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Euler systems in the Iwasawa theory of ordinary modular forms

To Gerhard Frey, with gratitude

By

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Abstract

We formulate conjectures linking Selmer structures for Hida-theoretic and Iwasawa-theoretic families of ordinary eigenforms to Euler systems built from generalized Heegner points on towers of modular curves. We prove part of these conjectures using the method of Euler systems, control theorems and descent for Selmer complexes.

§ 1. Introduction

Let p be an odd prime and E be an elliptic curve over $\mathbb{Q}$ of conductor N with good ordinary reduction at p. Let K be a quadratic imaginary number field such that all primes dividing N split in K and let K_∞ be the $\mathbb{Z}_p$-extension of K such that $\mathrm{Gal}(K_\infty/\mathbb{Q})$ is a pro-dihedral group. The complex L-function $L(E,s)$ of E being equal to the complex L-function of a modular form by [21], the completed L-function $L(E/K,s)$ is equal (up to the conventional choice of the central line) to the Rankin-Selberg L-function of the base change to K of the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ attached to f. Hence, it satisfies the functional equation:

$$L(E/K,s) = \epsilon(E/K,s)L(E/K,2-s).$$

The requirement on the behavior of primes dividing N forces $\epsilon(E/K,1)$ to be -1, and thus $L(E/K,1)$ to vanish at an odd order r. When r is exactly equal to 1, the

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general formalism of the Tamagawa Number Conjecture predicts that this vanishing is accounted for by a point z of infinite order on $E(K)$ and that the special value of $L'(E/K,s)$ at 1 is linked to the height of z, and it is indeed known by [7, 13] that the projection of Heegner points from the modular curve $X_0(N)$ to E give rises to such a point. Even when r is greater than 1, it was conjectured by Mazur and proved by C.Cornut and V.Vatsal that the generic order of vanishing of $L(E,\chi,s)$ at 1 is equal to 1, where $L(E,\chi,s)$ is the L-function of the Rankin-Selberg product of E with a finite order character of $\mathrm{Gal}(K_\infty/K)$. The Equivariant Tamagawa Number Conjecture (or ETNC for short) then predicts that the determinant of a well-chosen cohomology complex with coefficients in the anticyclotomic Iwasawa algebra $\Lambda_{anti}=\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ (here and elsewhere in the text, the subscript *anti* is meant to suggest the word anticyclotomic) should be trivialized by the p-adic height of a generic Heegner point. The Iwasawa Main Conjecture in this context, which is a p-adic variant of the Birch and Swinnerton-Dyer conjecture, then states that this trivialization should coincide with the Rankin-Selberg p-adic L-function $L'_p(E)\in\Lambda$.

Using again the modularity of E and appealing to Hida theory, we see that the G_K-representation T_pE also belongs to a p-adic family of modular Galois representations parametrized by weights, or equivalently that T_pE occurs as a specialization of the inverse limit on s of the ordinary part $e^{ord}H^1_{et}(X_1(Np^s),\mathbb{Z}_p)(1)$ of the étale cohmology of the tower of modular curves $X_1(Np^s)$. The ETNC then predicts again that there exists a p-adic L-function $L_p^{Hi}(k)$ interpolating the value at $s=k/2$ of $L(f_k,s)$, that L_p^{Hi} is an element of the ordinary Hecke algebra $\mathbf{T}^{\mathrm{ord}}_\infty$ and that it induces a trivialization of the determinant of a suitably chosen cohomology complex coinciding with the trivialization given by the p-adic height of a family of Heegner points.

One drawback of the ETNC is that it itself requires several hard conjectures even to be formulated, for instance the existence of a motivic cohomology theory with good properties and the non-degeneracy of certain height pairings. The aim of this article is to build a conceptual framework allowing for a study of these questions independently of any conjectures. The first section is devoted to recalling necessary facts mostly due to H.Hida and B.Howard. Then, we state variants of the ETNC involving Euler systems, examine their compatibilities by specialization in theorem 2.8 and give indications of their proofs in theorem 2.9. Theorem 2.8 is a consequence of [6], which is a joint work with T.Ochiai. The complete proof of 2.9, under the hypotheses of this introduction but also in a significantly more general context, is the object of [5]. In the last section, we mention generalizations to totally real fields F and to Galois representations coming from automorphic representation of the adelic point of the multiplicative group of a quaternion algebra over F split at at most one infinite place.

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§ 2. The ETNC for p-adic families of modular G_K-representations

§ 2.1. Galois representations attached to ordinary eigenforms

General notations Let $\widehat{\mathbb{Z}}$ be the pro-finite completion of $\mathbb{Z}$. For an abelian group G, let $\widehat{G}$ be $G\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$. Fix $p\geq 5$ a prime and embeddings of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and of $\bar{\mathbb{Q}}_p$ into $\mathbb{C}$. For F a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_\ell$, let G_F be the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. The symbol Fr denotes geometric Frobenius morphism.

Let N be a positive integer prime to p and $k\geq 2$ an integer. The ring of diamond operators of level p^s is the ring $\mathbb{Z}_p[(\mathbb{Z}/p^s\mathbb{Z})^\times]$. Let $\mathbf{T}_k(N,s)$ be the image of the $\mathbb{Z}_p$-algebra generated by all Hecke operators $T(\ell)$ with $\ell\nmid N$ and all diamond operators $<a>$ of level p^s inside the endomorphism ring of the cuspforms of weight k for the congruence subgroup $\Gamma_0(N)\cap\Gamma_1(p^s)$. Let ϕ be a Dirichlet character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $f\in S_k(\Gamma_0(Np),\phi)$ be an eigencuspform for the congruence group $\Gamma_0(N)\cap\Gamma_1(p)$ and let λ_f be the map such that for all $T(\ell)\in\mathbf{T}_k(N,s)$ and all $<a>$, $\lambda_f(T(\ell))=a_\ell(f)$ and $\lambda_f(<a>)=\phi(a)$. Let $L_{\mathfrak{p}}$ be a fixed finite extension of $\mathbb{Q}_p$ containing the image of λ_f, let $\mathcal{O}$ be its ring of integers and $\mathbb{F}$ the residue field of $\mathcal{O}$. Let $\mathbf{T}_k(N,s,\mathcal{O})$ be $\mathbf{T}_k(N,s)\otimes_{\mathbb{Z}_p}\mathcal{O}$. We assume that f is new outside p and that it is ordinary at p, *i.e* that $a_p(f)$ belongs to $\mathcal{O}^\times$.

Let K be a quadratic imaginary extension of $\mathbb{Q}$ in which all primes dividing N split. Let Σ be a finite set of finite places containing all places above Np and let $G_{K,\Sigma}$ be the Galois group of the maximal extension of K unramified outside Σ.

For G equal to a quotient of $G_{K,\Sigma}$ or G_{K_v} and for M a p-adic representation of G, let $C^\bullet_{\mathrm{cont}}(G,M)$ be the complex of continuous cochains with values in M. Whenever a complex appears in this article, it is a bounded complex whether or not this is explicitly mentioned.

Galois representations of eigenforms To f is attached an odd semi-simple residual $G_{\mathbb{Q}}$-representation $(\overline{T}(f),\bar{\rho}_f)_{\mathbb{F}}$ and an absolutely irreducible p-adic $G_{\mathbb{Q}}$-representation $(V(f),\rho_f)_{L_{\mathfrak{p}}}$. The $G_{\mathbb{Q}}$-representation $V(f)$ is unramified outside Np. For $\ell\nmid Np$, the trace and determinant of the geometric Frobenius $\mathrm{Fr}(\ell)$ are equal respectively to $a_\ell(f)$ and to $\phi(\ell)\chi_{cyc}^{k-1}(\ell)$, or equivalently to $\lambda_f(T(\ell))$ and $\lambda_f(<\ell>)$. Henceforth, we consistently assume that $\bar{\rho}_f$ is irreducible, hence absolutely irreducible, and that f does not acquire residual complex multiplication, that is to say that $\bar{\rho}_f$ restricted to G_K is still absolutely irreducible. Then there is a unique p-adic representation $(T(f),\rho_f)_{\mathcal{O}}$

with the same trace as $V(f)$. We note that $T(f)$ can also be considered as a free rank 2 module over $\mathbf{T}_k(N,s)$.

