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Kyoto University
UNIMODULAR FOURIER MULTIPLIERS
ON MODULATION SPACES $M^{p,q}$ FOR $0 < p < 1$

NAOHITO TOMITA

1. Introduction

In this note, we consider the boundedness of the Fourier multiplier operator $e^{i|D|^{\alpha}}$ on modulation spaces, where $\alpha > 0$ and $e^{i|D|^{\alpha}}$ is defined by

$$e^{i|D|^{\alpha}}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot \xi} e^{i|\xi|^{\alpha}} \hat{f}(\xi) d\xi.$$

In the case $\alpha = 2$, $u(t, x) = e^{it|D|^{2}}u_0(x)$ is the formal solution to the Schrödinger equation

$$\begin{cases}
i\frac{\partial u}{\partial t}(t, x) = \Delta_x u(t, x) & (t > 0, x \in \mathbb{R}^n), \\
u(0, x) = u_0(x) & (x \in \mathbb{R}^n). 
\end{cases}$$

Modulation spaces $M_{s}^{p,q}$ were introduced by Feichtinger [3, 4] (see also Gröchenig [5]). We recall the definition of modulation spaces. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and let $\psi \in S(\mathbb{R}^n)$ be such that

(1.1) \quad \text{supp } \psi \subset [-1, 1]^n \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n.

Then the modulation space $M_{s}^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M_{s}^{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\psi(D - k)f\|_{L^p}^q \right)^{1/q} < \infty,$$

where $\psi(D - k)f = \mathcal{F}^{-1}[\psi(\cdot - k) \hat{f}]$. If $s = 0$, we simply write $M^{p,q}(\mathbb{R}^n)$ instead of $M_{0}^{p,q}(\mathbb{R}^n)$. We remark that $M_{s}^{2,2}$ coincides with the Sobolev space $W^{s,2}$.

It is known that $e^{i|D|^2}$ is bounded on $L^p$ if and only if $p = 2$ (Hörmander [7]). However, $e^{i|D|^2}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$ (see Gröchenig-Heil [6], Toft [10], Wang-Zhao-Guo [11], Bényi-Gröchenig-Okoudjou-Rogers [1]). This is one of differences between $L^p$-spaces and modulation spaces. Bényi-Gröchenig-Okoudjou-Rogers ([1]) proved that if $0 \leq \alpha \leq 2$ then $e^{i|D|^\alpha}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$. Furthermore, in the case $\alpha > 2$, Miyachi-Nicola-Rivetti-Tabacco-Tomita [9] showed that, for $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$, $e^{i|D|^\alpha}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq (\alpha - 2)n(1/p - 1/2)$ (see [9] for more general results). In particular, this says that if $\alpha > 2$ and $p \neq 2$ then $e^{i|D|^\alpha}$ is not bounded on modulation spaces $M^{p,q}$.

The purpose of this note is to consider the case $0 < p < 1$, and our main result is the following:

**Theorem 1.1.** Let $0 < p < 1$, $0 < q \leq \infty$, $\alpha > n(1/p - 1)$ and $s \in \mathbb{R}$. Then $e^{i|D|^\alpha}$ is bounded from $M_{s}^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha - 2\}n(1/p - 1/2)$.

We remark that Bényi-Okoudjou [2] considered the cases $0 \leq \alpha \leq 2$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$, and $\alpha \in \{1, 2\}$, $n/(n+1) < p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. In Remark 3.5, we also treat the case $\alpha \geq 0
We end this section by explaining the organization of this note. In Section 2, we give the relation between $L^p$-boundedness and $M^{p,q}$-boundedness. In Section 3, we give the proof of Theorem 1.1.

