Sharp maximal inequalities and bilinear estimates

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1 Introduction

This article is based on an unpublished paper of A. Miyachi. His ideas are collected in [24]. See [32], for the detail of this report.

In this report we discuss the bilinear estimate of the form;

$$\|f\nabla^m g\|_X \le c\Big(\|f\|_X\|\nabla^m g\|_{BMO} + \|\nabla^m f\|_{BMO}\|g\|_X\Big) \tag{1}$$

with a nonnegative integer m and a quasi-Banach space X. In particular, we consider cases in which X is Herz space, $\dot{K}_q^{\alpha,p}$, or Morrey space, $\mathcal{M}_q^p$. We refer to Section 2, for the definitions of these spaces.

This estimate, with $m = 1$ and $X = L^p, (1 < p < \infty)$, was used in [14] to prove that the weak solution $u = (u_1, \cdots, u_n)$ of the Navier-Stokes equation, with divergence-free initial data belonging to L^2, on $(0, T)$ satisfying

$$\text{rot } u := (\partial_j u_k - \partial_k u_j)_{1 \le j,k \le n} \in L^1((0,T); BMO(\mathbb{R}^n))^{n \times n}$$

or

$$\text{Def } u := (\partial_j u_k + \partial_k u_j)_{1 \le j,k \le n} \in L^1((0,T); BMO(\mathbb{R}^n))^{n \times n}$$

fulfills the energy equality;

$$\|u(t)\|_{L^2}^2 + 2\int_s^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \|u(s)\|_{L^2}^2 \quad \text{for all } 0 \le s \le t < T.$$

By using (1) with $m = 1$ and $X = L^p$ and the energy equality, Kozono and Taniuchi obtained results on the continuation of the strong solution and the regularity criterion on the weak solution in the term of rot u and Def u. Their proof of (1) with $m = 1$ and $X = L^p$ is based on the boundedness of the bilinear Fourier multipliers due to Coifman-Meyer [6]. There are other

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bilinear estimates for fg in several function spaces by using the paraproduct initiated by Bony, [5], [12], [13], etc...

Our purpose is to extend Kozono-Taniuchi's bilinear estimate, i.e. (1) from $m = 1$ and $X = L^p$ $(1 < p < \infty)$ to various function spaces with general order m, by following the ideas of Miyachi, i.e. by using the sharp maximal function which was studied by DeVore-Sharpley in [7]. We introduce the notations before we define the sharp maximal function. By a " cube " Q, we mean a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Its side length and center will be denoted by $l(Q)$ and $c(Q)$, respectively. Also, for $a > 0$, aQ means the cube with the same center as Q whose side length is a times that of Q. Let $\mathcal{Q}$ be a set of all dyadic cubes. For a measurable set E, the slashed integral $\fint_E f dx$ denotes the average $f_E = \dfrac{1}{|E|} \int_E f dx$, where $|E|$ is the Lebesgue measure of E. For $0 < p < \infty$, we write $p_+ = \max(p, 1)$. Let $[s]$ denote the integer part of real number of s. The sharp maximal function $f_{l,\tau}^{\sharp(r)}$ is defined as follows; for $0 < r < \infty, l \in \mathbb{N}_{-1} := \mathbb{N} \cup \{0\} \cup \{-1\}, 0 \le \tau < \infty$ and an open set $\Omega \subset \mathbb{R}^n$,

$$f_{l,\tau}^{\sharp(r),\Omega}(x) := \sup_{x \in Q \subset \Omega} \inf_{P \in \mathbf{P}_l} |Q|^{-\tau/n} \left(\fint_Q |f(y) - P(y)|^r dy \right)^{1/r}$$

where the supremum is taken over all cubes Q containing x and included in Ω, and $\mathbf{P}_l$ is the space of all polynomials of degree at most l for $l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbf{P}_{-1} = \{0\}$. Also, we write $f_{l,\tau}^{\sharp(r)} = f_{l,\tau}^{\sharp(r),\mathbb{R}^n}$. This type maximal function was firstly introduced in [4]. See also [2], [16], [20] and [29], for several variants of the sharp maximal function. It is clear that for $r \le 1$ and $l \in \mathbb{N}_{-1}$, $f_{l,0}^{\sharp(r)} \le f_{l,0}^{\sharp(1)} \le f_{0,0}^{\sharp(1)} =: f^\sharp$ where $f^\sharp$ is the Fefferman-Stein sharp maximal function, [8]. Note that $\|f\|_{BMO} := \|f^\sharp\|_{L^\infty}$ and BMO strictly includes L^∞. Our main tools are the sharp maximal inequality with $f_{l,\tau}^{\sharp(r)}$ and the pointwise estimate of the sharp maximal function of $f \nabla^m g$. The sharp maximal inequality is an inequality of the form;