As $V(f)$ is of dimension 2 over $L_{\mathfrak{p}}$, its dual $V(f)^*$ is isomorphic to $V(f)(k-1)\otimes\phi$. The form f being non-trivial, the characters χ_{cyc}^{k-2} and ϕ have the same parity so the character $\chi_{cyc}^{2-k}\phi$ factors through a group with no 2-torsion. Fix a character ψ such that $\psi^2=\chi_{cyc}^{k-2}\phi^{-1}$. We let $(V,\rho)_{L_{\mathfrak{p}}}$, $(T,\rho)_{\mathcal{O}}$ and $(\overline{T},\bar{\rho})_{\mathbb{F}}$ be respectively the self-dual $G_{\mathbb{Q}}$-representation obtained by twisting $V(f)(1)$, $T(f)(1)$ and $\overline{T}(f)(1)$ by ψ.

Let v be a place of $\mathcal{O}_K$ above p. The fact that a_p belongs to $\mathcal{O}^\times$ implies that the local G_{K_v}-representation T fits in a short exact sequence of non-trivial $\mathcal{O}[G_{K_v}]$-modules:

$$0\longrightarrow T_v^+\longrightarrow T\longrightarrow T_v^-\longrightarrow 0.$$

We assume that $\bar{\rho}_f$ is p-distinguished, that is to say that the semi-simplification of $\bar{\rho}_f$ restricted to G_{K_v} for $v|p$ is not scalar.

Review of Hida theory Let Γ be the torsion-free part of $\varprojlim_s(\mathbb{Z}/p^s\mathbb{Z})^\times$ and Λ be the regular local ring $\mathcal{O}[[\Gamma]]$, which is also the torsion-free part of the inverse limit on s of the ring of diamond operators. Let γ be a topological generator of Γ. We allow ourselves to consider γ as an element of $G_{\mathbb{Q}}$ using the fact that Γ is isomorphic to the Galois group of the $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For $k\geq 2$ an integer and ϵ a finite order character of Γ, an arithmetic point of weight k and character ϵ of Λ is an $\mathcal{O}$-algebra morphism:

$$\begin{aligned}\phi:\ \Lambda&\longrightarrow\bar{\mathbb{Q}}_p\ ;\\ \gamma&\longmapsto\epsilon(\gamma)\chi_{cyc}^{k-2}(\gamma).\end{aligned}$$

Here, γ is considered as an element of $G_{\mathbb{Q}}$ via the identification of Γ with the Galois group of the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For a finite Λ-algebra R, an arithmetic point of R is an $\mathcal{O}$-algebra morphism whose restriction to Λ coincides with an arithmetic point of Λ and an arithmetic prime is the kernel of an arithmetic point.

The ring $\mathbf{T}_k(N,s,\mathcal{O})$ is a finite, flat and reduced $\mathcal{O}$-algebra and an $\mathcal{O}[(\mathbb{Z}/p^s\mathbb{Z})^\times]$-algebra by the inclusion of the diamond operators. Let e^{ord} be Hida's projector, this is to say the idempotent:

$$e^{ord}=\lim_{n\to\infty}T(p)^{n!}.$$

The Hida ordinary Hecke algebra $\mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})$ is the inverse limit on s of $e^{ord}\mathbf{T}_k(N,s,\mathcal{O})$. It is a torsion-free Λ-algebra independent of k, finite as Λ-module. The $\mathcal{O}$-algebra morphism λ_f is an arithmetic point of $\mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})$ and conversely, arithmetic points of weight k of $\mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})$ are attached to ordinary eigenforms in $S_k(\Gamma_0(N)\cap\Gamma_1(p^s))$.

Let $\mathcal{P}_{\min}$ be a minimal prime of $\mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})$ and let $R_{\min}$ be $\mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})/\mathcal{P}_{\min}$. As arithmetic primes of fixed weight containing $\mathcal{P}_{\min}$ are Zariski-dense in $\operatorname{Spec}R_{\min}$, the patching argument of [20, Lemma 2.2.3] shows that there is a continuous $G_{\mathbb{Q}}$-pseudo-representation $\tilde{\rho}_{\min}$ with values in $R_{\min}$ interpolating the traces of the $G_{\mathbb{Q}}$-representations attached to eigencuspforms g such that $\ker \lambda_g$ contains $\mathcal{P}_{\min}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbf{T}^{\mathrm{ord}}_{\infty}(N, \mathcal{O})$ and $\mathfrak{a}$ the minimal prime contained in $\mathfrak{m}$ such that λ_f factors through the local domain:

$$R = \mathbf{T}^{\mathrm{ord}}_{\infty}(N, \mathcal{O})_{\mathfrak{m}}/\mathfrak{a}. \tag{2.1}$$

Patching again the pseudo-representations $\tilde{\rho}_{\min}$ for all $\mathcal{P}_{\min}$ contained in $\mathfrak{m}$ defines a pseudo-representation $\tilde{\rho}_{\mathfrak{m}}$ with values in R. As $\bar{\rho}_f$ is absolutely irreducible, [18, Theorem 1] shows that there is a $G_{\mathbb{Q}}$-representation $(\mathcal{T}(f), \rho_{\mathfrak{m}})_R$ free of rank 2 over R such that $\operatorname{Tr}(\rho_{\mathfrak{m}})$ is equal to $\tilde{\rho}_{\mathfrak{m}}$. The $G_{\mathbb{Q}}$-representation $\mathcal{T}(f)$ is unramified outside Np and for $\ell \nmid Np$, the characteristic polynomial of $\operatorname{Fr}(\ell)$ acting on $\mathcal{T}(f)$ is the same as the characteristic polynomial of $\operatorname{Fr}(\ell)$ acting on $T(f)$ with λ_f replaced by localization at $\mathfrak{m}$ and reduction modulo $\mathfrak{a}$.

In particular, the determinant of $\mathcal{T}(f)$ evaluated at $\operatorname{Fr}(\ell)$ for $\ell \nmid Np$ is equal to $\lambda_{\mathfrak{m}}(<\ell>)\chi_{cyc}(\ell)$. Observing that the obstruction of $\lambda_{\mathfrak{m}}(<\cdot>)$ to be a square depends only on the tame part of this character and that the tame part is the same for all specializations of $\mathcal{T}(f)$, we see that $\mathcal{T}(f)$ admits a self-dual twist $\mathcal{T}$ interpolating the T exactly as $\mathcal{T}(f)$ interpolates the $T(f)$.

Let v be a place of $\mathcal{O}_K$ above p. The local G_{K_v}-representation $\mathcal{T}$ fits in a short exact sequence of non-trivial $R[G_{K_v}]$-modules:

$$0 \longrightarrow \mathcal{T}_v^+ \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_v^- \longrightarrow 0.$$

Geometric realization of $\mathcal{T}(f)$ and control theorem for $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}$ We review important and well-known commutative algebra properties of $\mathcal{T}(f)$ and $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}$.

Assume first that f is of weight 2, which is a very mild assumption to make as $\mathcal{T}(f)$ certainly has plenty of specializations of weight 2. Then, the fact that $\bar{\rho}_f$ is p-distinguished and absolutely irreducible implies by [21, Theorem 2.1] that there are isomorphisms of Hecke and $G_{\mathbb{Q}}$-modules

$$\overline{T}(f) \xrightarrow{\sim} e^{ord}_{\mathfrak{m}} H^1_{et}(X_1(Np^s) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O})[\mathfrak{m}],$$
$$T(f) \xrightarrow{\sim} e^{ord}_{\mathfrak{m}} H^1_{et}(X_1(Np^s) \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathcal{O}),$$

and that $\mathbf{T}^{\mathrm{ord}}_2(N, s)_{\mathfrak{m}}$ is a Gorenstein ring. These facts in turn imply the following control theorem.