### 2. Relation between $L^p$-boundedness and $M^{p,q}$-boundedness

Let $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F} f$ and the inverse Fourier transform $\mathcal{F}^{-1} f$ of $f \in S(\mathbb{R}^n)$ by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) \, d\xi.$$ 

For $m \in S'(\mathbb{R}^n)$, we define the Fourier multiplier operator $m(D)$ by

$$m(D) f = \mathcal{F}^{-1} [m \hat{f}] = [\mathcal{F}^{-1} m] * f \quad \text{for all } f \in S(\mathbb{R}^n).$$

To avoid the fact that $S(\mathbb{R}^n)$ is not dense in $M^{p,q}_0(\mathbb{R}^n)$ if $p = \infty$ or $q = \infty$, we use the following definition of the boundedness of Fourier multiplier operators on modulation spaces: We say that $m(D)$ is bounded from $M^{p,q}_0(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that $\|m(D)f\|_{M^{p,q}} \leq C \|f\|_{M^{p,q}_0}$ for all $f \in S(\mathbb{R}^n)$, and set

$$\|m(D)\|_{L(M^{p,q}_0,M^{p,q})} = \sup \{ \|m(D)f\|_{M^{p,q}} \mid f \in S(\mathbb{R}^n), \|f\|_{M^{p,q}_0} = 1 \}.$$ 

Similarly, we set

$$\|m(D)\|_{L(L^p,L^q)} = \sup \{ \|m(D)f\|_{L^q} \mid f \in S(\mathbb{R}^n), \|f\|_{L^p} = 1 \},$$

and simply write $\|m(D)\|_{L(L^p)} = \|m(D)\|_{L(L^p,L^q)}$ if $p = q$.

The notation $A \asymp B$ stands for $C^{-1} A \leq B \leq C A$ for some positive constant $C$ independent of $A$ and $B$. For $1 \leq p \leq \infty$, $p'$ is the conjugate exponent of $p$ (that is, $1/p + 1/p' = 1$). Throughout the rest of this note, $\psi \in S(\mathbb{R}^n)$ is the same as in (1.1).

**Lemma 2.1** ([8, Lemma 2.6]). Let $0 < p < 1$, and let $\Gamma$ be a compact subset of $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that

$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^p}$$

for all $f, g \in L^p(\mathbb{R}^n)$ with $\text{supp } \hat{f} \subset \xi + \Gamma$ and $\text{supp } \hat{g} \subset \xi' + \Gamma$, where $C > 0$ is independent of $\xi, \xi' \in \mathbb{R}^n$.

The following is on the relation between $L^p$-boundedness and $M^{p,q}$-boundedness which is a slight modification of [9, Lemma 2.2]:

**Lemma 2.2.** Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and $m \in S'(\mathbb{R}^n)$. Then $m(D)$ is bounded from $M^{s,q}_0(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if there exists a constant $C > 0$ such that

$$\|\psi(D - k)m(D)f\|_{L^p} \leq C(1 + |k|^s)\|\psi(D - k)f\|_{L^p}$$

for all $k \in \mathbb{Z}^n$ and $f \in S(\mathbb{R}^n)$.

**Proof.** We assume that (2.1) holds for some constant $C > 0$. Then, by our assumption,

$$\|m(D)f\|_{M^{p,q}} = \left( \sum_{k \in \mathbb{Z}^n} \|\psi(D - k)m(D)f\|_{L^p}^q \right)^{1/q} \leq C \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|^s)^q \|\psi(D - k)f\|_{L^p}^q \right)^{1/q} = C \|f\|_{M^{p,q}}$$

for all $f \in S$, and we obtain the boundedness of $m(D)$ from $M^{p,q}_0$ to $M^{p,q}$.
We next assume that $0 < p < 1$ and $m(D)$ is bounded from $M^{p,q}_{s}$ to $M^{p,q}$. Let $\varphi \in S$ be such that $\varphi = 1$ on supp $\psi$, supp $\varphi \subset [-2,2]^n$ and $|\sum_{k \in \mathbb{Z}^n} \varphi(x - k)| \geq C > 0$ for all $\xi \in \mathbb{R}^n$. Note that $\|f\|_{M^{p,q}_{s}} \leq \left( \sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\varphi(D - k)f\|_{L^p}^q \right)^{1/q}$. Since $\psi = \varphi \psi$, supp $\psi(\cdot - k + \ell) \subset k + \ell + [-1,1]^n$ and supp $\psi(\cdot - k) \subset k + [-1,1]^n$ for all $k, \ell \in \mathbb{Z}^n$, we have by Lemma 2.1 and the boundedness of $m(D)$ from $M^{p,q}_{s}$ to $M^{p,q}$

