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Iwasawa Theory for nearly ordinary Hida deformations of Hilbert modular forms

By

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Contents

§ 1. General overview and the motivation of our project

Let us start from a general overview of our project on a generalization of Iwasawa theory. To make clear the evolution of Iwasawa theory, it might be useful to understand it through three generations [1].

> **I. 1st generation (since 60's)**. Iwasawa theory for **class groups** over a $\mathbb{Z}_p$-extension ($\mathbb{Z}_p^d$-extension) of a number field F

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[1] In the article below, we consider only the commutative case where the ring of definition R of Galois representations is a commutative algebra. However, there is another important way of generalization called non-commutative Iwasawa theory studied actively by Coates and others. There they try to generalize $\mathbb{Z}_p$-extensions which appear in the second generation below to more general p-adic Lie extensions. Certainly, taking the "fiber product" of our generalization and such a non-commutative theory, one could discuss a further generalization.

II. 2nd generation (since 70's). Iwasawa theory for **ordinary p-adic Galois representations** over a $\mathbb{Z}_p$-extension ($\mathbb{Z}_p^d$-extension) of a number field F

$$\Downarrow$$

III. 3rd generation (since 90's). Iwasawa theory for **nearly ordinary p-adic Galois deformations** of $\mathrm{Gal}(\overline{F}/F)$ defined over a big local ring R

I. The origin of various researches of Iwasawa Theory goes back to Iwasawa's work on class groups over $\mathbb{Z}_p^d$-extensions. Iwasawa struck a rich vein of gold in the theory of cyclotomic fields and established various foundational results as well as the formulation of Iwasawa Main Conjecture for class groups, which was proved later by Mazur-Wiles. This is what we call the first generation here. Since we have already a lot of good references for Iwasawa theory of the first generation (see [CS], [L] and [Wa]), we do not discuss anymore about it.

II. Since then, the framework of Iwasawa theory has enlarged to more general objects other than class groups and to more general situations other than the one obtained by $\mathbb{Z}_p^d$-extension. Compared to the first generation, there are no written book and very few references on the second and third generations except those which discuss some restricted subjects. Also, these programs of a generalization of the Iwasawa theory is a motivation for the case of $GL(2)$ over totally real fields which we discuss here. So, it is also important to insist on the importance of the subject here. Hence, we will give a rough guide on the second generation and the third generation of Iwasawa theory [2].

Influenced by this successful theory for class groups, a lot of mathematicians tried to generalize the framework of the Iwasawa theory to more general Galois representations T ordinary at p, which we call the 2nd generation. Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ and let Γ be the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty / \mathbb{Q}$. We expect to introduce and study:

A. "algebraic ideal for T" in $\mathcal{O}[[\Gamma]]$ which is the characteristic ideal of a Selmer group (cf. [G89], [G91]).

B. "analytic ideal for T" in $\mathcal{O}[[\Gamma]]$ which is the p-adic L-function for T (cf. [CP89]).

C. the Iwasawa Main Conjecture which predicts the equality:

$$\text{"algebraic ideal for } T\text{"} = \text{"analytic ideal for } T\text{"}.$$

[2] However, I will not /can not give a complete list of references.

When $T = \mathbb{Z}_p(1)$, the theory is nothing but the previous theory for class groups. When $T = T_pE$ is the p-Tate-module of an elliptic curve E which has ordinary reduction at p, Mazur proposed the Iwasawa Theory for T_pE (see [Mz72] for the algebraic theory and [MTT86] for the analytic theory, for example), which motivated Greenberg, Perrin-Riou and Kato, etc. to work for conjectural framework for more general T as in the above A, B and C. For this case of $T = T_pE$, if E has complex multiplication, Iwasawa Main Conjecture is proved by Rubin. For E without complex multiplication, Kato proves an inequality

$$\text{“algebraic ideal for } T\text{”} \supset \text{“analytic ideal for } T\text{”}$$

by using the Euler system of Beilinson-Kato and, under certain conditions, Skinner-Urban announced [3] an inequality:

$$\text{“algebraic ideal for } T\text{”} \subset \text{“analytic ideal for } T\text{”}$$

assuming the conjectural existence of Galois representations for automorphic forms on $\mathrm{U}(2,2)$ by using the method of Eisenstein ideal for $\mathrm{U}(2,2)$. Basically, the results of Kato and Skinner-Urban mentioned above work on p-adic representations T_f associated to general elliptic modular forms f of weight ≥ 2. For other p-adic representations, there are no general results except a few cases like $\mathrm{Sym}^2 T_f$, which is related to "$R = \mathbb{T}$ Theorem".

III. By taking such evolution of Iwasawa theory into consideration, and also by introducing a new and important point of view of Galois deformation spaces, Greenberg [G94] proposed a generalization of the Iwasawa theory. For the setting for this third generation of Iwasawa theory, we are given the following things:

- $\mathcal{R}$: a Noetherian complete local domain with a finite residue field (for example, $\mathcal{R} = \mathcal{O}[[X_1, \cdots, X_d]]$ or its finite flat extension).

- $\mathcal{T}$: a free $\mathcal{R}$-module of finite rank on which the absolute Galois group $G_{\mathbb{Q}}$ acts continuously unramified outside a finite set of primes $\Sigma \supset \{p, \infty\}$.

- S: a Zariski dense subset of $\mathrm{Spec}_{\mathrm{cont}}(\mathcal{R}) = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{R}, \overline{\mathbb{Q}}_p)$.

As in [G94], we assume the following three conditions for $(\mathcal{R}, \mathcal{T}, S)$.

[3] At the moment, they publish no article for the proof nor the one which explains the statement of precise results.

(Geom) For any $\kappa \in S$, a usual p-adic representation $V_\kappa := (\mathcal{T} \otimes_{\mathcal{R}} \kappa(\mathcal{R})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the p-adic etale realization of a certain pure motive M_κ over $\mathbb{Q}$.

(Pan) There exists a D_p-stable filtration

$$0 \longrightarrow \mathrm{F}^+\mathcal{T} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathrm{F}^+\mathcal{T} \longrightarrow 0$$

of free $\mathcal{R}$-modules such that all Hodge-Tate weights of $\mathrm{F}^+V_\kappa := (\mathrm{F}^+\mathcal{T} \otimes_{\mathcal{R}} \kappa(\mathcal{R})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are positive and that all Hodge-Tate weights of $V_\kappa/\mathrm{F}^+V_\kappa$ are non-positive at every $\kappa \in S$.