Proposition 2.1. *Let λ be an arithmetic point of $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}$ of weight k and level p^s and let $\mathfrak{p}$ be the kernel of λ restricted to Λ. Then $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}} \otimes_{\Lambda} \Lambda/\mathfrak{p}$ is isomorphic to $\mathbf{T}^{\mathrm{ord}}_k(N, s)_{\mathfrak{m}}$.*

Proof. Assume first that $k=2$. For $s\geq 1$, the complex $\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$ is a bounded below complex of $\mathbf{T}_2^{\mathrm{ord}}(N,1)_{\mathfrak{m}}$-modules, for instance by [8, Proposition 4.5]. Let $C(s)$ be the complex $\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$ and $M(s)$ its first cohomology group. The $\mathbf{T}_2^{\mathrm{ord}}(N,1)_{\mathfrak{m}}$-module $M(1)$ is free and is the only non-trivial cohomology group of $C(1)$, which is thus a perfect complex. For $s'\geq s$, the isomorphism

$$\mathrm{R}\Gamma_{et}(X_1(Np^{s'})\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})\overset{\mathrm{L}}{\otimes}_{\mathcal{O}[(\mathbb{Z}/p^{s'}\mathbb{Z})]}\mathcal{O}[\mathbb{Z}/p^s\mathbb{Z}]\overset{\sim}{\longrightarrow}\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O}) \tag{2.2}$$

induced by the trace map shows that $C(s)$ is a perfect complex of $\mathbf{T}_2^{\mathrm{ord}}(N,s)$-modules for all $s\geq 1$. Because the first cohomology group $M(s)$ is the only non-trivial cohomology group of $C(s)$, $M(s)$ is a free $\mathbf{T}_2^{\mathrm{ord}}(N,s)$-module and $M(s')\otimes_{\mathcal{O}[\mathbb{Z}/p^{s'}\mathbb{Z}]}\mathcal{O}[\mathbb{Z}/p^s\mathbb{Z}]$ is isomorphic to $M(s)$ for $s'\geq s$ by (2.2). Hence, the complex $C(\infty)$

$$\varprojlim_s e_{\mathfrak{m}}^{ord}\,\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$$

is a perfect complex of $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$-modules and $M(\infty)$ is $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$-free of rank 2. This implies that $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$ is a Gorenstein ring by the arguments of [14, Lemma 15.1]. Moreover $M(\infty)\otimes_{\Lambda}\Lambda/\mathfrak{p}$ is isomorphic to $M(s)$ so $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}\otimes_{\Lambda}\Lambda/\mathfrak{p}$ is isomorphic to $\mathbf{T}_k^{\mathrm{ord}}(N,s)_{\mathfrak{m}}$.

Now $\mathfrak{p}$ is of arbitrary weight. Let $\mathcal{F}_k$ be the sheaf $j_*\operatorname{Sym}^{k-2}(R^1\pi_*\mathcal{O})$ where π is the universal elliptic curve over the affine modular curve and j the inclusion of the affine modular curve in the compact modular curve. The sheaf $\mathcal{F}_k$ is pure of weight $k-2$. The ordinary étale cohomology complex satisfies the independence of weight property:

$$\varprojlim_s e_{\mathfrak{m}}^{ord}\,\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})\overset{\sim}{\longrightarrow}\varprojlim_s e_{\mathfrak{m}}^{ord}\,\mathrm{R}\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{F}_k).$$

The isomorphism (2.2) with coefficient sheaf $\mathcal{F}_k$ shows that $M(\infty)\otimes_{\Lambda}\Lambda/\mathfrak{p}$ is isomorphic to $e_{\mathfrak{m}}^{ord}H^1_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{F}_k)$. The fact that

$$e_{\mathfrak{m}}^{ord}H^1_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{F}_k\otimes_{\mathcal{O}}L_{\mathfrak{p}})$$

is $\mathbf{T}_k^{\mathrm{ord}}(N,s)\otimes_{\mathcal{O}}L_{\mathfrak{p}}$-free of rank 2 show that the $\Lambda/\mathfrak{p}$-module $\mathbf{T}_{\infty}^{\mathrm{ord}}\otimes_{\Lambda}\Lambda/\mathfrak{p}$ is free, that it surjects onto $\mathbf{T}_k^{\mathrm{ord}}(N,s)$ and that this surjection becomes an isomorphism after inverting p. Hence, the kernel of this surjection is p-torsion. As $\mathbf{T}_{\infty}^{\mathrm{ord}}\otimes_{\Lambda}\Lambda/\mathfrak{p}$ is p-torsion free, the $\Lambda/\mathfrak{p}$-modules $\mathbf{T}_{\infty}^{\mathrm{ord}}\otimes_{\Lambda}\Lambda/\mathfrak{p}$ and $\mathbf{T}_k^{\mathrm{ord}}(N,s)$ are isomorphic. $\square$

Remark: We seize this opportunity to alert readers of a mistake in the literature originating in [15, Theorem 7]: contrary to what is apparently claimed in this theorem and in subsequent quotations, the Hecke algebra localized at an ordinary, non-Eisenstein, maximal ideal is not necessarily a Gorenstein ring (see the many counter-examples found by G.Wiese and L.Kilford). In other words, our hypothesis that $\bar{\rho}_f$ is p-distinguished is necessary.

Review of dihedral Iwasawa theory Recall that $\widehat{K}^\times$ denotes the finite ideles of K. Let τ be the complex conjugation. For c an integer, let Z_c be the order $\mathbb{Z}+c\mathcal{O}_K$ and $K[c]/K$ be the ring class field of conductor c, this is to say the abelian extension of K such that rec_K induces an isomorphism between $\widehat{K}^\times/K^\times\widehat{F}^\times\widehat{Z}_c^\times$ and $\mathrm{Gal}(K[c]/K)$. For instance, the field $K[1]$ is the Hilbert class field of K. If $c'|c$, the following short sequence is exact:

$$1\longrightarrow\frac{K^\times\cap\widehat{\mathbb{Q}}^\times\widehat{Z}_{c'}^\times}{K^\times\cap\widehat{\mathbb{Q}}^\times\widehat{Z}_c^\times}\longrightarrow\frac{\widehat{Z}_{c'}^\times}{\widehat{Z}_c^\times}\overset{\mathrm{rec}_K}{\longrightarrow}\mathrm{Gal}(K[c]/K[c'])\longrightarrow 1.$$

Let c' be such that $p\nmid c'$ and such that c' does not divide the lowest common multiple of all the $u-1$ for $u\in(\mathcal{O}_K^\times)_{\mathrm{tors}}$. Let $x=ab$ be in $K^\times\cap\widehat{\mathbb{Q}}^\times Z_{c'}$ with $a\in\widehat{\mathbb{Q}}^\times$ and $b\in Z_{c'}$. Then $x/\tau x$ belongs to the kernel of $N_{K/\mathbb{Q}}$ so is a torsion unit in $\mathcal{O}_K^\times$. Moreover, $x/\tau x$ reduces to 1 in $(\mathcal{O}_K/c'\mathcal{O}_K)^\times$. Our choice of c' implies that x is equal to 1 and so that x belongs to $\mathbb{Q}$. Fix such a c'. For any c such that $c'|c$, the previous exact sequence then reduces to the isomorphism:

$$\widehat{Z}_{c'}^\times/\widehat{Z}_c^\times\overset{\sim}{\longrightarrow}\mathrm{Gal}(K[c]/K[c']).\tag{2.3}$$

Let $K[c'p^\infty]$ be the union of the $K[c'p^n]$ for all n. Because $\widehat{Z}_{c'}^\times/\widehat{Z}_c^\times$ is of rank 1 as a $\mathbb{Z}_p$-module, it has a quotient Z isomorphic to $\mathbb{Z}_p$. Hence, there exists a sub-extension K_∞ of $K[c'p^\infty]$ such that $\mathrm{Gal}(K_\infty/K)$ is isomorphic to $\mathbb{Z}_p$ via the isomorphism (2.3) composed with the isomorphism between Z and $\mathbb{Z}_p$. We are grateful to the referee for pointing out that this isomorphism between $\mathrm{Gal}(K_\infty/K)$ and $\mathbb{Z}_p$ is not canonical: as described, it depends on a choice of identification between Z and $\mathbb{Z}_p$. The extension K_∞ is the unique $\mathbb{Z}_p$-extension of K such that the Galois group $\mathrm{Gal}(K_\infty/\mathbb{Q})$ is equal to the pro-dihedral group $\mathbb{Z}_p\rtimes\{1,\tau\}$ with $\tau\gamma\tau=\gamma^{-1}$ for all $\gamma\in\mathrm{Gal}(K_\infty/K)$. In particular, it does not depend on our choice of c'. Let K_n be the sub-extension of K_∞ with Galois group $\mathbb{Z}/p^n\mathbb{Z}$. Let Λ_{anti} be the 2-dimensional regular local ring $\mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$ endowed with the action of $G_{K,\Sigma}$ coming from the surjection of $G_{K,\Sigma}$ to $\mathrm{Gal}(K_\infty/K)$ and inclusion of $\mathrm{Gal}(K_\infty/K)$ inside $\Lambda_{anti}^\times$. For ϵ a finite order character of $\mathrm{Gal}(K_\infty/K)$, we define the arithmetic point ϕ of character ϵ of Λ_{anti} to be the $\mathcal{O}$-algebra morphism:

$$\begin{aligned}\phi:\ \Lambda_{anti}&\longrightarrow\bar{\mathbb{Q}}_p\ ;\\ \gamma\ &\longmapsto\epsilon(\gamma).\end{aligned}$$

Arithmetic points in this sense are called ring class characters in [2, Section 1.1] and anticyclotomic characters in [1, Introduction] and [16, Section 4]. Let R_{Iw} be the 3-dimensional Gorenstein ring $R[[\mathrm{Gal}(K_\infty/K)]]$, where R is defined by (2.1). We define arithmetic points of R_{Iw} to be $\mathcal{O}$-algebra morphisms whose restrictions to R and to Λ_{anti} are arithmetic. The weight of an arithmetic point λ of R_{Iw} is defined to be the weight of λ restricted to R. Arithmetic primes of R_{Iw} are kernels of arithmetic points. Let $\mathcal{T}_{\mathrm{Iw}}$ be the $G_{K,\Sigma}$-representation $\mathcal{T}\otimes_R R_{\mathrm{Iw}}$ with $G_{K,\Sigma}$ acting on both sides of the tensor product.

