \tau(z) \big\}.$$

Lemma 17.10. *(1) $M = B_n(M) \cup \omega$, $B_n(M) \cap \omega = \emptyset$.*
(2) $B_n(M)$ is an open set.
(3) ω is a closed set of measure 0 with no interior points.

Example 17.11. (1) Let $M = B^1 = \{|x| < 1\}$ equipped with the Euclidean metric. Then $\omega = \{0\}$, which is both a boundary cut point and the first focal point. In fact, letting $z = (\cos\theta, \sin\theta) \in \partial B^1$, we have $\gamma_z(t) = (1 - t)(-\cos\theta, -\sin\theta)$.
(2) Let M be the inside of an ellipse : $M = \{(x, y) \in \mathbf{R}^2; x^2/a^2 + y^2/b^2 < 1\}$, $(a > b > 0)$ equipped with the Euclidean metric. Then $\omega = \{(x, 0); |x| \leq (a^2 - b^2)/a\}$. The end points $(\pm(a^2 - b^2)/a, 0)$ are focal points (note that they are not the focus points in the sense of classical conic curves), and all the points in the open interval $\{(x, 0); |x| \leq (a^2 - b^2)/a\}$ are boundary cut points.

§ 17.5. Boundary normal coordinates

Definition 17.12. For $x \in B_n(M) = M \setminus \omega$, there exists a unique $z(x) \in \partial M$ such that $d(x, w) > d(x, z(x))$ if $z(x) \neq w \in \partial M$. By the *boundary normal coordinates* we mean the map

$$M \setminus \omega \ni x \to (z(x), d(z(x), x)). \tag{17.4}$$

In this case, $x = \gamma_{z(x)}(t)$ with $t = d(x, \partial M)$, i.e. x is on the boundary normal geodesic starting from $z(x)$.

§ 18. Boundary distance coordinates

§ 18.1. Conjugate point

The boundary cut locus is different from the standard notion of cut locus on the manifold without boundary. To study this difference is important to consider the differentiable structure near the boundary cut locus. In manifolds with boundary, the geodesic may hit the boundary. In this case, there occurs a difficulty in extending the notion of cut locus. To avoid it, we shall assume in this section that the manifold M is embedded in a complete manifold without boundary $\widetilde{M}$. This is the case for our application of asymptotically hyperbolic manifolds.

Definition 18.1. Let $c(t)$ $(a \leq t \leq b)$ be a geodesic on $\widetilde{M}$. Two points $c(a)$ and $c(b)$ are said to be *conjugate* along $c(t)$ if there exists a non-trivial Jacobi field $Y(t)$ along $c(t)$ such that $Y(a) = 0, Y(b) = 0$. We also say that $c(b)$ is conjugate to $c(a)$ along $c(t)$.

For $y \in \widetilde{M}$, let $\gamma_y(v, t) = \exp_y(tv)$ be the geodesic starting from y with initial direction $v \in S_y(\widetilde{M})$.

Lemma 18.2. *Let $c(t) = \gamma_y(v, t) = \exp_y(tv)$ $(0 \leq t \leq t_0, v \in S_y(\widetilde{M}))$ be a geodesic on $\widetilde{M}$. Then $c(t_0)$ is conjugate to y along $c(t)$ if and only if there exists $0 \neq k \in T_y(\widetilde{M}) = T_{t_0 v}(T_y(\widetilde{M}))$ such that*

$$d \exp_y \Big|_{t_0 v} k = 0.$$

Lemma 18.3. *Let $c(t)$ $(a \leq t \leq b)$ be a geodesic on $\widetilde{M}$. If there exists $a < \tau < b$ such that $c(\tau)$ is conjugate to $c(a)$ along $c(t)$, there is a curve with end points $c(a)$ and $c(b)$ which is strictly shorter than the geodesic $c(t)$ $(a \leq t \leq b)$.*

Definition 18.4. Let $y \in \widetilde{M}$ and $v \in S_y(\widetilde{M})$. By the *cut locus distance of (y, v) in the Riemannian normal coordinates*, we mean a number $\tau^R(y, v)$ such that if $t < \tau^R(y, v)$, $\gamma_y(v, \cdot)$ is the shortest path from $\gamma_y(v, t)$ to y, and for $t > \tau^R(y, v)$, there exists a strictly shorter path from $\gamma_y(v, t)$ to y.

Note that $d(y, \gamma_y(v, \tau^R(y,v)) = \tau^R(y,v)$. The point $\gamma_y(v, \tau^R(y,v))$ is called the *cut point* for y along the geodesic $\gamma_y(v,\cdot)$. This should not be confused with the boundary cut point of Definition 17.6, where we considered the distance to ∂M.