\[
\|\psi(D - k)m(D)f\|_{L^p} = \|\varphi(D - k)(m(D)\psi(D - k)f)\|_{L^p}
\]

\[
\leq \left( \sum_{k \in \mathbb{Z}^n} \|\varphi(D - k)(m(D)\psi(D - k)f)\|_{L^p}^q \right)^{1/q}
\]

\[
\leq C\|m(D)(\psi(D - k)f)\|_{M^{p,q}_{s}} \leq C\|m(D)\|_{L(M^{p,q}_{s},M^{p,q})}\|\psi(D - k)f\|_{M^{p,q}_{s}}
\]

\[
|\sum_{k \in \mathbb{Z}^n} \varphi(x - k)| \geq C > 0 
\]

\[
\sum_{k \in \mathbb{Z}^n} (1 + |k|)^{sq} \|\varphi(D - k)f\|_{L^p}^q \leq C\|m(D)\|_{L(M^{p,q}_{s},M^{p,q})}\|\psi(D - k)f\|_{M^{p,q}_{s}}
\]

for all $k \in \mathbb{Z}^n$ and $f \in S$, where we have used $(1 + |k + \ell|)^s \leq (1 + |k|)^s(1 + |\ell|)^s$. Hence, we obtain (2.1) with $0 < p < 1$. For $1 \leq p \leq \infty$, by using Young’s inequality ($\|f * g\|_{L^p} \leq \|f\|_{L^r}\|g\|_{L^r}$) instead of Lemma 2.1, we can prove (2.1) in the same way. \qed

### 3. Proof of Theorem 1.1

The proof of the following lemma is based on that of [9, Lemma 3.1]:

**Lemma 3.1.** Let $0 < p < 1$, $N = \lceil n(1/p - 1/2) \rceil + 1$ and $\alpha > n(1/p - 1)$, where $\lceil n(1/p - 1/2) \rceil$ stands for the largest integer $\leq n(1/p - 1/2)$. If $m$ is a $\mathcal{C}^N(\mathbb{R}^n \setminus \{0\})$-function with compact support satisfying

\[
|\partial^\beta m(\xi)| \leq C_\beta |\xi|^{\alpha - |\beta|}
\]

for all $\xi \neq 0$ and $|\beta| \leq N$, then $\mathcal{F}^{-1}m \in L^p(\mathbb{R}^n)$.

**Proof.** Assume that supp $m \subset \{|\xi| \leq 2^{j_0}\}$, where $j_0 \in \mathbb{Z}$. Let $\varphi \in S$ be such that supp $\varphi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\sum_{j \in \mathbb{Z}} \varphi(\xi/2^j) = 1$ for all $\xi \neq 0$. Since supp $\varphi(\cdot/2^j) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, we see that

\[
m(\xi) = \sum_{j = -\infty}^{j_0} \varphi(\xi/2^j) m(\xi) = \sum_{j = -\infty}^{j_0} m_j(\xi/2^j),
\]

where $m_j(\xi) = \varphi(\xi) m(2^j \xi)$. By using $p < 1$, we have

\[
\left\|\mathcal{F}^{-1}m\right\|_{L^p}^p \leq \sum_{j = -\infty}^{j_0} \left\|\mathcal{F}^{-1}m_j(2^j \cdot)\right\|_{L^p}^p = \sum_{j = -\infty}^{j_0} 2^{jn(p-1)} \left\|\mathcal{F}^{-1}m_j\right\|_{L^p}^p.
\]