Specializations of $\mathcal{T}$ Let S be an integral quotient of $R_{\mathrm{Iw}}\otimes_{\mathbb{Z}_p}\overline{\mathbb{Z}}_p$. Of particular interest will be the cases where S is R, Λ_{anti} or a discrete valuation ring containing $\mathcal{O}$. An S-specialization λ of R_{Iw} is an $\mathcal{O}$-algebra map with values in S. An $S[G_{K,\Sigma}]$-module T_λ is an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$ if T is equal to $\mathcal{T}_{\mathrm{Iw}}\otimes_{R,\lambda}S$ as $S[G_{K,\Sigma}]$-modules. A specialization T_λ is said to be arithmetic if λ is an arithmetic point of R_{Iw}. In that case, it is said to be arithmetic of weight k if λ restricted to R is an arithmetic point of R of weight k. It is said to contain an arithmetic specialization if T_λ has a quotient which is an arithmetic specialization. In particular, $\mathcal{T}_{\mathrm{Iw}}$, $\mathcal{T}$ and $T\otimes_{\mathcal{O}}\Lambda_{anti}$ all contain arithmetic specializations.

For $v|p$, the local G_{K_v}-representation T_λ is reducible and we define $T^+_{\lambda,v}$ to be $\mathcal{T}^+_{\mathrm{Iw},v}\otimes_{R_{\mathrm{Iw}},\lambda}S$.

§ 2.2. Heegner points and special values of L-function

Rankin-Selberg L-function Let S be a valuation ring. Let λ be an arithmetic S-specialization of R_{Iw} of weight 2 and V_λ be the $\mathrm{Frac}(S)[G_{K,\Sigma}]$-module $T_\lambda\otimes_S\mathrm{Frac}(S)$. Then there exists an eigenform f_λ of weight 2 and central character ϕ as well as a finite order character χ of $\mathrm{Gal}(K_\infty/K)$ such that V_λ is equal to $V(f_\lambda)(1)\otimes\phi^{-1/2}\chi$. The $G_{K,\Sigma}$-representation V_λ is self-dual and its completed complex L-function $L(V_\lambda,s)$ coincides with the Rankin-Selberg L-function $L(\pi(f)\times\chi,s+1/2)$. If χ is trivial or has a sufficiently large conductor, our hypotheses on the splitting of primes dividing N implies that $L(V_\lambda,0)$ vanishes at odd, and hence non-trivial, order. As V_λ occurs in the étale cohomology of the curve $X_1(Np^s)$, the Bloch-Kato conjecture then predicts that there are points on the Jacobian of $X_1(Np^s)$ rational over a finite sub-extension of K_∞ accounting for this vanishing. This observation remains true if λ restricted to R is fixed, or equivalently if $V(f_\lambda)$ is fixed, and if λ restricted to Λ_{anti} varies. Then, Mazur's conjecture predicts that the order of vanishing of $L(V_\lambda,0)$ is exactly 1 except for a finite number of λ. Symmetrically, we could fix χ and let f_λ vary among arithmetic points of weight 2, in which case R.Greenberg conjectures again that the order of vanishing of $L(V_\lambda,0)$ is exactly 1 except for a finite number of λ. We note that these conjectures are known in most cases thanks to works of V.Vatsal, C.Cornut and B.Howard.

This suggests that there should exist a non-zero p-adic L-function, or rather a leading term of a p-adic L-function, belonging to R_{Iw} and whose image by an arithmetic specialization λ interpolates $L'(V_\lambda,0)/\Omega(\lambda)$ for a suitable choice of period $\Omega(\lambda)$. Moreover, this p-adic L-function should be linked to the height of families of points rational over sub-extension of K_∞. To the best of knowledge of this author, this p-

adic L-function is not known to exist, but he surmises that the p-adic Rankin-Selberg convolution of the p-adic L-function of [3] should satisfy these properties.

Heegner points Since the seminal work [7], it is known that the special points alluded to in the previous paragraph should be linked with points on $X_1(Np^s)$ parametrizing isogeny between CM elliptic curves, or Heegner points as they have come to be known. In this paragraph, we recall the adelic construction of points in $X_1(Np^s)(K_n)$ verifying distribution relation reminiscing of those we expect of the p-adic L-function. These points were constructed by B.Howard in [11] who has moreover shown under very mild hypotheses that they are non-torsion if and only if certain L-function does not vanish, as expected.

We consider the tower of compact modular curves $\{X(N,s)\}_{s\geq 1}$ coming from the tower of compact open subgroups $U(N,s)$ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})$ defined by:

$$U(N,s)=\left\{g\in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})|g_\ell\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod \ell^{\mathrm{ord}_\ell N},\ g_p\equiv\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}\bmod p^s\right\}.$$

The complex points of $X(N,s)$ are given by the double coset:

$$X(N,s)(\mathbb{C})=\mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathbb{C}-\mathbb{R}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{(\infty)})/U(N,s)\right)^*.$$

Here, * denotes smooth compactification. The embedding of K inside $\mathcal{M}_2(\mathbb{Q})$ defines an action of $K^\times$ on $\mathbb{C}-\mathbb{R}$ which has a unique fixed point Z with positive imaginary part. The set of CM points is the set of complex points $[Z,b]\in X(N,s)(\mathbb{C})$. According to Shimura reciprocity law, CM points are in fact rational over abelian extensions of K. We consider the following family of CM points on the tower $\{X(N,s)\}_{s\geq 1}$:

$$\mathcal{X}(c,s)=\left\{x(c,s)=[Z,b(c,s)]\in X(N,s)|b(c,s)_\ell=\begin{pmatrix}\ell^{\mathrm{ord}_\ell N} & 0\\ 0 & 1\end{pmatrix},\ b(c,s)_p=\begin{pmatrix}p^s & 0\\ 0 & 1\end{pmatrix}\right\}.$$

The field of rationality of $x(c,s)$ is denoted by $K(c,s)$, the usual ring-class field of conductor c by $K[c]$ and $K[c](\mu_{p^s})$ by $K[c,s]$. Let

$$z(c,s)\in H^1(K[c],e^{ord}_{\mathfrak{m}}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\psi^{-1})$$

be the cohomology class constructed in the following way. First, the Kummer map twisted on the target space and composed with projection on the ordinary part sends $x(c,s)$ to $H^1(K(c,s),e^{ord}_{\mathfrak{m}}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\psi^{-1})$. The image $\Phi(x(c,s))$ of $x(c,s)$ under this map is easily seen to belong to the $\mathrm{Gal}(K[c,s]/K_0[cp^s])$-invariants. By purity of $e^{ord}_{\mathfrak{m}}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$, the Hochschild-Serre spectral sequence for the groups $G_{K[c,s]}$ and $G_{K_0(cp^s)}$ induces an isomorphism allowing us to consider $\Phi(x(c,s))$

as a class in $H^1(K[cp^s], e_{\mathfrak{m}}^{ord}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\chi^{-1})$. Then $z(c,s)$ is equal to $T(p)^{-s}\operatorname{Cor}\Phi(x(c,s))$ where Cor denotes corestriction from $K[cp^s]$ to $K[c]$. The classes $z(c,s)$ are equivariant under the action of the Hecke algebra and under the action of the Galois groups of ring-class fields in the sense that they satisfy the following Euler system relations.

Proposition 2.2. *[[11, Proposition 2.3.1 and Theorem 3.1.1]] Let $\mathscr{L}$ be the set of square-free products of rational primes inert in K with a power of p. Let $c\ell$ be in $\mathscr{L}$, let c^p be the p-power free part of c and let $s'\geq s$ be two integers. Let $\pi_{s'/s}$ be the projection from $X_1(N,s')$ to $X(N,s)$. Then:*

$$\pi_{s'/s}z(c,s')=z(c,s), \tag{2.4}$$

$$\operatorname{Cor}_{K[c\ell]/K[c]}z(cl,s)=T(\ell)z(c,s), \tag{2.5}$$

$$\operatorname{Cor}_{K[c]/K[c^p]}z(c,s)=z(c^p,s). \tag{2.6}$$

In particular, the inverse limit on s and t of $z(cp^t,s)$ composed with corestriction to K defines a class z_∞ in $H^1(K,\mathcal{T}_{\mathrm{Iw}})$. Moreover, the class z_∞ is not R_{Iw}-torsion.