(Crit) At each $\kappa \in S$, the motive M_κ is critical in the sense of Deligne-Shimura (cf. [De79]).

(NV) There is a $\kappa \in S$ such that the Hasse-Weil L-function $L(M_\kappa, s)$ does not vanish at $s = 0$.

We will propose three important conjectures for this setting of Iwasawa Theory of the third generation. The conjectures stated below are in some sense the modification and improvement of the conjectures stated in the section 4 of the article [G94]. However, as is explained in the introduction, after careful study via examples, we refine the conjectures. After stating the conjectures, we will come back again to historical notes around these conjectures. Let $\mathcal{A}$ be the discrete abelian group $\mathcal{T} \otimes_{\mathcal{R}} \mathcal{R}^{\mathrm{PD}}$ where $\mathcal{R}^{\mathrm{PD}}$ is the Pontrjagin dual of $\mathcal{R}$. Firstly, according to Greenberg, we define the Selmer group by

$$\mathrm{Sel}_{\mathcal{A}} = \mathrm{Ker}\left[H^1(\mathbb{Q}, \mathcal{A}) \longrightarrow H^1(I_p, \mathcal{A}/F^+\mathcal{A}) \times \prod_{l \nmid p} H^1(I_l, \mathcal{A}) \right]. \tag{1.1}$$

The Pontrjagin dual $\mathrm{Sel}_{\mathcal{A}}^{\mathrm{PD}}$ of $\mathrm{Sel}_{\mathcal{A}}$ is a compact $\mathcal{R}$-module. It is not difficult to see that $\mathrm{Sel}_{\mathcal{A}}^{\mathrm{PD}}$ is further a finitely generated $\mathcal{R}$-module. The first conjecture is as follows:

Conjecture A . $\mathrm{Sel}_{\mathcal{A}}^{\mathrm{PD}}$ is a torsion $\mathcal{R}$-module.

The following conjecture concerns the existence of the analytic p-adic L-function.

Conjecture B . There is an analytic p-adic L-function $L_{p,\mathcal{T}} \in \mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ which has the following interpolation property for every $\kappa \in S$:

$$\frac{\kappa(L_{p,\mathcal{T}})}{C_{\kappa,p}} = P_\kappa \cdot Q_\kappa \cdot \frac{L(M_\kappa, 0)}{C_{\kappa,\infty}}, \tag{1.2}$$

where

- $P_\kappa = \prod_i \left(1 - \dfrac{1}{p\alpha_i}\right) \times \prod_j (1-\beta_j)$ where α_i runs through the eigenvalues of the frobenius φ on $D_{\mathrm{crys}}(\mathrm{F}^+V_\kappa)$ and β_j runs through the eigenvalues of φ on the image of $D_{\mathrm{crys}}(V_\kappa)$ in $D_{\mathrm{crys}}(V_\kappa/\mathrm{F}^+V_\kappa)$.

- $Q_\kappa = (\prod_i \alpha_i^{-1})^*$ where $*$ is a non-negative integer determined by M_κ.

- $C_{\kappa,p} \in \mathbb{C}_p$ (resp. $C_{\kappa,\infty} \in \mathbb{C}$) is a p-adic period (resp. complex period) defined by using the determinant of the comparison isomorphism of p-adic Hodge theory (resp. Hodge theory over $\mathbb{C}$) proved by Faltings, Niziol and Tsuji [4]:

$$H_{\mathrm{Betti}}(M_\kappa)^+ \otimes_{\mathbb{Q}} B_{\mathrm{HT}} \longrightarrow \big(H_{\mathrm{dR}}(M_\kappa)/\mathrm{Fil}^0 H_{\mathrm{dR}}(M_\kappa)\big) \otimes_{\mathbb{Q}} B_{\mathrm{HT}}$$
$$\Big(\text{resp. } H_{\mathrm{Betti}}(M_\kappa)^+ \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \big(H_{\mathrm{dR}}(M_\kappa)/\mathrm{Fil}^0 H_{\mathrm{dR}}(M_\kappa)\big) \otimes_{\mathbb{Q}} \mathbb{C}\Big)$$

where $H_{\mathrm{Betti}}(M_\kappa)$ and $H_{\mathrm{dR}}(M_\kappa)$ are the Betti realization and the de Rham realization of the pure motive M_κ.

Remark 1.1.

1. Note that the terms like Gauss sums are hidden in the complex period in Conjecture B.

2. There is no canonical choice for a complex period $C_{\kappa,\infty}$ and a p-adic period $C_{\kappa,p}$. In fact, $C_{\kappa,\infty}$ and $C_{\kappa,p}$ for each motive M_κ depend on the choice of bases of $H_{\mathrm{Betti}}(M_\kappa)^+$ and $\mathrm{Fil}^0 H_{\mathrm{dR}}(M_\kappa)$ over some number fields. However, if we change these basis, both $C_{\kappa,\infty}$ and $C_{\kappa,p}$ are multiplied by the determinant of the matrix of this base change. Hence, the interpolation property (1.2) is well-defined.

Assuming Conjecture A, $\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}$ is a finitely generated torsion $\mathcal{R}$-module. If we denote by $(\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^{\mathrm{nor}}$ the integral closure of $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ in its fraction field, we denote by $\mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}) \subset (\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^{\mathrm{nor}}$ the characteristic ideal of the torsion $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$-module $\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}} \otimes_{\mathcal{R}} (\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^{\mathrm{nor}}$.

Conjecture C . Let $\mathcal{A}^*$ be $\mathrm{Hom}_{\mathrm{cont}}(\mathcal{T}, \mathbb{Q}_p(1)/\mathbb{Z}_p(1))$. We have the equality of ideals in $(\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^{\mathrm{nor}}$:

$$e_{\mathcal{A}} \cdot \mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}})\mathrm{char}(H^0(\mathbb{Q},\mathcal{A})^{\mathrm{PD}})^{-1}\mathrm{char}(H^0(\mathbb{Q},\mathcal{A}^*)^{\mathrm{PD}})^{-1} = (L_{p,\mathcal{T}}),$$ [5]

[4]The p-adic comparison map below restricted to $+$-part is expected to remain isomorphic after changing the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The complex comparison map below restricted to $+$-part remains isomorphic by the assumption (Crit).