Let $r$ be the conjugate exponent of $2/p$, and set $N = \lfloor n/(pr) \rfloor + 1$. Then $N = \lfloor n/(p-1/2) \rfloor + 1$. By Hölder’s inequality and Plancherel’s theorem,

\[
\left\|\mathcal{F}^{-1}m_j\right\|_{L^p} = \left\|(1 + |\xi|)^{-N}(1 + |\xi|)^N \mathcal{F}^{-1}m_j\right\|_{L^p}
\]

\[
\leq \left\|(1 + |\xi|)^{-pN}\right\|_{L^r}^{1/p} \left\|(1 + |\xi|)^N \mathcal{F}^{-1}m_j\right\|_{L^2} \leq C \sum_{|\beta| \leq N} \left\|\partial^\beta m_j\right\|_{L^2}
\]

The proof of Theorem 1.1 is completed.
for all \( j \in \mathbb{Z} \), where we have used the fact \( prN > n \). Since \( \text{supp} \varphi \subset \{2^{-1} \leq |\xi| \leq 2\} \), we have by our assumption

\[
|\partial^\beta m_j(\xi)| = \left| \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} (\partial^\beta_1 \varphi)(\xi) 2^{j|\beta_2|} (\partial^\beta_2 m)(2^j \xi) \right|
\]

(3.3)

\[
\leq \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} |(\partial^\beta_1 \varphi)(\xi)| 2^{j|\beta_2|} (C_{\beta_2} |2^j \xi|^{\alpha - |\beta_2|}) \leq C_{\beta} 2^{j\alpha}
\]

for all \( j \in \mathbb{Z} \) and \(|\beta| \leq N\). On the other hand, \( \text{supp} m \subset \{2^{-1} \leq |\xi| \leq 2\} \) for all \( j \in \mathbb{Z} \). Therefore, by (3.1)-(3.3),

\[
\| \mathcal{F}^{-1}m \|_{L^p}^p \leq \sum_{j=-\infty}^{j_0} 2^{jn(p-1)} \| \mathcal{F}^{-1}m_j \|_{L^p}^p \leq C \sum_{j=-\infty}^{j_0} 2^{-jn(1/p-1)p} \sum_{|\beta| \leq N} \| \partial^\beta m_j \|_{L^2}^p \leq C \sum_{j=-\infty}^{j_0} 2^{j(\alpha - n(1/p-1))p} < \infty.
\]

The proof is complete. \( \square \)

For \( \alpha > 0 \) and \( k \in \mathbb{Z}^n \), we set

\[
(3.4) \quad \sigma_\alpha(\xi) = |\xi|^\alpha \quad \text{and} \quad \tau_{\alpha,k}(\xi) = \sigma_\alpha(\xi + k) - \sigma_\alpha(k) - (\nabla \sigma_\alpha)(k) \cdot \xi.
\]

**Lemma 3.2.** Let \( 0 < p < 1 \) and \( \alpha > n(1/p - 1) \). Then there exists a constant \( C > 0 \) such that

\[
\| \psi(D - k)e^{i\sigma_\alpha(D)}f \|_{L^p} \leq C\| \psi(D - k)f \|_{L^p}
\]

for all \( |k| < 4\sqrt{n} \) and \( f \in \mathcal{S}(\mathbb{R}^n) \).

**Proof.** Let \( \eta \) be a Schwartz function with compact support. Then

\[
\| \partial^\beta e^{i\sigma_\alpha(\xi) - 1} \|_{L^p} \leq C_{\beta} |\xi|^{\alpha - |\beta|}
\]

for all \( \xi \neq 0 \) and \( \beta \). Hence, it follows from Lemma 3.1 that

\[
\mathcal{F}^{-1}[\eta e^{i\sigma_\alpha}] = \mathcal{F}^{-1}[\eta(e^{i\sigma_\alpha} - 1)] + \mathcal{F}^{-1}\eta \in L^p.
\]