§ 2.3. Selmer complexes and the ETNC

Review of determinants We review briefly the formalism of the determinant functor. Let S be a complete reduced local noetherian ring. A graded invertible S-module is the pair composed of a free S-module of rank 1 and a locally constant function from $\operatorname{Spec}(S)$ to $\mathbb{Z}$. For a finitely generated S-module M free of rank r, the determinant $\det_S M$ of M is the graded invertible S-module $(\bigwedge^r M, r)$. For a bounded complex of free S-modules C, the determinant $\det_S C$ is the graded invertible S-module:

$$\det_S C=\bigotimes_{i\in\mathbb{Z}}\det_S^{(-1)^i}C^i.$$

The determinant functor extends to the homotopy category of perfect complexes with morphisms restricted to quasi-isomoprhisms. If the cohomology groups $H^i(C)$ of a bounded complex C regarded as complexes in degree zero are themselves perfect complexes of S-modules, then C is a perfect complex and there is a canonical isomorphism:

$$\det_S C\xrightarrow{\sim}\bigotimes_{i\in\mathbb{Z}}\det_S^{(-1)^i}H^i(C).$$

If S is a regular ring and M is a torsion S-module, then M admits finite free resolution by the Auslander-Buchsbaum-Serre theorem so $\det_S M$ is well-defined. As $M\otimes\operatorname{Frac}(S)$ is trivial, $\det_{\operatorname{Frac}(S)}M\otimes\operatorname{Frac}(S)$ is canonically isomorphic to $\operatorname{Frac}(S)$. The image of $\det_S M$ inside $(\det_S M)\otimes_S\operatorname{Frac}(S)$ is identified by this canonical isomorphism with an invertible S-module inside $\operatorname{Frac}(S)$. This module is equal to $(\operatorname{char}_S M)^{-1}$.

Selmer complexes for specializations of $\mathcal{T}_{\mathrm{Iw}}$ Let λ be an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$. We define cohomological complexes which are conjecturally linked to arithmetic properties of T_λ when λ is arithmetic.

If $v \nmid p$, let M_λ be a free S-submodule of $T_\lambda^{I_v}$ with the same rank as $T_\lambda^{I_v}$. Let $C_f(G_{K_v}, T_\lambda)$ be the complex $C^\bullet_{\mathrm{cont}}(\mathrm{Fr}(v), M_\lambda)$. If $v|p$, let $C_f(G_{K_v}, T)$ be $C^\bullet_{\mathrm{cont}}(G_{K_v}, T^+_{\lambda,v})$. For all v, there is a morphism i_v from $C_f(G_{K_v}, T_\lambda)$ to $C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)$. This morphism is given by inclusion of M_λ inside $T_\lambda^{I_v}$ composed with inflation if $v \nmid p$ and by the inclusion of $T^+_{\lambda,v}$ inside T_λ if $v|p$.

Definition 2.3. Let $\mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$ be the object in the derived category corresponding to

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,\Sigma}, T_\lambda) \oplus \bigoplus_{v\in\Sigma} C_f(G_{K_v}, T_\lambda) \overset{\mathrm{res}_v - i_v}{\longrightarrow} \bigoplus_{v\in\Sigma} C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)\right)[-1]$$

and let $\tilde{H}^i_f(G_{K,\Sigma}, T_\lambda)$ be its i-th cohomology group.

The S-module T_λ is free of rank 2 by construction. The groups $G_{K,\Sigma}$ and the G_{K_v} have finite cohomological p-dimension bounded by 3 so the complex $\mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$ is a perfect complex of S-modules acyclic outside $[0,3]$. The S-module $\tilde{H}^0_f(G_{K,\Sigma}, T_\lambda)$ injects in $H^0(G_{K,\Sigma}, T_\lambda)$ so is trivial by absolute irreducibility of $\bar{\rho}$. Hence, the same is true of $\tilde{H}^3_f(G_{K,\Sigma}, T_\lambda)$ by self-duality of T_λ and [17, Statement (8.9.10)].

When $H^0(G_{K_v}, T^-_{\lambda,v})$ is zero for all $v|p$, the long exact sequence in cohomology induced by definition 2.3 shows that $\tilde{H}^1_f(G_{K,\Sigma}, T_\lambda)$ is equal to the usual compact Greenberg Selmer group:

$$H^1_{\mathrm{Gr}}(G_{K,\Sigma}, T_\lambda) = \ker\left(H^1(G_{K,\Sigma}, T_\lambda) \longrightarrow \bigoplus_{v\in\Sigma} H^1(G_{K,\Sigma}, T_\lambda)/H^1_{\mathrm{Gr}}(G_{K_v}, T_\lambda)\right).$$

Here $H^1_{\mathrm{Gr}}(K_v, T_\lambda)$ is equal to $H^1(K_v^{ur}, T_\lambda)$ if $v \nmid p$ and to $H^1(K_v, T^+_{\lambda,v})$ if $v|p$. We note that $H^0(G_{K_v}, T^-_{\lambda,v})$ is zero in particular if λ is arithmetic and if T_λ is a potentially crystalline G_{K_v}-representation, or if it is of weight different from 2. Hence, $H^0(G_{K_v}, \mathcal{T}^-)$ and $H^0(G_{K_v}, \mathcal{T}^-_{\mathrm{Iw}})$ are trivial. A specialization λ is said to be exceptional if there exists a finite extension L_w of K_v such that $H^0(G_{L_w}, T^-_\lambda)$ is not trivial.

When S is a discrete valuation ring, let $\mathrm{R}\Gamma_c(\mathrm{Spec}\,\mathcal{O}_K[1/\Sigma], T_\lambda)$ be the object in the derived category corresponding to

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(\mathrm{Spec}\,\mathcal{O}_K[1/\Sigma], T_\lambda) \overset{\mathrm{res}_v - i_v}{\longrightarrow} \bigoplus_{v\in\Sigma} C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)\right)[-1].$$

Special classes in $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$ According to proposition 2.2, the class z_∞ belongs to $H^1(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. Let λ be a specialization of R_{Iw}. Let z_λ be the image of z_∞ in $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$.

Proposition 2.4. *For all $c\in\mathscr{L}$, the class $z_\infty(c)$ belongs to $\tilde{H}^1_f(G_{K[c],\Sigma},\mathcal{T}_{\mathrm{Iw}})$. The class z_∞ belongs to $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. Let λ be an S-specialization of R_{Iw}. Then z_λ belongs to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$.*

Proof. The first two statements are contained in [11, Proposition 2.4.5]. If λ is an S-specialization which is not exceptional, the natural map from $H^1(K,\mathcal{T}_{\mathrm{Iw}})$ to $H^1(K,T_\lambda)$ induced by λ defines a class z_λ which belongs to $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$, and hence to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$. Even if λ is exceptional, the class z_λ belongs to $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$ so lifts to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$. The short exact sequence

$$0\longrightarrow\bigoplus_{v|p}H^0(G_{K_v},T^-_{\lambda,v})\longrightarrow\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)\longrightarrow H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)\longrightarrow 0$$

shows that this lift is defined up to an element of $\bigoplus_{v|p}H^0(G_{K_v},T^-_{\lambda,v})$. The fact that the inverse limit on n of $H^0(G_{K(n)_v},T^-_{\lambda,v})$ vanishes establish an isomorphism between $\tilde{H}^1_f(G_{K_\infty,\Sigma},T_\lambda)$ and $H^1_{\mathrm{Gr}}(G_{K_\infty,\Sigma},T_\lambda)$. As z_λ belongs to the image of this latter group inside $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$, it admits a canonical lift to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$. □

Thanks to the following results of B.Howard, we very often know that z_λ is not S-torsion.