Lemma 18.5. *The mapping $\tau^R(y,v) : T(\widetilde{M}) \to \mathbf{R}_+ \cup \infty$ is continuous.*

Lemma 18.6. *Let $z \in \partial M$, and ν be the inner unit normal to ∂M at z. Then $\tau^R(z,\nu) > \tau(z)$, where $\tau(z)$ is the distance from z to the boundary cut locus.*

Let $z \in \partial M$ and γ_z be the boundary normal geodesic from z. Then by Lemma 18.6, there exists $\epsilon > 0$ such that for $\tau(z) - \epsilon < t < \tau(z) + \epsilon$, $\gamma_z(\cdot)$ is still the shortest geodesic (lying inside M) from z to $\gamma_z(t)$.

Lemma 18.7. *The first conjugate point on a normal geodesic always appears strictly beyond the boundary cut point.*

§ 18.2. Hamilton's equation

Let $(g^{ij}) = (g_{ij})^{-1}$, and define a C^∞-function on $T^*(M)$ by $H(x,\xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j$. The equation of geodesic is rewritten as Hamiltons's canonical equation

$$
\begin{cases}
\dfrac{dx^i}{dt} = \dfrac{\partial H}{\partial \xi_i} = g^{ij}(x)\xi_j, \\[2mm]
\dfrac{d\xi_i}{dt} = -\dfrac{\partial H}{\partial x^i} = -\left(\dfrac{\partial g^{kl}(x)}{\partial x^i}\right)\xi_k\xi_l.
\end{cases}
\tag{18.1}
$$

Fix a point $y \in M$ and let $x(t),\ \xi(t)$ be the solution to (18.1) with initial data $x(0) = y,\ \xi(0) = \xi_0$, where ξ_0 satisfies $g^{ij}(y)\xi_{0i}\xi_{0j} = 1$. Then by the energy conservation law,

$$
g^{ij}(x(t))\xi_i(t)\xi_j(t) = 1. \tag{18.2}
$$

Let $v^i(t) = dx^i(t)/dt = g^{ij}(x(t))\xi_j(t)$, and put $v(t) = (v^1(t), \cdots, v^n(t))$, $v_0 = v(0)$. Then $x(t)$ is a geodesic starting from y with initial direction v_0. Assume that the map $:\ S_y(M) \times (0, t_0) \ni (v_0, t) \to x(t)$ is a diffeomorphism for some $t_0 > 0$. Then t and v_0 become functions of x depending on a parameter y : $t = t(x,y)$, $v_0 = v_0(x,y)$. Hence so is $\xi = \xi(x,y)$. Since $t(x,y) = \int_y^x \xi_i dx^i$, we have

$$
\frac{\partial t(x,y)}{\partial x^i} = \xi_i(x,y). \tag{18.3}
$$

This equality is rewritten as

$$
(\mathrm{grad}_x t(x,y))^i = g^{ij}(x)\frac{\partial t}{\partial x^j}(x,y) = v^i(x,y). \tag{18.4}
$$

§ 18.3. Boundary distance coordinates

Near the cut loci, we cannot use the boundary normal coordinates. However, the *boundary distance coordinates* constructed below can be used everywhere on M.

Lemma 18.8. *For any $x_0 \in M$, there exist points $z_1, \cdots, z_n \in \partial M$ such that the functions $(\rho_1(x), \cdots, \rho_n(x))$, where $\rho_i(x) = d(x, z_i)$, give local coordinates in a small neighborhhood of x_0.*

Example 18.9. Let M be a Euclidean sphere : $M = \{|x| < 1\}$. Then the boundary normal coordinate system is the polar coordinate with center at the origin. The center is the cut locus. To define the local coordinate around the origin, we have only to take n points $w_1, \cdots, w_n$ on ∂M such that the vectors $\overrightarrow{Ow_i}$, $i = 1, \cdots, n$, are linearly independent, and regard the distance from w_i as the coordinate function.

§ 18.4. Reconstruction of the metric

The following lemma is a key trick to reconstruct the Riemannian metric.

Lemma 18.10. *On M, we can recover the metric tensor $g_{ij}(x)$ from the boundary distance function $\partial M \ni w \to d(x, w)$.*

§ 19. Reconstruction of $R(M)$ from BSP

In this section, we shall prove that if two manifolds $M^{(1)}$ and $M^{(2)}$ have the same BSP, the spaces of boundary distance functions $R(M^{(1)})$ and $R(M^{(2)})$ coincide. We shall consider the wave equation (14.1) and make use of Blagovestcenskii identity to convert the knowledge of BSP to that of boundary normal geodesic.

We use the expression "BSP determines the quantity A" to mean the following: Let $A^{(1)}$ and $A^{(2)}$ be the quantities associated to the manifolds $M^{(1)}$ and $M^{(2)}$, respectively. Then if $M^{(1)}$ and $M^{(2)}$ have the same BSP, $A^{(1)} = A^{(2)}$ holds.