[5]When the residual representation of $\mathcal{T}$ is irreducible, the factors $\mathrm{char}(H^0(\mathbb{Q},\mathcal{A})^{\mathrm{PD}})$ and $\mathrm{char}(H^0(\mathbb{Q},\mathcal{A}^*)^{\mathrm{PD}})$ are trivial.

where $e_{\mathcal{A}}$ is given as follows (see also the remark below) [6]:

$$e_{\mathcal{A}} = \begin{cases} \mathrm{char}((\mathcal{A}/\mathrm{F}^{+}\mathcal{A})^{D_p})^{\mathrm{PD}} & \text{if } \mathcal{A}/\mathrm{F}^{+}\mathcal{A} \text{ is unramified at } p, \\ 0 & \text{otherwise.} \end{cases}$$

Remarks on Conjectures A, B, C. Firstly, all such conjectures are greatly influenced by the paper [G94] which motivated my research. However, the conjectures are modified at several points.

1. Firstly, the p-adic L-function is considered as an element of $\mathcal{R}$ in [G94], but, in our Conjecture B, we expect it as an element of $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$. If we try to find a p-adic L-function in $\mathcal{R}$, we will need to introduce some ambiguous choices caused by non-canonical choice of periods. (For example, [Ki94] and [GS93] needed to fix a basis of "Module of Λ-adic modular symbols" to define their p-adic error terms which appear in the interpolation property of their p-adic L-function in $\mathcal{R}$.) It seems better to extend the algebra to $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ and use the interpolation property with "real" p-adic periods defined by the p-adic Hodge theory. We recall that the general framework was first studied in [Pa91] and later further investigated by [H96]. We will discuss at another occasion, on the detail of a further refinement of Conjecture B, especially on some ambiguous points on the terms P_κ and Q_κ in previous references.

2. By a careful study through certain examples obtained by specializing Hida deformations (see §7 especially around Corollary 7.10 of [Oc06a]), we find a case where Conjecture C does not hold without modification factor $e_{\mathcal{A}}$. The necessity of such factor was not found in the article [G94] and it is one of our refinements of previous conjectures.

For the known cases of conjectures A, B and C, we recall that the case where the Galois module $\mathcal{T}$ is of rank one over $\mathcal{R}$ falls down to the 1st generation, in which case $\mathcal{R}$ is isomorphic to the cyclotomic Iwasawa algebra $\mathcal{O}[[\Gamma]]$ and $\mathcal{T}$ is "the cyclotomic deformation" of a p-adic representation of rank one associated to a Dirichlet character. (Conjecture A is a theorem of Iwasawa, Conjecture B has also been done by Kubota-Leopoldt, Iwasawa, Coleman etc. Conjecture C in this case is proved by Mazur-Wiles.) Hence, the first new example for the Iwasawa theory of the third generation appears when $\mathcal{T}$ is of rank two over $\mathcal{R}$.

[6] For those who are familiar with trivial zero conjecture as proposed by [MTT86] and solved in [GS93], we remark that $e_{\mathcal{A}}$, though it might appear to be a modification related to the trivial zero, has no relation to the trivial zero phenomena and is a modification which was not known before. In fact, Greenberg's Selmer group $\mathrm{Sel}_{\mathcal{A}}$ matches well with the trivial zero phenomena and we need not modify $\mathrm{char}(\mathrm{Sel}_{\mathcal{A}}^{\mathrm{PD}})$ with the trivial zero factor.

For the rank two case, the most universal $\mathcal{R}$ is the so called Hida's nearly ordinary Hecke algebra which is often isomorphic to $\mathcal{O}[[X]][[\Gamma]]$ with universal Galois representation $\mathcal{T} \cong \mathcal{O}[[X]][[\Gamma]]^{\oplus 2}$. Further, conjecturally, every ordinary geometric p-adic representation of rank-two is obtained as a specialization of such a nearly ordinary deformation $\mathcal{T}$ (and $\mathcal{R}$) [7]. For each $k \geq 2$, by specializing the variable X to $(1+p)^{k-2}-1$, $\mathcal{T}$ is specialized to a rank two Galois module over $\mathcal{O}[[\Gamma]]$ which is the cyclotomic deformation of a certain ordinary cusp form f_k of weight k. Hida's nearly ordinary deformation $\mathcal{T}$ is the first test case of the Iwasawa theory of the third generation and our main new results presented later treat the analogue for $\mathrm{Gal}(\overline{F}/F)$ in place of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For the results known for $F = \mathbb{Q}$, we will come back to them in §3 after formulating the detail of the setting in §2 and before stating the results for general totally real fields in §4. We believe that to explain the case for $F = \mathbb{Q}$ is important to state the current state of research for general F's.

Finally, we remark that few results are known for Galois representation of rank > 2. Exceptionally, for the rank three representation called adjoint type, which is a family of $\mathrm{Sym}^2 T_f$ where elliptic modular forms f vary, some results are known (cf. [HTU97])[8].

§ 2. Notation and the Hida theory of Hilbert modular forms

Here is the list of our notation which is fixed throughout the article:
F: a totally real number field with degree $d = [F : \mathbb{Q}]$,
r_F: the ring of integers of F,
$I_F = \{\iota_1, \cdots, \iota_d\}$: the set of embeddings $\iota : F \hookrightarrow \mathbb{R}$,
p: an odd prime number relatively prime to the discriminant D_F,
$\mathcal{O}$: a finite flat extension of $\mathbb{Z}_p$ which contains all conjugates of r_F.
We will always fix a complex embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and a p-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.
We also introduce:
$\widehat{r}_F := r_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$,
$Z := ((\widehat{r}_F)^\times \times (\widehat{r}_F)^\times) \Big/ \overline{r_F^\times}$, where $\overline{r_F^\times}$ is the p-adic closure of the group of units $r_F^\times$ embedded diagonally into $(\widehat{r}_F)^\times \times (\widehat{r}_F)^\times$. We have an isomorphism

$$((\widehat{r}_F)^\times \times (\widehat{r}_F)^\times) \Big/ \overline{r_F^\times} \cong (\widehat{r}_F)^\times \times \left((\widehat{r}_F)^\times \Big/ \overline{r_F^\times}\right) \tag{2.1}$$

[7] There should be a minor modification of this statement for residually reducible p-adic representations.

