RIMS Kôkyûroku Bessatsu **B17**
(2010), 37–53

Asymptotically affine and asymptotically conformal circle endomorphisms

By

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Abstract

We show that every uniformly asymptotically affine circle endomorphism has a uniformly asymptotically conformal extension.

Introduction

First we summarize basic properties of uniformly asymptotically affine circle degree $d > 1$ endomorphisms. Then we use the Beurling-Ahlfors extension to realize any uniformly asymptotically affine system as the restriction to the circle of a uniformly asymptotically conformal system. Theorem 1 is a well-known characterization of symmetric homeomorphisms of the real axis in terms of possible quasiconformal extensions. Theorem 2 is an exposition of calculations given by Cui in [4]. Theorem 3 is a special case of a theorem for any one-dimensional Markov map with bounded geometry in [8, 9] (see also [10]). Theorem 4 is the main new result. Since it is known that the Teichmüller space of uniformly asymptotically affine expanding maps is complete with Teichmüller's metric (see, for example [7, 5, 10]), this theorem shows that the uniformly asymptotically conformal expanding maps also form a complete metric space with Teichmüller's

2000 Mathematics Subject Classification(s): Primary 37F15, Secondary 37F30.
Key Words: *uniformly asymptotically affine (UAA), uniformly asymptotically conformal (UAC), quasisymmetric homeomorphism, symmetric homeomorphism, Beurling-Ahlfors extension.*
The research is partially supported by PSC-CUNY awards.
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metric. In a subsequent paper we will exploit this fact to construct a dual dynamical system corresponding to every UAA circle expanding map.

We take this opportunity to express our gratitude to the referee for several important and helpful corrections.

§ 1. Circle endomorphisms

Let $\mathbb{S}^1 = \{z \in \mathbb{C} \, : \, |z| = 1\}$ be the unit circle. The map $\pi : \mathbb{R} \to \mathbb{S}^1$ defined by

$$\pi(x) = e^{2\pi i x}$$

realizes $\mathbb{R}$ as the universal covering of $\mathbb{S}^1$ with covering π and covering group $\mathbb{Z}$. π induces an isomorphism from $\mathbb{R}/\mathbb{Z}$ onto $\mathbb{S}^1$.

Let m be the degree of an orientation preserving covering f from $\mathbb{S}^1$ onto itself and assume $1 < m < \infty$. f is an endomorphism of $\mathbb{S}^1$ and it necessarily has one fixed point p. By selecting an orientation preserving Möbius transformation A that preserves the unit disk with $A(p) = 1$, we may shift consideration of the map f to the map $\tilde{f} = A \circ f \circ A^{-1}$. $\tilde{f}$ has the same dynamical properties as f and it fixes the point 1. Therefore, without loss of generality, we may assume to begin with that f fixes the point $p = 1$. We denote the homeomorphic lift of f by F. F is uniquely determined by f if we assume it has the following properties:

i) F is a homeomorphism of $\mathbb{R}$,

ii) $\pi \circ F = f \circ \pi$,

iii) $F(0) = 0$.

Note that $F(x+1) = F(x) + m$. In this paper we refer either to f or to its unique corresponding lift F as a circle endomorphism. We denote the n-fold composition of f with itself by f^n. Similarly, F^n is the n-fold composition of F.

Suppose h is an orientation-preserving circle homeomorphism. Let H be the lift of h to $\mathbb{R}$ such that $0 \leq H(0) < 1$. Then $H(x+1) = H(x)+1$ and h and H are one-to-one correspondences.

Definition 1. A circle homeomorphism h is called M-quasisymmetric if there is a constant $M \geq 1$ such that for all real numbers x and for all $y > 0$,

$$\frac{1}{M} \leq \frac{H(x+y) - H(x)}{H(x) - H(x-y)} \leq M. \tag{1}$$

The expression

$$\rho_H(x,y) = \frac{H(x+y) - H(x)}{H(x) - H(x-y)}, \quad x,\ y \neq 0 \in \mathbb{R},$$

is called the quasisymmetric distortion function for H. We also need the skew quasisymmetric distortion function for H, which is defined by

$$\rho_H(x, y, k) = \frac{H(x+ky) - H(x)}{H(x) - H(x-y)}, \quad x,\ y \neq 0 \in \mathbb{R},\ \ 0 < k \leq 1.$$

The following lemma is well-known for a quasisymmetric homeomorphism of the real line. One can prove it by using quasiconformal mapping theory (see, for example, [11]). However, we need to use it for a quasisymmetric homeomorphism of a compact interval. For quasisymmetric homeomorphisms of compact intervals, the proof by using quasiconformal mapping theory does not work since a M-quasisymmetric homeomorphism of a compact interval is not necessarily a restriction of a M-quasisymmetric homeomorphism of the real line. However, there is a proof for a quasisymmetric homeomorphism of a compact interval by using elementary real analysis methods (see, for example, [10]). For the convenience of the reader we also give the proof here. First we define the notion of quasisymmetry for a homeomorphism of a compact interval. An orientation-preserving homeomorphism H of a closed interval $[a, b]$ is called M-quasisymmetric if

$$M^{-1} \leq \frac{H(x+t) - H(x)}{H(x) - H(x-t)} \leq M, \quad \forall\, x, x+t, x-t \in [a, b].$$

Lemma 1. *There is a function $\zeta(M) > 0$ satisfying $\zeta(M) \to 0$ as $M \to 1$ such that for any M-quasisymmetric homeomorphism H of $[0, 1]$ with $H(0) = 0$ and $H(1) = 1$,*

$$|H(x) - x| \leq \zeta(M), \quad \forall\, x \in [0, 1].$$

Proof. Consider points $x_n = 1/2^n$, $n = 0, 1, \ldots$. The M-quasisymmetry condition implies that

$$M^{-1} \leq \frac{H\left(\dfrac{1}{2^{n-1}}\right) - H\left(\dfrac{1}{2^n}\right)}{H\left(\dfrac{1}{2^n}\right) - H(0)} \leq M.$$

From this and the fact that $H(0) = 0$, we get

$$(1 + M^{-1}) H\left(\frac{1}{2^n}\right) \leq H\left(\frac{1}{2^{n-1}}\right) \leq (1 + M) H\left(\frac{1}{2^n}\right).$$

This gives

$$\frac{1}{1+M} H\left(\frac{1}{2^{n-1}}\right) \leq H\left(\frac{1}{2^n}\right) \leq \frac{1}{1+M^{-1}} H\left(\frac{1}{2^{n-1}}\right).$$

Using $H(1)=1$, we get

$$\left(\frac{1}{1+M}\right)^n \le H\left(\frac{1}{2^n}\right) \le \left(\frac{1}{1+M^{-1}}\right)^n, \quad \forall\, n \ge 1. \tag{2}$$

Furthermore, from M-quasisymmetry we claim that

$$\left(\frac{1}{1+M}\right)^n \le H\left(\frac{i}{2^n}\right) - H\left(\frac{i-1}{2^n}\right) \le \left(\frac{1}{1+M^{-1}}\right)^n \tag{3}$$

for all $n \ge 1$ and $1 \le i \le 2^n$. To justify this claim we use induction on n. First from (2), (3) is true for $n=1$, that is,

$$\frac{1}{1+M} \le H\left(\frac{1}{2}\right) \le \frac{1}{1+M^{-1}} \quad \text{and} \quad \frac{1}{1+M} \le 1 - H\left(\frac{1}{2}\right) \le \frac{1}{1+M^{-1}}.$$

Suppose (3) is true for $n-1 \ge 1$, that is, for all $1 \le j \le 2^{n-1}$,

$$\left(\frac{1}{1+M}\right)^{n-1} \le H\left(\frac{j}{2^{n-1}}\right) - H\left(\frac{j-1}{2^{n-1}}\right) \le \left(\frac{1}{1+M^{-1}}\right)^{n-1}. \tag{4}$$

Then for any $i/2^n$ with odd i (a similar argument works if $i-1$ is odd) $(i-1)/2^n$ and $(i+1)/2^n$ can be written as $(j-1)/2^{n-1}$ and $j/2^{n-1}$ for some $1 \le j \le 2^{n-1}$. Using M-quasisymmetry, we get

$$1+M^{-1} \le \frac{H\left(\frac{i+1}{2^n}\right) - H\left(\frac{i-1}{2^n}\right)}{H\left(\frac{i}{2^n}\right) - H\left(\frac{i-1}{2^n}\right)} = \frac{H\left(\frac{i+1}{2^n}\right) - H\left(\frac{i}{2^n}\right)}{H\left(\frac{i}{2^n}\right) - H\left(\frac{i-1}{2^n}\right)} + 1 \le 1+M.$$

Taking reciprocals, we get

$$\frac{1}{1+M} \le \frac{H\left(\frac{i}{2^n}\right) - H\left(\frac{i-1}{2^n}\right)}{H\left(\frac{i+1}{2^n}\right) - H\left(\frac{i-1}{2^n}\right)} \le \frac{1}{1+M^{-1}}.$$

Thus

$$\begin{aligned} \frac{1}{1+M}\left(H\left(\frac{j}{2^{n-1}}\right) - H\left(\frac{j-1}{2^{n-1}}\right)\right) &\le H\left(\frac{i}{2^n}\right) - H\left(\frac{i-1}{2^n}\right) \\ &\le \frac{1}{1+M^{-1}}\left(H\left(\frac{j}{2^{n-1}}\right) - H\left(\frac{j-1}{2^{n-1}}\right)\right). \end{aligned}$$

Thus (3) follows from (4) which proves the inductive step.

Let

$$\tau_n = \max\left\{\left(\frac{M}{M+1}\right)^n - \frac{1}{2^n},\ \frac{1}{2^n} - \left(\frac{1}{M+1}\right)^n\right\}, \quad n = 1, 2, \ldots.$$

Then for $n = 1$,
$$\left| H\left(\frac{1}{2}\right) - \frac{1}{2} \right| \leq \tau_1 = \frac{1}{2}\frac{M-1}{M+1},$$
and for any $n > 1$, we have
$$\max_{0\leq i\leq 2^n} \left| H\left(\frac{i}{2^n}\right) - \frac{i}{2^n} \right| \leq \max_{0\leq i\leq 2^{n-1}} \left| H\left(\frac{i}{2^{n-1}}\right) - \frac{i}{2^{n-1}} \right| + \tau_n$$
By summing over k for $1 \leq k \leq n$, we obtain
$$\max_{0\leq i\leq 2^n} \left| H\left(\frac{i}{2^n}\right) - \frac{i}{2^n} \right| \leq \delta_n = \sum_{k=1}^{n} \tau_k.$$
If we put $\zeta(M) = \sup_{1\leq n<\infty}\{\delta_n\}$, by summing geometric series, we obtain
$$\zeta(M) = \max_{1\leq n<\infty} \left\{ M - 1 + \frac{1}{2^n} - M\Big(\frac{M}{1+M}\Big)^n, 1 - \frac{1}{M} + \frac{1}{M}\Big(\frac{1}{M}\Big)^n - \frac{1}{2^n} \right\}.$$
Clearly, $\zeta(M) \to 0$ as $M \to 1$, and since the dyadic points
$$\{i/2^n \mid n = 1, 2, \ldots; 0 \leq i \leq 2^n\}$$
are dense in $[0, 1]$, we conclude
$$|H(x) - x| \leq \zeta(M) \quad \forall\, x \in [0, 1],$$
which proves the lemma. $\square$

Corollary 1. *Let $\vartheta(M) = M-1+M\zeta(M)$. Then for any homeomorphism H of $\mathbb{R}$ and any $x, y > 0 \in \mathbb{R}$, if H restricted to the interval $[x-y, x+y]$ is M-quasisymmetric, then*
$$\max\{|\rho_H(x, y, k) - k|, |\rho_H(x, -y, k) - k|\} \leq \vartheta(M), \quad 0 < \forall\, k \leq 1.$$

Proof. Consider $\hat{H}(k) = (H(x+ky) - H(x))/(H(x+y) - H(x))$. Then $\hat{H}(1) = 1$ and $\hat{H}(0) = 0$. Also, $\hat{H}$ is quasisymmetric because
$$\frac{\hat{H}(k+j) - \hat{H}(k)}{\hat{H}(k) - \hat{H}(k-j)} = \frac{H(x+ky+jh) - H(x+ky)}{H(x+ky) - H(x+ky-jy)}$$
for any $0 \leq k \leq 1$ and $j > 0$ such that $[k-j, k+j] \subset [0, 1]$ and this is bounded above by M and below by $1/M$ because H is M-quasisymmetric. So, from Lemma 1,
$$k - \zeta(M) \leq \frac{H(x+ky) - H(x)}{H(x+y) - H(x)} \leq k + \zeta(M).$$

Thus

$$(k-\zeta(M))\rho_H(x,y) \leq \frac{H(x+ky)-H(x)}{H(x)-H(x-y)} \leq (k+\zeta(M))\rho_H(x,y).$$

Since $1/M \leq \rho_H(x,y) \leq M$ and we are assuming that $0<k\leq 1$, this implies that

$$|\rho_H(x,y,k)-k| \leq \vartheta(M) = M-1+M\zeta(M).$$

Similarly, we have that

$$|\rho_H(x,-y,k)-k| \leq \vartheta(M) = M-1+M\zeta(M).$$

□

Definition 2. A bounded positive function $\epsilon(y)$ defined for positive values of t is called vanishing if $\epsilon(y) \to 0^+$ as $y \to 0^+$.

Definition 3. A quasisymmetric circle homeomorphism h is called symmetric (or asymptotically affine) if it is quasisymmetric and if there exists a vanishing function $\varepsilon(y)$ such that

$$\frac{1}{1+\varepsilon(y)} \leq \rho_H(x,y) \leq 1+\varepsilon(y), \tag{5}$$

for all real numbers x and all $y>0$.

We say asymptotically affine in this definition since the ratio in the middle expression in (5) is identically equal to one if and only if H is affine, that is, if $H(x)=ax+b$, for some $a\neq 0$.

Lemma 2. *A quasisymmetric circle homeomorphism is symmetric if, and only if, there is a vanishing function $\epsilon(y)$ such that*

$$\max\{|\rho_H(x,y,k)-k|, |\rho_H(x,-y,k)-k|\} \leq \epsilon(y) \tag{6}$$

for all real numbers x and $y>0$ and for all k with $0<k\leq 1$.

Proof. Since H is symmetric, by picking $y>0$ sufficiently small, we obtain a number M arbitrarily close to 1, such that

$$\frac{1}{M} \leq \rho_H(s,t) \leq M$$

for all numbers s and t for which $s-t, s$ and $s+t$ lie in the interval $[x-y,x+y]$. By Corollary 1, this implies that there is a vanishing function $\varepsilon'(y)$ such that

$$\max\{|\rho_H(x,y,k)-k|, |\rho_H(x,-y,k)-k|\} \leq \varepsilon'(y),$$

for all real numbers x, all $y > 0$ and all k with $0 \leq k \leq 1$.

Conversely, (6) with $k = 1$ implies

$$|\rho_H(x, y) - 1| \leq \varepsilon(y),$$

and this implies the existence of a vanishing function $\epsilon'(y)$ for which

$$\frac{1}{1 + \varepsilon'(y)} \leq \rho(x, y) \leq 1 + \varepsilon'(y).$$

□

Definition 4. A circle endomorphism f of degree d is called uniformly symmetric or uniformly asymptotically affine (UAA) if all of the inverse branches of f^n, $n = 1, 2, \ldots,$ are symmetric uniformly. More precisely, f^n is UAA if there is a vanishing function $\epsilon(y)$ such that, for all positive integers n and all real numbers x and $y > 0$,

$$\frac{1}{1 + \varepsilon(y)} \leq \rho_{F^{-n}}(x, y) = \frac{F^{-n}(x + y) - F^{-n}(x)}{F^{-n}(x) - F^{-n}(x - y)} \leq 1 + \varepsilon(y). \tag{7}$$

We say uniformly since the ratio in the middle expression in (7) approaches 1 when y approaches 0 independently of the number n of compositions of F.

§ 2. Beurling-Ahlfors Extensions

By definition, a homeomorphism G from a plane domain Ω onto another plane domain $G(\Omega)$ is quasiconformal if it is orientation preserving and if it has locally integrable distributional first partial derivatives $G_{\overline{z}}$ and G_z satisfying the inequality

$$|G_{\overline{z}}(z)| \leq k |G_z(z)| \tag{8}$$

for some number k with $0 \leq k < 1$ and for almost all z. The complex valued Beltrami coefficient $\mu = \mu_G$ for G is defined by the equation

$$G_{\overline{z}}(z) = \mu(z) G_z(z) \tag{9}$$

where $||\mu(z)||_\infty < 1$. It is standard to call the quantity

$$K_z(G) = \frac{|G_z(z)| + |G_{\overline{z}}(z)|}{|G_z(z)| - |G_{\overline{z}}(z)|} = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{10}$$

the dilatation of G at the point z, and to call

$$K(G) = \mathrm{essup}_{z \in \Omega} K_z(G) = \frac{1 + ||\mu||_\infty}{1 - ||\mu||_\infty}$$

the dilatation of G on the domain Ω. Thus, the homeomorphism G is quasiconformal on Ω if $K(G) < \infty$. We will also use the nonstandard notation $K(G,z)$ for the same fraction that appears in (10) without the absolute value signs, that is,

$$K(G,z) = \frac{G_z(z) + G_{\overline{z}}(z)}{G_z(z) - G_{\overline{z}}(z)}.$$

Note that $K(G,z)$ is complex valued and $|K(G,z)| \leq K(G)$. In all of these notations, $\mu_G, K_z(G), K(G)$ and $K(G,z)$, we omit reference to the mapping G if this is clear from the context.

Consider all possible extensions of quasisymmetric self-mappings H of the real axis to quasiconformal self-mappings $\tilde{H}$ of the upper half plane $\mathbb{H}$, and define $K(H)$ by the formula

$$K(H) = \inf\{K(\tilde{H}) : \tilde{H} \text{ extends } H\}.$$

From the theory of quasiconformal mappings, if $K(H) = 1$, then H is affine, that is, $H(x) = ax+b, a \neq 0$. Similarly, H is also affine if $M = M(H) = 1$ in the M-condition (1). Thus, we may take both $M(H)$ and $K(H)$ as measurements of the extent to which H fails to be affine. A well-known result of Beurling and Ahlfors [3] shows that $M(H)$ and $K(H)$ are simultaneously finite and there are estimates for $M(H)$ in terms of $K(H)$ and vice-versa. Moreover, $M(H)$ and $K(H)$ simultaneously approach 1.

The Beurling-Ahlfors extension procedure provides a canonical extension $\tilde{H}$ of any quasisymmetric homeomorphism H such that the Beltrami coefficient μ of $\tilde{H}$ satisfies $\|\mu\|_\infty < 1$. Furthermore, it satisfies the following well-known theorem [6].

Theorem 1. *The Beurling-Ahlfors extension of a quasisymmetric self-mapping H of the real axis has a Beltrami coefficient μ with $|\mu(x+iy)| \leq \eta(y)$ for some vanishing function $\eta(y)$ if, and only if, there is a vanishing function $\epsilon(y)$ such that*

$$\frac{1}{1+\epsilon(y)} \leq \rho_H(x,y) \leq 1+\epsilon(y).$$

In this paper we require a similar estimate that allows the comparison between distortion functions of two quasisymmetric homeomorphisms of the real axis H_0 and H_1 fixing 0, 1 and ∞. The following theorem which was stated in [4] would be a corollary of Theorem 1 if the Beurling-Ahlfors extension procedure defined a homomorphism. That is, if the Beurling-Ahlfors extension of H_1 composed with the Beurling-Ahlfors extension of H_2 were equal to the Beurling-Ahlfors extension of $H_1 \circ H_2$. But that is not the case. However, the result is still true and we will give a complete proof.

Theorem 2 (Cui [4]). *Suppose the skew quasisymmetric distortion functions $\rho_0(x,y,k)$ and $\rho_1(x,y,k)$ of H_0 and H_1 satisfy the inequality*

$$|\rho_0(x,y,k)-\rho_1(x,y,k)|,\ \ |\rho_0(x,-y,k)-\rho_1(x,-y,k)|\le \epsilon(y)$$

for $x,y>0\in\mathbb{R}$ and $0<k\le 1$, where $\epsilon(y)$ is a vanishing function. Suppose furthermore that μ_1 and μ_2 are the Beltrami coefficients of the Beurling-Ahlfors extensions $\tilde{H}_0$ and $\tilde{H}_1$, that is,

$$\mu_0(z)=\frac{\tilde{H}_{0\overline{z}}}{\tilde{H}_{0z}}\ \ \text{and}\ \mu_1(z)=\frac{\tilde{H}_{1\overline{z}}}{\tilde{H}_{1z}}.$$

Then there is a vanishing function $\eta(y)$ depending only on $\epsilon(y)$ such that

$$|\mu_0(x+iy)-\mu_1(x+iy)|\le\eta(y).$$

Conversely, given two quasiconformal maps $\tilde{H}_0$ and $\tilde{H}_1$ preserving the real axis and a vanishing function $\eta(y)$ such that

$$|\mu_0(z)-\mu_1(z)|\le\eta(y),$$

then there is a vanishing function $\epsilon(y)$ such that

$$|\rho_0(x,y,k)-\rho_1(x,y,k)|,\ \ |\rho_0(x,-y,k)-\rho_1(x,-y,k)|\le \epsilon(y)$$

for $x,y>0\in\mathbb{R}$ and $0<k\le 1$, where H_0 and H_1 are the restrictions of $\tilde{H}_0$ and $\tilde{H}_1$ to the real axis.

Proof. We take the following formulas as the definition of the Beurling-Ahlfors extension:

$$\widetilde{H}=U+iV,$$

where

$$U(x,y)=\frac{1}{2y}\int_{x-y}^{x+y}H(s)ds=\frac{1}{2}\int_{-1}^{1}H(x+ky)dk \tag{11}$$

and

$$V(x,y)=\frac{1}{y}\int_{x}^{x+y}H(s)ds-\frac{1}{y}\int_{x-y}^{x}H(s)ds. \tag{12}$$

In (11) and (12) we have chosen a normalization slightly different from the one given in [1]. It has the property that the extension of the identity is the identity and the extension is affinely natural, by which we mean that for affine maps A and B,

$$\widetilde{id_{\mathbb{R}}}=\ id_{\mathbb{C}}$$

and

$$\widetilde{A \circ H \circ B} = A \circ \widetilde{H} \circ B.$$

Note that

$$\int_0^1 \rho(x,y,k)dk = \frac{1}{H(x)-H(x-y)}\left(\frac{1}{y}\int_x^{x+y} H(s)ds - H(x)\right) \tag{13}$$

and

$$\int_0^1 \rho(x,-y,k)dk = \frac{1}{H(x+y)-H(x)}\left(H(x) - \frac{1}{y}\int_{x-y}^{x} H(s)ds\right). \tag{14}$$

Let

$$\begin{aligned} L \; &= H(x) - H(x-y) \\ R \; &= H(x+y) - H(x) \\ L' &= H(x) - \frac{1}{y}\int_{x-y}^{x} H(s)ds, \\ R' &= \frac{1}{y}\int_x^{x+y} H(s)ds - H(x). \end{aligned} \tag{15}$$

and let $\rho_+(x,y) = \displaystyle\int_0^1 \rho(x,y,k)dk$ and $\rho_-(x,y) = \displaystyle\int_0^1 \rho(x,-y,k)dk$. Let $\rho(x,y) = \rho_H(x,y)$. Then

$$\begin{aligned} \rho(x,y) \;\; &= R/L \\ \rho_+(x,y) &= R'/L \\ \rho_-(x,y) &= L'/R. \end{aligned} \tag{16}$$

Notice that for symmetric homeomorphisms the quantity ρ approaches 1 and the two quantities ρ_+ and ρ_- approach $1/2$ as y approaches zero. The complex dilatation of $\tilde{H}$ is given by

$$\mu(z) = \frac{K(z)-1}{K(z)+1}$$

where

$$\begin{aligned} K(z) &= \frac{\tilde{H}_z + \tilde{H}_{\overline{z}}}{\tilde{H}_z - \tilde{H}_{\overline{z}}} = \frac{(U+iV)_z + (U+iV)_{\overline{z}}}{(U+iV)_z - (U+iV)_{\overline{z}}} \\ &= \frac{(U+iV)_x - i(U+iV)_y + (U+iV)_x + i(U+iV)_y}{(U+iV)_x - i(U+iV)_y - (U+iV)_x - i(U+iV)_y} \\ &= \frac{U_x + iV_x}{V_y - iU_y}. \end{aligned}$$

Thus

$$K(z) = \frac{1+ia}{b-ic},$$

where $a = V_x/U_x, b = V_y/U_x$ and $c = U_y/U_x$.

To find estimates for these three ratios we must find expressions for the four partial derivatives of U and V in (11) and (12). In the notation of (15)

$$
\begin{aligned}
U_x &= \frac{1}{2y}(R+L), \\
V_x &= \frac{1}{y}(R-L), \\
V_y &= \frac{1}{y}(R+L) - \frac{1}{y}(R'+L'), \\
U_y &= \frac{1}{2y}(R-L) - \frac{1}{2y}(R'-L').
\end{aligned}
$$

Thus

$$
\begin{aligned}
a(1+\rho) &= 2\,\frac{R-L}{R+L}\cdot\frac{R+L}{L}, \\
b(1+\rho) &= 2\,\frac{R+L-R'-L'}{R+L}\cdot\frac{R+L}{L} = 2\left(R/L+1-R'/L-(R/L)(L'/R)\right), \\
c(1+\rho) &= \frac{R-L-R'+L'}{R+L}\cdot\frac{R+L}{L} = R/L-1-R'/L+(R/L)(L'/R).
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
a &= \frac{2(\rho-1)}{\rho+1}, \\
b &= \frac{2(\rho+1-\rho_+-\rho\rho_-)}{\rho+1}, \\
c &= \frac{\rho-1-\rho_++\rho\rho_-}{\rho+1}.
\end{aligned} \tag{17}
$$

Since $K(z) = (1+ia)/(b-ic)$, $K(z)+1 = (1+ia+b-ic)/(b-ic)$, we have

$$
\begin{aligned}
\mu_1(z)-\mu_0(z) &= \frac{K_1(z)-1}{K_1(z)+1} - \frac{K_0(z)-1}{K_0(z)+1} = 2\,\frac{K_1(z)-K_0(z)}{(K_1(z)+1)(K_0(z)+1)} \\
&= 2\,\frac{(1+ia_1)(b_0-ic_0)-(1+ia_0)(b_1-ic_1)}{(1+ia_1+b_1-ic_1)(1+ia_0+b_0-ic_0)} \\
&= 2\,\frac{(a_1-a_0)(ib_1+c_1)+(b_0-b_1)(1+ia_1)+(c_1-c_0)(i-a_1)}{(1+ia_1+b_1-ic_1)(1+ia_0+b_0-ic_0)}.
\end{aligned} \tag{18}
$$

From the equation for b in (17) and the inequalities $\rho_+ < \rho$ and $\rho\rho_- < 1$, we see that $b > 0$. Since this inequality is true for b_1 and for b_0, it follows that the denominator in (18) is greater than 1. These equations show that if a_0, b_0, c_0 converge to a_1, b_1, c_1, as y

approaches zero, then μ_0 approaches μ_1. Clearly ρ_0 approaches ρ_1 implies a_0 approaches a_1.

From the hypothesis

$$|\rho_0(x,y,k)-\rho_1(x,y,k)|,\ \ |\rho_0(x,-y,k)-\rho_1(x,-y,k)|\leq\epsilon(y)$$

for $x, y > 0 \in \mathbb{R}$ and $0 < k \leq 1$, we have that

$$|\rho_{1+}(x,y)-\rho_{0+}(x,y)|\leq\int_0^1|\rho_1(x,y,k)-\rho_0(x,y,k)|dk\leq\epsilon(y)$$

and

$$|\rho_{1-}(x,y)-\rho_{0-}(x,y)|\leq\int_0^1|\rho_1(x,-y,k)-\rho_0(x,-y,k)|dk\leq\epsilon(y).$$

This implies that b_0, c_0 converge to b_1, c_1, as y approaches zero. This completes the proof of the first half of the theorem.

Since the subsequent arguments do not require the second half, we only sketch the proof. Notice that if $\tilde{H}_0$ and $\tilde{H}_1$ are quasiconformal self-maps of the complex plane preserving the real axis with Beltrami coefficients μ_0 and μ_1 satisfying

$$|\mu_0(z)-\mu_1(z)|\leq\epsilon(y)$$

for a vanishing function $\epsilon(y)$, then the quasiconformal map $\tilde{H}_1\circ(\tilde{H}_0)^{-1}$ has Beltrami coefficient σ with

$$|\sigma(z)|\leq\epsilon'(y)$$

for another vanishing function $\epsilon'(y)$. Then $\tilde{H}_1\circ(\tilde{H}_0)^{-1}$ carries the extremal length problem for the family of curves joining $[-\infty, \tilde{H}_0(x-y)]$ to $[\tilde{H}_0(x), \tilde{H}_0(x+ky)]$ to the extremal length problem for the family of curves joining $[-\infty, \tilde{H}_1(x-y)]$ to $[\tilde{H}_1(x), \tilde{H}_1(x+ky)]$. If $\Lambda_0(x,y,k)$ and $\Lambda_1(x,y,k)$ are these two extremal lengths, then by the Grötzsch argument there is another vanishing function $\epsilon''(y)$ such that

$$\left|\log\frac{\Lambda_0(x,y,k)}{\Lambda_1(x,y,k)}\right|\leq\epsilon''(y).$$

In [2, pages 74-76] Ahlfors shows that if Λ is the extremal length of the curve family that joins the interval $[-\infty,-1]$ to $[0,m]$, Λ is an increasing real analytic function of m. In particular,

$$\left|\log\frac{m_0}{m_1}\right|<\epsilon \text{ if and only if } \left|\log\frac{\Lambda_0}{\Lambda_1}\right|<\epsilon' \tag{19}$$

and ϵ and ϵ' approach zero simultaneously.

Hence by (19) there is another vanishing function $\eta(y)$ such that

$$\left| \frac{H_0(x+ky) - H_0(x)}{H_0(x) - H_0(x-y)} - \frac{H_1(x+ky) - H_1(x)}{H_1(x) - H_1(x-y)} \right| \leq \eta(y).$$

Similarly, we have that

$$\left| \frac{H_0(x) - H_0(x-ky)}{H_0(x+y) - H_0(x)} - \frac{H_1(x) - H_1(x-ky)}{H_1(x+y) - H_1(x)} \right| \leq \eta(y).$$

This completes the proof of the second half of the theorem. □

§ 3. The UAA Teichmüller space

The endomorphism $p(z) = z^m$ of $\mathbb{S}^1$ is a degree m circle endomorphism and its lift via the covering mapping π is $P(x) = m\ x$. That is, $P(0) = 0$ and $\pi \circ P = p \circ \pi$. Obviously, p^n is UAA with constant $M = 1$. In fact, the restriction to the unit circle of the ratio of Blaschke products,

$$f(z) = \frac{\prod_{j=1}^{k+m} \dfrac{z - \alpha_j}{1 - \overline{\alpha_j} z}}{\prod_{j=1}^{k} \dfrac{z - \beta_j}{1 - \overline{\beta_j} z}},$$

for sufficiently small $|\alpha_j|$ and $|\beta_j|$, is also a degree m UAA circle endomorphism. The following theorem has been proved for any one-dimensional Markov maps with bounded geometry. A UAA map of the circle is a Markov map with bounded geometry. The reader may refer to [8, 9, 10] for a detailed proof. However, for the convenience of the reader, we outline a proof.

Theorem 3. *Given any degree m UAA circle endomorphism f, there exists a unique quasisymmetric map h such that $h \circ p \circ h^{-1} = f$, where $p(z) = z^m$.*

Proof. We begin by using the dynamics of the iterations of p and f to construct a self map H of $\mathbb{R}$ satisfying

i) $H(0) = 0$,

ii) $H \circ T = T \circ H$ and

iii) $H \circ P = F \circ H$.

From $H(0) = 0$ and $H \circ T^k(0) = T^k \circ H(0)$, we conclude that $H(k) = k$. Note that $F \circ T(0) = T^m \circ F(0)$ and $F(0) = 0$ implies that $F(1) = m$. Also, $F \circ T = T^m \circ F$ implies

$$F^n \circ T(0) = T^{m^n}(0)$$

and so $F^n(1) = m^n$. Since F is an increasing homeomorphism, $F(0) = 0$ and $F^n(1) = m^n$, we may select numbers $a_{j,n}$ between 0 and 1 such that $F^n(a_{j,n}) = j$ for integers j and n with $0 < j < m^n$. Then, by definition, if we put $H(j/m^n) = a_{j,n}$, we obtain

$$H \circ P^n(j/m^n) = H(j) = j \quad \text{and} \quad F^n \circ H(j/m^n) = F^n(a_{j.n}) = j.$$

This defines H on a dense set of the unit interval with the property that for points x in the dense set $H \circ P(x) = F \circ H(x)$. We extend H to a dense subset of the interval $[k-1, k]$ by requiring that $H \circ T^k = T^k \circ H$. If we put $x = j/m^n$ and $t = 1/m^n$, then we obtain

$$\frac{1}{M} \leq \frac{a_{j+1,n} - a_{j,n}}{a_{j,n} - a_{j-1,n}} \leq M.$$

Now let c be any number of the form j/m^n and t any positive number. Select k so that $1/m^k \leq t \leq 1/m^{k-1}$. Then

$$\frac{H(c+t) - H(c)}{H(c) - H(c-t)} \leq \frac{H(c + 1/m^{k-1}) - H(c)}{H(c) - H(c - 1/m^k)} \leq M',$$

where $M' = 1 + M + M^2 + \cdots + M^m$, which depends only on m and M. The same type of argument yields the lower bound

$$\frac{1}{M'} \leq \frac{H(c+t) - H(c)}{H(c) - H(c-t)}.$$

Since H is continuous and the set of points j/m^n for variable integers j and n is dense, we conclude that H is quasisymmetric. □

Definition 5. The Teichmüller space $\mathcal{T}(m)$ consists of all UAA circle endomorphisms of degree $m > 1$ factored by an equivalence relation. Two endomorpisms f_0 and f_1 representing elements of $\mathcal{T}(m)$ are equivalent if, and only if, there is a symmetric homeomorphism h of $\mathbb{S}^1$ such that $h \circ f_0 \circ h^{-1} = f_1$. Since the dynamics of the mappings T, P and F uniquely determine the points $a_{j,n}$, the mapping H is unique.

