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Quasiarcs and the outside of the asymptotic Teichmüller space

By

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Abstract

In this paper, we give a version of the Cricket theorem due to K. Astala and F. Gehring for the asymptotic Teichmüller space. Using this, we show that for the complement Ω of some quasiarc α, the asymptotic class of the Schwarzian derivative of the Riemann mapping for Ω is not in the closure of the asymptotic Teichmüller space of the unit disk. The quasiarc α we give here is known to be a simple zipper in the sense of Thurston. This deduces that the image of the asymptotic Bers map is not dense in the set of asymptotic classes of Schwarzian derivatives of univalent functions, which was already observed by the author in a different method.

§ 1. Introduction

Let $AT(\mathbb{D})$ be the asymptotic Teichmüller space of unit disk $\mathbb{D}$. It is known that the asymptotic Teichmüller space is canonically embedded in the quotient space $\hat{B}(\mathbf{1})$ of the space $B(\mathbf{1})$ of bounded quadratic differentials on $\mathbb{D}$. The equivalence class of the differential is called its *asymptotic class* and the canonical embedding is said to be the *asymptotic Bers map* (cf. §2.1).

Let Ω be a simply connected domain in $\hat{\mathbb{C}}$. Let $f_\Omega : \mathbb{D}^* := \{|z| > 1\} \cup \{\infty\} \to \Omega$ be the Riemann mapping and $\varphi_\Omega \in B(\mathbf{1})$ is the Schwarzian derivative of f_Ω. In this paper, we focus on the following problem.

Problem. *When is the asymptotic class $[\varphi_\Omega]$ of φ_Ω contained in the closure of the image $\mathcal{AT}_{\mathbf{1}}$ of $AT(\mathbb{D})$ under the asymptotic Bers map?*

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This problem is closely related to the *density problem* (or the *density conjecture*) posed by L. Bers for Teichmüller spaces. In fact, we will solve the density problem in the negative for the case of the asymptotic Teichmüller space of unit disk. In §4.2.3, we discuss briefly the density problem for (asymptotic) Teichmüller spaces.

We will study our problem stated above for simply connected domains which are complements of quasiarcs, where by a *quasiarc*, we mean the image of a closed interval in $\mathbb{R} \cup \{\infty\}$ under a quasiconformal mapping on the Riemann sphere.

These kinds of domains were deeply studied by K. Astala and F. Gehring [4]. Indeed, they obtained a necessary and sufficient condition for quasiarcs, so called the *Cricket theorem*, which decides whether the Schwarzian derivatives of the Riemann mappings of their complements are in the closure of the universal Teichmüller space or not (cf. Theorem 3.1). In §3 we will obtain a version of the Cricket theorem for the asymptotic Teichmüller space (cf. Theorem 3.2). To show this, we will apply necessary and sufficient conditions for the Schwarzian derivative of a local univalent function that determine whether its asymptotic class is in the image $\mathcal{AT}_{\mathbf{1}}$ or in the closure of the image. These conditions are obtained in [14], and will be recalled in §2.2.

In §4, we will give three properties for quasiarcs. The asymptotic classes of the Schwarzian derivatives with the first two properties are in the Bers boundary, and the asymptotic classes of derivatives with the third property are not. Using the third property, we observe that when for the complement Ω of some simple zipper α in the sense of Thurston, the asymptotic class $[\varphi_\Omega]$ for Ω is not in the closure of $\mathcal{AT}_{\mathbf{1}}$ (cf. Corollary 4.4).

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§ 2. Notation

Throughout of this paper, we let $D(z_0, r) = \{|z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$. For a round disk $B = D(z_0, r)$ and $b > 0$, we set $bB = D(z_0, br)$. Let $\mathbb{D} = D(0, 1)$, $\mathbb{D}^* = \{|z| > 1\} \cup \{\infty\}$ and $\partial\mathbb{D} = \partial\mathbb{D}^* = \mathbb{S}^1$. For a set X, by $\mathrm{cl}(X)$ we mean the closure of X in the ambient space. For a quasiarc α in $\hat{\mathbb{C}}$, we denote by $\partial\alpha$ the endpoints of α.

§ 2.1. Teichmüller spaces of $\mathbb{D}$

The reader may consult [8],[9] and [10] for references and further details.

Let $\mathcal{QC}$ be the set of quasiconformal mappings on $\mathbb{D}$ which fixes $-i$, 1 and i. Let $\mathcal{AC}$ be the subset of $\mathcal{QC}$ consisting of asymptotically conformal mappings, where $h \in \mathcal{QC}$ is called *asymptotically conformal* if for any $\epsilon > 0$ there is a compact set C such that the Beltrami differential $\mu[h]$ of h satisfies $|\mu[h](z)| < \epsilon$ a.e. on $\mathbb{D} \setminus C$. Two f_1 and f_2 in $\mathcal{QC}$ are said to be *T-equivalent* if $f_1 = f_2$ on $\mathbb{S}^1$, and to be *AT-equivalent* if $h \circ f_1 = f_2$ on $\mathbb{S}^1$ for some $h \in \mathcal{AC}$. The set of T-equivalence classes and AT-equivalence classes of $\mathcal{QC}$ are called the *universal Teichmüller space* and the *asymptotic Teichmüller space* of unit disk $\mathbb{D}$ which are denoted by $T(\mathbb{D})$ and $AT(\mathbb{D})$, respectively. By $[f]_T$ and $[f]_{AT}$, we mean the T-equivalence class and the AT-equivalence class of $f \in \mathcal{QC}$, respectively. From the definition there is a canonical projection

$$T(\mathbb{D}) \ni [f]_T \mapsto [f]_{AT} \in AT(\mathbb{D}).$$

Let E be an open set in $\hat{\mathbb{C}}$ whose complement contains at least two points. Let $\lambda_E = \lambda_E(z)|dz|$ be the hyperbolic metric on E with curvature -4. For a holomorphic function φ on E, we denote by

$$\|\varphi\|_E = \sup_{z\in E} \lambda_E(z)^{-2}|\varphi(z)|.$$

Let $B(\mathbf{1})$ be a complex Banach space of holomorphic mappings φ on $\mathbb{D}^*$ satisfying $\|\varphi\|_{\mathbb{D}^*} < \infty$. Occasionally, elements of $B(\mathbf{1})$ are recognized as holomorphic quadratic differentials on $\mathbb{D}^*$, because of the transformation rule of Schwarzian derivatives under the pre-composition with Möbius transformations.

We say that $\varphi \in B(\mathbf{1})$ *vanishes at infinity* if for any $\epsilon > 0$ there is a compact set C in $\mathbb{D}^*$ such that

$$(|z|^2 - 1)^2|\varphi(z)| < \epsilon$$

for all $z \in \mathbb{D}^* \setminus C$. We denote by $B_0(\mathbf{1})$ a subspace of $B(\mathbf{1})$ consisting of holomorphic quadratic differentials vanishing at infinity. Then, $B_0(\mathbf{1})$ is a closed subspace, and hence the quotient space $\hat{B}(\mathbf{1}) = B(\mathbf{1})/B_0(\mathbf{1})$ is also a complex Banach space. Let $\|\cdot\|_{\hat{\mathbb{D}}^*}$ be the quotient norm on $\hat{B}(\mathbf{1})$. We call the equivalence class $[\varphi] \in \hat{B}(\mathbf{1})$ of $\varphi \in B(\mathbf{1})$ the *asymptotic class* of φ.

