Some mixed norm estimates of free Schrödinger waves

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Abstract

In this paper we consider mixed norm estimates of linear Schrödinger waves. In [13] Shao obtained some estimates under spherical symmetry condition. We generalize them and show that the symmetry condition can be substituted by angular regularity.

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1 Introduction

The free Schrödinger wave is the solution to the Cauchy problem

$$iu_t - \Delta u = 0 \ \text{ in } \ \mathbb{R}^{1+n}, \quad u(0) = \varphi \ \text{ in } \ \mathbb{R}^n, n \geq 2. \tag{1.1}$$

It can be written as

$$u(t,x) = (e^{-it\Delta}\varphi)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{\varphi}(\xi)\, d\xi.$$

Here $\mathcal{F}(\cdot) = \widehat{(\cdot)}$ is the Fourier transform defined by

$$\mathcal{F}(\varphi)(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\, dx$$

and its inverse is given by

$$\mathcal{F}^{-1}(\varphi)(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(\xi)\, d\xi.$$

We are concerned with mixed norm estimates of free Schrödinger waves, especially with ones in Sobolev type spaces, which are defined in spherical coordinates as follows:

$$\begin{aligned} &\dot{H}^{s,p}_r H^{\alpha,\ell}_\sigma = \ \text{ the closure of } \ C_0^\infty \ \text{ w.r.t. the norm} \\ &\|f\|_{\dot{H}^{s,p}_r H^{\alpha,\ell}_\sigma} = \||\nabla|^s D^\alpha_\sigma f\|_{L^p_r L^\ell_\sigma}, |s| < n/p, \alpha \in \mathbb{R}, \end{aligned} \tag{1.2}$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$, $D_\sigma = \sqrt{1-\Delta_\sigma}$, Δ_σ is the Laplace-Beltrami operator defined on the unit sphere and

$$\|f\|_{L^p_r L^\ell_\sigma} = \left(\int_0^\infty \left(\int_{S^{n-1}} |f(r\sigma)|^\ell \, d\sigma\right)^{\frac{p}{\ell}} r^{n-1}\, dr\right)^{\frac{1}{p}}, \ \ 1 \leq p, \ell < \infty.$$

Here we denoted $L^p(r^{n-1}dr)$ by L^p_r. We also use the time-space normed spaces,

$$\|v\|_{L^q_t \dot{H}^{s,p}_r H^{\alpha,\ell}_\sigma} = \left(\int_{\mathbb{R}} \|v(\cdot,t)\|^q_{\dot{H}^{s,p}_r H^{\alpha,\ell}_\sigma}\, dt\right)^{\frac{1}{q}}, \ \ 1 \leq q \leq \infty.$$

If $p = 2$ and $\ell = 2$, then we will use $\dot{H}^s_r H^\alpha_\sigma$ for $\dot{H}^{s,2}_r H^{\alpha,2}_\sigma$. We remark here that if $\alpha = 0$ and $p = \ell$, then the mixed norm is just Schtrichartz one (see Remark 1 and 2 for definition).

If $\ell = 2$, then one can expand any $v \in L^q_t \dot{H}^{p,s}_r H^\alpha_\sigma$ by the spherical harmonics of orthonomal basis $\{Y^l_k\}, k \geq 0, 1 \leq l \leq d(k)$ ($d(k)$ is the dimension of spherical harmonics of order k) such that there exists a unique sequence of measurable functions $a_k^l(t,r) \in L_t^q L_r^p$ satisfying that

$$|\nabla|^s v(t,r,\sigma) = \sum_{k\geq 0, 1\leq l\leq d(k)} a_k^l(t,r) Y_k^l(\sigma) \ \text{ in } \ L_t^q L_r^p H_\sigma^\alpha$$

and

$$\|v\|_{L_t^q \dot{H}_r^{p,s} H_\sigma^\alpha} = \left\| \left(\sum_{k,l} (1+k(k+n-2))^\alpha |a_k^l|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_r^p}.$$

Here, we used the identity $-\Delta_\sigma Y_k^l = k(k+n-2)Y_k^l$.

Now let us introduce our main theorem.

