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Modulus of continuity, a Hardy-Littlewood theorem and its application

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§1. Introduction

Let $D$ be a simply connected proper domain in $\mathbb{C}$ and $\varphi : \Delta \to D$ a Riemann mapping from the unit disk $\Delta = \{|z| < 1\}$ onto $D$. The geometric function theory gives us various informations of the mapping $\varphi$. For example, if $D$ is a quasi-disk, then we have

(1.1) \[ |\varphi'(z)| = O((1 - |z|)^{-\kappa}) \]

for some $\kappa \in [0, 1)$ as $|z| \to 1$ (cf. [8]). On the other hand, if a simply connected domain $D$ is an invariant component of a finitely generated Kleinian group $G$, we can say much more on the Riemann mapping $\varphi$. In fact, if $D$ is a Jordan domain, then $G$ must be a quasi-Fuchsian group by a theorem of Maskit ([3]). Hence, $D$ is a quasi-disk, and the inequality (1.1) holds. Recently ([9]), we have shown that the converse is also true. Namely, we have shown the following;

**Theorem 1.1.** Let $D \ni \infty$ be a simply connected invariant component of a finitely generated non-elementary Kleinian group $G$ and $\varphi$ a Riemann mapping from the unit disk onto $D$. Then the following are equivalent.

1. $G$ is a quasi-Fuchsian group and $D$ is a quasi-disk.
2. (1.1) holds for some $\kappa \in [0, 1)$ as $|z| \to 1$.

In other words, the growth rate of the derivatives of the Riemann mappings characterizes quasi-Fuchsian groups.

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Remark. The condition (2) implies that $D$ is a Hölder domain. It is known that every quasi-disk is a Hölder domain. Thus (1) implies (2). But the converse is not true in general.

It is a natural question what happens for $\varphi$ if $D$ is a simply connected invariant component of $G$ other than a quasi-Fuchsian group. In fact, we have obtained the growth rate of $|\varphi'(z)|$ of Riemann mappings $\varphi$ for regular $b$-groups and Kleinian groups with bounded geometry. Particularly, when $G$ is a regular $b$-group, we have estimated the modulus of continuity of $\varphi$ on the unit circle and we have shown the local connectivity of the limit set of $G$.

In this note, we will show a Hardy-Littlewood theorem to estimate the growth rate of $|\varphi'(z)|$ from the modulus of continuity and as a corollary, the growth rate of $|\varphi'(z)|$ for Kleinian groups with bounded geometry. It is an alternative proof of a result obtained in our previous paper [9].

§ 2. A Hardy-Littlewood theorem

Let $f$ be a continuous function on the unit circle. The modulus of continuity of $f$ is the function

$$\omega(t) = \sup_{|\theta_1 - \theta_2| \leq t} |f(e^{i\theta_1}) - f(e^{i\theta_2})|.$$

In 1932, Hardy and Littlewood [2] shows the following theorem called a Hardy-Littlewood theorem.

**Theorem 2.1** (cf. [1] p. 74). Let $f$ be a holomorphic function on the unit disk $\Delta$ and continuous on $\overline{\Delta} = \Delta \cup \partial \Delta$. Suppose that there exists $\alpha \in (0, 1]$ such that

$$|f(e^{i\theta_1}) - f(e^{i\theta_2})| = O(|\theta_1 - \theta_2|^{\alpha}).$$

Then

$$|f'(z)| = O((1 - |z|)^{\alpha - 1})$$

holds as $|z| \to 1$.

In this section, we shall show the following theorem of Hardy-Littlewood type for holomorphic functions whose modulus of continuity is $|\log |\theta||^{-\alpha}$.

**Theorem 2.2.** Let $f$ be a holomorphic function on the unit disk $\Delta$ and continuous on $\overline{\Delta} = \Delta \cup \partial \Delta$. Suppose that there exists $\alpha > 0$ such that

(2.1) $$|f(e^{i\theta_1}) - f(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-\alpha}),$$
if $|\theta_1 - \theta_2| < \delta$ for some $\delta \in (0, 1)$. Then,

$$|f'(z)| = O((1 - |z|)^{-1} \log(1 - |z|)^{-\alpha})$$

holds as $|z| \to 1$.