For any $[f]_T \in T(\mathbb{D})$, there is a quasiconformal mapping W_f on $\hat{\mathbb{C}}$ such that $\mu[W_f] = \mu[f]$ on $\mathbb{D}$ and W_f is conformal on $\mathbb{D}^*$. Then, the mapping

$$\beta : T(\mathbb{D}) \ni [f]_T \mapsto \mathcal{S}(W_f \mid_{\mathbb{D}^*}) \in B(\mathbf{1})$$

is a well-defined embedding, so called the *Bers embedding*, where $\mathcal{S}(\cdot)$ denotes the Schwarzian derivative. We denote by $\mathcal{T}_\mathbf{1}$ the image of β.

Correspondingly, in the case of the asymptotic Teichmüller space, there is also an embedding $\hat{\beta} : AT(\mathbb{D}) \to \hat{B}(\mathbf{1})$ satisfying the following commutative diagram

$$\begin{array}{ccc} T(\mathbb{D}) & \xrightarrow{\beta} & B(\mathbf{1}) \\ \downarrow & & \downarrow \\ AT(\mathbb{D}) & \xrightarrow{\hat{\beta}} & \hat{B}(\mathbf{1}), \end{array} \tag{2.1}$$

where the vertical directions are the canonical projections. The embedding $\hat{\beta}$ is called the *asymptotic Bers map.* We denote by $\mathcal{AT}_{\mathbf{1}}$ the image of $AT(\mathbb{D})$ under the asymptotic Bers map.

§ 2.2. Characterizations for the interior and the closure

Let E be a domain in $\hat{\mathbb{C}}$. A mapping $g : E \to \hat{\mathbb{C}}$ is called a *locally univalent K-quasiregular mapping* if any $z_0 \in E$ has a neighborhood U in E where the restriction $g\mid_U$ is K-quasiconformal. A *quasiloop* is the image of a mapping $\gamma : \mathbb{S}^1 \to \hat{\mathbb{C}}$ such that for any $z_0 \in \mathbb{S}^1$, there is a neighborhood U_0 of z_0 in $\mathbb{C}$ such that $\gamma\mid_{\mathbb{S}^1\cap U_0}$ is the restriction of a quasiconformal mapping on U_0. By definition, the image of $\mathbb{S}^1$ under a locally univalent quasiregular mapping is a quasiloop.

It is known that for $\varphi \in B(\mathbf{1})$, there is a locally univalent mapping $f_\varphi : \mathbb{D}^* \to \hat{\mathbb{C}}$, called the *developing mapping associated to φ*, satisfying $\mathcal{S}(f_\varphi) = \varphi$.

Theorem 2.1 (cf. Theorem 1 in [14]). *For $\varphi \in B(\mathbf{1})$, the following conditions are equivalent.*

(a) *The asymptotic class $[\varphi] \in \hat{B}(\mathbf{1})$ is contained in the image $\mathcal{AT}_{\mathbf{1}}$ of $AT(\mathbb{D})$ under the asymptotic Bers map.*

(b) *The developing mapping f_φ associated to φ is extended as a locally univalent quasiregular mapping on a neighborhood of* $\mathrm{cl}(\mathbb{D}^*)$.

(c) *f_φ has a continuous extension on* $\mathrm{cl}(\mathbb{D}^*)$ *with the property that f_φ is locally injective on* $\mathrm{cl}(\mathbb{D}^*)$ *and the image $f_\varphi(\mathbb{S}^1)$ is a quasiloop.*

We next give a characterization for the closure of $\mathcal{AT}_{\mathbf{1}}$. Though we have already obtained a characterization for locally univalent functions whose Schwarzian derivative is in $B(\mathbf{1})$, we need here the following version.

Theorem 2.2 (cf. Theorem 3 in [14]). *Let Ω be a simply connected domain and φ_Ω the Schwarzian derivative of the Riemann mapping $f_\Omega : \mathbb{D}^* \to \Omega$. Then, the asymptotic class $[\varphi_\Omega]$ of φ_Ω is in the closure* $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$ *of $\mathcal{AT}_{\mathbf{1}}$ in $\hat{B}(\mathbf{1})$ if and only if for any $K > 1$ and $L > 1$, there are a K-quasiconformal mapping g on Ω onto a quasidisk and a compact set C in Ω such that for any round disk B in $\Omega - C$, the restriction $g\mid_B : B \to \hat{\mathbb{C}}$ admits an L-quasiconformal extension on $\hat{\mathbb{C}}$.*

§ 3. Cricket theorem for asymptotic classes

Let α be a quasiarc and $\Omega = \hat{\mathbb{C}} \setminus \alpha$. We first recall the (original) Cricket theorem due to K. Astala and F. Gehring to compare with our theorem.

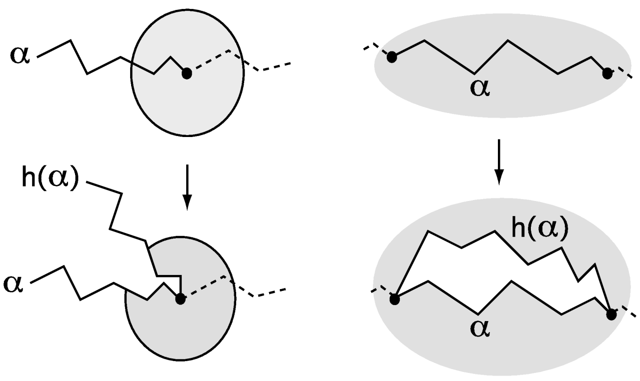

Figure 1. Schematic illustrations representing "openable at an endpoint" (the left) and "globally openable" (the right). The dashed lines stand for the remaining arc $\gamma \setminus \alpha$.

Theorem 3.1 (Cricket theorem in [4]). *Let γ be a quasicircle which contains α as a subarc. Then the Schwarzian derivative φ_Ω of the Riemann mapping of Ω is in the closure of $\mathcal{T}_{\mathbf{1}}$ if and only if for each $K > 1$, there exists a sense preserving K-quasiconformal self-mapping h of $\hat{\mathbb{C}}$ such that $h(z) = z$ in $\gamma \setminus \alpha$ and $h(\alpha) \cup \alpha$ is a quasicircle.*

§ 3.1. Cricket theorem for asymptotic classes

Notice from Theorem 2.1 that the asymptotic class of the Schwarzian derivative of the Riemann mapping f_Ω of Ω is not contained in $\mathcal{AT}_{\mathbf{1}}$, since f_Ω can not extend injectively around the preimage of $\partial\alpha$. Our main theorem in this paper is as follows.

Theorem 3.2 (Cricket theorem for asymptotic classes). *Let α be a quasiarc and set $\Omega = \hat{\mathbb{C}} \setminus \alpha$. Let γ be a quasicircle which contains α as a subarc and D_1 and D_2 components of $\hat{\mathbb{C}} \setminus \gamma$. Then, the asymptotic class $[\varphi_\Omega]$ is in the Bers boundary $\partial_b\mathcal{AT}_{\mathbf{1}}$ of $\mathcal{AT}_{\mathbf{1}}$ if and only if for any $K > 1$, there is a locally univalent K-quasiregular mapping h on Ω with the following three properties.*

(1) *h fixes D_1 pointwise.*

(2) *$h\,|_{D_2}$ has a K-quasiconformal extension on $\hat{\mathbb{C}}$.*

(3) *$h \circ f_\Omega$ is locally injective near $\mathbb{S}^1$ and the image $h \circ f_\Omega(\mathbb{S}^1)$ is a quasiloop.*

We give a comment on the comparizon between Theorem 3.1 and our theorem above. In our theorem, we deeply care about the structure around points of $\partial\alpha$, which is a *local* structure of α, while K. Astala and F. Gehring care about the *global* structure of the quasiarc. Actually, our conditions imply that when the asymptotic class $[\varphi_\Omega]$