Proposition 2.5. *[[11, Theorem 3.1.1] and [12, Corollary 5]] Let λ be an arithmetic point of R. Let T be $\mathcal{T}_{\mathrm{Iw}}$ or $T_\lambda\otimes_S\Lambda_{anti}$. Then the image of z_∞ in $\tilde{H}^1_f(G_{K,\Sigma},T)$ is not torsion. Assume that there exists an arithmetic point μ of R of weight 2 such that $L'(V_\mu,0)$ does not vanish. Then the image of z_∞ in $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T})$ is not torsion and z_λ is not torsion for almost all arithmetic points λ of R. The set of specialization of R_{Iw} such that z_λ is torsion is of codimension at least 1.*

An Equivariant Tamagawa Number Conjecture When S is a discrete valuation ring and λ is arithmetic, the Tamagawa Number Conjecture predicts that there exists an isomorphism of free S-modules

$$\zeta_{\lambda,c}:\ S\longrightarrow\det{}^{-1}_S\mathrm{R}\Gamma_c(\operatorname{Spec}\mathcal{O}_K[1/\Sigma],T_\lambda)$$

such that $\zeta_{\lambda,c}\otimes_S\mathrm{Frac}(S)$ expresses the p-part of the algebraic part of the leading term of the complex L-function of V_λ. If this isomorphism exists, then it can be modified to give an isomorphism of free S-modules:

$$\zeta_{\lambda,f}:\ S\longrightarrow\det{}^{-1}_S\mathrm{R}\Gamma_f(G_{K,\Sigma},T_\lambda).$$

The point of this modification is to link $\zeta_{f,\lambda}$ with the value of the p-adic L-function rather than with the complex L-function. The philosophy of the ETNC for the family of motives $\{h^1(X_1(Np^s))\otimes h^0(\mathrm{Spec}(K_n))\}_{s,n}$ predicts that there exists an isomorphism of free R_{Iw}-modules

$$\zeta_f:\ R_{\mathrm{Iw}}\longrightarrow \det\nolimits^{-1}_{R_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$$

verifying the change of ring property that $\zeta_f\otimes_{R_{\mathrm{Iw}},\lambda}S$ is equal to $\zeta_{\lambda,f}$ and such that $\zeta_f\otimes\mathrm{Frac}(R_{\mathrm{Iw}})$ is linked with the p-part of the algebraic part of the leading term of the conjectural p-adic L-function L_p.

Without assuming any conjecture, we know there is a non-torsion element z_∞ inside $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. The isomorphism

$$\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})\overset{\sim}{\longrightarrow}\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$$

of [17, Theorem 8.9.11] defines a non-zero element z^*_∞ inside $\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})$ $\otimes\,\mathrm{Frac}(R_{\mathrm{Iw}})$. Choose an isomorphism:

$$\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})\overset{\sim}{\longrightarrow}\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}}).$$

Let z^{**}_∞ be the image of z^*_∞ under this isomorphism. Let $d\in\mathrm{Frac}(R_{\mathrm{Iw}})$ and $z'_\infty\in\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$ be such that $z^{**}_\infty=d^{-1}z'_\infty$ in $\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. The maps sending $1\in R_{\mathrm{Iw}}[-1]$ to z_∞ and $1\in R_{\mathrm{Iw}}[-2]$ to z'_∞ induce a morphism of complexes:

$$[R_{\mathrm{Iw}}\longrightarrow R_{\mathrm{Iw}}]\longrightarrow \mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}).$$

We also consider the morphism

$$R_{\mathrm{Iw}}\overset{d}{\longrightarrow}R_{\mathrm{Iw}}$$

viewed as a morphism of complexes concentrated in degree 0. Let $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ be the product of the determinants of the cones of these morphisms of complexes. Then $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ does not depend on our choice of a submodule M_λ of $\mathcal{T}_{\mathrm{Iw}}^{I_v}$ for $v|N$, on our choice of isomorphism between $\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$ and its dual, nor on our choice of d and z'_∞. The isomorphism

$$\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})\overset{\sim}{\longrightarrow}\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$$

induces an isomorphism from $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$ to $\mathrm{Frac}(R_{\mathrm{Iw}})$, which we take to be an indentification in all that follows. Let $\mathscr{I}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ be the image of $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ in $\mathrm{Frac}(R_{\mathrm{Iw}})$ induced by this identification.

More generally, we define in the same way $\mathscr{X}_f(T_\lambda,S)$ and $\mathscr{I}_f(T_\lambda,S)$ for all S-valued λ with z_λ not S-torsion. When in addition S is a discrete valuation ring and λ is a non-exceptional S-specialization, the method of Euler systems shows that $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$

is free of rank 1 and hence that $\tilde{H}^2_f(G_{K,\Sigma},T_\lambda)$ is of rank 1. The isomorphism between $\mathscr{X}_f(T_\lambda,S)\otimes\mathrm{Frac}(S)$ and $\mathrm{Frac}(S)$ then identifies $\mathscr{I}_f(T_\lambda,S)$ with:

$$\frac{|\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)/z_\lambda|^2}{|\tilde{H}^2_f(G_{K,\Sigma})_{\mathrm{tors}}|}S.$$

The TNC suggests that this module is included in S, and is equal to S if z_λ is sufficiently well optimized. Generalizing the case of discrete valuation ring, we are led to the following two conjectures for a specialization λ containing an arithmetic point and such that z_λ is not S-torsion.

Definition 2.6. Let $\mathbf{Conj}(T_\lambda,S)$ be the statement that $\mathscr{I}_f(T_\lambda,S)$ is included in S.

Definition 2.7. Let $\mathbf{StrongConj}(T_\lambda,S)$ be the statement that $\mathscr{I}_f(T_\lambda,S)$ is equal to S.

We consider especially the following special cases of conjectures 2.6 and 2.7, in which we implicitly conjecture that z_λ is not torsion: $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$, $\mathbf{Conj}(\mathcal{T},R)$, $\mathbf{Conj}(T\otimes\Lambda_{anti},\Lambda_{anti})$and $\mathbf{Conj}(T,\mathcal{O})$. When λ is an arithmetic specialization of weight 2, $\mathbf{StrongConj}(T_\lambda,\Lambda_{anti})$ is reminiscent of a conjecture of B.Perrin-Riou. When R is regular, $\mathbf{StrongConj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ is a conjecture of B.Howard.

The author confesses that he is inclined to a certain dose of scepticism towards $\mathbf{StrongConj}(T_\lambda,S)$ as stated, if only because he feels the question of whether Heegner points correspond to improved or to usual p-adic L-function has not been sufficiently explored in what precedes.

§ 2.4. Main theorem

We now state the two main theorems of this article.

Theorem 2.8. *If* $\boldsymbol{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ *is true, then* $\boldsymbol{Conj}(T_\lambda,S)$ *is true for all specializations* λ *containing an arithmetic specialization and such that* z_λ *is not torsion. In particular, provided* z_λ *is not torsion for these specializations, the conjectures* $\boldsymbol{Conj}(\mathcal{T},R)$, $\boldsymbol{Conj}(T\otimes\Lambda_{anti},\Lambda_{anti})$ *and* $\boldsymbol{Conj}(T,\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N,s))$ *are then true. Under the same hypothesis, if there exists a specialization* λ *containing an arithmetic specialization such that* $\boldsymbol{StrongConj}(T_\lambda,S)$ *is true, then* $\boldsymbol{StrongConj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ *and* $\boldsymbol{StrongConj}(T_\mu,S)$ *for all* μ *containing an arihtmetic specialization and such that* z_μ *is not torsion are also true.*

Theorem 2.9. *Let* T_λ *be an arithmetic specialization of* $\mathcal{T}$ *with coefficients in the discrete valuation ring* S*. Then:*

1. *Conjecture* $\textbf{\textit{Conj}}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$ *is true.*

2. *Conjecture* $\textbf{\textit{Conj}}(T_\lambda, S)$ *is true provided* z_λ *is not torsion.*

Assume that R *is a regular ring. Then:*

1. *Conjecture* $\textbf{\textit{Conj}}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ *is true*

2. *Conjecture* $\textbf{\textit{Conj}}(\mathcal{T}, R)$ *is true provided that the image* z *of* z_∞ *in* $\tilde{H}^1_f(G_{K,\Sigma}, \mathcal{T})$ *is not torsion.*

Theorem 2.8 roughly states that our conjectures are compatible with change of base ring, while theorem 2.9 corresponds in usual cases to a divisibility of characteristic ideals in the Iwasawa Main Conjecture. Because of their similar statement apart from increasing generality, the reader might believe that $\mathbf{Conj}(T, S)$, $\mathbf{Conj}(T \otimes \Lambda_{anti}, \Lambda_{anti})$, $\mathbf{Conj}(\mathcal{T}, R)$ and $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ are proved in that order and by roughly the same method. In fact, this is far from true: we first establish a weaker version of $\mathbf{Conj}(T, S)$ for many non-necessarily arithmetic specializations T, then $\mathbf{Conj}(T \otimes \Lambda_{anti}, \Lambda_{anti})$ for the same set of T, then $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ and eventually $\mathbf{Conj}(T, S)$ and $\mathbf{Conj}(\mathcal{T}, R)$ using theorem 2.8.

§ 3. Elements of proofs of theorems 2.8 and 2.9

This section presents the outlines of a proof of our two main theorems. As indicated in the introduction, complete proofs of them can be found in [5].