[8] At the talk at RIMS conference and at another talks, I explained that the Iwasawa Main Conjecture in the third generation is already proved in [HTU97] thanks to celebrated "$R = T$ theorem". However, as the authors of [HTU97] remark in page 11122, they can only show the Iwasawa Main Conjecture by replacing the analytic p-adic L-function by another function whose relation to the real p-adic L-function is not clear to them. Hence, Theorem C in the next section might be the only supporting result for Conjecture C in Iwasawa Theory of the third generation known at the moment.

induced by

$$(\widehat{r}_F)^\times \times (\widehat{r}_F)^\times \xrightarrow{\sim} (\widehat{r}_F)^\times \times (\widehat{r}_F)^\times,\ (a,b) \mapsto (ab^{-1}, a).$$

We will denote the first factor $(\widehat{r}_F)^\times$ by G_1 and denote the second factor $\left((\widehat{r}_F)^\times \big/ \overline{r_F^\times}\right)$ by G_2 in the right hand side of (2.1). We remark that, if we assume the Leopoldt conjecture, the p-Sylow subgroup of G_2 is naturally identified with the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of F. If we denote by Z_{tor} the largest finite subgroup of Z, we have:

$$\mathcal{O}[[Z]] \cong \mathcal{O}[Z_{\mathrm{tor}}] \otimes_{\mathcal{O}} \mathcal{O}[[Z/Z_{\mathrm{tor}}]]$$

The complete group algebra $\Lambda := \mathcal{O}[[Z/Z_{\mathrm{tor}}]]$ is non-canonically isomorphic to a power series algebra $\mathcal{O}[[X_1, \cdots, X_d, Y_1, \cdots, Y_{1+\delta}]]$ where δ is the Leopoldt defect for F and p which is conjectured to be zero by Leopoldt conjecture.

Let us fix an ideal $\mathfrak{N} \subset r_F$ which is prime to (p). Hida ([H88], [H89]) constructs an algebra $\mathcal{H}_{\mathfrak{N}}$ which is finite and torsion-free over Λ and is called the nearly ordinary Hecke algebra of level $\mathfrak{N}p^\infty$. $\mathcal{H}_{\mathfrak{N}}$ is a semi-local algebra

$$\mathcal{H}_{\mathfrak{N}} = \prod \mathcal{H}_{\overline{\rho}}$$

indexed by the set of mod p Hecke eigen systems $\overline{\rho}$ of level $\mathfrak{N}p$ of $GL(2)_{/F}$. Note that these $\overline{\rho}$ are not necessarily the reduction modulo p of a certain ρ. However, according to custom, we will denote a given mod p representation by $\overline{\rho}$ even if we have no specific choice of a lifting ρ.

We denote by $\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}$ the quotient of $\mathcal{H}_{\overline{\rho}}$ corresponding to forms which are new at all primes dividing $\mathfrak{N}$. $\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}$ is a local Noetherian ring without nilpotent elements which is finite and torsion-free over Λ. The theory is different between mod p Hecke eigen systems $\overline{\rho}$ which are congruent to an Eisenstein series and those which are non-Eisenstein. From now on, we choose and fix a non-Eisenstein mod p Hecke eigen system $\overline{\rho}$ and we study the Iwasawa theory on a standard Galois deformation on $\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}$ [9].

In order to introduce the Hida deformation, we recall the basic notion of the arithmetic points.

Definition 2.1. Let $(w_1, \cdots, w_d, j) \in \mathbb{Z}^d \times \mathbb{Z}$. A ring homomorphism

$$\kappa :\ \mathcal{O}[[G_1 \times G_2]] \longrightarrow \overline{\mathbb{Q}}_p$$

is called an arithmetic point of weight $(w_1, \cdots, w_d, j)$ if it satisfies the following conditions:

[9] For the Iwasawa theory for eigen systems which are Eisenstein mod p, we refer to [Oc08] for some results and problems.

1. $\kappa|_{G_1}$ coincides with the character:

$$(\widehat{r}_F)^\times = \varprojlim_n (r_F/p^n)^\times \longrightarrow \varprojlim_n (\overline{\mathbb{Z}}/p^n)^\times \longrightarrow \varprojlim_n (\mathcal{O}_{\mathbb{C}_p}/p^n)^\times, \ x \mapsto \prod (x^{\iota_i})^{w_i},$$

modulo a finite character of $\left(G_1 \Big/ \overline{r_F^\times}\right)$. Here, x is regarded as a projective system $\{x_n\}$ of elements $x_n \in r_F$ such that $x_n \equiv x_{n+1} \bmod p^n$.

2. $\kappa|_{G_2}$ coincides with $\chi^j \psi_\kappa$ where χ is the cyclotomic character and ψ_κ is a finite character of G_2.

For an algebra R finite over $\mathcal{O}[[G_1 \times G_2]]$, a ring homomorphism $\kappa: \ R \longrightarrow \overline{\mathbb{Q}}_p$ is called an arithmetic point of weight $(w_1, \cdots, w_d, j)$ if $\kappa|_{\mathcal{O}[[G_1 \times G_2]]}$ is an arithmetic point of weight $(w_1, \cdots, w_d, j)$.

Theorem 2.2 (Hida). *Suppose that $\overline{\rho}$ is a non-Eisenstein mod p Hecke eigen system. Then, there is a free $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$-module $\mathcal{T}_{\overline{\rho}}^{\mathrm{new}}$ of rank two on which Galois group G_F acts continuously and $\mathcal{T}_{\overline{\rho}}^{\mathrm{new}}$ satisfies the following properties:*

1. *For each arithmetic point $\kappa \in \mathrm{Spec}_{\mathrm{cont}}(\mathcal{H}_{\overline{\rho}}^{\mathrm{new}})$ of weight $(w_1, \cdots, w_d, j) \in \mathbb{Z}^d \times \mathbb{Z}$ such that $w_i \geq 0$ and that w_i's have the same parity for all i, there exists an ordinary eigen cuspform f_κ of weight $(k_1, \cdots, k_d) = (w_1, \cdots, w_d) + (2, \cdots, 2)$ and $\mathcal{T}_{\overline{\rho}}^{\mathrm{new}} \otimes_{\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}} \kappa(\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}) \cong T_{f_\kappa} \otimes \kappa|_{G_2}$ where T_{f_κ} is the Galois representation associated to f_κ.*

2. *For each prime $\wp$ of F dividing p, there is a filtration stable under the decomposition group $D_\wp$:*
$$0 \longrightarrow F_\wp^+ \mathcal{T}_{\overline{\rho}}^{\mathrm{new}} \longrightarrow \mathcal{T}_{\overline{\rho}}^{\mathrm{new}} \longrightarrow \mathcal{T}_{\overline{\rho}}^{\mathrm{new}} / F_\wp^+ \mathcal{T}_{\overline{\rho}}^{\mathrm{new}} \longrightarrow 0,$$
where $F_\wp^+ \mathcal{T}_{\overline{\rho}}^{\mathrm{new}}$ and $\mathcal{T} / F_\wp^+ \mathcal{T}_{\overline{\rho}}^{\mathrm{new}}$ are free of rank one over $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$.

We will remark on the relation of the Hida deformation introduced in Theorem 2.2 to the setting III of §1. Note that $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$ has an injection:

$$\mathcal{H}_{\overline{\rho}}^{\mathrm{new}} \hookrightarrow \prod_i R_i$$

where R_i runs through quotients of $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$ by prime ideals of height 0. The number of such R_i's are finite and each R_i is a local domain which is finite over Λ. We call such an R_i *a branch of* $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$. Note that any arithmetic point $\kappa \in S$ of $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$ factors through one of R_i.

We put $\mathcal{R}$ to be a branch of $\mathcal{H}_{\overline{\rho}}^{\mathrm{new}}$ and S to be the set of arithmetic points of R whose weights $(w_1, \cdots, w_d, j)$ satisfy the inequality:

$$\frac{w_{\max} - w_{\min}}{2} \leq j - 1 \leq \frac{w_{\max} + w_{\min}}{2},$$

where $w_{\max}$ (resp. $w_{\min}$) is the maximal one (resp. minimal one) among $\{w_i\}_{1\le i\le d}$. We define $\mathcal{T}$ to be $\mathcal{T}^{\mathrm{new}}_{\overline{\rho}} \otimes_{\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}} \mathcal{R}$.

For this triple $(\mathcal{R}, \mathcal{T}, S)$, S is Zariski dense in $\mathrm{Spec}_{\mathrm{cont}}(\mathcal{R})$. For the condition (**Geom**), motives corresponding to T_f are constructed by [BR93] for large class of Hilbert modular forms. For $F = \mathbb{Q}$, the condition (**Geom**) is always true thanks to Scholl. The conditions (**Pan**) and (**Crit**) are true by Theorem 2.2.

§ 3. Known results for the 3rd generation when $F = \mathbb{Q}$

For Conjecture A, we have the following theorem:

Theorem A ([Oc01], [Oc06a]). Let $\mathcal{R}$ be a branch of $\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}$. Suppose that $F = \mathbb{Q}$. Then, $\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}$ is a torsion $\mathcal{R}$-module.

Outline of Proof. For any arithmetic point $\kappa \in S$, we have the restriction map:

$$\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}} \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow \mathrm{Sel}^{\mathrm{PD}}_{A_{f_\kappa} \otimes \kappa|_{G_2}}, \tag{3.1}$$

where $\mathrm{Sel}_{A_{f_\kappa} \otimes \kappa|_{G_2}}$ is the Selmer group for a single ordinary cuspform f_κ defined in the same way as in the case of the Selmer group $\mathrm{Sel}_{\mathcal{A}}$ for family of ordinary cuspforms. By "the control theorem of the Selmer group for Hida deformation" proved in [Oc01] and [Oc06a] which is a generalization of Mazur's control theorem [Mz72] for the cyclotomic deformation of elliptic curves, the kernel and the cokernel of (3.1) are finite except the case when f_κ has weight 2 and it is Steinberg at p. On the other hand, Kato [Ka04] proves that $\mathrm{Sel}^{\mathrm{PD}}_{A_{f_\kappa} \otimes \kappa|_{G_2}}$ is finite when the special value $L(f_\kappa, \kappa|_{G_2}, 0)$ is non-zero. Note that $L(f_\kappa, \kappa|_{G_2}, 0) \neq 0$ when the weight of the Hecke character $\kappa|_{G_2}$ is different from the half of the weight of the cusp form f_κ and that the nearly ordinary Hida deformation contains always such $\kappa \in S$. Thus, we prove Theorem A by Nakayama's lemma. □

For Conjecture B on the existence of p-adic L-function, we proved in [Oc06b] that we can modify Kitagawa's p-adic L-function (cf. [Ki94]) by a unit of $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ so that the obtained p-adic L-function satisfies a more canonical interpolation property replacing Kitagawa's p-adic periods by p-adic periods defined by the comparison isomorphism of the p-adic Hodge theory.

Theorem B ([Oc06b]). Let $\mathcal{R}$ be a branch of $\mathcal{H}^{\mathrm{new}}_{\overline{\rho}}$. Assume the following condition:

(SL) The image of the residual representation $G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ of $\mathcal{T}$ contains $SL_2(\mathbb{F})$.

Then, the extension of Kitagawa's p-adic L-function $L_{p,\mathcal{T}} \in \mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ originally constructed in $\mathcal{R}$ satisfies the following interpolation property for each $\kappa_{f_\kappa,\chi^j\phi} \in S$ associated to a cusp form f_κ of weight $w+2$, an integer j and a finite character of $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with $1 \leq j \leq w+1$:

$$\frac{\kappa_{f_\kappa,\chi^j\phi}(L_{p,\mathcal{T}})}{C_{f_\kappa,p}} = (-1)^j(j-1)!U(f_\kappa,j,\phi)G(\phi^{-1}\omega^{1-j})\frac{L(f_\kappa,\phi\omega^{1-j},j)}{(2\pi\sqrt{-1})^j C_{f_\kappa,\infty}}$$

where $C_{f_\kappa,p} \in \mathcal{O}_{\mathbb{C}_p}^\times$ (resp. $C_{f_\kappa,\infty} \in \mathbb{C}^\times$) is a p-adic period (resp. a complex period) for f_κ introduced in Conjecture B for the motive $M_\kappa = M_{f_\kappa}$ and $G(\phi\omega^{-j})$ is the Gauss sum. Here, $U(f_\kappa,j,\phi)$ is defined as follows:

$$U(f_\kappa,j,\phi) = \begin{cases} \left(1 - \dfrac{p^{j-1}}{a_p(f_\kappa)}\right) & \text{if } \phi \text{ is trivial,} \\ \left(\dfrac{p^{j-1}}{a_p(f_\kappa)}\right)^{\mathrm{ord}_p \mathrm{Cond}(\phi)} & \text{otherwise.} \end{cases}$$

Remark 3.1. There are two-variable p-adic L-functions in $\mathcal{R}$ (or in the fraction field of $\mathcal{R}$) similar to that of [Ki94] by Greenberg-Stevens[GS93] and Ohta (unpublished) by modular symbol method and by Panchishkin, Fukaya[Fu03] and myself [Oc03] by Rankin-Selberg method. However, because of subtle but essential problems on the definition of complex periods as remarked in the introduction of [Oc03], it is not clear that these p-adic L-functions coincide with the one obtained in Theorem B modulo units of $\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$.

In general there is a notion of *a Hida deformation with complex multiplication* (CM) which is defined (or characterized) by the behavior of the Fourier coefficients with respect to the twist by Dirichlet characters or by the size of the image of the Galois representation for $\mathcal{T}$. For the Iwasawa Main Conjecture (Conjecture C), Hida deformations $\mathcal{T}$ with complex multiplication and Hida deformations $\mathcal{T}$ without complex multiplication are studied by completely different approach, while Conjectures A and B are insensitive to such a difference.

Though Hida deformations with complex multiplication are easier than Hida deformations without complex multiplication, even the case with complex multiplication is not completely understood yet [10].

Theorem C ([Oc03], [Oc05], [Oc06a], [Oc06b] and [OP])**.**

1. **(CM case)** Suppose that our Hida deformation $\mathcal{T}$ has complex multiplication by an imaginally quadratic field K. Let us assume that there is an arithmetic point

[10] We remark that the equivalence between Rubin's theorem on Two-variable Iwasawa Main Conjecture and our Two-variable Iwasawa Main Conjecture (Conjecture C) in the setting of Hida deformation is not yet established, which is pointed out in [Oc07] and [OP]

$\kappa \in S$ such that the Iwasawa μ-invariant of the cyclotomic p-adic L-function for f_κ constructed by Mazur-Tate-Teitelbaum is trivial. Then, Two-variable Iwasawa Main Conjecture for $\mathbb{Z}_p^2$-extension of K proved by Rubin is equivalent to Two-variable Iwasawa Main Conjecture formulated by Kitagawa's two-variable p-adic L-function.

2. (non CM case) Suppose the condition (**SL**) and the following condition:

(Reg) $\mathcal{R}$ is a regular local ring.

Then, we have the following inequality [11]:

$$\mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}) \supset (L_{p,\mathcal{T}}).$$ [12]

Outline of Proof. For the proof in the CM case, we refer the reader to the papers [OP] and [Oc07]. However, we recall that the main issue (which is (d) in the diagram below) is to show that, our Two-variable p-adic L-function for $\mathcal{T}$ is equal to Katz's Two-variable p-adic L-function for imaginary quadratic field K under the above assumption. The situation is summarized in the following diagram:
(3.2)

$$\begin{tikzcd}
\text{Char. ideal for certain Galois gp.} \arrow[r, equal, "(a)"] \arrow[d, equal, "(c)"'] & \text{Katz's Two-variable p-adic L-funct.} \arrow[d, dotted, dash, "(d)"] \\
\mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}) \arrow[r, dotted, dash, "(b)"'] & (L_{p,\mathcal{T}}).
\end{tikzcd}$$

For non-CM case, we use the Beilinson-Kato Euler system which is extended to Hida deformations. In order to relate two objects of totally different nature, we need an intermediate object:

$$\mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}) \overset{(1)}{\cdots\cdots} \text{intermidiate object} \overset{(2)}{\cdots\cdots} (\mathcal{R}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})/(L_{p,\mathcal{T}}). \tag{3.3}$$

In our case, we consider $H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1))/\mathcal{Z}$ as an intermediate object in question, where $\mathcal{Z}$ is a projective limit of linear combinations of Beilinson-Kato element over modular curves of p-power level, which is defined in [Oc06a]. The element $\mathcal{Z}$ is known to be sent to the p-adic L-function $L_{p,\mathcal{T}}$ via a generalized Perrin-Riou map $H^1_{/f}(\mathbb{Q}_p, \mathcal{T}^*(1)) \longrightarrow \mathcal{R}$ constructed in [Oc03]. Thus, we prove the equality for (2) of the diagram (3.3).

Beilinson-Kato element $\mathcal{Z} = \mathcal{Z}(1)$ is a part of system $\{\mathcal{Z}(r) \in H^1_{/f}(\mathbb{Q}_p(\zeta_r), \mathcal{T}^*(1))\}_r$ where r runs through a set of square-free natural numbers and $\mathcal{Z}(r)$ satisfies a certain

[11] In [Del08, p. 250], Delbourgo gives an erroneous comment that our result on Two-variable Iwasawa Main Conjecture is incomplete because of delicate problems on periods posed by ourself in [Oc03]. However, these problems are already solved by ourself in [Oc06b, §6.3].

[12] Note that the modification factor $e_{\mathcal{A}}$ is trivial in this case.

norm compatible condition. Kolyvagin's method of Euler system generalized by [Ka99], [Pe98] and [R] allows us to bound the size of Selmer group associated to the cyclotomic deformation of a p-adic Galois representation. However, the proof of [Ka99], [Pe98] and [R] work only for cyclotomic deformations and can not be applied to the analogous statement for more general Galois deformations like our two-variable Hida deformations.[13] When $\mathcal{R}$ is an Iwasawa algebra of one-variable, Mazur-Rubin [MR04, Chapter 5] and our result [Oc05] provide independently a technique to establish the Euler system theory which work for non-cyclotomic deformations. (The method of [MR04, Chapter 5] and that of [Oc05] are essentially the same in the case of one-variable.) However, when the number of variables of $\mathcal{R}$ is greater than one, the method for the case of one-variable does not work and we find no other references. We develop a different method for the proof of these general cases, for which we refer the reader to [Oc05]. □

We remark that we have the following immediate corollary to Theorem C using control theorem for Hida deformation (cf. [Oc06a, Corollary 7.5]):

Corollary 3.2 ([Oc06a]). *Assume the conditions* (**SL**) *and* (**Reg**) *for our* $\mathcal{T}$. *Then, we have the following:*

1. *If the cyclotomic Iwasawa Main conjecture holds for a single cuspform* f_0 *in the Hida family* $\mathcal{T}$, *the following equality of Two-variable Iwasawa Main Conjecture is true:*
$$\mathrm{char}(\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}}) = (L_{p,\mathcal{T}}).$$
2. *If the cyclotomic Iwasawa Main conjecture holds for a single cuspform* f_0 *in the Hida family* $\mathcal{T}$, *the cyclotomic Iwasawa Main conjecture holds for every cuspforms* f *in the Hida family* $\mathcal{T}$.

Note that [EPW06] also obtains the second statement of the above corollary. The difference is that [EPW06] essentially requires to assume $\mu = 0$ conjecture for f_0, but in our case we assume no assumption on the μ-invariant.

§ 4. Results over general totally real fields F

Iwasawa Theory for nearly ordinary Hida deformations for $F = \mathbb{Q}$, though it is not completely solved yet, seems well-understood through the works introduced in the

[13] In fact, the cyclotomic deformation is regarded as a family of $H^1(\mathbb{Q}(\zeta_{p^n}), T^*(1))$ for a usual p-adic Galois representation over a discrete valuation ring where Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^n}))$ varies. The advantage of this case is that the coefficient ring of Galois cohomology is always a DVR, for which [Ka99], [Pe98] and [R] can use the Chebotarev density theorem to choose carefully a sequence of square-free numbers r related to the order of torsion elements in the Galois cohomology. For more general deformations which are not cyclotomic, the base ring $\mathcal{R}$ is not a DVR anymore and the Chebotarev density theorem can not count the structure of $\mathcal{R}$.

previous section. We are also interested in generalizing some of results to general totally real fields F. Deformation spaces of such generalizations have a much bigger dimension as is explained in §2 and each index of multi-weight of Hilbert modular forms can move separately. Hence studying such a generalization seems important and interesting.

On the other hand, there are essential difficulties when we pass from $\mathbb{Q}$ to general F. We recall two of the biggest difficulties:

1. Firstly, Beilinson-Kato elements play an important role for our results for $F = \mathbb{Q}$. However, there are essential difficulties on an analogous construction of these Beilinson-Kato elements for general totally real fields.

2. Secondly, Hida theory over totally real fields is much more complicated than Hida theory over $\mathbb{Q}$. Over $\mathbb{Q}$, the nearly ordinary deformation (which is of two variables) is nothing but the composite of the ordinary deformation (which is of one variable) and the cyclotomic deformation (which is of one variable). However, over a totally real field F of degree d (assuming the Leopoldt conjecture for simplicity), the nearly ordinary deformation (which is of $d+1$ variables) is greater than the composite of the ordinary deformation (which is of one variable) and the cyclotomic deformation (which is of one variable), which makes the study of the nearly ordinary deformation more difficult for general F. We remark that the difference above is also related to the existence of global units of F.

From now, we will review our results and idea in relation with such difficulties.

For Conjecture A, we have a conditional result which is a joint work with Olivier Fouquet. As we discussed in the case of $F = \mathbb{Q}$, establishing Control theorem of Selmer group is an important step to prove Conjecture A.

Theorem 4.1 (Fouquet-Ochiai). *Suppose that $\mathcal{R}$ is regular. Let κ be an arithmetic point of $\mathcal{R}$. The kernel and the cokernel of*

$$\mathrm{Sel}^{\mathrm{PD}}_{\mathcal{A}} \otimes_{\mathcal{R}} \kappa(\mathcal{R}) \longrightarrow \mathrm{Sel}^{\mathrm{PD}}_{A_{f_\kappa} \otimes \kappa|_{G_2}}$$

are finite except the case where the weight of f_κ is $(2, \cdots, 2)$ and f_κ is locally of Steinberg type at one of the primes over p.

Now, using Control theorem above, we expect to generalize our Theorem A of the previous section to Hilbert modular Hida family of general totally real fields. One of the problems for this goal is that it seems difficult to construct the analogue of Beilinson-Kato Euler system for various geometric reasons. On the other hand, Euler system of Heegner points which also existed in the elliptic modular cases are generalized to the Hilbert modular cases. Unfortunately, Euler system of Heegner points exists only on the central critical arithmetic points κ where the weight of the Hecke character $\kappa|_{G_2}$ is equal to the half of the weight of the cusp form f_κ.

Corollary 4.2. *Suppose that there exists an arithmetic point κ such that*

1. *f_κ is of weight $(2, \cdots, 2)$.*
2. *$L(f_\kappa, 1) \neq 0$.*

Then, Conjecture A is true.

The proof goes in the same way as the proof of Theorem A in the previous section. For general totally real fields F, we find a κ satisfying the above conditions only when the sign of the functional equation for modular forms in our Hida family is $+1$.

In order to state our result on Conjecture B, we introduce the following conditions:

($\star$) ρ is congruent to a cusp form of weight $(w_1, \cdots, w_d)+(2, \cdots, 2)$ whose conductor is prime to p and which satisfies the following inequality:

$$\sum_{1 \leq i \leq d} (w_i + 1) < p - 1.$$

($\star\star$) By abuse of notation, let us denote also by $\overline{\rho}$ the mod p Galois representation of G_F associated to the mod p Hecke eigen system $\overline{\rho}$. The representation $\underset{\tau \in J_F}{\otimes} \overline{\rho}(\tau^{-1} \cdot \tau)$ of the absolute Galois group $G_{\widetilde{F}}$ is irreducible of order divisible by p, where $\widetilde{F}$ denotes the Galois closure of F in $\overline{\mathbb{Q}}$.

