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A synopsis of the Dirichlet principle for measured foliations

By

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Abstract

We present the minimum norm or Dirichlet principle for measured foliations on a Riemann surface of finite type. In this setting the principle says that if you minimize total energy in a given measure class, you will find a unique representative which is harmonic and represented by the imaginary part of a holomorphic quadratic differential. We include as part of this principle the notion of extremal length of a measured foliation and the extremal length functional on Teichmüller space. We show that this functional is differentiable and that its derivative is represented by the unique holomorphic quadratic differential whose heights are equal to the heights of the initially given measured foliation.

In previous publications, [6], [7], [8], [9], [10], [12], [13], [14], some of them more than twenty years old, a Dirichlet principle for measured foliations has been developed. But nowhere has the principle been fully stated in a single theorem. Moreover, its analogy to the solution of the classical Dirichlet problem for finding a harmonic function with given boundary values is not explained. In this largely expository paper we take the opportunity to emphasize these points. There is also a part which is not expository and involves the introduction of the new concept of a *partial measured foliation.* A partial measured foliation does not necessarily determine leaves. Roughly speaking, it is only a family of real valued functions v_j defined on open subsets U_j of a Riemann surface R with the property that

(1) $$v_j = \pm v_k + c_{jk}$$

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where c_{jk} is constant on overlapping sets $U_j \cap U_k$. We do not require that the open sets U_j to form a covering of R, but we do require that the family $\{v_j\}$ has finite Dirichlet integral. That is, we require

$$\int\int_R (v_x^2 + v_y^2)\, dxdy < \infty.$$

Since R is a Riemann surface and the family $\{v_j\}$ satisfies the cocycle condition (1), this integral is well defined.

Another part of this exposition that has not previously been emphasized is that the solution of this Dirichlet problem implies the theorem of Hubbard and Masur [10], which says that associated to any compact Riemann surface R and measured F foliation on that surface, there is a unique holomorphic quadratic differential whose horizontal trajectories and vertical measure realize the measure class of F. The Dirichlet principle has the advantage that it yields this result for any surface of finite analytic type, a case that was not included in Hubbard and Masur's original paper. It is likely that the principle also applies to surfaces of infinite type, but we do not discuss this situation here.

Before elaborating further, we first review the idea for the Dirichlet problem. A continuous function is defined on the smooth boundary of a multiply connected plane domain or on the boundary of a Riemann surface. The problem is to find a harmonic function in the interior whose continuous extension to the boundary coincides with the given function. For purposes of illustration temporarily we assume we have a plane domain bounded by a finite number of analytic curves. We are given a continuous function f on the boundary and the Dirichlet problem is to find a function $u(z)$ continuous on the closure of D satisfying

$$u_{xx}(z) + u_{yy}(z) = 0 \quad \text{for } z \text{ in the interior of } D \quad \text{and} \tag{2}$$

$$u(p) = f(p) \quad \text{for } p \text{ on the boundary of } D. \tag{3}$$

Solutions to this problem appear in many introductory books on complex analysis, see for example [1]. If the domain is the unit disc, one can use the Poisson integral formula:

$$u(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, f(e^{i\theta})\, d\theta. \tag{4}$$

Note that

$$2\,\mathrm{Re}\,\frac{1+z}{1-z} = \left(\frac{1+z}{1-z} + \frac{1+\overline{z}}{1-\overline{z}}\right) = 2\,\frac{1-r^2}{1-2r\cos\theta+r^2},$$

and so the Poisson kernel in (4) is the real part of a holomorphic function. Since the formula is the convolution of this kernel with the function $f(e^{i\theta})$, it produces a harmonic function $u(x, y)$, that is, $u_{xx}(z) + u_{yy}(z) = 0$. Moreover, if $z = g(\zeta)$ is a conformal map from a simply connected domain onto $\{|z| < 1\}$, then $(u \circ g)_{\xi\xi}(\zeta) + (u \circ g)_{\eta\eta}(\zeta) = 0$, where $\zeta = \xi + i\eta$. Since Riemann mappings on Jordan domains extend continuously to the boundary, this allows one to transport a solution in the unit disc to a solution in an arbitrary Jordan domain.

Perron's principle and the method of subharmonic functions provides another approach that applies to arbitrary plane domains as well as to Riemann surfaces. This approach involves many steps. First one finds a subharmonic function in the domain with the desired boundary values. Then one replaces the subharmonic function by a new subharmonic function which coincides with the old function outside of an open disc with smooth boundary and is harmonic inside. The new function is again subharmonic and one applies this replacement procedure successively infinitely many times over variable discs with smooth boundary and contained in the domain. Convergence relies on Harnack's principle.

Another approach is called the Schwarz alternating procedure [3]. It yields a solution on the union of two domains with non-empty intersection provided there is a certain geometric condition on the way their boundaries intersect and provided it is already known how to find the solution on each of the domains separately.

Riemann [15] proposed a different method. For simplicity, we suppose that D is a multiply connected plane domain bounded by a finite number of smooth simple closed curves. Riemann considers the family $\mathcal{F}$ of $u(z)$ with $z = x + iy$ defined and continuous on the closure of D such that

- $u(p) = f(p)$ for every point p in the boundary of D and
- $u(z)$ has continuous second partial derivatives in D.

From among functions with these properties one finds a function u such that its Dirichlet integral,

$$\iint_D (u_x^2 + u_y^2)\; dxdy, \tag{5}$$

takes the smallest possible value. The claim is that if such a function can be found, it will be the unique harmonic function in D that has the boundary values prescribed by f. This method has an important physical interpretation. Since the Dirichlet integral represents the net energy of a physical system, it means that while minimizing energy with given boundary values, the system will converge towards harmonic equilibrium.

If the method is to succeed, one must show that there is a function $u(z)$ in $\mathcal{F}$ that realizes the minimum in (5), that its boundary values coincide with the values of the given function f, and finally, that the minimizing function is harmonic, that is, that it satisfies the equation $u_{xx}(z) + u_{yy}(z) = 0$ for all z in the domain.

This last step is surprisingly simple. If (5) is minimized by a function $u(z)$, we select an arbitrary C^1 function $\delta u(z)$ that vanishes in a neighborhood of the boundary of D. If that is the case, then $u(z)+t\delta u(z)$ is a competing function for every real number t, and so

$$\int\int_D (u_x^2 + u_y^2)dxdy \leq \int\int_D ((u + t\delta u)_x^2 + (u + t\delta u)_y^2)dxdy. \tag{6}$$

Expanding out the right hand side of (6) one obtains

$$0 \leq 2t \int\int_D (u_x \delta u_x + u_y \delta u_y)\ dxdy + t^2 \int\int_D (\delta u_x^2 + \delta u_y^2)\ dxdy.$$

Dividing by t and letting t approach zero first through positive and then through negative values yields

$$\int\int_D (u_x \delta u_x + u_y \delta u_y)\ dxdy = 0.$$

Finally, since δu has compact support, integration by parts leads to

$$\int\int_D (u_{xx} + u_{yy})\delta u\ dxdy = 0 \tag{7}$$

for every C^1 function δu with compact support in D. This implies

$$u_{xx} + u_{yy} = 0 \text{ in } D. \tag{8}$$

We omit further discussion of these steps because they are useful only insofar as they provide an analogy to a similar problem for measured foliations.

Before describing the problem for measured foliations, we want to mention a direction for research. It concerns discrete versions of the Dirichlet problem. A nice introduction to this topic is given in the book by Doyle and Snell [4] where there is an exposition of Polya's famous result concerning the qualitative difference between random walks on a rectilinear grid in two and three dimensions. Estimates that would show that when the Dirichlet integral is nearly extremal the resulting function u is nearly harmonic would be interesting. A formulation and solution to this type of problem for measured foliations would also be interesting.

Unlike functions, measured foliations do not have boundary values and it is not possible to evaluate a measured foliation at a point. On the other hand, the measure of a measured foliation makes it possible to evaluate the vertical distance between level sets and this is what makes it possible to set up an extremal problem for measured foliations analogous to the extremal problem in (5).

To this end, we make the following definition.

Definition. A *partial measured foliation* $|dv|$ on a Riemann surface R consists of two things:

(i) a family of open subsets U_i of R,

(ii) an assignment of a real valued, Lipschitz continuous function v_i defined on U_i such that if there is any non-empty intersection of two of the sets U_i and U_j, then for every z in $U_i \cap U_j$,

$$v_i(z) = \pm v_j(z) + c_{ij},$$

for some constant c_{ij}.

Remark. Note that we do not require the subsets U_i to form a covering of R.

Let $U = \bigcup_j U_j$ and

$$\int_\gamma |dv| = \int_{\gamma \cap U} |dv| + \int_{\gamma \cap (R-U)} |dv|.$$

By definition, put the second integral equal to zero and we are left with defining the first integral. Let γ be parameterized by t with $0 \leq t \leq 1$ and let $\mathcal{P}$ be a partition of the unit interval $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ and let $I_k = \{t : t_{k-1} \leq t \leq t_k\}$. We define the integral over the partition $\mathcal{P}$ by

$$\int_{\mathcal{P}} |dv| = \sum |v(\gamma(t_k)) - v(\gamma(t_{k-1}))|,$$

where the sum is over those terms for which the entire arc $\gamma(I_k)$ is contained in one of the open sets U_j. If for any such term the arc $\gamma(I_k)$ is contained in another one of the open sets U_ℓ, then because of property (ii)

$$|v_k(\gamma(t_k)) - v_k(\gamma(t_{k-1}))| = |v_\ell(\gamma(t_k)) - v_\ell(\gamma(t_{k-1}))|.$$

As usual, we define the mesh of the partition $\mathcal{P}$ by

$$mesh(\mathcal{P}) = \max_{1 \leq k \leq n} (t_k - t_{k-1}),$$

and finally, by definition,

$$\int_{\gamma \cap U} |dv| = \limsup_{mesh(\mathcal{P}) \to 0} \int_{\mathcal{P}} |dv|.$$

We define the line integral of the free homotopy class $[\gamma]$ of γ in R by

$$\int_{[\gamma]} |dv| = \inf_{\tilde{\gamma}} \int_{\tilde{\gamma}} |dv|,$$

where the infimum is taken over all curves or arcs $\tilde{\gamma}$ in R freely homotopic to γ, and we call this infimum the height of $|dv|$ with respect to $[\gamma]$. That is, by definition,

$$ht(|dv|, \gamma) = \int_{[\gamma]} |dv| = \inf_{\tilde{\gamma} \text{ homotopic to } \gamma} \int_{\tilde{\gamma}} |dv|. \tag{9}$$

In this definition, we need to define free homotopy. Let $\mathbb{S}^1$ be the unit circle and I be the unit interval. Any other closed curve $\tilde{\gamma}$ is freely homotopic to γ if there is a continuous function $h(s,t)$ defined on $\mathbb{S}^1 \times I$ and mapping into R such that $h(s,0) = \gamma(s)$, $h(s,1) = \tilde{\gamma}(s)$. Two arcs $\tilde{\alpha}$ and α with endpoints on boundary contours of R are freely homotopic if there is a continuous function $h(s,t)$ defined on $I \times I$ and mapping into $R \cup \partial R$ such that $h(s,0) = \alpha(s)$, $h(s,1) = \tilde{\alpha}(s)$, $h(0,t)$ lies on the same boundary contour for every $t \in I$, and $h(1,t)$ lies on the same boundary contour for every $t \in I$.

In the version of the Dirichlet principle we are going to state we will assume R is of finite type. This means R can be obtained by deleting a finite number of points and open discs from a compact Riemann surface $\tilde{R}$ of finite genus. We let the genus of R be g, the number of deleted discs be r and the number of deleted points be n.

If $r > 0$, then by Fuchsian uniformization one can see that R has a natural double $R^d = R \cup j(R)$ where j is an anti-conformal involution. In fact, if we uniformize R by a Fuchsian group acting on the upper half plane, then j is realized by $j(z) = \overline{z}$. Every simple arc α with endpoints on a boundary curve of R reflects to an arc $j(\alpha)$ and $\alpha \cup j(\alpha)$ is a simple closed curve on R^d. And every system of multi-curves and multi-arcs on R is the restriction of a j-invariant system of closed multicurves on R^d.

By definition the Dirichlet norm of a partial measured foliation $|dv|$ defined on R is equal to

$$Dir(|dv|) = \int\int_R \left(v_x^2 + v_y^2\right)\, dxdy. \tag{10}$$

The star operation on 1-forms extends to a well defined operation on partial measured foliations. In particular, after choosing a representative v of $|dv|$ up to plus or minus sign and an additive constant, one puts

$$*dv = *\left(v_x dx + v_y dy\right) = -v_y dx + v_x dy,$$

which is also defined only up to plus or minus sign. Then

$$\begin{aligned} dv + i * dv &= v_x dx + v_y dy + i(-v_y dx + i v_x dy) = (v_x - i v_y)dx + (i v_x + v_y)dy \\ &= (v_x - i v_y)(dx + i dy). \end{aligned}$$

Using the complex notation,

$$\frac{\partial v}{\partial z} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}\right),$$

we see that if we put $q_v = (dv + i * dv)^2$, then

$$q = \left(2\frac{\partial v}{\partial z}dz\right)^2 = \left((v_x - iv_y)\,(dx + idy)\right)^2 = (dv + i * dv)^2.$$

Thus $q = q_v$ is a quadratic differential on R, although not necessarily holomorphic. Moreover,

$$\begin{aligned}
||q_v|| &= \int\int_R |q| = \int\int_R |\sqrt{q}\sqrt{\bar{q}}| \\
&= \int\int_R |[(v_x - iv_y)dx + i(v_x - iv_y)dy] \wedge [(v_x + iv_y)dx - i(v_x + iv_y)dy]| \\
&= \int\int_R |-i(v_x - iv_y)(v_x + iv_y)dxdy - i(v_x - iv_y)(v_x + iv_y)dxdy| \\
&= 2\int\int_R \left(v_x^2 + v_y^2\right) dxdy = 2Dir(|dv|).
\end{aligned}$$

So we see that the L^1-norm of the quadratic differential q_v is equal to twice the Dirichlet norm of the partial measured foliation $|dv|$. Although the partial measured foliations do not form a linear space, the correspondence $|dv| \mapsto q_v$ realizes the partial measured foliations as a subset of the vector space of quadratic differentials. Furthermore, any problem of minimizing the Dirichlet integral $Dir(|dv|)$ under certain conditions is equivalent to the problem of minimizing the norm $||q_v||$ of the quadratic differential q_v under corresponding conditions.

Having made these observations and definitions, we can now state

THE DIRICHLET PROBLEM.

For a given partial measured foliation $|d\tilde{v}|$ on a Riemann surface of finite analytic type, find a measured foliation $|dv|$ with the same heights as the heights of $|d\tilde{v}|$ on all simple closed curves γ such that $|dv|$ has the smallest possible Dirichlet integral.

A version of this problem is discussed by Ahlfors in [2, pages 65-70] but there the infimum of the Dirichlet integrals is taken over functions and partial measured foliations are not considered. Because the class over which the infimum is taken includes only functions, the richness of the topological possibilities for the trajectory structure of the level lines of the foliations is lost.