§ 3.1. Control theorem for Selmer complexes

Proposition 3.1. *If* λ *is an* S*-specialization of* R_{Iw} *containing an arithmetic point, then there is a canonical isomorphism identifying* $(\det_{R_{\mathrm{Iw}}} \mathrm{R}\Gamma_f(G_{K,\Sigma}, \mathcal{T}_{\mathrm{Iw}})) \otimes_{R_{\mathrm{Iw}}} S$ *with* $\det_S \mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$.

Proof. It is enough to prove a comparable base-change statement for the determinant of the complexes involved in the definition of $\mathrm{R}\Gamma_f(G_{K,\Sigma}, \mathcal{T}_{\mathrm{Iw}})$. The following base-change results follow from the fact that taking continuous cochains is a triangulated way-out functor:

$$C^\bullet_{\mathrm{cont}}(G_{K,\Sigma}, \mathcal{T}_{\mathrm{Iw}}) \overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda} S \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_{K,\Sigma}, T_\lambda),$$
$$C^\bullet_{\mathrm{cont}}(G_{K_v}, \mathcal{T}_{\mathrm{Iw}}) \overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda} S \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda),$$
$$C^\bullet_{\mathrm{cont}}(G_{K_v}, \mathcal{T}^+_{\mathrm{Iw},v}) \overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda} S \xrightarrow{\sim} C^\bullet_{\mathrm{cont}}(G_{K_v}, T^+_{\lambda,v}).$$

To conclude, it is thus enough to prove that $\det_{R_{\mathrm{Iw}}} \mathrm{R}\Gamma_f(G_{K_v}, \mathcal{T}_{\mathrm{Iw}})$ satisfies the same base-change property for $v|N$. This in turn would follow from the fact that $\mathcal{T}_{\mathrm{Iw}}^{I_v} \otimes_{R_{\mathrm{Iw}},\lambda} S$ and $T_\lambda^{I_v}$ have the same rank over S. This is indeed true because the action of G_{K_v} on Λ_{anti} is unramified and because the automorphic type of all arithmetic points of R are the same at v. See [6] for details. □

Sketch of proof of theorem 2.8. Let λ be an S-specialization of R_{Iw} containing an arithmetic specialization and such that z_λ is not torsion. According to the preceding proposition, the determinant of the complex $\mathrm{R}\Gamma_f(G_{K,\Sigma}, \mathcal{T}_{\mathrm{Iw}}) \overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda} S$ is equal to the determinant of $\mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$. By construction $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}}) \otimes_{R_{\mathrm{Iw}},\lambda} S$ is thus equal to $\mathscr{X}_f(T_\lambda, S)$. As an element of R_{Iw} specializes to an element of S and to an element of $S^\times$ if and only if it is in $R_{\mathrm{Iw}}^\times$, theorem 2.8 follows. □

§ 3.2. The method of Euler systems

Assume in this sub-section that S is a discrete valuation ring and that λ is a non exceptional S-specialization of R_{Iw} such that z_λ is not S-torsion. Note that we do not assume λ to be arithmetic.

Proposition 3.2. *The S-module $\tilde{H}^1_f(G_{K,\Sigma}, T_\lambda)$ is free of rank 1 and $\tilde{H}^2_f(G_{K,\Sigma})$ is of rank 1. Let $C(\lambda)$ be:*

$$C(\lambda) = \left(\prod_{v|N} |H^1(G_{K_v}, T_\lambda)/H^1(\mathrm{Fr}(v), T_\lambda^{I_v})|\right)^2 \left(\prod_{v|p} |H^1(G_{K_v}, T_\lambda^-)_{\mathrm{tors}}|\right)^2.$$

Then:

$$|\tilde{H}^2_f(G_{K,\Sigma}, T_\lambda)_{\mathrm{tors}}| \le C(\lambda) |\tilde{H}^1_f(G_{K,\Sigma}, T_\lambda)/z|^2. \tag{3.1}$$

Equivalently, $\mathscr{I}_f(T_\lambda, S)$ is included in $C(\lambda)^{-1}S$.

Proof. This is a slightly non-standard presentation of the standard results given by the method of Euler systems on bound of Selmer groups, which we obtain by the methods of [13, 9]. In these articles, it is assumed that the image of ρ_λ contains homotheties. We observe that in the axiomatic of [9], this is used only to establish hypothesis H.2. Because hypothesis H.2 is an hypothesis on the residual representation $\bar{\rho}_\lambda$, it is true for all λ if and only if it is true for $\bar{\rho}$. In the appendix, we prove the group-theoretic lemma that hypothesis H.2 remains true under our standing assumptions that $\bar{\rho}$ is absolutely irreducible after restriction to G_K. □

Let T_λ be a specialization of $T \otimes_{\mathcal{O}} \Lambda_{anti}$ with coefficients in a discrete valuation ring S. Then $T_\lambda^{I_v}$ is equal to $(T^{I_v} \otimes \Lambda_{anti}) \otimes_{\mathcal{O}} S$ for $v \nmid p$ and $|H^1(G_{K_v}, T_\lambda^-)_{\mathrm{tors}}|$ is bounded by $|H^0(G_{K_{\infty,v}}, T^- \otimes \mathrm{Frac}(S)/S)|$. As long as S is fixed, $C(\lambda)$ is thus bounded independently of λ. We will use proposition 3.2 to show that $\mathbf{Conj}(T_\lambda, S)$ is uniformly not too false when λ varies among the S-specializations of $T \otimes_{\mathcal{O}} \Lambda_{anti}$.

§ 3.3. Descent for Selmer complexes

We assume henceforth that R_{Iw} is a regular ring. Let S be a specialization of R_{Iw} which is a regular ring of Krull dimension at least 2 and which contains an arithmetic point λ. Let T be an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$. Let S' be a discrete valuation ring and $T_{\lambda'}$ an S'-specialization of T. Then, the counterpart for the couple T and $T_{\lambda'}$ of proposition 3.1 is not true because $T^{I_v} \otimes_S S'$ has not in general the same rank as $T_{\lambda'}^{I_v}$ for $v|N$ when λ' is not arithmetic. Nevertheless, the R_{Iw}-rank of $\mathcal{T}_{\mathrm{Iw}}^{I_v}$ is equal to the rank of $T_\lambda^{I_v}$, and so both ranks are equal to the S-rank of T^{I_v}.

Moreover, if the ranks of $T^{I_v} \otimes_S S'$ and $T_{\lambda'}^{I_v}$ differ the following assertions are true: the S'-module $T^{I_v} \otimes_S S'$ is of rank 1, after restriction to an open subgroup if necessary the inertia group I_v acts on T through an infinite pro-cyclic unipotent group P, the S'-module T' is unramified at v. In that case, let $e \in T$ be such that T^{I_v} and e generates the $\mathrm{Frac}(S)$-vector space $T \otimes_S \mathrm{Frac}(S)$. Let σ be a generator of P. Then, $\sigma e = au + e$ with $a \neq 0$. The element a thus belongs to the kernel of λ'.

Hence the ranks of $T^{I_v} \otimes_S S'$ and of $T_{\lambda'}^{I_v}$ are equal outside a finite union of irreducible components of codimension 1. Thus, there exists a $\mathfrak{m}_S$-adically dense subset of regular specialization such that $(\det_S \mathrm{R}\Gamma_f(G_{K,\Sigma}, T)) \otimes_{S,\lambda'} S'$ is equal to $\det_{S'} \mathrm{R}\Gamma_f(G_{K,\Sigma}, T_{\lambda'})$. Together with the following lemma, this will imply proposition 3.4, which is very useful to relate $\mathbf{Conj}(T, S)$ to $\mathbf{Conj}(T', S')$.