Theorem 1.1. *Let $1/2 \leq \alpha \leq n/2$ and $n \geq 2$. Suppose that q and α are numbers such that*

$$(2n+6\alpha-2)/(n-1+\alpha) < q \leq 6,$$
$$\widetilde{\alpha} = (3\alpha+2)/q - (\alpha+3)/2.$$

Then for any $\varphi \in L^2$ the solution u of (1.1) *satisfies*

$$\|u\|_{L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}} \lesssim \|\varphi\|_{L^2}, \tag{1.3}$$

where $s = \frac{n+2}{q} - \frac{n}{2}$.

Theorem 1.1 is a generalization of Shao's results [13] for spherically symmetric data. In particular, if we take $\alpha = 1/2$, then we can recover his result for $q > \frac{4n+2}{2n-1}$. If $\alpha > (3q-4)/(6-q)$ for $q \neq 6$, then we can take a positive $\tilde{\alpha}$. Hence we get a slight spatial and angular regularity gain for $\widetilde{\alpha} > 0$, $(2n+6\alpha-2)/(n-1+\alpha) < q < (2n+4)/n$ and $n \geq 6$. For another angular smoothing effects of Strichartz estimate see [10, 19], in which two dimensional endpoint case was treated.

Remark 1. Applying the Christ-Kiselev lemma (see [4, 19, 1]) and Strichartz estimate (see [7] for instance), it is possible to consider an inhomogeneous estimate. Let $q, \tilde{\alpha}, s$ be as above. Then we have

$$\| \int_0^t e^{-(t-t')\Delta} F(t')\, dt' \|_{L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}} \lesssim \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \tag{1.4}$$

for any pairs $(\widetilde{q}, \widetilde{r})$ such that $2/\widetilde{q} + n/\widetilde{r} = n/2, 2 \leq \widetilde{q} \leq \infty$ and $(\widetilde{q}, \widetilde{r}) \neq (2, \infty)$ if $n = 2$. We call such pair admissible one.

Remark 2. We now apply the estimates (1.3) and (1.4) to the mass critical Schrödinger equations;

$$iu_t - \Delta u = V(u)u, u(0) = \varphi \in L^2, V(u) = \pm |u|^{\frac{4}{n}} \text{ or } \pm |x|^{-2} * |u|^2.$$

The existence of local or small data or spherical symmetric global solutions is well-known (see [8, 11, 20]). The Strichartz estimate is main tool for that problem. Actually one can find a solution u in $S_T \equiv \sup_{(\widetilde{q},\widetilde{r}):admissible} L^{\widetilde{q}}_{[0,T]} L^{\widetilde{r}}_x$. Then the estimate (1.4) and standard nonlinear estimate for critical nonlinearity give us that the solution u is in $L_t^q \dot{H}_q^s H_\sigma^{\tilde{\alpha}}$. Hence if $n \ge 6$, $\tilde{\alpha} > 0$ and $(2n + 6\alpha - 2)/(n - 1 + \alpha) < q < (2n+4)/n$, then the solution u obtain a spatial and angular regularity.

Theorem 1.1 follows directly from dyadic decomposition and interpolation between the estimates of linear operator T_R defined as

$$T_R f(t,x) = \chi_R(x) \int e^{i(x\cdot\xi + t|\xi|^2)} f(\xi)\, d\xi,$$

where $R > 0$ is dyadic number and χ_R is characteristic function on the annulus $\{R \le |x| \le 2R\}$.

Proposition 1.2. *Let f be supported in the annulus $\{1 \le |\xi| \le 2\}$. Then we have*
(1) *for $f \in L^2$ and $1/2 \le \alpha \le n/2$*

$$\|T_R f\|_{L_t^2 L_r^2 H_\sigma^{\alpha - 1/2}} \lesssim \min(R^\alpha, R^{\frac{n}{2}}) \|f\|_{L^2}. \tag{1.5}$$

(2) *for $f \in L^p$ for $1 < p \le 2$ and $q = 3p'$*

$$\|T_R f\|_{L_t^q L_r^q L_\sigma^2} \lesssim \min(R^{-(n-1)(1/2 - 1/q)}, R^{n/q} \|f\|_{L_r^p H_\sigma^{1+1/q}}. \tag{1.6}$$

The estimates (1.5) and (1.6) can be used to get local or weighted global smoothing estimates which were obtained by many authors [3, 6, 5, 12, 14, 17, 18, 21]. It is also possible to consider the end point cases $(q,p) = (4,4)$ and $(\infty, 1)$. But we will append them in the last section since they are not essential in proving the main theorem (see Proposition 3.1 below).