Before giving a solution, we need several more definitions. If one is given a holomorphic quadratic differential q on R, it induces a *natural parameter* $w_j = u_j + iv_j$ determined up to plus or minus sign and an additive constant by the equation

$$w_j = u_j + iv_j = \pm \int \left(\sqrt{q_j(z)} \right) dz.$$

Thus there is an open covering U_j of $R \setminus \{$ the zeroes of $q\}$ and for this open covering u_j and v_j satisfy the equations

$$\begin{aligned} u_j &= \pm u_k + \text{ Re } c_{jk} \quad \text{and} \\ v_j &= \pm v_k + \text{ Im } c_{jk} \end{aligned}$$

in any overlapping set $U_j \cap U_k$. Because u_j and v_j are locally the real and imaginary parts of a holomorphic function, they are harmonic, that is,

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \quad \text{and} \\ v_{xx} + v_{yy} &= 0. \end{aligned}$$

The level sets determined by the equation $v_j = a\ constant$ are called the horizontal trajectories for $q = q(z)(dz)^2$, $|dv|$ is its vertical measure and sets determined by the equation $u_j = a\ constant$ are called the vertical trajectories and $|du|$ is its horizontal measure.

Definition. A *measured foliation* on a surface R is a partial measured foliation such that at every point p in R there is a neighborhood N_p for which there is a homeomorphism h from N_p onto a neighborhood of the origin in the complex plane carrying p onto 0 and carrying the level sets of $|dv|$ onto the horizontal trajectories of $\text{Im }\left(\sqrt{z^n}dz\right)$. If $n = 0$, p is called a regular point, and if $n > 0$, p is called an $n + 2$ pronged singularity.

Remark. Locally a level set of a measured foliation is an arc or several arcs joined at singular points. This is not necessarily the case for a partial measured foliation. Moreover, a partial measured foliation assigns values on certain open subsets, but these open sets do not necessarily form a covering of R.

THE DIRICHLET PRINCIPLE makes three assertions:

I. *For a measured foliation $|d\tilde{v}|$ on a Riemann surface of finite analytic type, the infimum*

$$M(|d\tilde{v}|) = \inf \left\{ \int\int_R (v_x^2 + v_y^2)\ dxdy \right\}, \tag{11}$$

taken over all partial measured foliations $|dv|$ *such that*

$$ht(|dv|, \gamma) \geq ht(|d\tilde{v}|, \gamma) \text{ for all simple closed curves } \gamma \text{ in } R,$$

is realized by a unique measured foliation $|dv_0|$. *Moreover, all of the heights of homotopy classes of simple closed curves with respect to* $|dv_0|$ *coincide with the heights of the given partial measured foliation* $|d\tilde{v}|$.

II. *The minimizing measured foliation* $|dv_0|$ *in* (11) *is the vertical part of a holomorphic quadratic differential* q_0 *and this differential is uniquely determined by the heights of* $|d\tilde{v}|$ *on homotopy classes of simple closed curves.*

III. *The quantity* $M(|d\tilde{v}|)$ *defined by* (11) *is a differentiable function on Teichmüller space* $T(R)$ *of* R *and, if the Riemann surface* R *is chosen as the base point for* $T(R)$ *and if* $R_{t\mu}$ *is the surface* R *with variable conformal structure determined by* $t\mu$ *where* μ *is a bounded Beltrami coefficient multiplied by a small complex factor* t, *then*

$$\log M_{t\mu}(|d\tilde{v}|) = \log M(|d\tilde{v}|) + \frac{2}{||q_0||} \text{ Re } t \int\int_R q_0 \mu \, dxdy + o(t). \tag{12}$$

Furthermore, $\log M_\tau(|d\tilde{v}|)$ *is a* C^1 *function of the point* τ *in Teichmüller space.*

In a subsequent paper we give further details for the proofs of these results and discuss their consequences for the the extremal length geometry of Teichmüller space as developed by Gardiner and Masur in [9] and Miyachi in [14].

Several closely related extremal problems for extremal metrics and for quadratic differentials are considered by Strebel [16] and Jenkins [11] but neither of these papers introduces the rôle of measured foliations. In some physical applications consideration of quadratic differentials with double poles and infinite norm is important [19]. Formulation of the minimum norm principle in this setting requires introducing a notion of reduced norm.

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