**Proof.** By Cauchy’s integral formula,

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z)^2} \, dz'$

Thus, we have

$$|f'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\varphi)}) - f(e^{i\varphi})|}{1 - 2r \cos t + r^2} \, dt.$$

Since

$$1 - 2r \cos t + r^2 \geq (1 - r)^2 + \frac{4rt^2}{\pi^2},$$

it follows from (2.1) that

$$|f'(z)| \leq \frac{A}{2\pi} \int_{-\delta}^{\delta} \frac{|\log|t||^\alpha}{(1 - r)^2 + 4r(t/\pi)^2} \, dt + B.$$

Setting $C_r = \pi^2(1 - r)^2/4r$ and $t = \sqrt{C_r} \tan \theta$, we have

$$I(z) := \frac{A}{2\pi} \int_{-\delta}^{\delta} \frac{|\log|t||^\alpha}{(1 - r)^2 + 4r(t/\pi)^2} \, dt = \frac{A\pi}{8r} \int_{-\beta_r}^{\beta_r} \frac{d\theta}{\log\sqrt{C_r} + \log|\tan \theta|^\alpha},$$

where $\beta_r = \arctan \frac{\delta}{\sqrt{C_r}} \in (0, \frac{\pi}{2})$. As $r = |z| \to 1$, $C_r \to 0$ and $\beta_r \to \frac{\pi}{2}$. We take $r > 0$ sufficiently close to 1 so that $C_r < 1$.

When $\theta \in (0, \frac{\pi}{4})$, $\tan \theta \in (0, 1]$. Hence

$$\log \sqrt{C_r} + \log|\tan \theta| \leq \log \sqrt{C_r} < 0,$$

and

$$\left| \log \sqrt{C_r} + \log|\tan \theta| \right|^{-\alpha} \leq \left| \log \sqrt{C_r} \right|^{-\alpha}.$$

Thus, we have

$$\int_{0}^{\pi/4} \frac{d\theta}{\left| \log \sqrt{C_r} + \log|\tan \theta| \right|^\alpha} = O \left( |\log(1 - |z|)|^{-\alpha} \right),$$

(2.4)
because $C_r = O((1 - |z|)^2)$.

Next, we take a constant $\lambda \in (\frac{1}{2}, 1)$ and put $\gamma_r := \arctan \left( \frac{1}{\sqrt{C_r}} \right)^\lambda$. We may assume that $\gamma_r < \beta_r$. When $\theta \in (\frac{\pi}{4}, \gamma_r]$, $\tan \theta \in (1, C_r^{-\lambda/2})$ and we have

$$\log \sqrt{C_r} + \log |\tan \theta| \leq (1 - \lambda) \log \sqrt{C_r} < 0.$$  

This implies

$$\left| \log \sqrt{C_r} + \log |\tan \theta| \right|^{-\alpha} \leq (1 - \lambda)^{-\alpha} \left| \log \sqrt{C_r} \right|^{-\alpha}$$

and we have

$$\int_{\gamma_r}^{\beta_r} \frac{d\theta}{|\log \sqrt{C_r} + \log |\tan \theta||^\alpha} = O \left( |\log(1 - |z|)|^{-\alpha} \right).$$

Finally, we consider the case where $\theta \in (\gamma_r, \beta_r]$. Since $\arctan x = \int_0^x \frac{1}{t^2+1}dt$, we have

$$\beta_r - \gamma_r = \int_{(\sqrt{C_r})^{-\lambda}}^{\delta/\sqrt{C_r}} \frac{dx}{x^2 + 1} \leq \left( \frac{\delta}{\sqrt{C_r}} - \frac{1}{\sqrt{C_r^\lambda}} \right) \frac{\sqrt{C_r^{2\lambda}}}{\sqrt{C_r^{2\lambda}} + 1} = O \left( C_r^{\lambda - 1/2} \right).$$

On the other hand,

$$\log \sqrt{C_r} + \log |\tan \theta| \leq \log \delta < 0,$$

because $\tan \theta \leq \frac{\delta}{\sqrt{C_r}}$. Therefore, we conclude

$$\int_{\gamma_r}^{\beta_r} \frac{d\theta}{|\log \sqrt{C_r} + \log |\tan \theta||^\alpha} \leq (\beta_r - \gamma_r) |\log \delta|^{-\alpha}$$

$$= O((1 - |z|)^{2\lambda - 1}).$$

Combining (2.4), (2.5) and (2.6), we have

$$I(z) = O \left( (1 - |z|)^{-1} |\log(1 - |z|)|^{-\alpha} \right).$$

Thus, we complete the proof of the theorem. \(\square\)

§ 3. Conformal mappings on invariant components of Kleinian groups

Let $G$ be a finitely generated non-elementary Kleinian group. The group $G$ is said to have \textit{bounded geometry} if there exists a constant $\varepsilon > 0$ such that the injectivity radius with respect to the hyperbolic metric at any point in $\mathbb{H}^3/G$ is greater than $\varepsilon$. 
We also assume that $G$ has a simply connected invariant component $D$ and denote by $\varphi$ a Riemann mapping from the unit disk $\Delta$ onto $D$ as before. Many things are known for Kleinian groups with bounded geometry (cf. [5]). For example, the limit set of $G$ is locally connected whenever it is connected. Particularly, H. Miyachi ([6]) shows the following;

**Proposition 3.1.** Let $G$ be a Kleinian group with bounded geometry having a simply connected invariant component $D$ and $\varphi : \Delta \to D$ a Riemann mapping. Then, $\varphi$ has a continuous extension to $\partial \Delta$ and

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-\alpha})$$

holds as $|\theta_1 - \theta_2| \to 0$.

From this proposition and Theorem 2.1, we immediately obtain a theorem which is shown in [9] by a different method;

**Theorem 3.2.** Let $G, D$ and $\varphi$ be the same ones as in Proposition 3.1. Then,

$$|\varphi'(z)| = O\left((1 - |z|)^{-1}|\log(1 - |z|)|^{-\alpha}\right)$$

holds as $|z| \to 1$.

**Remark.** In [9], we have also shown that if $G$ is a regular $b$-group, then

$$|\varphi'(z)| = O\left((1 - |z|)^{-1}|\log(1 - |z|)|^{-2}\right)$$

and we obtain the modulus of continuity on $\partial \Delta$,

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O\left(|\log |\theta_1 - \theta_2||^{-1}\right).$$

by using (3.2). From Theorem 2.1 it seems to be difficult to show (3.2) from (3.3). Actually, Nolder and Oberlin [7] show the following;

**Proposition 3.3.** Let $\omega(t)$ be a differentiable non-negative increasing function on $[0, \infty)$ having the decreasing derivative $\omega'(t)$. The following are equivalent:

1. If $f$ is a holomorphic function with the modulus of continuity $\theta$, then

$$|f'(z)| = O(\omega'(1 - |z|)).$$

2. $\limsup_{t \to 0^+} \frac{\omega(t)}{tw'(t)} < \infty$. 
In our case, (3.3) implies $\omega(t) = (- \log t)^{-1}$ for small $t > 0$ and $\omega'(t) = t^{-1}(\log t)^{-2}$. However,

$$\limsup_{t \to 0^+} \frac{\omega(t)}{t \omega'(t)} = \lim_{t \to 0^+} (- \log t) = +\infty.$$ 

Hence, the second condition is not satisfied and we can not apply the above proposition to get (3.3) from (3.2).

We have also characterized quasi-Fuchsian groups in terms of the growth of derivatives of Riemann mappings of invariant components.

**Proposition 3.4 ([9]).** Let $G$ be a Kleinian group having a simply connected invariant component $D$ with $\partial D \subset \mathbb{C}$ and $\varphi$ a conformal mapping of the unit disk $\Delta$ onto $D$. Suppose that $D/G$ has no punctures. Then, the following conditions are equivalent.

1. There exist constants $\alpha > 0$, $A > 0$ and a point $\zeta_0 \in D$ such that for any $z \in \varphi^{-1}(G\zeta_0) \setminus \varphi^{-1}(\infty)$,

$$|\varphi'(z)| \leq \frac{A}{(1 - |z|) \log(1 - |z|)^{2 + \alpha}}$$

holds.

2. $G$ is a quasi-Fuchsian group.

From Theorem 2.1 and the above one, immediately we have;

**Theorem 3.5.** Let $G$, $D$ and $\varphi$ be the same ones as above. Suppose that $\varphi : \Delta \to D$ has the continuous extension to $\partial \Delta$. If the extension $\varphi$ on $\partial \Delta$ satisfies

$$|\varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})| = O(|\log |\theta_1 - \theta_2||^{-2 - \alpha})$$

for some $\alpha > 0$. Then $G$ is a quasi-Fuchsian group.

**References**

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