$$\|f\|_X \le c \|f^\sharp\|_X. \tag{2}$$

See [8], [22], for this inequality in the case $X = L^p$. We extend (2) in Section 3. The sharp maximal function $f^\sharp$ is useful for pointwise estimates of several operators, for example, Calderón-Zygmund operators, pseudodifferential operators and commutators. In particular, for a Calderón-Zygmund operator T the pointwise estimate

$$(Tf)^\sharp(x) \le c M_r f(x)$$

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holds, where $M_r f = M(|f|^r)^{1/r}$ and M is the Hardy-Littlewood maximal operator, and the sharp maximal inequality (2) with $X = L^p$ lead us directly to the L^p-boundedness of T without the weak type estimate and interpolation theory. See [1], [15] and [23], for applications of the sharp maximal function to the theory of partial differential equations.

We explain the structure of this report. In Section 2, we define the two function spaces and their Hardy type variants. These four spaces are used as X in (1) in Section 4. In Section 3, we recall some properties of the general sharp maximal function $f_{l,\tau}^{\sharp(r)}$ and state the sharp maximal inequality with $f_{l,\tau}^{\sharp(r)}$ which is stronger than (2), and pointwise estimate of the sharp maximal function of $f\nabla^m g$. In Section 4, we apply the results in Section 3 to bilinear estimates for $f\nabla^m g$ by following Miyachi's idea. The main result in Section 4 are more general than what we discussed in the actual seminar delivered in the conference, because we can take the exponent $q \leq 1$ which appears in $\dot{K}_q^{\alpha,p}$ and $\mathcal{M}_q^p$. It seems that the parameter q in $\dot{K}_q^{\alpha,p}$ and $\mathcal{M}_q^p$ describes local regularity.

2 Function spaces

In this section, we introduce the function spaces which extend L^p. Moreover, we define the Hardy spaces for these spaces which are used as substitutes for these spaces in the case $q \leq 1$ in our bilinear estimates. We begin to recall the definition of Herz space and Morrey space.

Definition 2.1. Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. One defines the (homogeneous) Herz space $\dot{K}_q^{\alpha,p}$ as

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) := \{f \in L_{loc}^q(\mathbb{R}^n\backslash\{0\}); \|f\|_{\dot{K}_q^{\alpha,p}} := \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|f\|_{L^q(A_k)}^p\Big)^{1/p} < \infty\},$$

where $A_k := (-2^k, 2^k)^n\backslash(-2^{k-1}, 2^{k-2})^n$.

Definition 2.2. Let $0 < q \leq p < \infty$. One defines the Morrey space $\mathcal{M}_q^p(\mathbb{R}^n)$ as

$$\mathcal{M}_q^p(\mathbb{R}^n) := \{f \in L_{loc}^q; \|f\|_{\mathcal{M}_q^p} := \sup_Q |Q|^{1/p}\Big(\fint_Q |f|^q dy\Big)^{1/q} < \infty\},$$

where the supremum is taken over all cubes.

A geometric observation shows $\|f\|_{\mathcal{M}_q^p} \approx \sup_{Q \in \mathcal{Q}} |Q|^{1/p} \left(\fint_Q |f|^q dy \right)^{1/q}$.

The two function spaces above extends Lebesgue spaces; for all $0 < p < \infty$,

$$\dot{K}_p^{0,p} = L^p = \mathcal{M}_p^p.$$

Furthermore, Herz spaces include also the Lebesgue spaces with power weight, i.e.

$$\dot{K}_p^{\alpha,p} = L^p(|x|^{\alpha p} dx).$$

Note that, for the nonhomogeneous Herz space, we have

$$K_p^{\alpha,p} = L^p(\langle x \rangle^{\alpha p} dx)$$

where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

Morrey spaces have the following inclusion property;

$$\mathcal{M}_p^p = L^p \subset L^{p,\infty} \subset \mathcal{M}_{q_1}^p \subset \mathcal{M}_{q_2}^p \quad \text{for } 0 < q_2 \leq q_1 < p < \infty.$$

Here, $L^{p,\infty}$ is the Lorentz space which is defined as the set of all measurable functions f such that

$$\|f\|_{L^{p,\infty}} := \sup_{\lambda > 0} \lambda |\{x \in \mathbb{R}^n; |f(x)| > \lambda\}|^{1/p} < \infty.$$