Take \( \varphi \in \mathcal{S} \) such that \( \text{supp} \varphi \) is compact and \( \varphi = 1 \) on \( \text{supp} \psi \). Then, by Lemma 2.1 and the first part of this proof with \( \eta = \varphi(\cdot - k) \), for \( |k| < 4\sqrt{n} \),

\[
\| \psi(D - k)e^{i\sigma_\alpha(D)}f \|_{L^p} = \| \varphi(D - k)e^{i\sigma_\alpha(D)}\psi(D - k)f \|_{L^p} \leq C \| \mathcal{F}^{-1}[\varphi(\cdot - k)e^{i\sigma_\alpha}] \|_{L_p} \| \psi(D - k)f \|_{L^p}
\]

(3.5)

\[
\leq C \| \psi(D - k)f \|_{L^p}
\]

for all \( f \in \mathcal{S} \). This completes the proof. \( \square \)

**Lemma 3.3.** Let \( 0 < p < 1 \). Then there exists a constant \( C > 0 \) such that

\[
\| \psi(D - k)e^{i\sigma_\alpha(D)}f \|_{L^p} \leq C |k|^{\max\{0, \alpha - 2\}n(1/p - 1/2)} \| \psi(D - k)f \|_{L^p}
\]

for all \( |k| \geq 4\sqrt{n} \) and \( f \in \mathcal{S}(\mathbb{R}^n) \).
Proof. Throughout this proof, we assume that \(|k| \geq 4\sqrt{n}\) and \(f \in S\). Let \(\varphi \in S\) be such that \(\text{supp} \varphi \subset [-2, 2]^n\) and \(\varphi = 1\) on \(\text{supp} \psi\). Then, by Lemma 2.1,

\[
\| \psi(D-k)e^{i\sigma_\alpha(D)f} \|_{L^p} = \| \varphi(D-k)e^{i\sigma_\alpha(D)\psi(D-k)f} \|_{L^p} 
\]

\[
= \| (\mathcal{F}^{-1}[\varphi(-k)e^{i\sigma_\alpha}]) * \psi(D-k)f \|_{L^p} 
\]

\[
\leq C \| \mathcal{F}^{-1}[\varphi(-k)e^{i\sigma_\alpha}] \|_{L^p} \| \psi(D-k)f \|_{L^p} 
\]

\[
= C \| \mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}] \|_{L^p} \| \psi(D-k)f \|_{L^p} 
\]

(3.6)

Let us estimate \(\| \mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}] \|_{L^p}\). Since

\[
\mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}](x) = e^{i\sigma_\alpha(k)} \mathcal{F}^{-1}[\varphi e^{i\tau_\alpha,k}](x + (\nabla \sigma_\alpha)(k)),
\]

where \(\tau_\alpha,k\) is defined by (3.4), we see that

(3.7)

\[
\| \mathcal{F}^{-1}[\varphi e^{i\sigma_\alpha(k)}] \|_{L^p} = \| \mathcal{F}^{-1}[\varphi e^{i\tau_\alpha,k}] \|_{L^p} 
\]

By Taylor's formula,

\[
\tau_\alpha,k(\xi) = 2 \sum_{|\beta|=2} \xi^\beta \int_0^1 (1-t) (\partial^\beta \sigma_\alpha)(k+t\xi) \ dt.
\]

(3.8)

If \(\xi \in [-2, 2]^n\), then \(|k+t\xi| \approx |k|\) for all \(0 \leq t \leq 1\). Since \(|\partial^\gamma \sigma_\alpha(\eta)| \leq C_\gamma |\eta|^{\alpha-|\gamma|}\), we have by (3.8)

\[
|\partial^\gamma (\varphi(\xi) e^{i\tau_\alpha,k}(\xi))| \leq C_\gamma \sum_{|\gamma| \leq N} \sum_{\mu_1+\cdots+\mu_N=|\gamma|} \| \partial^\mu \varphi \|_{L^\infty} (C_{\mu_1} |k|^{\alpha-2}) \cdots (C_{\mu_N} |k|^{\alpha-2}) 
\]

\[
\leq C_\gamma \sum_{|\gamma| \leq N} \sum_{\mu_1+\cdots+\mu_N=|\gamma|} \| \partial^\mu \varphi \|_{L^\infty} (C_{\mu_1} |k|^{\alpha-2}) \cdots (C_{\mu_N} |k|^{\alpha-2}) 
\]

\[
\leq C_\gamma |k|^{\max\{0,\alpha-2\}|\gamma|}. 
\]