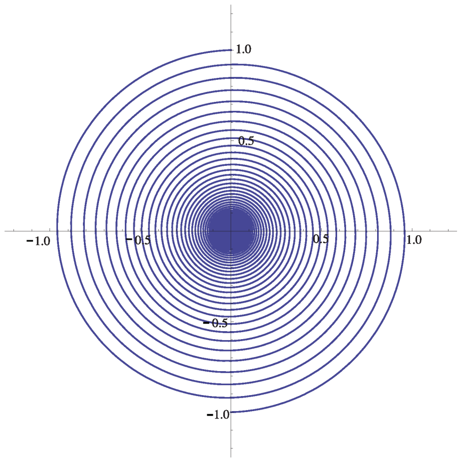

Figure 2. Gehring's spiral.

is in the Bers boundary $\partial_b \mathcal{AT}_{\mathbf{1}}$, the endpoints are "openable" by a locally univalent quasiregular mapping with arbitrary small dilatation (cf. the left figure in Figure 1). From Theorem 3.1, when φ_Ω is in the Bers boundary, $h(\alpha) \cup \alpha$ is a quasicircle. In other words, Astala and Gehring's condition means, from our viewpoint, that when $\varphi_\Omega \in \partial_b \mathcal{T}_{\mathbf{1}}$, the quasiarc α itself needs a kind of globally openable condition by a quasiconformal mapping with arbitrary small dilatation (cf. the right figure in Figure 1).

For the simplicity, let us suppose that α is the Gehring's spiral (cf. Figure 2). In [12], F. Gehring observed that φ_Ω is not in the closure of $\mathcal{T}_{\mathbf{1}}$. However, its asymptotic class is in the Bers boundary $\partial_b \mathcal{AT}_{\mathbf{1}}$ (cf. §.4.1.2 or [14]). One reason why the difference occurs is that the Gehring's spiral is real analytic except for the origin, and its complement has two mutually intricately-intertwining (spiraling) ends around the origin.

Indeed, Theorem 3.1 and Gehring's observation in [12] tell us that these two ends can not be unwreathed (or α is not globally openable) by quasiconformal mappings with arbitrary small dilatation.

On the other hand, according to Theorem 2.1, $\mathcal{AT}_{\mathbf{1}}$ consists of the asymptotic classes of Schwarzian derivatives such that the associated developing mappings admit locally univalent quasiregular extensions. It is natural to consider that any locally univalent function on $\mathbb{D}^*$ whose Schwarzian derivative is near φ_Ω are realized by composing a locally univalent function on Ω with the Riemann mapping f_Ω of Ω. By definition, a locally univalent quasiregular mapping need not be injective. Furthermore, since α is a quasiarc, the Riemann mapping f_Ω extends quasiconformally along open circular arcs $\mathbb{S}^1 \setminus f_\Omega^{-1}(\partial\alpha)$, and any locally univalent function on Ω with small Schwarzian derivative can extend quasiconformally at neighborhoods of points in $\alpha \setminus \partial\alpha$. Hence, to find an interior point in $\mathcal{AT}_{\mathbf{1}}$ near the asymptotic class of φ_Ω, we need not pay attention to unwreathing these spiral ends of α. This situation concerns that h in Theorem 3.2 is a "locally" univalent quasiregular mapping.

As noted above, we should care about the (analytic and geometric) structure of endpoints. In the case of the Gehring's spiral, it is real analytic around endpoints. Such endpoints can be openable by locally univalent quasiregular mappings with arbitrary small dilatation, and hence we can find a locally univalent function on $\mathbb{D}^*$ with locally univalent quasiregular extensions whose Schwarizian derivative is near φ_Ω. (See §4.1.2. See also the proof of Proposition 7.1 of [14].)

§ 3.2. Proof of Theorem 3.2

In this section, we shall give the proof of Theorem 3.2.

Proof of the necessity in Theorem 3.2. Suppose that the asymptotic class $[\varphi_\Omega]$ is in the closure $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$. By Theorem 2.1 for any $\epsilon > 0$, there is a locally univalent mapping g on $\mathbb{D}^*$ such that g_ϵ admits a locally univalent quasiregular extension on the closure $\mathrm{cl}(\mathbb{D}^*)$ and satisfies

$$\|\mathcal{S}(g_\epsilon) - \varphi_\Omega\|_{\mathbb{D}^*} < \epsilon. \tag{3.1}$$

Let $K > 1$. Let $h_{\epsilon,i} = g_\epsilon \circ (f_\Omega)^{-1}\mid_{D_i}$ for $i = 1, 2$. Notice from the Schwarz lemma that

$$\|\mathcal{S}(h_{\epsilon,i})\|_{D_i} \leq \|\mathcal{S}(g_\epsilon) - \varphi_\Omega\|_{\mathbb{D}^*} < \epsilon.$$

Since D_i is a quasidisk, when we take ϵ to be sufficiently small, each $h_{\epsilon,i}$ admits a K-quasiconformal extension H_i on whole $\hat{\mathbb{C}}$ from Gehring's theorem [11]. We can easily check that $h = H_1^{-1} \circ g_\epsilon \circ (f_\Omega)^{-1}$ has the desired properties. □

Proof of the sufficiency in Theorem 3.2. We may assume that α connects 0 and ∞. We here recall the following two lemmas, the first is due to K. Astala and F. Gehring in [4] and the second is obtained by combining a result in [4] and results due to P. Tukia and J. Väisälä in [17] (see also the last paragraph of the proof of the sufficiency in Theorem 3.1 of [4]).

Lemma 3.3 (Lemma 3.4 in [4]). *For each $0 < a < \infty$, there exists $0 < b = b(a) < \infty$ with the following property: If g is M_1-quasiconformal in $\hat{\mathbb{C}}$ with $1 \leq M_1 \leq e^{a/b}$ and fixes z_1, z_2 and ∞, then*

$$|g(z) - z| \leq b|z_1 - z_2| \log M_1$$

for $z \in D(z_1, a|z_1 - z_2|)$.

Lemma 3.4 (Lemma 3.6 in [4] and Theorems 5.23 and 2.6 in [17]). *For $M_2 > 1$, there exist constants $t > 0$ and $M_3 > 1$ with the following property: If g is M_3-quasiconformal and satisfies $|g(z) - z| \leq tr$ in $D(z_0, 2r)$, then the restriction of g to $D(z_0, r)$ admits M_2-quasiconformal extension on $\hat{\mathbb{C}}$.*

For any $K > 1$ and $L > 1$, we take $K_1 > 1$ in temporary such that $K_1 \leq \min\{\sqrt{K}, \sqrt{L}\}$. We will modify K_1 to be more smaller later, however, the modification will depend only on K, L and γ (cf. (3.4)). We take a locally univalent quasiregular mapping h with the properties in Theorem 3.2 for K_1.

Assume that γ is a K_2-quasicircle. From the condition (2) in Theorem 3.2, the restriction $h\mid_{D_2}$ extends to a K_1-quasiconformal mapping $\hat{H}_2$ on $\hat{\mathbb{C}}$ which fixes $\gamma \setminus \alpha$ pointwise. Since $\hat{H}_2(\gamma)$ is a K_1K_2-quasicircle passing through ∞ and $K_1 \leq \sqrt{K}$, Ahlfors' three points principle tells us that there is a constant $C_1 > 0$ depending only on K and K_2 such that every three points $\zeta_1, \zeta_2, \zeta_3 \in \hat{H}_2(\gamma)$ which follow each other in this order satisfy

$$|\zeta_1 - \zeta_2| \leq C_1|\zeta_1 - \zeta_3| \tag{3.2}$$

(cf. [1]). We take $a_1 > 0$ satisfying $a_1 < 1$ and $2a_1(1-a_1)^{-1}C_1 < 1$. Let $L' > 1$ be a constant satisfying

$$L \geq \min\{\sqrt{K}, \sqrt{L}\}\left\{2\left\{1 - \frac{108}{a_1^2}\frac{L'^2 - 1}{L'^2 + 1}\right\}^{-1} - 1\right\}. \tag{3.3}$$

Indeed, since the right-hand side tends to $\min\{\sqrt{K}, \sqrt{L}\} < L$ as $L' \to 1$, a constant $L' > 1$ satisfying (3.3) exists. We take an absolute constant $b > 0$ for $a = 3$ by Lemma 3.3, and constants $M_3 > 1$ and $t > 0$ for $M_2 = L'$ by Lemma 3.4.

We redefine $K_1 > 1$ satisfying

$$1 < K_1 \leq \min\{L', M_3, e^{3/b}, e^{t/b}, \sqrt{K}, \sqrt{L}\}. \tag{3.4}$$

Since h is locally univalent K_1-quasiregular on Ω, there is a K_1-quasiconformal self-mapping h_1 of $\mathbb{D}^*$ such that $h \circ f_\Omega \circ h_1$ is locally univalent. Furthermore, by the condition (3) above, $h \circ f_\Omega \circ h_1$ is locally injective near $\mathbb{S}^1$ and $h \circ f_\Omega \circ h_1\mid_{\mathbb{S}^1}$ is a quasiloop. Hence, by Theorem 2.1 the asymptotic class of the Schwarzian derivative of $h \circ f_\Omega \circ h_1$ is in $\mathcal{AT}_{\mathbf{1}}$. Therefore, from the commutative diagram (2.1), we find a quasidisk E and a conformal mapping $f_1 : \mathbb{D}^* \to E$ such that $f_2 := (h \circ f_\Omega \circ h_1) \circ f_1^{-1}$ is locally univalent on E and its Schwarzian derivative vanishes at infinity.

Let $h_2 = f_1 \circ h_1^{-1} \circ (f_\Omega)^{-1} = (f_\Omega \circ h_1 \circ f_1^{-1})^{-1} : \Omega \to E$. Since h_1 is K_1-quasiconformal, so is h_2. For $\epsilon > 0$, there is a compact set C'_ϵ in E such that

$$\sup_{z \in E \setminus C'_\epsilon} \lambda_E(z)^{-2}|\mathcal{S}(f_2)(z)| < \epsilon. \tag{3.5}$$

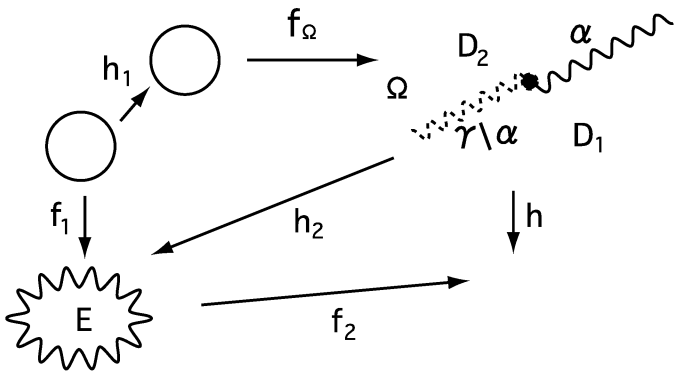

Figure 3. Mappings in the proof of the sufficiency in Theorem 3.2

Let $C_\epsilon = h_2^{-1}(C'_\epsilon) \subset \Omega$. Let $B := D(w_0, r)$ be a round disk in Ω with center w_0 and radius r. We will show that h_2 and C_ϵ with sufficiently small ϵ satisfy the sufficient condition for being $[\varphi_\Omega] \in \mathrm{cl}(\mathcal{AT}_1)$ in Theorem 2.2. To check this, we claim the following lemma.

Lemma 3.5. *There is an $\epsilon_0 > 0$ such that for $\epsilon \le \epsilon_0$, $3r/a_1 \le \mathrm{dist}(w_0, \alpha)$ and $B \subset \Omega \setminus C_\epsilon$, the restriction $h_2 |_B$ extends to an L'^2-quasiconformal mapping on $\hat{\mathbb{C}}$.*

Proof of Lemma 3.5. Notice from the definition that $h \circ h_2^{-1} = f_2$ on E.

Case 1 : $B \subset D_1$. Since h is the identity on D_1, $f_2 = h_2^{-1}$ on $h_2(B)$, and hence h_2 is holomorphic on B whose Schwarzian derivative satisfies

$$\begin{aligned}\sup_{w\in B} \lambda_B(w)^{-2}|\mathcal{S}(h_2)(w)| &= \sup_{z\in h_2(B)} \lambda_{h_2(B)}(z)^{-2}|\mathcal{S}(f_2)(z)| \\ &\le \sup_{z\in E\setminus C'_\epsilon} \lambda_E(z)^{-2}|\mathcal{S}(f_2)(z)| < \epsilon\end{aligned}$$

by (3.5). Therefore, when $\epsilon \le \epsilon_1 := 2(L'^2 - 1)/(L'^2 + 1)$, $h_2 |_B$ extends to an L'^2-quasiconformal mapping on $\hat{\mathbb{C}}$ by Ahlfors-Weill's theorem in [2].

Case 2 : $B \subset D_2$. Recall that the restriction $h |_{D_2}$ admits a K_1-quasiconformal extension $\hat{H}_2$ on $\hat{\mathbb{C}}$, and hence $h(B)$ is a K_1-quasidisk. Since $f_2 = (h \circ f_\Omega \circ h_1)^{-1} \circ f_1^{-1}$, the restriction $f_2^{-1} |_{h(B)}$ of the inverse of f_2 to $h(B)$ is a well-defined univalent function on $h(B)$ satisfying $f_2^{-1}(h(B)) = h_2(B)$. Therefore, by (3.5) we have

$$\begin{aligned}\sup_{\zeta\in h(B)} \lambda_{h(B)}(\zeta)^{-2}|\mathcal{S}(f_2^{-1} |_{h(B)})(\zeta)| &= \sup_{z\in h_2(B)} \lambda_{h_2(B)}(z)^{-2}|\mathcal{S}(f_2)(z)| \\ &\le \sup_{z\in E} \lambda_E(z)^{-2}|\mathcal{S}(f_2)(z)| < \epsilon.\end{aligned}$$

Since $h(B)$ is a K_1-quasidisk and $K_1 \leq L'$, by Gehring's theorem [11], there is an $\epsilon_2 = \epsilon_2(L')$ such that when $\epsilon \leq \epsilon_2$, $f_2^{-1} \mid_{h(B)}$ admits a L'-quasiconformal extension on $\hat{\mathbb{C}}$, and hence $h_2 \mid_B = (f_2^{-1} \mid_{h(B)}) \circ h$ extends to an ${L'}^2$-quasiconformal mapping on $\hat{\mathbb{C}}$.

Case 3 : $B \cap (\gamma \setminus \alpha) \neq \emptyset$. This is the most interesting case. Let $B = D(w_0, r)$ with $3r/a_1 \leq \mathrm{dist}(w_0, \alpha)$ as above. Fix $w_1 \in B \cap (\gamma \setminus \alpha)$. Let $B' = 3B = D(w_0, 3r) \subset \Omega$. Notice from the definition that

$$3r \leq a_1 \mathrm{dist}(w_0, \alpha) \leq a_1 |w_0|.$$

Let $z \in \hat{H}_2(\alpha)$ and $w \in B'$. Notice that $|w - w_0| \leq a_1|w_0|$, $|w_1 - w_0| \leq r < a_1|w_0|$ and

$$|w_1| \geq |w_0| - |w_1 - w_0| \geq (1 - a_1)|w_0|.$$

Since $\hat{H}_2(\alpha)$ and w_1 are divided at $0 \in \gamma$, by applying Ahlfors' three points principle (3.2) for w_1,0 and z, we obtain

$$\begin{aligned} |w - w_1| &\leq |w - w_0| + |w_1 - w_0| \\ &\leq 2a_1|w_0| \leq 2a_1(1 - a_1)^{-1}|w_1| \\ &\leq 2a_1(1 - a_1)^{-1}C_1|z - w_1| < |z - w_1|, \end{aligned}$$

and hence

$$B' \cap h \mid_{\mathrm{cl}(D_2)} (\alpha) = \emptyset, \tag{3.6}$$

where $h \mid_{\mathrm{cl}(D_2)}$ is the continuous extension of $h \mid_{D_2}$ to the closure $\mathrm{cl}(D_2)$. We claim

Claim. *h is injective on B'.*

Proof of Claim. Let $z_1, z_2 \in B'$ with $h(z_1) = h(z_2)$. When both z_1 and z_2 are in $B' \cap D_i$ for some $i = 1, 2$, $z_1 = z_2$ since $h \mid_{D_i}$ extends to a quasiconformal mapping on $\hat{\mathbb{C}}$. Suppose to the contrary that $z_1 \neq z_2$ and $z_i \subset B' \cap D_i$ for $i = 1, 2$. Then $z_1 = h(z_1) = h(z_2)$ since h fixes D_1 pointwise. Therefore, $h(D_2) \cap (B' \cap D_1) \neq \emptyset$.

Let B_1' be a component of $B' \cap D_1$ with $B_1' \cap h(D_2) \neq \emptyset$. Since

$$\partial h(D_2) = \hat{H}_2(\gamma) = (\gamma \setminus \alpha) \cup h \mid_{\mathrm{cl}(D_2)} (\alpha),$$

from (3.6), we have $B_1' \cap \partial h(D_2) = \emptyset$. Hence, $B_1' \subset h(D_2)$.

Let γ_1 be a component of $\gamma \cap B' = (\gamma \setminus \alpha) \cap B'$ contained in $\partial B_1'$. Since γ is a Jordan curve dividing D_1 and D_2, there is a component B_2' of $D_2 \cap B'$ which shares γ_1 with B_1' in their boundaries. Since h fixes $\gamma \setminus \alpha$ pointwise, $h(D_2)$ contains the B_2'-side of γ_1. On the other hand, as discussed above, B_1' is contained in $h(D_2)$. Therefore, $h(D_2)$ also contains the B_1'-side of γ_1. Thus, a Jordan domain $h(D_2)$ contains both sides of a subarc γ_1 ($\subset (\gamma \setminus \alpha)$) of its boundary $\partial h(D_2)$. This is a contradiction. $\square$

We now continue the proof of Case 3 in Lemma 3.5. Since $\gamma \setminus \alpha$ is unbounded and $B \cap (\gamma \setminus \alpha) \neq \emptyset$, there is $w_2 \in \gamma \setminus \alpha$ with $|w_1 - w_2| = r$. Since $w_1, w_2 \in B(w_0, 2r) \cap (\gamma \setminus \alpha)$ and h fixes $\gamma \setminus \alpha$ pointwise, by Lemma 3.3 and (3.4),

$$|h(w) - w| = |\hat{H}_2(w) - w| \leq b|w_1 - w_2| \log K_1 \leq tr \tag{3.7}$$

for $w \in D(w_0, 2r) \cap D_2$. Since h also fixes D_1 pointwise, the inequality (3.7) also holds on whole $D(w_0, 2r)$. From (3.4), h is M_3-quasiconformal on $D(w_0, 2r) = 2B$. Hence, by Lemma 3.4, the restriction of h to B has an L'-quasiconformal extension on $\hat{\mathbb{C}}$. Therefore, by applying the same argument as that in the proof of Case 2 above, we conclude that $h_2 |_B = (f_2^{-1} |_{h(B)}) \circ h$ extends to an ${L'}^2$-quasiconformal mapping on $\hat{\mathbb{C}}$, when ϵ is less than an appropriate constant $\epsilon_3 = \epsilon_3(L')$ depending only on L'. Thus, we complete the proof of Lemma 3.5. □

Let us return to proving the sufficiency in Theorem 3.2. Let B be a round disk in $\Omega \setminus C_\epsilon$ for $\epsilon \leq \epsilon_0$, where ϵ_0 is the constant in the claim above. We have already checked that the restriction of h to $(a_1/3)B$ has an ${L'}^2$ -quasiconformal extension on $\hat{\mathbb{C}}$. Notice from (3.4) that $K_1 \leq \min\{\sqrt{K}, \sqrt{L}\}$, and hence, h_2 is a K-quasiconformal mapping onto a quasidisk E. Thus, from (3.3) and Lemma 3.6 below, for any round disk $B \subset \Omega \setminus C_\epsilon$, the restriction $h_2 |_B$ admits an L-quasiconformal extension. This implies that the asymptotic class $[\varphi_\Omega]$ is in the closure $\mathrm{cl}(\mathcal{AT}_\mathbf{1})$ by Theorem 2.2. □

To accomplish the proof of Theorem 3.2, we need to check the following lemma. (We apply the lemma for $\Omega_0 = \Omega \setminus C_\epsilon$.)

Lemma 3.6. *Let Ω_0 be a domain in $\mathbb{C}$ and g a K-quasiconformal mapping on Ω_0 onto a domain in $\hat{\mathbb{C}}$. Suppose that there is a constant $0 < b < 1$ such that for any round disk $B \subset \Omega_0$, the restriction of g to bB extends to an L'-quasiconformal mapping on $\hat{\mathbb{C}}$. Then, for any round disk $B \subset \Omega_0$, the restriction of g to B also admits an L-quasiconformal extension on $\hat{\mathbb{C}}$, where*

$$L = K\left\{2\left\{1 - \frac{12}{b^2}\frac{L'-1}{L'+1}\right\}^{-1} - 1\right\}. \tag{3.8}$$

Proof. This lemma might be well-known. However, we give the proof for the completeness.

Since g is K-quasiconformal, there is a K-quasiconformal mapping W on $\hat{\mathbb{C}}$ such that $W \circ g$ is conformal on Ω_0. Let B be a round disk in Ω_0. By applying the similarity on $\mathbb{C}$, we may assume that B is the unit disk $\mathbb{D}$.

Let $z_0 \in B$. Since $B' = D(z_0, 1 - |z_0|) \subset B \subset \Omega_0$, from the assumption, the restriction of $W \circ g$ on bB' admits an L'-quasiconformal extension on $\hat{\mathbb{C}}$. Therefore,

$$(b(1 - |z_0|))^2 |\mathcal{S}(W \circ g)(z_0)| \leq \|\mathcal{S}(W \circ g)\|_{bB'} \leq 6(L' - 1)/(L' + 1).$$

Thus, if L'' is at least the right hand side of (3.8), then

$$\begin{aligned}(1-|z_0|^2)^2|\mathcal{S}(W\circ g)(z_0)| &\le 4(1-|z_0|)^2|\mathcal{S}(W\circ g)(z_0)| \\ &\le 24b^{-2}(L'-1)/(L'+1) \\ &\le 2(L''-1)/(L''+1)\end{aligned}$$

for all $z_0 \in B$. Hence, by Ahlfors-Weill's theorem ([2]), the restriction $(W\circ g)\mid_B$ admits an L''-quasiconformal extension on $\hat{\mathbb{C}}$, which implies what we desired. □

§ 4. Examples

This section is devoted to discuss three properties for quasiarcs α. In §4.1, we introduce two properties which imply the asymptotic class $[\varphi_\Omega]$ for $\Omega = \hat{\mathbb{C}} \setminus \alpha$ is in the closure $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$. The second property seems to be more comprehensible than the first one, though a quasiarc with the second property satisfies the first (Corollary 4.2). In §4.2, following [4], we recall a notion of interlocking sequences, and prove that when $\infty \in \partial\alpha$ and α contains an interlocking sequence converges to ∞, the asymptotic class $[\varphi_\Omega]$ for Ω is not in the closure $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$.

§ 4.1. Quasiarcs associated to points in the Bers boundary

4.1.1. Asymptotically conformal quasiarcs at endpoints A quasiarc α is said to be *asymptotically conformal at endpoints* if for any $L > 1$, there is a quasiconformal mapping g on $\hat{\mathbb{C}}$ such that g is L-quasiconformal on neighborhoods of 0 and 1 and $\alpha = g([0,1])$.

Then, we have the following theorem.

Theorem 4.1. *When a quasiarc α is asymptotically conformal at endpoints, the asymptotic class $[\varphi_\Omega]$ for $\Omega = \hat{\mathbb{C}} \setminus \alpha$ is in the Bers boundary $\partial_b\mathcal{AT}_{\mathbf{1}}$.*

Proof. We may assume that $\partial\alpha \subset \mathbb{C}$. Fix $K > 1$. Let g be a quasiconformal mapping on $\hat{\mathbb{C}}$ such that $\alpha = g([0,1])$ and g is L-quasiconformal on neighborhoods V_1 and V_2 of 0 and 1, respectively. Take $0 < r_0 < 1/2$ such that $D(0,r_0) \subset V_1$ and $D(1,r_0) \subset V_2$. We will construct a locally univalent K-quasiregular mapping h with the conditions in Theorem 3.2 for appropriately small L.

Fix a sufficiently small constant $l > 0$. For $t \in [0,1]$, let $v(t,y)$ be a function on $[-2,2]$ defined by

$$v(t,y) = \begin{cases} lt + \dfrac{lt+2}{2}y & (-2 \le y \le 0) \\ lt + \dfrac{2-lt}{2}y & (0 \le y \le 2). \end{cases}$$

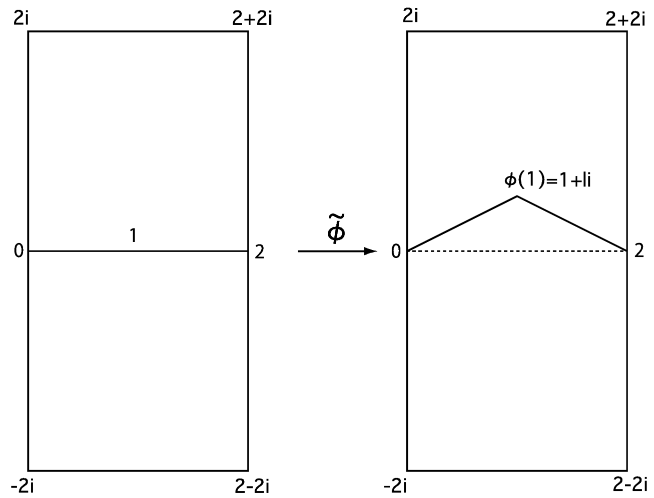

Figure 4. The mapping $\tilde{\phi}$ on the rectangle

Then, we define a quasiconformal mapping $\tilde{\phi}$ on the rectangle $[0,2]\times[-2,2]$ by

$$\tilde{\phi}(x+iy)=\begin{cases} x+iv(x,y) & (0\le x\le 1)\\ \\ x+iv(2-x,y) & (1\le x\le 2).\end{cases}$$

By definition, $\tilde{\phi}$ extends to a quasiconformal mapping on $\hat{\mathbb{C}}$ by putting the identity mapping outside the rectangle (see Figure 4). It is easy to check that the maximal dilatation of $\tilde{\phi}$ is $1+O(l)$ as $l\to 0$.

Let ϕ be the restriction of $\tilde{\phi}$ to the square $[0,2]\times[0,2]$. Then, ϕ also admits an extension by setting $\phi(z)=z$ on the outside of square (but the extension is not continuous along the lower edge of the square). Define

$$f_l(z)=\begin{cases} T_1^{-1}\circ\phi\circ T_1(z) & \text{on } D(0,r_0)\\ T_2^{-1}\circ\phi\circ T_2(z) & \text{on } D(1,r_0)\\ z & \text{otherwise,}\end{cases}\tag{4.1}$$

where $T_1 : D(0,r_0)\to D(0,2\sqrt{2})$ and $T_2 : D(1,r_0)\to D(2,2\sqrt{2})$ are affine mappings defined by $T_1(z)=(2\sqrt{2}/r_0)z$ and $T_2(z)=(2\sqrt{2}/r_0)(z-1)+2$. Then f_l is a quasiconformal mapping on $\hat{\mathbb{C}}\setminus[0,1]$ with the same maximal dilatation as that of ϕ. Notice from the definition that $f_l(z)=z$ on the lower-half plane $\mathbb{H}_-$ and $f_l(\mathbb{H}_+)\subset\mathbb{H}_+$ where $\mathbb{H}_+$ is the upper-half plane. Hence $f_l(\hat{\mathbb{C}}\setminus[0,1])\subset\hat{\mathbb{C}}\setminus[0,1]$. By a simple calculation, we can check that the maximal dilatation f_l tends to 1 as $l\to 0$. Therefore, we may choose $l>0$ such that f_l is L-quasiconformal.

Set $h=g\circ f_l\circ g^{-1}$. Then, h is an L^3-quasiconformal mapping on Ω, and hence it is in particular a locally univalent L^3-quasiregular mapping on Ω. Let $\gamma=g(\mathbb{R}\cup\{\infty\})$,

$D_1 = g(\mathbb{H}_-)$ and $D_2 = g(\mathbb{H}_+)$. We shall check that h satisfies the three conditions in Theorem 3.2 for K when L is sufficiently small.

(1) It follows from the definition of f_l that h fixes D_1 pointwise.

(2) Since the maximal dilatation of $\tilde{\phi}$ tends to 1 as $l \to 0$, we can choose l to be sufficiently small such that $f_l \mid_{\mathbb{H}_+}$ admits an L-quasiconformal extension $\tilde{f}_l$ on $\hat{\mathbb{C}}$, which defined by the same equations in (4.1) for f_l except for exchanging ϕ for $\tilde{\phi}$. Since $\tilde{f}_l$ coincides with the identity outside disks $D(0, r_0)$ and $D(1, r_0)$, the conjugation $g \circ \tilde{f}_l \circ g^{-1}$ is an L^3-quasiconformal extension of $h \mid_{D_2}$ on the Riemann sphere.

(3) Recall that f_Ω is the Riemann mapping on $\mathbb{D}^*$ onto Ω. Note that ϕ is defined by the angle-opening procedure at $z = 0$ and 2 and $g^{-1} \circ f_\Omega$ is a quasiconformal mapping on $\mathbb{D}^*$ onto $\hat{\mathbb{C}} \setminus [0, 1]$. Thus, the image of $\mathbb{S}^1$ under $f_l \circ g^{-1} \circ f_\Omega$ is a polygonal curve in $\mathbb{C}$. Hence, we can check that for any $z_0 \in \mathbb{S}^1$, $f_l \circ g^{-1} \circ f_\Omega$ admits a quasiconformal extension on a neighborhood of z_0. Therefore, $f_l \circ g^{-1} \circ f_\Omega$ is locally injective near $\mathbb{S}^1$ and the image $f_l \circ g^{-1} \circ f_\Omega(\mathbb{S}^1)$ is a quasiloop. Since g is quasiconformal on the Riemann sphere, $h \circ f_\Omega = g \circ f_l \circ g^{-1} \circ f_\Omega$ satisfies the desired properties. □

Remark. Asymptotically conformal quasiarcs at endpoints in this section are comparable objects to asymptotically conformal quasiarcs in [4], where a quasiarc α is, by definition, *asymptotically conformal* if for any $K > 1$, there is a quasiconformal mapping g on $\hat{\mathbb{C}}$ such that $g([0, \infty]) = \alpha$ and g is K-quasiconformal on a neighborhood of the interval $[0, \infty]$. K. Astala and F. Gehring showed that for the complement Ω of asymptotically conformal quasiarc, the corresponding Schwarzian derivative φ_Ω is in the closure of the universal Teichmüller space (cf. Theorem 4.1 in [4]). Clearly, asymptotically conformal quasiarc is asymptotically conformal at endpoints, but the converse does not hold in general. For instance, the Gehring's spiral is not asymptotically conformal, but is asymptotically conformal at endpoints.

4.1.2. Real analytic quasiarcs at endpoints Let us introduce a typical asymptotically conformal quasiarc at endpoints. Suppose that $\partial\alpha \subset \mathbb{C}$. We say that α is *real analytic at endpoints* if for each point $w_1 \in \partial\alpha$, there is a real analytic mapping g on $(-1, 1)$ and a neighborhood V at w_1 such that $g(0) = w_1$, $g'(0) \neq 0$, and $g([0, 1)) = \alpha \cap V$.

The following corollary implies that the asymptotic class of the Schwarzian derivative of the Riemann mapping for the complement of the Gehring spiral is in the Bers boundary of $\mathcal{AT}_{\mathbf{1}}$.

Corollary 4.2. *When a quasiarc α is real analytic at endpoints, the asymptotic class $[\varphi_\Omega]$ of the Schwarzian derivative for $\Omega = \hat{\mathbb{C}} \setminus \alpha$ is in the Bers boundary of $\mathcal{AT}_{\mathbf{1}}$.*

Proof. By virtue of Theorem 3.2, it suffices to show that α is asymptotically conformal at endpoints.

Let $\partial\alpha = \{w_1, w_2\} \subset \mathbb{C}$. Since every real analytic function on an interval admits a holomorphic extension on a neighborhood of the interval, from the assumption, there is a $\delta > 0$ such that for each point $w_i \in \partial\alpha$, there is a conformal mapping g_i on $D(0,\delta)$ satisfying $g_i(0) = w_i$ and $g_i([0,\delta)) = \alpha \cap V_i$ for some neighborhood V_i of w_i. We may assume that $\delta < 1/2$.

Consider a mapping g on $D(0,\delta) \cup D(1,\delta)$ defined by $g(z) = g_1(z)$ on $D(0,\delta)$ and $g(z) = g_2(1-z)$ on $D(1,\delta)$. Since α is a quasiarc, when we replace δ by a sufficiently small constant if necessary, we can see that g extends to a quasiconformal mapping on $\hat{\mathbb{C}}$ with $g([0,1]) = \alpha$. Since g is conformal near 0 and 1, we conclude the assertion. □

§ 4.2. Quasiarcs associated to points not in the Bers boundary

4.2.1. Interlocking sequences From now on, let α be a quasiarc connecting 0 and ∞. Let γ be a quasicircle which contains α as a subarc. For consecutive three points z_{j-1}, z_j, z_{j+1} in a sequence $\mathbf{z} = \{z_j\}_{j=1}^{\infty}$, we set $r_j(\mathbf{z}) = (z_{j+1}-z_j)/(z_{j-1}-z_j)$. Following Astala and Gehring [4], a sequence $\mathbf{z} = \{z_j\}_{j=1}^{\infty}$ in α is called *interlocking* if there is a constants $0 < a, b < \infty$, disjoint neighborhood V_j of z_j, and a component G of $\hat{\mathbb{C}} \setminus \gamma$ such that

(1) $|r_j(\mathbf{z})| \leq a$ for $j \geq 2$

(2) Let $\mathbf{w} = \{w_j\}_{j=1}^{\infty}$ be a sequence. If $w_{j-1} \in V_{j-1} \cap G$, $w_j \in V_j \cap G$, $w_{j+1} \in G$ and $|r_j(\mathbf{w}) - r_j(\mathbf{z})| \leq b$ for some $j \geq 2$, then $w_{j+1} \in V_{j+1}$.

(3) $\mathrm{diam}(V_j)/|z_j| \to 0$ as $j \to \infty$,

where $\mathrm{diam}(V)$ denotes the Euclidean diameter of a set V (see also [3] and [16]).

The following theorem is proved by the similar argument as that of Theorem 4.9 of [4]. However, we give the proof for the completeness.

Theorem 4.3. *Let α be a quasiarc connecting* 0 *and* ∞*, and set* $\Omega = \hat{\mathbb{C}} \setminus \alpha$*. If α contains an interlocking sequence $\{z_j\}_{j=1}^{\infty}$ which converges to ∞, then the asymptotic class $[\varphi_\Omega]$ is not in the Bers boundary $\partial_b \mathcal{AT}_{\mathbf{1}}$.*

Proof. Without loss of generality, we may assume that $1 \in \gamma \setminus \alpha$. Let D_2 be the component of $\hat{\mathbb{C}} \setminus \gamma$ in the definition of the interlocking sequence $\mathbf{z} = \{z_j\}_{j=1}^{\infty}$. Let D_1 be the other component. Let α_i be the side of α facing D_i for $i = 1, 2$. Let f_Ω is the Riemann mapping of $\mathbb{D}^* \to \Omega$ with $f_\Omega(\{-1,1\}) = \{0,\infty\}$ and $f_\Omega(1) = \infty$. For the simplicity of the argument, we assume that the lower-half part of the unit circle $\mathbb{S}^1$

corresponds to the side α_2 under f_Ω. Let $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ be the sequences in the upper and lower-half parts of the unit circle $\mathbb{S}^1$ satisfying $f_\Omega(x_j) = f_\Omega(y_j) = z_j$.

Suppose to the contrary that $[\varphi_\Omega] \in \mathrm{cl}(\mathcal{AT}_\mathbf{1})$. Fix $K > 1$ and let h be a locally univalent K-quasiregular mapping for α satisfying properties in Theorem 3.2. We suppose that h fixes a component D_1 of $\hat{\mathbb{C}} \setminus \gamma$ pointwise. Since $h \circ f_\Omega$ extends locally to a quasiconformal mapping near 1, there is a quasicircle γ' containing 0 and $h \circ f_\Omega(\mathbb{S}^1 \cap D(1,\delta))$ as a subarc for some sufficiently small δ. Furthermore, since h fixes D_1 pointwise, we may assume that all $w_j = h \circ f_\Omega(y_j)$ is contained in D_2. Notice that $h \circ f_\Omega(x_j) = z_j$ for $j \geq 1$. Set $\mathbf{w} = \{w_j\}_{j=1}^\infty$.