2 Proof of Proposition 1.2

2.1 Case $R \gtrsim 1$; Proof of (1)

We first expand f as $f(\xi) = f(\rho\sigma) = \sum_{k \ge 0, 1 \le l \le d(k)} a_k^l(\rho) Y_k^l(\sigma)$. Then for the proof of (1.5) we may assume that $f \in L_r^2 H_\sigma^{1/2-\alpha}$ and a_k^l are supported in

$\{1 \le \rho \le 2\}$ for all k, l. Using the Fourier transform of spherical harmonic functions (see [16])

$$\widehat{Y_k^l}(\rho\sigma) = c_{n,k}\rho^{-\frac{n-2}{2}} J_\nu(\rho) Y_k^l(-\sigma), \nu = \nu(k) = \frac{n-2+2k}{2},$$

we have

$$T_R f(t, r\sigma) = \sum_{k,l} c_{n,k}\chi_R(r) r^{-\frac{n-2}{2}} \int e^{it\rho^2} a_k^l(\rho)\rho^{\frac{n}{2}} J_\nu(r\rho)\, d\rho Y_k^l(-\sigma).$$

It should be noticed that $|c_{n,k}| \le C$ for all k and C does not depend on k. By the change of variables $T_R f$ is converted into

$$\frac{1}{2}\sum_{k,l} c_{n,k}\chi_R(r) r^{-\frac{n-2}{2}} \int e^{it\rho} a_k^l(\sqrt{\rho})\rho^{\frac{n}{4}-\frac{1}{2}} J_\nu(r\sqrt{\rho})\, d\rho Y_k^l(-\sigma).$$

Hence taking $L_t^2 L_\sigma^2$ norm, the Plancherel's theorem and the reverse change variables give

$$\|T_R f(\cdot, r)\|_{L_t^2 L_\sigma^2} \lesssim \left(\sum_{k,l} \chi_R(r) r^{-(n-2)} \int |a_k^l(\rho)|^2 \rho^{n-3} |J_\nu(r\rho)|^2\, d\rho \right)^{\frac{1}{2}}. \tag{2.1}$$

To estimate L_r^2 norm of RHS we are going to use some asymptotic behavior of Bessel function. For this purpose we choose smooth cut-off functions ψ_1, ψ_2 and ψ_3 so that $\psi_1(r) = 1$ on $\{|r| < \frac{1}{4}\}$, $\psi_1(r) = 0$ on $\{|r| > \frac{1}{2}\}$, $\psi_2(r) = 1$ on $\{1/2 < |r| < 1\}$, $\psi_2(r) = 0$ on $\{|r| \le 1/4$ or $|r| \ge 2\}$, $\psi_3 = 0$ on $\{|s| < 2\}$, $\psi_3 = 1$ on $\{|s| > 3\}$, and $\psi_1 + \psi_2 + \psi_3 = 1$. Now we introduce four types of asymptotic behavior of Bessel function as follows: for $\nu \ge 1$

$$|J_\nu(r)| \le C\exp(-C\nu), \quad \text{if} \quad 0 \le r \le \frac{\nu}{2}, \tag{2.2}$$

$$|J_\nu(r)| \le C\nu^{-\frac{1}{3}}(1 + \nu^{-\frac{1}{3}}|r-\nu|)^{-\frac{1}{4}} \quad \text{for all } \frac{\nu}{2} < r < 2\nu, \tag{2.3}$$

$$|J_\nu(r)| \le Cr^{-\frac{1}{2}} \quad \text{for all } r \ge 2\nu, \tag{2.4}$$

$$J_\nu(r)\psi_3(\frac{r}{\nu}) = r^{-\frac{1}{2}}(b_+e^{ir} + b_-e^{-ir})\psi_3(\frac{r}{\nu}) + \Phi_\nu(r)\psi_3(\frac{r}{\nu}), \tag{2.5}$$

where $|\Phi_\nu(r)| \le C\frac{\nu^{\frac{3}{2}}}{r}$, $|b_\pm| \le C$ and the constant C is independent of ν. For the proof of (2.2), one can use the Poisson representation formula (2.7) and Stirling's formula below. Invoking the Schläfli's integral representation (see p.176 in [22])

$$J_\nu(r) = \frac{1}{\pi}\int_0^\pi e^{i(r\sin\theta - \nu\theta)}\, d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - r\sinh\tau}\, d\tau,$$

the two asymptotic behaviors (2.3) and (2.4) follow from the easy estimate

$$\left|\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - r\sinh\tau}\,d\tau\right| \le \frac{C}{\nu+r}$$

and the method of stationary phase for $\frac{1}{\pi}\int_0^\pi e^{i(r\sin\theta-\nu\theta)}\,d\theta$ when $\nu/2 < r < 2\nu$ or $r \ge 2\nu$. For instance see the page 1478 of [19] and Lemma 3 of [2]. For (2.5), see 5.2 on the page 356 of [15].

Now taking L^2_r norm on both sides of (2.1) and then changing variables $r \mapsto r\rho$, we get

$$\|T_R f\|^2_{L^2_t L^2_r L^2_\sigma} \lesssim \sum_{k,l} \int |a^l_k(\rho)|^2 \rho^{n-5} (\star)\,d\rho,$$

where

$$\begin{aligned}
(\star) &= \int \chi_{R\rho}(r) r |J_\nu(r)|^2\,dr \\
&= \int \chi_{R\rho}(r) r |J_\nu(r)|^2 (\psi_1(r/\nu) + \psi_2(r/\nu) + \psi_3(r/\nu))\,dr \equiv I_1 + I_2 + I_3.
\end{aligned}$$

For the Bessel function estimates we may assume that $\nu \ge 1$. By (2.2),

$$\begin{aligned}
I_1 &\lesssim \int_R^{\min(4R,\nu/2)} e^{-C\nu} r\,dr \\
&\lesssim Re^{-CR} \int_R^{\min(4R,\nu/2)} e^{-C\nu/2}\,dr \\
&\lesssim Re^{-C\nu/2} \lesssim R^{2\alpha} e^{-C\nu/2} \ \text{ because } \ \alpha \ge 1/2.
\end{aligned}$$

By (2.3),

$$\begin{aligned}
I_2 &\lesssim \int_{\max(R,\nu/2)}^{\min(4R,2\nu)} \nu^{-2/3}\nu^{1/6}|r-\nu|^{-1/2} r\,dr \\
&\lesssim R^{2\alpha}\nu^{1/2-2\alpha} \int_{\nu/2}^{2\nu} |r-\nu|^{-1/2}\,dr \\
&\lesssim R^{2\alpha}\nu^{1-2\alpha}.
\end{aligned}$$

By (2.4),

$$I_3 \lesssim \int_{\max(R,2\nu)}^{4R} 1\,dr \lesssim R^{2\alpha}\nu^{1-2\alpha}.$$

Substituting these estimates into $(\star)$, we get the desired estimate.

2.2 Case $R \gtrsim 1; Proof of (2)$

Similarly to the case (1), we expand f as $f(\rho\sigma) = \sum_{k,l} a_k^l(\rho) Y_k^l(\sigma)$ and assume that $f \in L_r^2 H_s^{1+1/q}$ and a_k^l are supported in $\{1 \le \rho \le 2\}$. Then we need only to show that

$$\begin{aligned}
&\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho)\, d\rho\|_{L_t^q L_r^q} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{1+1/q} \|a_k^l\|_{L_r^p}
\end{aligned}$$