§ 19.1. Projection to the domain of influence

For a subset $\Gamma \subset \partial M$ and $\tau > 0$, we put

$$M(\Gamma, \tau) = \{x \in M \,;\, d(x, \Gamma) < \tau\}.$$

We also define for $z \in \partial M$

$$M(z, \tau) = \{x \in M \,;\, d(x, z) < \tau\}.$$

Let $\chi_{M(\Gamma,\tau)}(x)$ be the characteristic function of $M(\Gamma, \tau)$. We define the projection on $L^2(M)$ by

$$P_{\Gamma,\tau} f(x) = \chi_{M(\Gamma,\tau)}(x) f(x), \quad f \in L^2(M).$$

For a domain $\Omega \subset M$, we regard $L^2(\Omega)$ as a subspace of $L^2(M)$ by extending the elements to be 0 outside Ω. Let $u^f(t)$ be the solution to IBVP (14.1).

Lemma 19.1. *Let $f \in C_0^\infty(\partial M \times (0,\infty))$ and $\tau, t > 0$. Let $\Gamma \subset \partial M$ be an open set. Then BSP determines the sequence $f_j \in C_0^\infty(\Gamma \times (0,\tau))$ such that $u^{f_j}(t) \to P_{\Gamma,\tau}u^f(t)$.*

Lemma 19.2. *Let $f, h \in C_0^\infty(\partial M \times (0,\infty))$ and $\tau_1, \tau_2, t, s > 0$.*
(1) Let $\Gamma_1, \Gamma_2 \subset \partial M$ be open sets. Then BSP determines the inner product

$$\left(P_{\Gamma_1,\tau_1}u^f(t), P_{\Gamma_2,\tau_2}u^h(s)\right)_{L^2(M)}.$$

(2) Let $z_1, z_2 \in \partial M$. Then BSP determines the inner product

$$\left(P_{z_1,\tau_1}u^f(t), P_{z_2,\tau_2}u^h(s)\right)_{L^2(M)}.$$

§ 19.2. Domain of influence and $R(M)$

We trace the boundary normal geodesic along solutions to IBVP (14.1).

Lemma 19.3. *Let $\gamma_y(\cdot)$ be the boundary normal geodesic starting from $y \in \partial M$, and $s > 0$. Then the following 3 assertions are equivalent.*
(1) $d(\gamma_y(s), y) = d(\gamma_y(s), \partial M)$.
(2) $M(\Gamma, s) \setminus M(\partial M, s - \epsilon) \neq \emptyset$ for any $\epsilon > 0$ and any neighborhood $\Gamma \subset \partial M$ of y.
(3) For any neighborhood $\Gamma \subset \partial M$ of y, there exists $h \in C_0^\infty(\Gamma \times (0,s))$ such that $\|u^h(s)\| > \|P_{\partial M, s-\epsilon}u^h(s)\|$.

Lemma 19.4. *Let $\gamma_y(\cdot)$ be the boundary normal geodesic starting from $y \in \partial M$, and $s > 0$ be such that $d(\gamma_y(s), y) = d(\gamma_y(s), \partial M)$. Let $z \in \partial M$ and $t > 0$. Then the following 3 assertions are equivalent.*
(1) $t > d(\gamma_y(s), z)$.
(2) There exist a neighborhood $\Gamma \subset \partial\Omega$ of y and $\epsilon > 0$ such that

$$M(\Gamma, s) \subset M(\partial M, s - \epsilon) \cup M(z, t - \epsilon).$$

(3) There exist a neighborhood $\Gamma \subset \partial\Omega$ of y and $\epsilon > 0$ such that for any $h \in C_0^\infty(\Gamma \times (0,s))$

$$\|u^h(s)\|^2 = \|P_{\partial M,s-\epsilon}u^h(s)\|^2 + \|P_{z,t-\epsilon}u^h(s)\|^2 - (P_{\partial M,s-\epsilon}u^h(s), P_{z,t-\epsilon}u^h(s)).$$

§ 19.3. Main theorem

We are now in a position to prove the following theorem.

Theorem 19.5. *Let (M, g) be a connected Riemannian manifold with compact boundary. Suppose we are given the boundary spectral projections of the Neumann Laplacian on M. Then these data determine (M, g) uniquely.*

Proof. We take $y \in \partial M$ and solve IBVP (14.1) with data in a small neighborhood of y. Then by Lemma 19.3, we can determine whether or not $\gamma_y([0,s])$ is the shortest geodesic to ∂M by using BSP. By Lemma 19.4, if $\gamma_y([0,s])$ is the shortest geodesic to ∂M, we can compute $d(\gamma_y(s),z)$ for any $z \in \partial M$ by using BSP. Lemma 17.10 shows that by varying $y \in \partial M$ and $s \in [0,\tau(y)]$, i.e. for s such that $\gamma_y([0,s])$ is the shortest geodesic to ∂M, we can recover all points $x \in M$. With the aid of Lemma 19.4, one can compute $d(x,z)$ for all $x \in M$ and $z \in \partial$, i.e. all boundary distance functions by BSP. We can then reconstruct M topologically using BSP by Lemma 16.1. By Lemma 18.10, we can recover the metric by BSP. $\square$

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