§ 4. UAC Endomorphisms

If f is a UAA circle endomorphism, it is possible that f has a reflection invariant extension $\tilde{f}$ defined in a small annulus $r < |z| < 1/r$ such that

$$\tilde{f}(1/\overline{z}) = 1/\overline{\tilde{f}(z)},$$

and such that for every $\epsilon > 0$ there exists a possibly smaller annulus $U = \{z : r' < |z| < 1/r'\}$ such that

$$K_z(\tilde{f}^{-n}) < 1 + \epsilon \tag{20}$$

for all z in U. Here $K_z(g)$ is the dilatation of g at z and inequality (20) is meant to hold for almost every z in U and for all positive integers n. If such an extension exists, then $\tilde{f}$ is called a uniformly asymptotically conformal (UAC) dynamical system.

Lemma 3. *If $\tilde{f}$ is a UAC degree m map defined in a neighborhood of $\mathbb{S}^1$, then the restriction f of $\tilde{f}$ to $\mathbb{S}^1$ is a degree m UAA circle endomorphism.*

Lemma 4. *For any degree m UAC map $\tilde{f}$ acting on a neighborhood of $\mathbb{S}^1$ with $\tilde{f}(1) = 1$, there is a unique lift $\tilde{F}$ to an infinite strip containing $\mathbb{R}$ and bounded by lines parallel to $\mathbb{R}$ such that*

1. *$\pi \circ \tilde{F} = \tilde{f} \circ \pi$,*
2. *$\tilde{F}(0) = 0$*
3. *$\tilde{F} \circ T = T^m \circ \tilde{F}$, and*
4. *$\tilde{F}$ preserves the real axis and $\tilde{F}(\overline{z}) = \overline{\tilde{F}(z)}$.*

Lemma 5. *In the notation of the previous lemma, if $\tilde{f}$ is UAC then $\tilde{F}$ is UAC in the sense that for every $\epsilon > 0$, there is a $\delta > 0$ such that if the absolute value of $y = Im\ z$ is less than δ, then*

$$K_z(\tilde{F}^{-n}) < 1 + \epsilon. \tag{21}$$

Conversely, if $\tilde{F}$ is UAC in the sense that (21) is satisfied and $\tilde{F}(T(z)) = T^m \circ \tilde{F}(z)$, then the induced map $\tilde{f}$ satisfying $\pi \circ \tilde{F} = \tilde{f} \circ \pi$ is UAC.

Theorem 4. *If f is a UAA circle endomorphism, then there exists a UAC map $\tilde{f}$ of a neighborhood of the circle such that the restriction of $\tilde{f}$ to the circle is equal to f.*

Proof. Let F be the lift to the real axis of f such that $F(0) = 0$, $F \circ T = T^m \circ F$ and such that $\pi \circ F = f \circ \pi$. By Theorem 3 there is a quasisymmetric homeomorphism H of $\mathbb{R}$ fixing 0 and 1 such that

i) $H \circ P \circ H^{-1} = F$ where $P(x) = m\ x$, and

ii) $H \circ T \circ H^{-1} = T$ where $T(x) = x + 1$.

By Lemma 5 it will suffice to find an extension $\tilde{F}$ of F such that

i) $\tilde{F} \circ T(z) = T^m \circ \tilde{F}(z)$

and

ii) the Beltrami coefficients $\mu_{\tilde{F}^{-n}}$ of $\tilde{F}^{-n}$ satisfy

$$|\mu_{\tilde{F}^{-n}}(x+iy)| \leq \epsilon(y)$$

where $\epsilon(y)$ is independent of n and x.

We define $\tilde{F}$ to be $\tilde{H} \circ P \circ \tilde{H}^{-1}$. Since $\tilde{H}$ extends H, clearly $\tilde{F}$ extends F.

Suppose that $\rho_1(x,y,k)$ and $\rho_2(x,y,k)$ are the skew quasisymmetric distortions of $F^{-n} \circ H$ and H. By Corollary 1, there is a vanishing function $\epsilon(y)$ such that

$$|\rho_1(x,y,k) - \rho_2(x,y,k)|, \ \ |\rho_1(x,-y,k) - \rho_2(x,-y,k)| \leq \epsilon(y)$$

for all real numbers x, all $y > 0$, all k with $0 < k \leq 1$ and all $n \geq 1$. Applying Theorem 2, there is another vanishing function $\eta(y)$ such that the Beltrami coefficients $\mu_{\widetilde{F^{-n} \circ H}}$ and $\mu_{\tilde{H}}$ satisfy

$$|\mu_{\widetilde{F^{-n} \circ H}}(z) - \mu_{\tilde{H}}(z)| \leq \eta(y), \quad \forall\, n > 0.$$

Since

$$\widetilde{F^{-n} \circ H} = \widetilde{H \circ P^{-n}} = \tilde{H} \circ P^{-n},$$

we conclude that

$$|\mu_{\tilde{H}}(m^{-n}z) - \mu_{\tilde{H}}(z)| \leq \eta(y).$$

Also, since the Beurling-Ahlfors extension is affinely natural, $\mu_{\tilde{H}}(T(z)) = \mu_{\tilde{H}}(z)$ and $\tilde{H} \circ T \circ \tilde{H}^{-1}(z) = T(z)$. We conclude that $\tilde{F} = \tilde{H} \circ M \circ \tilde{H}^{-1}$ is a uniformly asymptotically conformal circle endomorphism of degree m. $\square$

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