Using cut-off functions and asymptotic behaviors of Bessel functions, LHS is bounded by

$$\begin{aligned}
&\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho)\psi_1(r\rho/\nu)\, d\rho\|_{L_t^q L_r^q} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho)\psi_2(r\rho/\nu)\, d\rho\|_{L_t^q L_r^q} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2}(b_+ e^{ir\rho} + b_- e^{-ir\rho})\, d\rho\|_{L_t^q L_r^q} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2}(b_+ e^{ir\rho} + b_- e^{-ir\rho})(1-\psi_3(r\rho/\nu))\, d\rho\|_{L_t^q L_r^q} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) \Psi_\nu(r\rho)\psi_3(r\rho/\nu)\, d\rho\|_{L_t^q L_r^q} \\
&= I\!I_1 + I\!I_2 + I\!I_3 + I\!I_4 + I\!I_5.
\end{aligned}$$

Taking L_t^q norm first and then L_r^q norm, we get the bounds

$$\begin{aligned}
I\!I_1 &\lesssim \|a_k^l(\rho)\| \chi_{R\rho} r^{-(n-2)/2} J_\nu(r)\psi_1(r/\nu)\|_{L_r^q}\|_{L_\rho^{q'}} \\
&\lesssim e^{-C\nu} R^{-(n-2)/2+(n-1)/q} (\min(4R, \nu/2) - R)^{\frac{1}{q}} \|a_k^l\|_{L_\rho}^{q'} \\
&\lesssim \nu^{\frac{1}{q}} e^{-C\nu} R^{-(n-1)(1/2-1/q)} \|a_k^l\|_{L_\rho^{q'}},
\end{aligned}$$

for some $0 < \delta < 4/q$

$$\begin{aligned}
I\!I_2 &\lesssim \|a_k^l(\rho)\| \chi_{R\rho} r^{-(n-2)/2} J_\nu(r)\psi_2(r/\nu)\|_{L_r^q}\|_{L_\rho^{q'}} \\
&\lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \nu^{\frac{1}{2}} \left(\int_{\max(R,\nu/2)}^{\min(4R,2\nu)} \nu^{-q/3+\delta q/12} |r-\nu|^{-\delta q/4}\, dr \right)^{\frac{1}{q}} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{\frac{1-\delta}{6}+\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}},
\end{aligned}$$

$$
\begin{aligned}
I\!I_4 &\lesssim \|a_k^l(\rho)\|\chi_{R\rho} r^{-(n-1)/2}(1-\psi_3(r/\nu))\|_{L_r^q}\|_{L_\rho^{q'}} \\
&\lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \left(\int_R^{\min(4R,2\nu)} 1\,dr \right)^{\frac{1}{q}} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}},
\end{aligned}
$$

$$
\begin{aligned}
I\!I_5 &\lesssim \|a_k^l(\rho)\|\chi_{R\rho} r^{-n/2}\psi_3(r/\nu)\|_{L_r^q}\|_{L_\rho^{q'}} \\
&\lesssim \|a_k^l\|_{L_\rho^{q'}} R^{-(n-1)(1/2-1/q)} \nu^{3/2} R^{-1/2} \left(\int_{\max(R,2\nu)}^{4R} 1\,dr \right)^{\frac{1}{q}} \\
&\lesssim R^{-(n-1)(1/2-1/q)} \nu^{1+\frac{1}{q}} \|a_k^l\|_{L_\rho^{q'}}.
\end{aligned}
$$

It should be noted that $q' = (3p')' < p$ for $1 < p \le 2$.

For $I\!I_3$ one can use 2-d oscillatory integral estimate of [15] or Proposition 3.6 of [13] to get

$$
\begin{aligned}
I\!I_3 &\lesssim R^{-(n-2)(1/2-1/q)} \| \int e^{it\rho^2} \rho^{\frac{n-1}{2}} a_k^l(\rho)(b_+ e^{ir\rho} + b_- e^{-ir\rho})\,d\rho\|_{L_t^q L_r^q} \\
&\lesssim R^{-(n-2)(1/2-1/q)} \|a_k^l\|_{L_\rho^p}.
\end{aligned}
$$

This completes the proof of the case $R \gtrsim 1$ of (2).