For the sake of convenience for readers, we supply the proof of $L^{p,\infty} \subset \mathcal{M}_{q_1}^p$; for any cube Q,

$$|Q|^{1/p} \left(\fint_Q |f|^{q_1} dx \right)^{1/q_1}$$

$$= |Q|^{1/p + 1/q_1} \left(\int_0^\infty q_1 \lambda^{q_1 - 1} |\{x \in Q; |f(x)| > \lambda\}| d\lambda \right)^{1/q_1}$$

$$\leq |Q|^{1/p + 1/q_1} \left(\int_0^\infty q_1 \lambda^{q_1 - 1} \min(|Q|, \|f\|_{L^{p,\infty}}^p \lambda^p) d\lambda \right)^{1/q_1}$$

$$\leq |Q|^{1/p + 1/q_1} \left(\int_0^A q_1 \lambda^{q_1 - 1} |Q| d\lambda \right.$$

$$\left. + \int_A^\infty q_1 \lambda^{q_1 - p - 1} \|f\|_{L^{p,\infty}}^p d\lambda \right)^{1/q_1} \quad (A := |Q|^{-1/p} \|f\|_{L^{p,\infty}})$$

$$\leq c \|f\|_{L^{p,\infty}}.$$

Many authors studied the mapping properties of several operators on Herz spaces and Morrey spaces, see [3], [17], etc... For example, the boundedness

of the powered Hardy-Littlewood maximal operator; for $0 < r < \infty$ and open subset $\Omega \subset \mathbb{R}^n$

$$M_r^\Omega f(x) := \sup_{x \in Q \subset \Omega} \left(\fint_Q |f|^r dy \right)^{1/r}, \ for \ 0 < r < \infty$$

is well-known. Note that $M_r = M_r^{\mathbb{R}^n}$ and $M = M_1^{\mathbb{R}^n}$.

We invoke the following results on boundedness of the maximal operator M.

Proposition 2.1 ([17]). *M is a bounded operator on $\dot{K}_q^{\alpha,p}$ if $0 < p \leq \infty$, $1 < q \leq \infty$ and $-n/q < \alpha < n(1 - 1/q)$.*

Proposition 2.2 ([3]). *M is a bounded operator on $\mathcal{M}_q^p$ if $1 < q \leq p < \infty$.*

To pass from these function spaces to the Hardy-type variants, we fix a test function $\phi \in C_0^\infty$ which is supported in the unit ball $B(0,1)$ and whose integral is not zero. For a distribution $f \in \mathscr{D}'$ we define the radial maximal function $\phi_+(f)$ by

$$\phi_+(f)(x) = \sup_{0 < t < \infty} |\langle f, \phi_t(x - \cdot) \rangle|$$

where $\phi_t(x) = t^{-n}\phi(x/t)$.

Definition 2.3 ([17], [18], [19]). *Let p and q be the same as in Definition 2.2 and $-n/q < \alpha < \infty$. One defines the Herz-type Hardy space $H\dot{K}_q^{\alpha,p}$ as*

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) := \{f \in \mathscr{D}'; \|f\|_{H\dot{K}_q^{\alpha,p}} := \|\phi_+(f)\|_{\dot{K}_q^{\alpha,p}}\}.$$

Definition 2.4 ([10], [25]). *Let p and q be the same as in Definition 2.3. One defines the Hardy-Morrey space $H\mathcal{M}_q^p$ as*

$$H\mathcal{M}_q^p(\mathbb{R}^n) := \{f \in \mathscr{D}'; \|f\|_{H\mathcal{M}_q^p} := \|\phi_+(f)\|_{\mathcal{M}_q^p}\}.$$

A helpful remark on the properties of the function spaces may be in order.

Remark 2.1. *1. $H\dot{K}_q^{\alpha,p}$ and $H\mathcal{M}_q^p$ are independent of the choice of $\phi \in C_0^\infty(B(0,1))$ with $\int \phi \, dx \neq 0$.*

2. If $0 < p \leq \infty, 1 < q \leq \infty, -n/q < \alpha < n(1 - 1/q)$, then $H\dot{K}_q^{\alpha,p} = \dot{K}_q^{\alpha,p}$. Similarly, if $1 < q \leq p < \infty$, then $H\mathcal{M}_q^p = \mathcal{M}_q^p$. These facts are consequences of Propositions 2.1 and 2.2.

3. If $0 < p < \infty$, then $H\dot{K}_p^{0,p} = H^p = H\mathcal{M}_p^p$, where H^p is the usual Hardy space, [8]. In paticular, if $1 < p$, then $H\dot{K}_p^{0,p} = L^p = H\mathcal{M}_p^p$

See [10], [18], [19], [21], [31], etc... for these Hardy-type spaces.

3   Properties of the sharp maximal function and main tools

Feffereman-Stein's inequality (2) asserts that the magnitude of $f^\sharp$ can control that of f. The general sharp maximal function $f_{l,\tau}^{\sharp(r)}$ informs us of the smoothness of function f and can dominate the magnitude of the derivative of f. The following properties are known. In paticular, (iii) was shown by Miyachi in the case $m = 1$ and plays an important role in Section 4.