Then, setting \(\varphi_{\alpha,k}(\xi) = \varphi(\xi) e^{i\tau_\alpha,k}(\xi)\), we have

(3.9)

\[
|\partial^\gamma (\varphi_{\alpha,k}(\xi/|k|^{\max\{0,\alpha-2\}}))| \leq C_\gamma \chi_{[-2|k|^{\max\{0,\alpha-2\}},2|k|^{\max\{0,\alpha-2\}}]^n}(\xi)
\]

for all multi-indices \(\gamma\), where \(\chi_A\) denote the characteristic function of \(A\). Therefore, by (3.2), (3.6), (3.7) and (3.9),

\[
\| \psi(D-k)e^{i\sigma_\alpha(D)f} \|_{L^p} \leq C \| \mathcal{F}^{-1}[\varphi_{\alpha,k}] \|_{L^p} \| \psi(D-k)f \|_{L^p} 
\]

\[
= C |k|^{\max\{0,\alpha-2\}n(1/p-1)} \| \mathcal{F}^{-1}[\varphi_{\alpha,k}(\cdot/|k|^{\max\{0,\alpha-2\}})] \|_{L^p} \| \psi(D-k)f \|_{L^p} 
\]

\[
\leq C |k|^{\max\{0,\alpha-2\}n(1/p-1)} \left( \sum_{|\gamma| \leq N} \| \partial^\gamma (\varphi_{\alpha,k}(\cdot/|k|^{\max\{0,\alpha-2\}})) \|_{L^2}^2 \right) \| \psi(D-k)f \|_{L^p} 
\]

5
\[ \leq C|k|^\max\{0, \alpha - 2\}n(1/p - 1/2)\|\psi(D - k)f\|_{L^p}, \]

where \( N = [n(1/p - 1/2)] + 1 \) and \( C > 0 \) is independent of \( k \) satisfying \( |k| \geq 4\sqrt{n} \). The proof is complete. \( \square \)

Before proving Theorem 1.1, we give the following remark on the case \( 0 \leq \alpha \leq 2 \):

**Remark 3.4.** Let \( 0 \leq \alpha \leq 2 \) and \( 1 \leq p \leq \infty \). In this case, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q}(\mathbb{R}^n) \) to \( M^{p,q}(\mathbb{R}^n) \) only if \( s \geq 0 \).

We first consider the case \( p = 2 \). By Plancherel’s theorem,

\[ \|e^{i\sigma_\alpha(D)} f\|_{M_{2,q}} = \left( \sum_{k \in \mathbb{Z}^n} \|\psi(D - k)e^{i\sigma_\alpha(D)} f\|_{L^2}^q \right)^{1/q} \]

\[ = \left\{ \sum_{k \in \mathbb{Z}^n} \left( \frac{1}{(2\pi)^{n/2}} \|\psi(\cdot - k)e^{i\sigma_\alpha f}\|_{L^2}^2 \right)^q \right\}^{1/2} \]

Hence, the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M_s^{2,q} \) to \( M^{2,q} \) implies the embedding \( M_s^{2,q} \hookrightarrow M^{2,q} \). Therefore, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q} \) to \( M^{p,q} \) only if \( s \geq 0 \).