Lemma 3.3. *Let I be an invertible ideal which is not included in S.*

1. *For all $X \in \mathfrak{m}_S$ except possibly those contained in a locus of dimension 1 and all $n_0 \geq 0$ there exists a specialization S' of S such that S' is regular, the dimension of S' is less than the dimension of S and $I \otimes_S S'$ is an invertible ideal which does not belong to $S'[1/X^{n_0}]$.*

2. *There exists a discrete valuation ring A of residual characteristic p such that for all $n_0 \geq 0$, there exists a set of A-specializations Φ of codimension zero such that for all $\phi \in \Phi$, the A-module $I_{\otimes,\phi} A$ is an invertible ideal which does not belong to $\frac{1}{\mathfrak{m}_A^{n_0}} A$*

Proof. Let $(a, b) \in S^2$ be such that $I = \frac{a}{b} S$. Because S is Cohen-Macaulay, the prime divisors of b have height 1. Because $a \notin bS$, there exists a prime $\mathfrak{p}|b$ such that $a \notin bS_{\mathfrak{p}}$. Because $S_{\mathfrak{p}}$ is a discrete valuation ring, the $\mathfrak{p}$-valuation of a is less than the $\mathfrak{p}$-valuation of b. Because S is factorial, we can then assume that $\mathfrak{p} \nmid aS$. Let $X \in \mathfrak{m}_S$ be an element not contained in $\mathfrak{p}$ and let ϖ be a generator of $\mathfrak{p}$. For all $n > 0$, let

$\phi_{X,n}$ be the surjection from S to $S_{X,n} = S/(\varpi - X^n)$. Then $\phi_{X,n}(b)$ belongs to (X^n) for all $n \geq 0$. For n large enough, $\phi_{X,n}(b) \neq 0$. Because $a \notin \mathfrak{p}$, for n large enough $\phi_{X,n}(a) \notin (X^n)$. Consequently, for all $n_0 \geq 0$, there exists an n_1 such that for all $n \geq n_1$ the following properties are true: the element $\varpi + X^n$ is part of a system of generators, $\phi_{X,n}(a/b)$ is well-defined and does not belong to $\frac{1}{X^{n_0}} S_{X,n}$.

Because the previous procedure can be carried on with an arbitrary $X \in \mathfrak{m}_S$, we can ensure that $S_{X,n}$ has residual characteristic p. Proceeding by descending induction on dimension if necessary, we construct in this way a discrete valuation ring A and specializations ϕ_n such that $I \otimes_{S,\phi_n} A$ is an invertible ideal and such that for all n_0, there exists an n_1 such that $I_{\otimes_{S,\phi_n}} A$ does not belong to $\frac{1}{\mathfrak{m}_A^{n_0}} A$ for all $n \geq n_1$. Let ϕ_n be such a specialization and let ϕ be a specialization such that for all $x \in \ker \phi$, there exists a $y \in \ker \phi_n$ such that $x - y \in \mathfrak{m}_S^m$. For m large enough, ϕ has values in A by Hensel's lemma and ϕ satisfies the same properties as ϕ_n. This set of ϕ is of codimension 0. $\square$

Proposition 3.4. *Assume that the image of $\mathscr{I}_f(T, S)$ inside* $\mathrm{Frac}(S)$ *is not contained in S. Then there exists a discrete valuation ring A with residual characteristic p such that for all $M \in \mathbb{N}$, there exists a non-exceptional A-specialization ϕ such that z_ϕ is not torsion and such that $\mathscr{I}_f(T_\phi, A)$ does not belong to $\frac{1}{\mathfrak{m}_A^M} A$. Moreover, the set of such specializations is of codimension 0.*

Remark: This proposition aims to convey the idea that if $\mathscr{I}_f(T, S)$ is not integral, there exists many specializations which are arbitrarily non-integral.

Proof. Let M be an integer. According to lemma 3.3, there exists a discrete valuation ring A of residual characteristic p and a set Φ^* of A-specializations of codimension 0 such that $\mathscr{I}_f(T, S) \otimes_S A$ is not $\mathrm{Frac}(S)$ and does not belong to $\frac{1}{\mathfrak{m}_A^M} A$. The set of $\phi \in \Phi^*$ such that ϕ is exceptional, or $\phi(z)$ is torsion or the rank of T^{I_v} is not the same as the rank of $T_\phi^{I_v}$ for some v is contained in the union of finitely many components all of codimension at least 1. Hence, the set $\Phi \subset \Phi^*$ such that $\phi \in \Phi$ is not exceptional, $\phi(z)$ is not torsion and such that $\mathscr{I}_f(T, S) \otimes_{S,\phi} A = \mathscr{I}_f(T_\phi, A)$ is of codimension 0. In particular, Φ is not empty. $\square$

We now sketch the strategy to complete the proof of theorem 2.9.

Sketch of proof of theorem 2.9. For ease of notation, when S is a discrete valuation ring containing $\mathcal{O}$, we write $\otimes_S \Lambda_{anti}$ for $\otimes_S S[[\mathrm{Gal}(K_\infty/K)]]$ in the following proof.

Let λ be an arithmetic specialization of R with values in a discrete valuation ring S. First, we prove $\mathbf{Conj}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$. Assume $\mathbf{Conj}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$ is false. According to proposition 3.4, for all M there exists a discrete valuation ring A and a non-exceptional A-specialization μ such that $\mathscr{I}_f(T_\mu, A)$ is not equal to $\mathrm{Frac}(S)$ and does not belong to $\frac{1}{\mathfrak{m}_A} A$. According to proposition 2.5, the set of A-specialization of Λ_{anti}

such that z_ϕ is torsion is finite. Hence, there exists an A-specialization μ as above with z_μ not torsion. Choosing M larger than C a uniform bound on the $C(\mu)$ contradicts proposition 3.2. Hence $\mathbf{Conj}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$ is true.

Next, we remove the hypothesis that λ is arithmetic. In that case, it is not known in general that $z \in \tilde{H}^1_f(G_{K,\Sigma}, T_\lambda \otimes_S \Lambda_{anti})$ is non torsion, so we make this supplementary hypothesis. Let λ be a non-exceptional specialization of R with values in a discrete valuation ring S and such that z_λ is not torsion. Then, $z_{\lambda,anti} \in \tilde{H}^1_f(G_{K,\Sigma}, T_\lambda \otimes_S \Lambda_{anti})$ is not torsion so the set of A-specialization μ of Λ_{anti} such that z_μ is torsion is finite. Assume $\mathbf{Conj}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$ is false. According to proposition 3.4, for all M there exists a discrete valuation ring A and a non-exceptional A-specialization μ such that $\mathscr{I}_f(T_\mu, A)$ is not equal to $\mathrm{Frac}(S)$ and does not belong to $\frac{1}{\mathfrak{m}_A}A$. Hence, there exists an A-specialization μ as above with z_μ not torsion. Choosing M larger than C a uniform bound on the $C(\mu)$ contradicts proposition 3.2. Hence $\mathbf{Conj}(T_\lambda \otimes_S \Lambda_{anti}, \Lambda_{anti})$ is true.

We wish to apply theorem 2.8 to deduce that $\mathbf{Conj}(T_\lambda, S)$ is true for all non-exceptional λ with z_λ not torsion. Theorem 2.8 does not apply *verbatim* because λ is not necessarily arithmetic, so does not necessarily contain an arithmetic specialization. However, the hypotheses that λ contains an arithmetic specialization is used only to show that descent with respect to the variable in R induces no error term. Descent with respect to the variables in Λ_{anti}, which is the only one intervening here, induces no error because, as already mentioned in the proof of theorem 2.8, $(T_\lambda \otimes_S \Lambda_{anti})^{I_v}$ is equal to $T_\lambda^{I_v} \otimes_S \Lambda$. Hence $\mathbf{Conj}(T_\lambda, S)$ is true for all non-exceptional λ with z_λ not torsion.

We observe that the set of specialization λ which are exceptional or such that z_λ is torsion is of codimension at least 1, and thus that the set of specialization λ such that $\mathbf{Conj}(T_\lambda, A)$ is false is of codimension at least 1. Assume that $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ is false. Using proposition 3.4, for all integer M, we can choose a discrete valuation ring A and an A-specialization μ such that $\mathscr{I}_f(T_\mu, A)$ is neither $\mathrm{Frac}(A)$ nor contained in $\frac{1}{\mathfrak{m}_A^M}A$. In particular $\mathbf{Conj}(T_\mu, A)$ is false. Because the set of A-specialization λ for which $\mathbf{Conj}(T_\lambda, A)$ is false is of codimension at least 1, for all M', there exists an A-specialization λ for which $\mathbf{Conj}(T_\lambda, A)$ is true and such that $\ker(\lambda - \mu)$ belongs to $\mathfrak{m}_{R_{\mathrm{Iw}}}^{M'}$. Choosing M' large enough, we find a specialization λ for which $\mathbf{Conj}(T_\lambda, A)$ is both true and false. This is absurd, so we established $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$.

By theorem 2.8, we deduce that if $z \in \tilde{H}^1_f(G_{K,\Sigma}, \mathcal{T})$ is not torsion, then $\mathbf{Conj}(\mathcal{T}, R)$ is true. Finally, if λ contains an arithmetic specialization, $\mathbf{Conj}(T_\lambda, S)$ is true by theorem 2.8. $\square$

We remark that this proves a bound on the size of Selmer groups of modular forms in anticyclotomic tower even in the case of multiplicative reduction.

§ 4. Perspectives

§ 4.1. