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An application of Penner's coordinates of Teichmüller space of punctured surfaces

By

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Abstract

We apply R. C. Penner's coordinates for the Teichmüller space of once punctured surfaces and also his rational representation of the mapping class group to obtain certain Diophantine equations which have infinitely many integer solutions. The family of these equations can be thought as a generalization of the classical Markoff equation.

§ 1. Introduction

The Markoff equation

$$m_1^2 + m_2^2 + m_3^2 = 3m_1m_2m_3 \tag{1.1}$$

is preserved by the Markoff maps, which are iterative composites of

$$(m_1, m_2, m_3) \to (m_1, m_3, 3m_1m_3 - m_2), \tag{1.2}$$

$$(m_1, m_2, m_3) \to (m_3, m_2, 3m_2m_3 - m_1) \tag{1.3}$$

and their inverses. Since a Markoff map is a polynomial map with positive integer coefficients, it sends the solution $(1, 1, 1)$ to a positive integer solution. It is known that all positive integer solutions of (1.1) are found in the orbits of $(1, 1, 1)$ under the group generated by Markoff maps.

Let $\{A, B\}$ be a canonical generator system of a once punctured torus subgroup of $SL(2, \mathbb{R})$. Since $\mathrm{tr}ABA^{-1}B^{-1} = -2$, $(\frac{\mathrm{tr}A}{3}, \frac{trB}{3}, \frac{\mathrm{tr}AB}{3})$ is a solution of (1.1). The changes

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of canonical generators $\{A, B\} \to \{A, BA\}$ and $\{A, B\} \to \{AB, B\}$ induce the maps (1.2) and (1.3), respectively. Hence the group of Markoff maps acting on the solutions of the Markoff equation (1.1) can be understood to be the mapping class group $\mathcal{M}_{1,1}$ acting on the Teichmüller space $\mathcal{T}_{1,1}$ of once punctured torus. In [4, §7], R. C. Penner treated Markoff maps as mapping classes in $\mathcal{M}_{1,1}$ acting on the λ length coordinates of $\mathcal{T}_{1,1}$.

In this paper we pursue an analogy of the Markoff equation and present some Diophantine equations which admit infinitely many positive integer solutions. Each of these Diophantine equations is a model of the Teichmüller space of a once punctured surface represented in Penner's λ length coordinate space. We will show that the images of a special positive integer solution under the mapping class group acting on the Teichmüller space are infinitely many positive integer solutions of the Diophantine equation.

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§ 2. Distance between horocycles

Let $\mathbb{H} = \{z : \mathrm{Im}[z] > 0\}$ denote the hyperbolic plane equipped with the metric

$$\frac{dx^2 + dy^2}{y^2}. \tag{2.1}$$

The distance defined by (2.1) is denoted by $d(\cdot, \cdot)$. The circle at infinity $\partial\mathbb{H}$ is the boundary of $\mathbb{H}$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$. For two distinct points p, q of $\partial\mathbb{H}$, $l(p, q)$ denotes the hyperbolic geodesic line between p and q.

Let p be a point of $\partial\mathbb{H}$. A horocycle h at p is a Euclidean circle in $\mathbb{H}$ tangent at p to $\partial\mathbb{H}$ if $p \neq \infty$ or a horizontal line in $\mathbb{H}$ if $p = \infty$. The point p is called the *base point* of h. Let h_1 and h_2 be horocycles based at distinct points p_1 and p_2. Let

$$\lambda(h_1, h_2) = e^{\delta/2} \tag{2.2}$$

where δ is the signed length of the portion of the geodesic $l(p_1, p_2)$ intercepted between the two horocycles h_1 and h_2, $\delta > 0$ if h_1 and h_2 are disjoint and $\delta < 0$ otherwise.

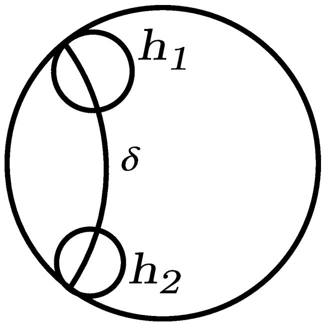

Figure 2.1

We quote two important lemmas from [4]. For the sake of completeness we also give proofs of the lemmas. But our proofs do not involve calculus in the three dimensional Minkowski space as in employed in the first two sections of [4].

Lemma 2.1 (the half horocyclic length [4]). *Let (p_1, p_2, p_3) be a sequence of three distinct points of $\partial\mathbb{H}$ which agrees with the positive orientation with respect to $\mathbb{H}$. Let h_i, $i = 1, 2, 3$, be a horocycle based at p_i. If $a = \lambda(h_1, h_2)$, $b = \lambda(h_2, h_3)$ and $c = \lambda(h_3, h_1)$, then the length of the portion of h_1 intercepted between the two geodesic lines $l(p_1, p_2)$ and $l(p_1, p_3)$ is $b/(ac)$.*

Proof. We may assume that $p_1 = \infty$, $p_2 = 0$ and $h_1 = \{z : \mathrm{Im}[z] = 1\}$. Let $t = p_3 > 0$. Then we need to show that $t = b/(ac)$. The horocycles h_2 and h_3 are the circles

$$x^2 + (y - \frac{1}{2a^2})^2 = \frac{1}{4a^4}, \quad (x-t)^2 + (y - \frac{1}{2c^2})^2 = \frac{1}{4c^4},$$

and meet $l(p_2, p_3)$, the upper semicircle defined by $x^2 - tx + y^2 = 0$, at

$$P = \frac{t}{1 + a^4t^2} + i\frac{a^2t^2}{1 + a^4t^2}, \qquad Q = \frac{t^3c^4}{1 + c^4t^2} + i\frac{c^2t^2}{1 + c^4t^2},$$

respectively. By definition $b = \exp(d(P, Q)/2)$ if $\mathrm{Re}[P] \leq \mathrm{Re}[Q]$ or $b = \exp(-d(P, Q)/2)$ if $\mathrm{Re}[P] > \mathrm{Re}[Q]$. Note that $\mathrm{Re}[P] \leq \mathrm{Re}[Q]$ if and only if $act \geq 1$. If $b = \exp(d(P, Q)/2)$, then by [1, Theorem 7.2.1]

$$b = \sinh(d(P, Q)/2) + \cosh(d(P, Q)/2) = \frac{|P - Q| + |P - \bar{Q}|}{2(\mathrm{Im}[P]\mathrm{Im}[Q])^{1/2}} = act.$$

We obtain the same result for the case where $b = \exp(-d(P, Q)/2)$. □

We remark that the quantity $\lambda(h_3, h_1)\lambda(h_1, h_2)^{-1}\lambda(h_2, h_3)^{-1}$ is twice the h-length defined in [4, p.313].

Lemma 2.2 (Proposition 2.6 in [4]). *Let (p_1, p_2, p_3, p_4) be a sequence of four distinct points of $\partial\mathbb{H}$ which agrees with the positive orientation with respect to $\mathbb{H}$. Let h_i,*

$i = 1, 2, 3, 4$, be a horocycle based at p_i. If $\lambda_a = \lambda(h_1, h_2)$, $\lambda_b = \lambda(h_2, h_3)$, $\lambda_c = \lambda(h_3, h_4)$, $\lambda_d = \lambda(h_4, h_1)$, $\lambda_e = \lambda(h_1, h_3)$ and $\lambda_f = \lambda(h_2, h_4)$, then

$$\lambda_e \lambda_f = \lambda_a \lambda_c + \lambda_b \lambda_d. \tag{2.3}$$

Proof. We assume again that $p_1 = \infty$, $p_2 = 0$ and $h_1 = \{z : \mathrm{Im}[z] = 1\}$. Then the segment s in h_1 between $l(p_1, p_2)$ and $l(p_1, p_4)$ is divided into the subsegment s_1 between $l(p_1, p_2)$ and $l(p_1, p_3)$ and the one s_2 between $l(p_1, p_3)$ and $l(p_1, p_4)$. The length of s is the sum of the lengths of s_1 and s_2. This is by Lemma 2.1

$$\frac{\lambda_f}{\lambda_a \lambda_d} = \frac{\lambda_b}{\lambda_a \lambda_e} + \frac{\lambda_c}{\lambda_e \lambda_d},$$

which is (2.3). □

A collection of pairwise disjoint geodesic lines in $\mathbb{H}$ is called a *geodesic ideal triangulation* if they divide $\mathbb{H}$ into ideal triangles. Let $\tilde{\Delta}$ be a geodesic ideal triangulation and $\mathcal{P}$ the set of end points of geodesic lines in $\tilde{\Delta}$. Suppose that for each point p of $\mathcal{P}$ a horocycle h_p based at p is given. For a geodesic line c such that both of its end points p and q are in $\mathcal{P}$, we define $\lambda(c) = \lambda(h_p, h_q)$.

Lemma 2.3. *Let e be a geodesic line with both end points in $\mathcal{P}$. Suppose that e meets the arcs a_1, a_2, ..., a_n of $\tilde{\Delta}$. Then*

$$\lambda(e) = \frac{P_e}{\lambda(a_1)\lambda(a_2)\cdots\lambda(a_n)},$$

where P_e is a homogeneous polynomial of degree $n+1$ in $\{\lambda(a) : a \in \tilde{\Delta}\}$ with positive integer coefficients.

Proof. We prove the lemma by the induction on n. Let p and q be the end points of c. We regard e as a directed line from p to q and suppose that e meets a_1, a_2, ..., a_n in this order. See Figure 2.2. Let q_L and q_R be the end points of a_n, chosen so that q_L lies on the left of e. Since a_n is the last arc in $\tilde{\Delta}$ which meets e, $b = l(q_R, q)$ and $c = l(q_L, q)$ are arcs of $\tilde{\Delta}$.

If $n = 1$, then $a = l(p, q_R)$ and $d = l(p, q_L)$ are arcs of $\tilde{\Delta}$, too. Then by Lemma 2.2

$$\lambda(e) = \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(a_1)}.$$

So the lemma is true for this case.

If $n > 1$, then there exists an $m < n - 1$ such that a_{m+1},..., a_n have q_L or q_R as a common end point. Without loss of generality we assume that the common end point is q_L. See Figure 2.2. Then $a = l(p, q_R)$ meets a_1, ..., a_{n-1} and $d = l(p, q_L)$ meets a_1, ..., a_m. Therefore, assuming that

$$\lambda(a) = \frac{P_a}{\lambda(a_1)\lambda(a_2)\cdots\lambda(a_{n-1})} \quad \text{and} \quad \lambda(d) = \frac{P_d}{\lambda(a_1)\lambda(a_2)\cdots\lambda(a_m)}$$

with homogeneous polynomials P_a of degree n and P_d of degree $m+1$, we have

$$\lambda(e) = \frac{\lambda(a)\lambda(c) + \lambda(b)\lambda(d)}{\lambda(a_n)} = \frac{P_a\lambda(c) + P_d\lambda(b)\lambda(a_{m+1})\cdots\lambda(a_n)}{\lambda(a_1)\lambda(a_2)\cdots\lambda(a_{n-1})}.$$

The numerator of the last expression is a homogeneous polynomial of degree $n+1$ with positive integer coefficients. $\square$

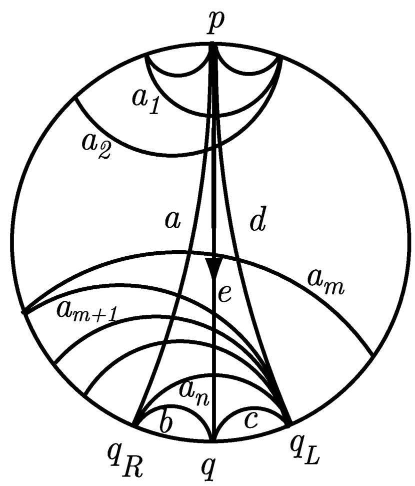

Figure 2.2

§ 3. Coordinates for the Teichmüller space of a once punctured surface

§ 3.1. Teichmüller space of a once punctured surface

Let F_g denote the oriented closed surface of genus $g \geq 1$ and p a point of F_g. Let $\dot{F}$ denote the punctured surface $F_g - \{p\}$. The fundamental group $G_{g,1}$ of $\dot{F}$ has the following presentation:

$$G_{g,1} = \langle a_1, b_1, ..., a_g, b_g, d : (\prod_{k=1}^{g} a_k b_k a_k^{-1} b_k^{-1})d = 1\rangle.$$

A point of the *Teichmüller space* $\mathcal{T} = \mathcal{T}_{g,1}$ is a class of faithful and finite covolume Fuchsian representations of $G_{g,1}$ into $SL(2, \mathbb{R})$. Points of $\mathcal{T}$ are represented by *marked* groups Γ_m, where Γ is a Fuchsian group and $m : G_{g,1} \to \Gamma$ is an isomorphism. Let $\mathcal{P}(\Gamma)$ denote the set of all parabolic fixed points of Γ. Then $(\mathbb{H} \cup \mathcal{P}(\Gamma))/\Gamma$ is a closed surface. We denote this surface by $\widehat{\mathbb{H}/\Gamma}$.

§ 3.2. λ-length of an ideal arc

An *ideal arc* c of the pointed surface (F, p) is a homotopically nontrivial path joining p to itself in $\dot{F}$. An ideal arc c is *simple* if $c \cap \dot{F}$ is a simple arc.

We fix a positive number α. For each point Γ_m of $\mathcal{T}_{g,1}$, $D = m(d)$ is a parabolic transformation in Γ. Let p be the fixed point of D. We choose a horocycle h based at p so that D acts on h by the translation of distance α. Let $\mathcal{H}(\Gamma) = \{\gamma(h) : \gamma \in \Gamma\}$, a Γ-invariant set of horocycles.

By Nielsen's theorem [5, Satz V.9], the marking m of Γ_m is induced by an orientation preserving homeomorphism

$$f_m : \dot{F} \to \mathbb{H}/\Gamma,$$

which extends to a homeomorphism of F onto $\widehat{\mathbb{H}/\Gamma}$. We denote this map again by f_m.

Let c be an ideal arc of (F, p) and send it by f_m to an arc connecting the puncture on $\mathbb{H}/\Gamma$ to itself. A lift of this arc to $\mathbb{H}$ connects two parabolic fixed points p_1 and p_2 of Γ. Let h_1, h_2 be horocycles of $\mathcal{H}(\Gamma)$ based at p_1 and p_2. We define

$$\lambda(c, \Gamma_m) = \lambda(h_1, h_2)$$

and call it the λ *length* of c with respect to Γ_m. The value $\lambda(c, \Gamma_m)$ does not depend on the choice of a lift of $f(c)$.

§ 3.3. Penner's coordinates for the Teichmüller space

An *ideal triangulation* $\Delta = (c_1, c_2, ..., c_q)$ of $\dot{F}$ is a maximal system of simple ideal arcs of (F, p) such that

(1) c_i and c_j are not homotopic in F relative to p, and

(2) c_i and c_j do not intersect in $\dot{F}$,

if $i \neq j$. Since Δ is a maximal system, each complementary component of arcs in Δ is bounded by three ideal arcs. We call the component a *triangle* in Δ. The number q of ideal arcs in Δ necessarily equals $6g - 3$ and the number of ideal triangles is $4g - 2$.

Let $\Gamma_m \in \mathcal{T}_{g,1}$. Then $f_m(\Delta) = (f_m(c_1), ..., f_m(c_q))$ is an ideal triangulation of $(\widehat{\mathbb{H}/\Gamma}, f_m(p))$. We deform $f(c_i \cap \dot{F})$ in its homotopy class into a geodesic arc in $\mathbb{H}/\Gamma$. Then we obtain a geodesic ideal triangulation $\Delta(\Gamma_m)$ of $\dot{F}$.

We define a map $\lambda_\Delta : \mathcal{T}_{g,1} \to \mathbb{R}^q_+$ by

$$(\lambda_1, ..., \lambda_q) = \lambda_\Delta(\Gamma_m) = (\lambda(c_1, \Gamma_m), ..., \lambda(c_q, \Gamma_m)).$$

Let $\{T_1, T_2, ..., T_{4g-2}\}$ be the set of ideal triangles in Δ and (c_{i1}, c_{i2}, c_{i3}) be the sides of T_i, $i = 1, ..., 4g-2$. Then $\{c_{i1}, c_{i2}, c_{i3}\}$ is a subset of Δ. Let $\lambda_{ik} = \lambda(c_{ik}, \Gamma_m)$, $k = 1, 2, 3$. The following theorem is an immediate consequence of Lemma 2.1 and it shows that the image of λ_Δ is a real algebraic variety determined by a zero locus of an algebraic equation. See also [3, Section 5.1], where the equation (3.1) in the theorem is obtained as a limit of real algebraic representations for Teichmüller spaces of surfaces with cone points.

Theorem 3.1. *For all* $\Gamma_m \in \mathcal{T}_{g,1}$*,*

$$\sum_{i=1}^{4g-2} \left(\frac{\lambda_{i1}}{\lambda_{i2}\lambda_{i3}} + \frac{\lambda_{i2}}{\lambda_{i1}\lambda_{i3}} + \frac{\lambda_{i3}}{\lambda_{i1}\lambda_{i2}} \right) = \alpha. \tag{3.1}$$

Proof. To simplify the notation, we identify $\dot{F}$ with $\mathbb{H}/\Gamma$ and Δ with $\Delta(\Gamma_m)$. We consider a small circle β around the puncture, positively directed with respect to the orientation of F_g, and let S_1, S_2, ..., S_r, $r = 3(4g-2)$, be the triangles in Δ which β meets in this order. We may assume that Γ contains the matrix

$$P = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.$$

P acts on the horocycle $h = \{z : \mathrm{Im}[z] = 1\}$ as the translation $z \to z + \alpha$. Let $\tilde{S}_i$ be a lift of S_i, chosen so that $\tilde{S}_i$ has vertices ∞, p_{i-1} and p_i with

$$0 = p_0 < p_1 < p_2 < \cdots < p_r = P(0).$$

Let h_i be the horocycle of $\mathcal{H}(\Gamma)$ based at p_i, $i = 0, 1, ..., r$. Then by Lemma 2.2,

$$\alpha = \sum_{i=1}^{r} (p_i - p_{i-1}) = \sum_{i=1}^{r} \frac{\lambda(h_{i-1}, h_i)}{\lambda(h, h_{i-1})\lambda(h, h_i)}. \tag{3.2}$$

Since each triangle T_i meets the circle β at three different ends, T_i appears three times in the sequence S_1, ..., S_r, and then contributes the term

$$\frac{\lambda_{i1}}{\lambda_{i2}\lambda_{i3}} + \frac{\lambda_{i2}}{\lambda_{i1}\lambda_{i3}} + \frac{\lambda_{i3}}{\lambda_{i1}\lambda_{i2}}$$

to the right-hand side of (3.2). Therefore we obtain (3.1). □

The equation (3.1) can be written as

$$P_\Delta(\lambda_1, \lambda_2, ..., \lambda_q) - \alpha\lambda_1\lambda_2 \cdots \lambda_q = 0, \tag{3.3}$$

where $P_\Delta(\lambda_1, \lambda_2, ..., \lambda_q)$ is a sum of $12g-6$ monomials of degree $q-1$. We define

$$\mathcal{A}_\Delta = \{(\lambda_1, ..., \lambda_q) \in \mathbb{R}_+^q : P_\Delta(\lambda_1, \lambda_2, ..., \lambda_q) - \alpha\lambda_1, \lambda_2 \cdots \lambda_q = 0\}.$$

The map λ_Δ above is the restriction to $\mathcal{T}_{g,1}$ of the real analytic diffeomorphism from the decorated Teichmüller space $\tilde{\mathcal{T}}_{g,1}$ to $\mathbb{R}_+^q$ in [4, Theorem 3.1]. Hence

$$\lambda_\Delta : \mathcal{T}_{g,1} \to \mathcal{A}_\Delta \tag{3.4}$$

is also a real analytic diffeomorphism.

Theorem 3.2. *Let $\Delta = (c_1, c_2, ..., c_q)$ be an ideal triangulation of (F, p). Let Γ_m be an arbitrary point of $\mathcal{T}_{g,1}$ and define $\lambda_i = \lambda(c_i, \Gamma_m)$, $i = 1, ..., q$. Then for any ideal arc c in $\dot{F}$,*

$$\lambda(c, \Gamma_m) = \frac{P_c(\lambda_1, \lambda_2, ..., \lambda_q)}{\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_q^{m_q}}, \tag{3.5}$$

where m_i is the geometric intersection number of c and c_i in $\dot{F}$. P_c is a homogeneous polynomial of degree $m_1 + m_2 + \cdots + m_q + 1$ with positive integer coefficients.

Proof. Let $\tilde{\Delta}(\Gamma_m)$ be the lift of $\Delta(\Gamma_m)$. Take a lift of $f(c)$ and let e be the geodesic line which connects the end points of the lift. Then Lemma 2.3 applied to e and $\tilde{\Delta}$ shows that $\lambda(c, \Gamma_m)$ has the form (3.5). □

§ 4. Integer solutions of a Diophantine equation

Let $\mathcal{M}_{g,1}$ denote the mapping class group of $\dot{F}$. Each element φ of $\mathcal{M}_{g,1}$ acts on $\mathcal{T}_{g,1}$ by changing the marking m to $m \circ \varphi_*^{-1}$, where φ_* is the automorphism of the surface group $G_{g,1}$ induced by φ.

We fix an ideal triangulation $\Delta = (c_1, c_2, ..., c_q)$ of (F, p) and consider Penner's coordinate-system $\lambda_\Delta : \mathcal{T}_{g,1} \to \mathbb{R}_+^q$. Then, by definition,

$$\lambda_\Delta(\varphi(\Gamma_m)) = (\lambda(\varphi^{-1}(c_1), \Gamma_m), ..., \lambda(\varphi^{-1}(c_q), \Gamma_m)).$$

By [4, Corollary 7.4] each entry $\lambda(\varphi^{-1}(c_i), \Gamma_m)$ is a rational function. Moreover, Theorem 3.2 shows that it is of degree 1 of the form as is described in (3.5). Therefore we obtain a rational map $R_\varphi : \mathbb{R}^q \to \mathbb{R}^q$. Penner showed in [4] that the correspondence $\varphi \to R_\varphi$ is a faithful representation of $\mathcal{M}_{g,1}$ to a group of rational transformations in $\mathbb{R}^q$. Since $R_\varphi \circ \lambda_\Delta = \lambda_\Delta \circ \varphi$, R_φ preserves the algebraic equation (3.3).

If $\alpha = 12g - 6$, then $(\lambda_1, \lambda_2, ..., \lambda_q) = (1, 1, ..., 1)$ is a solution of the equation (3.3). For all $\varphi \in \mathcal{M}_{g,1}$, the entries of R_φ are of the form as in (3.5). Therefore $R_\varphi(1, 1, ..., 1)$ are positive integer solutions of (3.3).

If $\Lambda = (\lambda_1, ..., \lambda_q)$ is fixed by a $\varphi \in \mathcal{M}_{g,1}$, then φ is the class of a conformal automorphism of the Riemann surface corresponding to Λ. By Wiman's theorem the order of the group of conformal automorphisms of a Riemann surface of type $(g, 1)$, $g \geq 1$, does not exceed $2(2g + 1)$ if $g > 1$ or 3 if $g = 1$ [2]. This means that only a finite number of elements in $\mathcal{M}_{g,1}$ send $(1, 1, ..., 1)$ to itself, and hence the orbit space $\{R_\varphi(1, 1, ..., 1) : \varphi \in \mathcal{M}_{g,1}\}$ contains infinitely many points.

Proposition 4.1. *If $\alpha = 12g - 6$, then there are infinitely many positive integer solutions of the equation* (3.3) .

Since R_φ with $\varphi \in \mathcal{M}_{g,1}$ is a rational map, an integer solution of (3.3) may not necessarily be sent to an integer solution by R_φ. This part is different from the case of Markoff maps which are polynomial maps of positive integer coefficients and hence we cannot employ Markoff's method which concludes that the set of all positive integer solutions of the Markoff equation coincides with the orbit of $(1, 1, 1)$ under all Markoff maps. The author does not know whether all positive integer solutions of (3.3) are in the orbit of $(1, 1, ..., 1)$ under $\mathcal{M}_{g,1}$. In [3] positive integer solutions of a Diophantine equation which arises from the Teichmüller space of twice punctured torus are considered. For this case there are positive integer solutions such that their orbits under the mapping class group $\mathcal{M}_{1,2}$ contain non integral points.

Let Δ_1 and Δ_2 be two ideal triangulations of (F_g, p). Then $\lambda_{\Delta_2} \circ \lambda_{\Delta_1}^{-1}$ is a rational map whose entries are of the form (3.5). Let $(1, 1, ..., 1)$ be a solution of

$$P_{\Delta_1} - (12g - 6)\lambda_1\lambda_2 \cdots \lambda_q = 0,$$

which is the equation (3.3) with $\Delta = \Delta_1$ and $\alpha = 12g - 6$. Then $\lambda_{\Delta_2} \circ \lambda_{\Delta_1}^{-1}(1, 1, ..., 1)$ is a positive integer solution of

$$P_{\Delta_2} - (12g - 6)\lambda_1\lambda_2 \cdots \lambda_q = 0. \tag{4.1}$$

The author does not know whether this point belongs to the orbits of $(1, 1, ..., 1)$ as a solution of (4.1) under $\mathcal{M}_{g,1}$.

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