Proposition 3.1. (i), [[7, Theorem 5.6]]; *For $1 < p \le \infty$, $m \in \mathbb{N}$ and $f \in W^{m,p}$,*
$$c^{-1}|\nabla^m f| \le f_{m-1,m}^{\sharp(1)} \le cM(|\nabla^m f|), \ \ a.e. \ x.$$

(ii), [[7, Theorem 2.5]]; *If f be a locally integrable function and $0 < \tau \le 1$, then*
$$|f(x) - f(y)| \le c|x - y|^\tau (f_{0,\tau}^{\sharp(1)}(x) + f_{0,\tau}^{\sharp(1)}(y)), \ \ a.e. \ x, y.$$

(iii); *Let f be a locally integrable function and $m \in \mathbb{N}$. If $\partial^\alpha f \in L_{loc}^1$ for all $|\alpha| = m$, then*
$$f_{m,m}^{\sharp(1)} \le c \sum_{|\alpha|=m} (\partial^\alpha f)_{0,0}^{\sharp(1)}.$$

For the sake of convenience for reader, we give the proof of (ii) above. Let x and y be Lebesgue points of f and $h := y - x$. Let R the smallest cube containing x and $y = x + h$. It is well known that
$$\lim_{Q \to \{x+jh\}} f_Q = f(x + jh), \ \ j = 0, 1.$$

For a cube $Q^j \subset R$ containing $x + jh$, we take a family of cubes $\{Q_k^j\}_{k=1}^{m+1}$ such that
$$Q^j = Q_1^j \subset Q_2^j \subset \cdots \subset Q_{m+1}^j = R,$$
$|Q_{k+1}^j| = 2^n |Q_k^j|, (k = 1, \cdots, m-1)$ and $|Q_{m+1}^j| = |R| \le 2^n |Q_m^j|$. Then, we have

$$|f_{Q^j} - f_R| \le \sum_{k=1}^m |f_{Q_k^j} - f_{Q_{k+1}^j}| \le \sum_{k=1}^m \fint_{Q_k^j} |f - f_{Q_{k+1}^j}| dy$$

$$\le 2^n \sum_{k=1}^m \fint_{Q_{k+1}^j} |f - f_{Q_{k+1}^j}| dy \le 2^n f_{0,\tau}^{\sharp(1)}(x + jh) \sum_{k=1}^m |Q_{k+1}^j|^{\tau/n}$$

$$\le c|R|^{\tau/n} f_{0,\tau}^{\sharp(1)}(x + jh)$$

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from which one obtains

$$|f(x+jh) - f_R| \leq \lim_{Q^j \to \{x+jh\}} |f_{Q^j} - f_R| \leq c|R|^{\tau/n} f_{0,\tau}^{\sharp(1)}(x+jh).$$

Therefore, we have the desired inequality

$$|f(x) - f(x+h)| \leq |f(x) - f_R| + |f(x+h) - f_R|$$
$$\leq c|R|^{\tau/n}(f_{0,\tau}^{\sharp(1)}(x) + f_{0,\tau}^{\sharp(1)}(x+h)).$$

Remark 3.1. *1.* (i) *and the boundedness of M on L^p imply that $\|\nabla^m f\|_{L^p} \approx \|f_{m-1,m}^{\sharp(1)}\|_{L^p}$.*
2. Thanks to (ii)*, it is not hard to see that $f \in Lip_\tau$ is equivalent to $f_{0,\tau}^{\sharp(1)} \in L^\infty$.*

The function spaces which are defined by $f_{l,\tau}^{\sharp(r)}$ have the some embedding relation to the homogeneous Besov spaces $\dot{B}_{p,q}^s$. Also, in [27] and [30], it was proved that the function spaces, which are defined by the more general function than $f_{l,\tau}^{\sharp(r)}$, are equivalent to the homogeneous Triebel-Lizorkin spaces $\dot{F}_{p,q}^s$.

Proposition 3.2 ([7, Theorem 7.1]). *Let $1 \leq p \leq \infty$ and $s > 0$. Then we have the embedding inequalities;*

$$c^{-1}\|f\|_{\dot{B}_{p,\infty}^s} \leq \|f_{[s],s}^{\sharp(1)}\|_{L^p} \leq c\|f\|_{\dot{B}_{p,p}^s}.$$

Next, we state the sharp maximal inequality with $f_{l,\tau}^{\sharp(r)}$ which is the one of main tools. We omit the proofs of theorems whose key is the good-λ inequality.

Theorem 3.1. *Let $0 < p \leq \infty$, $0 < q, r, s < \infty$, $-n/q < \alpha < \infty$ and $l \in \mathbb{N}_0$. Then, there exists a constant c such that for $f \in L_{loc}^r$ satisfying $\left(\fint_{A_j} |f|^r dy\right)^{1/r} \to 0$ as $j \to \infty$,*

$$\|f\|_{\dot{K}_q^{\alpha,p}} \leq c\|f_{l,0}^{\sharp(r)}\|_{\dot{K}_q^{\alpha,p}}.$$

Theorem 3.2. *Let $0 < q \leq p < \infty$, $0 < r, s < \infty$ and $l \in \mathbb{N}_0$. Then, there exists a constant c such that for $f \in L_{loc}^r$ satisfying that $\left(\fint_{2^j Q} |f|^r dy\right)^{1/r} \to 0$ as $j \to \infty$, for some cube Q,*

$$\|f\|_{\mathcal{M}_q^p} \leq c\|f_{l,0}^{\sharp(r)}\|_{\mathcal{M}_q^p}.$$

Theorems above are immediate consequences of the following lemmas.

Lemma 3.1 (Miyachi, [24], [32]). *Let $0 < r < \infty$ and $l \in \mathbb{N}_0$. Then, there exist $B = B(n, r, l) > 1$, $c = c(n, r, l) > 0$ so that for each $Q \in \mathcal{Q}$, $f \in L^r_{loc}$, $\lambda > \left(\fint_Q |f|^r dy\right)^{1/r}$ and $0 < \delta \leq 1$*

$$\left|\{x \in Q; \widetilde{M}^Q_r f(x) > B\lambda, \ \widetilde{f}^{\sharp(r),Q}_{l,0} \leq \delta\lambda\}\right| \leq c\left(\frac{\delta}{B}\right)^r \left|\{x \in Q; \widetilde{M}^Q_r f(x) > \lambda\}\right|,$$

where $\widetilde{M}^Q_r f$ and $\widetilde{f}^{\sharp(r),Q}_{l,0}$ mean the dyadic maximal function and the dyadic sharp maximal function of f, respectively.

Lemma 3.2 ([32]). *Let $0 < p \leq \infty$, $0 < q, r < \infty$, $-n/q < \alpha < \infty$ and $l \in \mathbb{N}_0$. Then, there exists a constant c so that for $f \in L^r_{loc}$ satisfying that $\left(\fint_{A_j} |f|^r dy\right)^{1/r} \to 0$ as $j \to \infty$,*

$$\|f\|_{\dot{K}^{\alpha_0,p}_r} \leq c\|f^{\sharp(r)}_{l,0}\|_{\dot{K}^{\alpha,p}_q},$$

where $\alpha_0 = \alpha + n(1/q - 1/r)$.

Lemma 3.3 ([26], [32]). *Let $0 < q \leq p < \infty$, $0 < r \leq p$ and $l \in \mathbb{N}_0$. Then, there exists a constant c so that for $f \in L^r_{loc}$ satisfying $\left(\fint_{2^j Q} |f|^r dy\right)^{1/r} \to 0$ as $j \to \infty$, for some cube Q,*

$$\|f\|_{\mathcal{M}^p_r} \leq c\|f^{\sharp(r)}_{l,0}\|_{\mathcal{M}^p_q}.$$

For example, because, by using Lemma 3.1, we can prove

$$\|f\|_{\dot{K}^{\alpha,p}_q} \leq c\left(\|f^{\sharp(r)}_{l,0}\|_{\dot{K}^{\alpha,p}_q} + \|f\|_{\dot{K}^{\alpha+n(1/q-1/r),p}_r}\right),$$

Theorem 3.1 follows from this inequality and Lemma 3.2. See [32], for the detail.

Remark 3.2. *1. Theorem 3.1 still holds for the non-homogeneous Herz spaces $K^{\alpha,p}_q$.*

2. In [11], Komori showed the inequality (ii) *in the context of the nonhomogeneous Herz space with $\alpha = -n/q$ in the following sense; for $1 < q < \infty$ and $f \in L^q_{loc}$,*

$$\|f\|_{CMO^q} \leq c\|f^{\sharp(1)}_{0,0}\|_{K^{-n/q,\infty}_q},$$

where CMO^q is the space of all functions of central mean oscillation and equipped with the norm

$$\|f\|_{CMO^q} := \sup_{R>1}\Big(\fint_{B(0,R)}|f - f_{B(0,R)}|^q dy\Big)^{1/q}.$$

3. Since we can take $r \leq 1$ in Theorems 3.1 and 3.2, these theorems are stronger than classical Fefferman-Stein's inequality (2).

4. Theorems 3.1 and 3.2 in the case $\dot{K}_p^{0,p} = \mathcal{M}_p^p = L^p$ were proved by Miyachi in his unpublished paper. Also, in the case $1 < q \leq p < \infty, r = 1$ and $l = 0$, that inequality in Theorem 3.2 was shown by Sawano-Tanaka [26] in the context of non-doubling measures.

5. The decay condition in Theorem 3.2 was firstly introduced by Fujii in [9].

To formulate our new sharp maximal operator control, we define the grand maximal function f_k^*.

Definition 3.1. Let $k \in \mathbb{N}_0$. For $x \in \mathbb{R}^n$ and $0 < t < \infty$, we define $\mathcal{T}_k(x,t)$ as the set of all functions $\phi \in C_0^\infty$ so that supp $\phi \subset B(x,t)$ and $\|\partial^\alpha \phi\|_{L^\infty} \leq t^{-n-|\alpha|}$ for $|\alpha| \leq k$. For $f \in \mathscr{D}'$, we set

$$f_k^*(x) = \sup\{|\langle f, \phi\rangle|; \phi \in \cup_{0<t<\infty}\mathcal{T}_k(x,t)\}.$$

Using above notation, we can prove pointwise estimates of the sharp maximal function of $f\nabla^m g$. See [24] or [32], for the proof of the proposition below.

Proposition 3.3. Let $0 < r < s < \infty$, $1/r = 1/u + 1/v$, $m \in \mathbb{N}$ and β a multi-index with $|\beta| = m$. Then, there exists a constant c such that the following (i) and (ii) hold.

(i): If $f \in L_{loc}^u$ and $g \in L_{loc}^v$, then

$$(fg)_{0,0}^{\sharp(r)}(x) \leq c\Big(M_u f(x)g_{0,0}^{\sharp(v)}(x) + f_{0,0}^{\sharp(u)}(x)M_v g(x)\Big).$$

(ii): If $f \in L_{loc}^u$ and $g \in \mathscr{D}'$ with $\partial^\beta g \in L_{loc}^{v+}$, then for $k \gg 1$,

$$(f\partial^\beta g)_{m,0}^{\sharp(r)}(x) \leq c\Big(M_u f(x)(\partial^\beta g)_{0,0}^{\sharp(v+)}(x) + f_{m,m}^{\sharp(r)}(x)g_k^*(x)\Big).$$

Remark 3.3. In the case $|\beta| = 1$, Miyachi proved the following pointwise estimate in his unpublished paper;

$$(f\partial^\beta g)_{1,0}^{\sharp(r)}(x) \leq c\Big(M_s f(x)(\partial^\beta g)_{0,0}^{\sharp(v+)}(x) + f_{1,1}^{\sharp(r)}(x)Mg(x)\Big).$$

This is slightly weaker than (ii), because we have $g_k^* \leq cMg$.

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4 Application to bilinear estimates

We shall prove our bilinear estimates below by using Theorems 3.1 and 3.2 and Proposition 3.3.

Theorem 4.1. (I): *Let* $0 < p \le \infty$, $0 < q < \infty$ *and* $-n/q < \alpha < \infty$.
(i): *There exists a constant* c *such that for any* $f, g \in \dot{K}_q^{\alpha,p} \cap BMO$,

$$\|fg\|_{\dot{K}_q^{\alpha,p}} \le c\Big(\|f\|_{\dot{K}_q^{\alpha,p}}\|g\|_{BMO} + \|f\|_{BMO}\|g\|_{\dot{K}_q^{\alpha,p}}\Big).$$

(ii): *Let* $m \in \mathbb{N}$. *There exists a constant* c *such that for any* $f \in \dot{K}_q^{\alpha,p}$ *with* $\nabla^m f \in BMO$ *and* $g \in H\dot{K}_q^{\alpha,p}$ *with* $\nabla^m g \in BMO$,

$$\|f\nabla^m g\|_{\dot{K}_q^{\alpha,p}} \le c\Big(\|f\|_{\dot{K}_q^{\alpha,p}}\|\nabla^m g\|_{BMO} + \|\nabla^m f\|_{BMO}\|g\|_{H\dot{K}_q^{\alpha,p}}\Big).$$

(II): *Let* $0 < q \le p < \infty$.
(i): *There exists a constant* c *such that for any* $f, g \in \mathcal{M}_q^p \cap BMO$,

$$\|fg\|_{\mathcal{M}_q^p} \le c\Big(\|f\|_{\mathcal{M}_q^p}\|g\|_{BMO} + \|f\|_{BMO}\|g\|_{\mathcal{M}_q^p}\Big).$$

(ii): *Let* $m \in \mathbb{N}$. *There exists a constant* c *such that for any* $f \in \mathcal{M}_q^p$ *with* $\nabla^m f \in BMO$ *and* $g \in H\mathcal{M}_q^p$ *with* $\nabla^m g \in BMO$,

$$\|f\nabla^m g\|_{\mathcal{M}_q^p} \le c\Big(\|f\|_{\mathcal{M}_q^p}\|\nabla^m g\|_{BMO} + \|\nabla^m f\|_{BMO}\|g\|_{H\mathcal{M}_q^p}\Big).$$

Finally we make a remark on our main results.