Let $\hat{H}_2$ be an K-quasiconformal extension of the restriction $h\mid_{D_2}$. Since h fixes $\gamma \setminus \alpha$ pointwise, so does $\hat{H}_2$. Hence, $\hat{H}_2$ converges to the identity when $K \to 1$. Thus, by Lemma 3.3, we can choose $K > 1$ such that $w_i = \hat{H}_2(z_i) \in V_i$ for $i = 1, 2$, and any K-quasiconformal mapping g fixing $0, 1, \infty$ satisfies

$$|g(w) - w| \leq b \tag{4.2}$$

for $w \in D(0,a)$. We consider a quasiconformal mapping

$$g_j(w) = \frac{1}{w_{j-1} - w_j}\left(\hat{H}_2(z_j + (z_{j-1} - z_j)w) - w_j\right)$$

for $j \in \mathbb{N}$. Then, each g_j is a K-quasiconformal mapping fixing $0, 1, \infty$ and satisfying $g(r_j(\mathbf{z})) = r_j(\mathbf{w})$. By definition, $|r_j(\mathbf{z})| \leq a$ for $j \in \mathbb{N}$. Hence, by (4.2) we have

$$|r_j(\mathbf{w}) - r_j(\mathbf{z})| = |g_j(r_j(\mathbf{z})) - r_j(\mathbf{z})| \leq b.$$

Thus, since $w_1 \in V_1$ and $w_2 \in V_2$, by virtue of the condition (2) of the definition of interlocking sequences, we obtain inductively that $w_j \in V_j$ for all $j \in \mathbb{N}$. Therefore, we conclude

$$|w_j - z_j|/|z_j| \leq \mathrm{diam}(V_j)/|z_j| \to 0 \tag{4.3}$$

as $j \to \infty$, from the condition (3) of the definition of interlocking sequences.

On the other hand, since $z_j = h \circ f_\Omega(x_j)$ and $w_j = h \circ f_\Omega(y_j)$ are in γ' for sufficiently large j, z_j and w_j are divided at 0 and ∞ in γ'. Thus, (4.3) contradicts to Ahlfors' three points principle and the assumption that γ' is a quasicircle. □

4.2.2. Simple zippers in the sense of Thurston The following example is given in §4.12 of [4].

Fix θ with $\pi/3 < \theta < 2\pi/3$. For $j \geq 1$, let α_j denote the polygonal arc formed by joining successively the points $2j-2$, $2j-1$, $2j-1+e^{i(\pi-\theta)}$, $2j+e^{i\theta}$, $2j$ with linear segments. Next let

$$\alpha = \left(\bigcup_{j=1}^\infty \alpha_j\right) \cup \{\infty\}$$

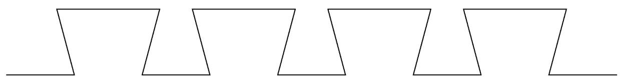

Figure 5. A simple zipper in the sense of Thurston

and $\gamma = (-\infty, 0) \cup \alpha$. Then, γ is a quasicircle and hence α is a quasiarc. K. Astala and F. Gehring proved that when $\theta < \pi/2$, the points $z_j = j$ are interlocking in α. (Notice from the definition that α contains all positive integers.) Such quasiarc α with $\theta < \pi/2$ is known to be a *simple zipper* in the sense of Thurston (cf. §3 of [16]).

Corollary 4.4. *Let α be a quasiarc defined as above with $\theta < \pi/2$. Then, the asymptotic class $[\varphi_\Omega]$ for $\Omega = \hat{\mathbb{C}} \setminus \alpha$ is not in the closure* $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$.

4.2.3. Density Problem Let $\mathcal{U}$ be the set of differentials $\varphi \in B(\mathbf{1})$ whose developing mapping f_φ is univalent. The set $\mathcal{U}$ is closed and contains the universal Teichmüller space $\mathcal{T}_{\mathbf{1}}$ as an open set. L. Bers raised a problem which asked whether $\mathcal{T}_{\mathbf{1}}$ is dense in $\mathcal{U}$ or not (cf. §1.7 of [5]). This problem is recently called the *Bers' density problem* or the *Bers' density conjecture*. As we have already noted in §3.1, F. Gehring showed that the Schwarzian derivative of the Riemann mapping for the complement of the Gehring spiral is not in the closure of $\mathcal{T}_{\mathbf{1}}$, which solved the Bers density problem in the negative. At the present time, the Bers' density problem is stated for Teichmüller spaces of arbitrary Fuchsian groups, and solved in several cases. For instance, see [6], [7], [13] or [15].

Let $\hat{\mathcal{U}}$ be the set of asymptotic classes of all $\varphi \in \mathcal{U}$. From the commutative diagram (2.1), the image $\mathcal{AT}_{\mathbf{1}}$ is contained in $\hat{\mathcal{U}}$. Hence, we can also formulate the density problem for asymptotic Teichmüller spaces of Riemann surfaces[1] (cf. §9 of [14]). In the case of asymptotic Teichmüller spaces, Corollary 4.4 tells us that $\mathcal{AT}_{\mathbf{1}}$ is not dense in $\hat{\mathcal{U}}$, and therefore, the density problem for this case is solved in the negative. The author have already observed this phenomenon by giving a different simply connected domain. (See §7.2 of [14].) However, to the author's knowledge, the density problem is open for arbitrary Riemann surfaces at the moment.

Let us mention the density problem for asymptotic Teichmüller spaces from a different point of view. As we noted, the Schwarzian derivative, we denote here by φ_{Geh}, for the Gehring spiral is not in $\mathrm{cl}(\mathcal{T}_{\mathbf{1}})$, but its asymptotic class is in $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$, and the Schwarzian derivative, denoted by φ_{SZ}, for a simple zipper and its asymptotic class are not in $\mathrm{cl}(\mathcal{T}_{\mathbf{1}})$ and $\mathrm{cl}(\mathcal{AT}_{\mathbf{1}})$, respectively. From these observations, we find a difference between the locations of both Schwarzian derivatives φ_{Geh} and φ_{SZ} in $B(\mathbf{1})$. Namely, when we look $\mathcal{T}_{\mathbf{1}}$ from an appropriate point of $B_0(\mathbf{1})$ where is very far from the origin,

[1]Though we define the asymptotic Teichmüller space only for unit disk in this paper, asymptotic Teichmüller spaces are actually defined for arbitrary Riemann surfaces. See [8] and [9].

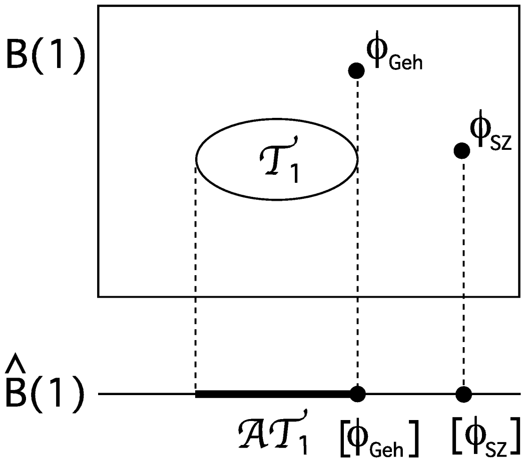

Figure 6. A schematic illustration representing the locations of φ_{Geh} and φ_{SZ}.

the derivative φ_{Geh} is behind $\mathrm{cl}(\mathcal{T}_\mathbf{1})$ like a lunar eclipse, but φ_{SZ} is not. (See Figure 6.) The author expects that the study for the density problem for (asymptotic) Teichmüller spaces contributes to elucidate how points in $\mathcal{U} \setminus \mathcal{T}_1$ encircle $\mathcal{T}_1$.

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