The following result for the existence of the p-adic L-function is the analogue of Kitagawa's result [Ki94] over $\mathbb{Q}$, with which we solved Conjecture B under certain conditions (cf. Theorem B):

Theorem 4.3 (Dimitrov-Ochiai [DO])**.** *Suppose that ρ satisfies ($\star$) and ($\star\star$). Let us fix a basis of Module of Λ-adic modular symbols* [14] *over certain d-variable Hecke algebra. Then, there exists a p-adic analytic L-function $L_{p,\mathcal{T}} \in \mathcal{R}$ satisfying the following interpolation property:*
For every arithmetic point $\kappa_{f_\kappa, \chi^j \phi}$ of $\mathcal{R}$ corresponding to f_κ of weight $k = (w_1, \cdots, w_d)+(2, \cdots, 2)$, a finite character ϕ of the p-Sylow subgroup of $\mathrm{Cl}_F^+(p^\infty)$ and an integer j satisfying the condition:

$$\frac{w_{\max} - w_{\min}}{2} + 1 \leq j \leq \frac{w_{\max} + w_{\min}}{2} + 1, \tag{4.1}$$

we have the following interpolation property:

$$\frac{\kappa_{f_\kappa, \chi^j \phi}(L_{p,\mathcal{T}})}{\Omega^\epsilon_{f_\kappa, p}} = (-1)^{dj^*} \Gamma_{f_\kappa}(j) \times \prod_{\mathfrak{p} \mid p} U_{\mathfrak{p}}(f_\kappa, j, \phi) \times G(\phi^{-1}\omega^{-j^*}) \frac{L(f_\kappa, \phi\omega^{-j^*}, j)}{(2\pi\sqrt{-1})^{dj^*} C^\epsilon_{f_\kappa, \infty}},$$

[14] Since the definition of Module of Λ-adic modular symbols is not essential to understand the result, we omit the definition here.

where $j^* = j - \left(\dfrac{w_{\max} - w_{\min}}{2} + 1\right)$, $\Omega^{\epsilon}_{f_\kappa, p} \in \overline{\mathbb{Z}_p}^{\times}$ *is a* p*-adic error term,* $\Gamma_{f_\kappa}(s)$ *is the* Γ*-factor for* f_κ *and* $U_{\mathfrak{p}}(f_\kappa, j, \phi)$ *is defined as follows*:

$$U_{\mathfrak{p}}(f_\kappa, j, \phi) = \begin{cases} \left(1 - \dfrac{N_{F/\mathbb{Q}}(\mathfrak{p})^{j^*}}{a_{\mathfrak{p}}(f_\kappa)}\right) & \text{if } \mathfrak{p} \nmid \mathrm{Cond}(\phi), \\ \left(\dfrac{N_{F/\mathbb{Q}}(\mathfrak{p})^{j^*}}{a_{\mathfrak{p}}(f_\kappa)}\right)^{\mathrm{ord}_{\mathfrak{p}} \mathrm{Cond}(\phi)} & \text{if } \mathfrak{p} \mid \mathrm{Cond}(\phi). \end{cases}$$

Our method of proof is based on the interpolation of the higher dimensional modular symbols on Hilbert modular variety, which is the analogue of classical modular symbol on modular curves (see [Od82] and [Mn76] for the references on modular symbols on Hilbert modular variety).

Remark 4.4.

1. The p-adic error term $\Omega^{\epsilon}_{f,p}$ depends on the choice of bases of $H_{\mathrm{Betti}}(M_f)^+$ and $\mathrm{Fil}^0 H_{\mathrm{dR}}(M_f)$ as well as fixed basis of Module of Λ-adic modular symbols. However, a pair $(\Omega^{\epsilon}_{f,p}, C^{\epsilon}_{f,\infty})$ has the same kind of cancelation property as in Remark 1.1. Hence the interpolation given in Theorem 4.3 is well-defined independently of the choice of bases of $H_{\mathrm{Betti}}(M_f)^+$ and $\mathrm{Fil}^0 H_{\mathrm{dR}}(M_f)$.

2. The p-adic L-function $L_{p,\mathcal{T}} \in \mathcal{R}$ depends on a fixed basis of "Module of Λ-adic modular symbols". However, if we change this basis, $L_{p,\mathcal{T}} \in \mathcal{R}$ is multiplied only by a unit of $\mathcal{R}$.

3. We expect to improve this p-adic L-function into the p-adic L-function with "real" p-adic periods as in Theorem B in the case of elliptic modular Hida deformation.

We will also remark on the proof and other known results:

Remark 4.5.

1. We recall the following known results:
   (a) For the one-variable (cyclotomic) p-adic L-function of Hilbert modular forms, Manin [Mn76] (resp. Dabrowski [Da94]) constructs it by the method of higher dimensional modular symbols on Hilbert modular variety (resp. by the Rankin-Selberg method).
   (b) Mok [Mo07] constructs a two-variable p-adic L-function on the two-variable quotient of $\mathcal{R}$ which represents the ordinary family of Hilbert modular forms of parallel weight (of one variable) and its cyclotomic deformation (of one variable). The construction of [Mo07] is done by Rankin-Selberg method using a family of Eisenstein series.

2. The idea which enables us to treat the whole nearly ordinary deformation (of $d+1$ variables) not only on the subspace of two variables is the use of the Hida deformation of the level structure $ZK_{11}(p^m)$ which contains the center $Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in (r_F \otimes \mathbb{Z}_p)^\times \right\}$.
3. The two assumptions ($\star$) and ($\star\star$) are used to show the vanishing of the torsion part of (certain part of) the Betti cohomology $H^d(Y_{11}(\mathfrak{N}p^m), \mathcal{L}(w, v; \mathcal{O}))$ and the vanishing of (certain part of) $H^i(Y_{11}(\mathfrak{N}p^m), \mathcal{L}(w, v; \mathcal{O}))$ ($i \neq d$) where $Y_{11}(\mathfrak{N}p^m)$ is the Hilbert modular variety of dimension d with level $K_{11}(\mathfrak{N}p^m)$ and $\mathcal{L}(w, v; \mathcal{O})$ is the local system on $Y_{11}(\mathfrak{N}p^m)$ corresponding to $\bigotimes_{\tau \in J_F} \mathrm{Sym}^{w_\tau} \otimes \det^{v_\tau}$. Such a vanishing theorem was shown in [Di05].

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§ 5. References

In this section, we list up references for Iwasawa Theory, especially those who are related to the subject of this article.

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