2.3 Case $R \ll 1$

Now we consider the case when $R \ll 1$. More generally we will get for all $2 \le q \le \infty, \alpha > 0$

$$
\|T_R f\|_{L_t^q L_r^q H_\sigma^\alpha} \lesssim R^{\frac{n}{q}} \|f\|_{L^{q'}}. \tag{2.6}
$$

Using spherical harmonic expansion as above, we have only to show that

$$
\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^i(\rho) J_\nu(r\rho)\,d\rho\|_{L_t^q L_r^q} \lesssim R^{\frac{n}{q}} e^{-\nu} \|a_k^l\|_{L_\rho^{q'}}.
$$

In fact, by Hausdorff-Young's inequality we have

$$
\begin{aligned}
LHS &\lesssim \|\chi_R r^{-(n-2)/2} \|a_k^l(\rho) J_\nu(r\rho)\|_{L_\rho^{q'}}\|_{L_r^q} \\
&\lesssim \|a_k^l(\rho)\|\chi_{R\rho} r^{-(n-2)/2} J_\nu(r)\|_{L_r^q}\|_{L_\rho^{q'}}.
\end{aligned}
$$

And since $\nu \geq (n-2)/2$, the inner integral is bounded by

$$\begin{aligned}
&\left(\int_R^{4R} r^{-q(n-2)/2+n-1}|J_\nu(r)|^q\,dr\right)^{\frac{1}{q}} \\
&\quad\lesssim R^{-(n-2)/2+n/q}\frac{(2R)^\nu}{\Gamma(\nu+1/2)} \\
&\quad\lesssim R^{n/q}R^{-(n-2)/2+\nu}\left(\frac{2e}{\nu+\frac{1}{2}}\right)^\nu \\
&\quad\lesssim R^{n/q}e^{-\nu}.
\end{aligned}$$

Here we used the Poisson representation formula [15, 22]

$$J_\nu(r) = \frac{(\frac{r}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{irs}(1-s^2)^{\nu-\frac{1}{2}}ds \tag{2.7}$$

and the Stirling's formula [9] $\Gamma(t) \sim \sqrt{2\pi}t^{t-\frac{1}{2}}e^{-t}$ for large t.

2.4 Proof of Theorem 1.1

Interpolating (1.5) and (1.6) with $(q,p) = (6,2)$, we get for $2 \leq q \leq 6$

$$\|T_R f\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \min(R^{-\frac{n-1+\alpha}{2}+\frac{3\alpha+n-1}{q}}, R^{\frac{n}{q}})\|f\|_{L^2},$$

where $\tilde{\alpha} = 3(\alpha+2/3)/q - (\alpha+3)/2$. Thus $\tilde{\alpha}$ satisfies the hypothesis and for $q > \frac{2n+6\alpha-2}{n-1+\alpha}$ the dyadic sum $\sum_{R:dyadic}\|T_R f\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}}$ is bounded by $\|f\|_{L^2}$.

If $\widehat{\varphi}$ is supported in $\{1 \leq |\xi| \leq 2\}$, then

$$\|u\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \sum_{R:dyadic}\|T_R(\widehat{\varphi})\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim \|\varphi\|_{L^2}.$$

If $\mathrm{supp}(\widehat{\varphi}) \subset \{N \leq |\xi| \leq 2N\}$, by rescaling we get

$$\|u\|_{L_t^q L_r^q H_\sigma^{\tilde{\alpha}}} \lesssim N^{-(n+2)/q+n/2}\|\varphi\|_{L^2}.$$

Now we decompose the solution u dyadically in frequency space. For this we use frequency projection operator P_N whose symbol is supported in $\{N \leq |\xi| \leq 2N\}$ and then $P_N u = e^{-it\Delta}P_N\varphi$. Thus by summing w.r.t. dyadic frequency we get the desired estimate.

3 Endpoint cases

Proposition 3.1. *Let f be supported in the annulus $\{1 \le |\xi| \le 2\}$. Then we have*
(1) *for $f \in L^4$ and $0 < \varepsilon \ll 1$*

$$\|T_R f\|_{L_t^4 L_r^4 L_\sigma^2} \lesssim \min(R^{-(n-1)/4+\varepsilon}, R^{\frac{n}{4}})\|f\|_{L_r^2 H_\sigma^{\frac{1}{4}}}. \tag{3.1}$$

(2) *for $f \in L^1$*

$$\|T_R f\|_{L_t^\infty L_r^\infty L_\sigma^2} \lesssim \min(R^{-(n-1)/2}, 1)\|f\|_{L_r^1 H_\sigma^\alpha}, \tag{3.2}$$

where $\alpha \ge 1/6$.