Remark 4.1. *1. It is easy to see that the inequality*

$$\|f\nabla^m g\|_{\dot{K}_q^{\alpha,p}} \le c\Big(\|f\|_{BMO}\|\nabla^m g\|_{\dot{K}_q^{\alpha,p}} + \|\nabla^m f\|_{H\dot{K}_q^{\alpha,p}}\|g\|_{BMO}\Big).$$

dose not generally hold for $f \in BMO$ *with* $\nabla^m f \in \dot{K}_q^{\alpha,p}$ *and* $g \in BMO$ *with* $\nabla^m g \in H\dot{K}_q^{\alpha,p}$. *As the example* $f \equiv 1$ *shows that the inequality above fails.*
2. The following local substitute can be shown; for $0 < q, r < \infty$, *each cube* Q *and* $f, g \in L^q(Q)$ *with* $\nabla^m f, \nabla^m g \in BMO(Q)$,

$$\|f\nabla^m g\|_{L^q(Q)} \le c\Big(\|f\|_{L^q(Q)}\|\nabla^m g\|_{BMO(Q)} + \|\nabla^m f\|_{BMO(Q)}\|g\|_{L^q(Q)}$$
$$+ |Q|^{1/q-1/r}\|f\nabla^m g\|_{L^r(Q)}\Big)$$

with a constant c *not depending on* f, g *and* Q. *Needless to say, this substitute inequality is significant only when* $0 < r \le q < \infty$.

Proof. Because the proof of (II) is similar to that of (I), we omit the proof of (II).

(I-i): Let $r < q/2$. Since

$$\left(\fint_{A_j} |fg|^r dy \right)^{1/r} \leq 2^{-2j(\alpha+n/q)} \|f\|_{\dot{K}_q^{\alpha,p}} \|g\|_{\dot{K}_q^{\alpha,p}} \to 0, \, as \, j \to \infty,$$

we can use Theorem 3.1 and hence, combining Proposition 2.1 with (i) of Proposition 3.3, we have the required inequality.

(I-ii): Since $\nabla^m g \in BMO$, as we can find in [[28] pp.141];

$$\fint_{B(0,2^k)} |h - h_{B(0,1)}| dx \leq ck\|h\|_{BMO}, \, for \, all \, k > 0,$$

we have $\fint_{A_j} |\nabla^m g| dy = \mathcal{O}(j)$ as $j \to \infty$, from which it follows that for any $\varepsilon > 0$, $2^{-\varepsilon j} \fint_{A_j} |\nabla^m g| dy \to 0$ as $j \to \infty$. Consequently, one obtains that

$$\left(\fint_{A_j} |f\nabla^m g|^r dy \right)^{1/r} \to 0$$

as $j \to \infty$ for all $0 < r < \min(1, q/(q+1))$. Indeed, if $1/r = 1/q + 1$, then we have

$$\left(\fint_{A_j} |f\nabla^m g|^r dy \right)^{1/r} \leq 2^{-jn/q} \|f\|_{L^q(A_j)} \fint_{A_j} |\nabla^m g| dx$$

$$\leq 2^{-j(\alpha+n/q)} \|f\|_{\dot{K}_q^{\alpha,p}} \fint_{A_j} |\nabla^m g| dx \to 0, \, as \, j \to \infty.$$

Here we have used the condition $-n/q < \alpha$. Hence, with the decay condition established, we are in the position of applying our sharp maximal inequality for $f\nabla^m g$ to obtain;

$$\|f\nabla^m g\|_{\dot{K}_q^{\alpha,p}} \leq c\|(f\nabla^m g)_{m,0}^{\sharp(r)}\|_{\dot{K}_q^{\alpha,p}}$$

with sufficiently small r. Combining the above inequality with (iii) of Proposition 3.1 and (ii) of Proposition 3.3 lead us to the following estimate; for sufficiently large k,

$$\|f\nabla^m g\|_{\dot{K}_q^{\alpha,p}} \leq c\left(\|f\|_{\dot{K}_q^{\alpha,p}} \|\nabla^m g\|_{BMO} + \|\nabla^m f\|_{BMO} \|g_k^*\|_{\dot{K}_q^{\alpha,p}} \right).$$

Finaly, the estimate $\|g_k^*\|_{\dot{K}_q^{\alpha,p}} \leq c\|g\|_{H\dot{K}_q^{\alpha,p}}$ follows from Uchiyama's pointwise estimate in [33];

$$g_k^*(x) \leq cM_{n/(n+k)}(\phi_+(g))(x).$$

$\square$

Remark 4.2. *Thanks to Miyachi's idea, we can establish the same bilinear estiomates as Theorem 4.1 for function spaces X with a quasi-norm $\|\cdot\|_X$ satisfying*

($A1$). the sharp maximal inequality holds

($A2$). the Hardy-Littlewood maximal operator M_r is bounded for some
$r \in (0, \infty)$

($A3$). quasi-norm $\|\cdot\|_X$ fulfills that if $|f| \leq |g|$ *a.e.* then $\|f\|_X \leq \|g\|_X$.

For example, some Orlicz spaces satisfy these properties.