We next consider the case \( 1 \leq p \leq \infty \) and \( p \neq 2 \). If \( m(D) \) is bounded from \( M_s^{p,q} \) to \( M_s^{p,q} \), then \( m(D) \) is also bounded from \( M_s^{p,q} \) to \( M_s^{p,q} \). This follows from the facts that \( m(D) \) is bounded from \( M_s^{p,q} \) to \( M^{p,q} \) if and only if \( \sup_{k \in \mathbb{Z}^n}(1 + |k|)^{-s}\|\psi(D - k)m(D)\|_{L(L^p)} < \infty \) ([9, Lemma 2.2]) and \( \|\psi(D - k)m(D)\|_{L(L^p)} = \|\psi(D - k)m(D)\|_{L(L^{p'})} \). Then, by interpolation, if \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q} \) to \( M_s^{p,q} \) for some \( s < 0 \), then \( e^{i\sigma_\alpha(D)} \) is also bounded from \( M_s^{2,q} \) to \( M^{2,q} \). Therefore, \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q} \) to \( M_s^{p,q} \) only if \( s \geq 0 \).

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( 0 < p < 1 \), \( 0 < q \leq \infty \), \( \alpha > n(1/p - 1) \) and \( s \in \mathbb{R} \).

We first assume that \( s \geq \max\{0, \alpha - 2\}n(1/p - 1/2) \). By Lemmas 2.2 and 3.3,

\[ \|\psi(D - k)e^{i\sigma_\alpha(D)} f\|_{L^p} \leq C(1 + |k|)^\max\{0, \alpha - 2\}n(1/p - 1/2)\|\psi(D - k)f\|_{L^p} \]

\[ \leq C(1 + |k|)^s\|\psi(D - k)f\|_{L^p} \]

for all \( k \in \mathbb{Z}^n \) and \( f \in \mathcal{S} \). Hence, by Lemma 2.2, we have the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M_s^{p,q} \) to \( M_s^{p,q} \).

We next assume that \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q} \) to \( M_s^{p,q} \). By Lemma 2.2, we may assume \( q \geq 1 \). We note that \( e^{i\sigma_\alpha(D)} \) is bounded on \( M^{2,q} \) (see Remark 3.4). Hence, it follows from interpolation with the boundedness on \( M^{2,q} \) that, if \( s < \max\{0, \alpha - 2\}n(1/p - 1/2) \), then \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{2,q} \) to \( M_s^{p,q} \), where \( 1 < \tilde{p} < 2 \) and \( \tilde{s} < \max\{0, \alpha - 2\}n(1/p - 1/2) \). However, in the case \( 1 < \tilde{p} < 2 \), \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{2,q} \) to \( M_s^{p,q} \) only if \( \tilde{s} \geq \max\{0, \alpha - 2\}n(1/p - 1/2) \) (see Remark 3.4 and [9]). Therefore, \( s \) must satisfy \( s \geq \max\{0, \alpha - 2\}n(1/p - 1/2) \).

We end this note by giving the following remark on the case \( 1 \leq p \leq \infty \) and \( 0 < q < 1 \):

**Remark 3.5.** Let \( \alpha \geq 0 \), \( 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \). Lemma 2.2 says that \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q}(\mathbb{R}^n) \) to \( M^{p,q}(\mathbb{R}^n) \) for some \( 0 < q \leq \infty \) if and only if \( e^{i\sigma_\alpha(D)} \) is bounded from \( M_s^{p,q}(\mathbb{R}^n) \) to \( M^{p,q}(\mathbb{R}^n) \) for all \( 0 < q \leq \infty \). In particular, the boundedness of \( e^{i\sigma_\alpha(D)} \) from \( M_s^{p,q}(\mathbb{R}^n) \) to \( M_s^{p,q}(\mathbb{R}^n) \) with \( 0 < q < 1 \) is equivalent to that with \( 1 \leq q \leq \infty \). On the other
hand, by [1, 9] and Remark 3.4, $e^{i\sigma_{\alpha}(D)}$ is bounded from $M^p_q(R^n)$ to $M^p_q(R^n)$ if and only if $s \geq \max\{0, \alpha - 2\}n|1/p - 1/2|$, where $0 \leq q \leq \infty$. Combining these facts, we see that $e^{i\sigma_{\alpha}(D)}$ is bounded from $M^p_q(R^n)$ to $M^p_q(R^n)$ if and only if $s \geq \max\{0, \alpha - 2\}n|1/p - 1/2|$, where $0 < q < 1$.

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References


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