Proof. Write f as $\sum_{k,l} a_k^l Y_k^l$. Then in view of Section 2.3 we have only to consider the case $R \gtrsim 1$ and to show that

$$\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho)\, d\rho\|_{L_t^4 L_r^4} \lesssim R^{-(n-1)/4+\varepsilon} \nu^{\frac{7}{4}} \|a_k^l\|_{L_\rho^2}.$$

Using cut-off functions and asymptotic behaviors of Bessel functions, LHS is bounded by

$$\begin{aligned}
&\|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_1(r\rho/\nu)\, d\rho\|_{L_t^4 L_r^4} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) J_\nu(r\rho) \psi_2(r\rho/\nu)\, d\rho\|_{L_t^4 L_r^4} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho})\, d\rho\|_{L_t^4 L_r^4} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) (r\rho)^{-1/2} (b_+ e^{ir\rho} + b_- e^{-ir\rho})(1 - \psi_3(r\rho/\nu))\, d\rho\|_{L_t^4 L_r^4} \\
&+ \|\chi_R r^{-(n-2)/2} \int e^{it\rho^2} \rho^{\frac{n}{2}} a_k^l(\rho) \Psi_\nu(r\rho) \psi_3(r\rho/\nu)\, d\rho\|_{L_t^4 L_r^4} \\
&= I\!I\!I_1 + I\!I\!I_2 + I\!I\!I_3 + I\!I\!I_4 + I\!I\!I_5.
\end{aligned}$$

The terms $I\!I\!I_i, i = 1, 2, 4, 5$ are treated similarly to $I\!I_i, i = 1, 2, 4, 5$ and their sum actually has the bound

$$R^{-(n-1)/4} \nu^{\frac{7}{4}} \|a_k^l\|_{L_\rho^{\frac{4}{3}}}.$$

As for $I\!I\!I_3$ one can follow the proof of Proposition 3.5 in [13] and can get

$$I\!I\!I_3 \lesssim R^{-(n-1)/4+\varepsilon} \|a_k^l\|_{L_\rho^2}.$$

This proves (1).

For the proof of (2) we show that

$$\|\chi_R r^{-(n-2)/2}\int e^{it\rho^2}\rho^{\frac{n}{2}}a_k^l(\rho)J_\nu(r\rho)\,d\rho\|_{L_t^\infty L_r^\infty}\lesssim R^{-(n-1)/2}\nu^\alpha\|a_k^l\|_{L_\rho^1}.$$

We bound LHS by

$$\begin{aligned}&\|\chi_R r^{-(n-2)/2}\int \rho^{\frac{n}{2}}|a_k^l(\rho)|e^{-C\nu}|\psi_1(r\rho/\nu)\,d\rho\|_{L_r^\infty}\\ &+\|\chi_R r^{-(n-2)/2}\int \rho^{\frac{n}{2}}|a_k^l(\rho)|\nu^{-1/3}\psi_2(r\rho/\nu)\,d\rho\|_{L_r^\infty}\\ &+\|\chi_R r^{-(n-2)/2}\int \rho^{\frac{n}{2}}|a_k^l(\rho)|(r\rho)^{-1/2}\,d\rho\|_{L_r^\infty}.\end{aligned}$$

The first term is bounded by

$$\|a_k^l\|_{L_\rho^1}e^{-C\nu}\|\chi_(R,4R)(r)r^{-(n-2)/2}\psi_1(r/\nu)\|_{L_r^\infty}\lesssim R^{-(n-1)/2}\nu^{\frac{1}{2}}e^{-C\nu}\|a_k^l\|_{L_\rho^1},$$

the second one by

$$R^{-(n-1)/2}\nu^{\frac{1}{6}}\|a_k^l\|_{L_\rho^1}\ \text{ because }\ R\sim\nu$$

and the last one by $R^{-(n-1)/2}\|a_k^l\|_{L_\rho^1}$. This completes the proof of Proposition 3.1. □

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