References

[1] H. Aimar, I. Gómez and B. Iaffei, Parabolic mean values and maximal estimates for gradients of temperatures, J. Funct. Anal. **255**, 1939-1956, (2008).

[2] B. Bojarski, Sharp maximal operator of fractional order and Sobolev imbedding inequalities, Bull. Polish Acad. Sci. Math. **33**, (1985), 7-16.

[3] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. **7**, 273-279, (1987).

[4] A. P. Calderón and R. Scott, Sobolev type inequalities for $p > 0$, Studia Math. **62**, 75-92, (1978).

[5] F. M. Christ and M. I. Weinstein, Dispersion of small amplitude solutions of the generalized Kortewg-de Vries equation, J. Func. Anal. **100**, 87-107, (1991).

[6] R. R. Coifman and Y. Meyer, Au-delá des opérateurs pseudo-différentiels, Astérisque **57**, (1978).

[7] R. A. DeVore and R. C. Sharpley, Maximal functions measuring smoothness, Mem. Amer. Math. Soc. **47**, (1984).

[8] C. Fefferman and E. Stein, H^p spaces of several variables, Acta Math. **129**, 137-193, (1972).

[9] N. Fujii, A proof of the Fefferman-Stein-Strömberg inequality for the sharp maximal functions, Proc. Amer. Math. Soc. **106**, 371-377, (1989).

[10] J. Houyu and W. Henggeng, Decomposition of Hardy-Morrey spaces, J. Math. Anal. Appl. **354**, 99-110, (2009).

[11] Y. Komori, Notes on interpolation theorem between B^p and CMO^p, Sci. Math. Jpn. **60**, 107–111, (2004).

[12] H. Kozono, T. Ogawa and Y. Taniuchi, Navier-Stokes equations in the Besov space near L^∞ and BMO, Kyusyu J. Math. **57**, 303-324, (2003).

[13] H. Kozono and Y. Shimada, Bilinear estimates in homogeneous Triebel-Lizorkin spaces and the Navier-Stokes equations, Math. Nachr. **276**, 63-74, (2004).

[14] H. Kozono and Y. Taniuchi, Bilinear estimates in BMO and the Navier-Stokes equations, Math. Z. **235**, 173-194, (2000).

[15] N. V. Krylov, Parabolic and elliptic equations with VMO coefficients, Comm. Partial Differential Equations. **32**, 453–475, (2007).

[16] A. K. Lerner, Weighted norm inequalities for the local sharp maximal function, J. Fourier Anal. Appl. **10**, 465-474, (2004).

[17] X. Li and D. Yang, Boundedness of some sublinear operators on Herz spaces, Illinois J. Math. **40**, 484-501, (1996).

[18] S. Lu, K. Yabuta and D. Yang, Boundedness of some sublinear operators in weighted Herz-type spaces, Kodai Math. J. **23**, 391-410, (2000).

[19] S. Lu and D. Yang, The weighted Herz-type Hardy spaces and its applications, Science in China (Ser. A), **38**, 662-673, (1995).

[20] J. Martell, Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications, Studia Math. **161**, 113-145, (2004).

[21] A. Miyachi, Remarks on Herz-type Hardy spaces, Acta Math. Sinica. English Series. **17**, 339-360, (2001).

[22] A. Miyachi and K. Yabuta, On good λ-inequalities, Bull. Fac. Sci. Ibaraki Univ. Ser. A. **16**, 1-11, (1984).

[23] K. Rogers and A. Seeger, Endpoint maximal and smoothing estimates for Schrödinger equations, J. Reine Angew. Math. to appear.

[24] T. Sato, Master thesis, (Japanese), Tohoku University. (2004).

[25] Y. Sawano, A Note on Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, Acta Math. Sinica. English Series. **25**, 1223-1242, (2009).

[26] Y. Sawano and H. Tanaka, Sharp maximal inequalities and commutators on Morrey spaces with non-doubling measures, Taiwanese J. **11**, 1091-1112, (2007).

[27] A. Seeger, A note on Triebel-Lizorkin spaces, Approximation and Function Spaces, Banach Center Publ. **22**, PWN-Polish Sci. Publ. Warsaw 391-400, (1989).

[28] E. Stein, Harmonic analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton (1993).

[29] J-O. Strömberg and A. Torchinsky, Weighted Hardy spaces, Lecture Note in Mathematics **1381**, (1989).

[30] H. Triebel, Theory of function spaces II, Birkhäuser, Basel (1992).

[31] Y. Tsutsui, Pseudo-differential operators of class $S_{0,0}^m$ on the Herz-type spaces, Hokkaido Math. J. **38**, 283-302, (2009).

[32] Y. Tsutsui, Sharp maximal inequalities and its application to some bilinear estimates, J. Fourier Anal. Appl. accepted.

[33] A. Uchiyama, On the radial maximal function of distributions, Pacific J. Math., **121**, 467-483, (1986).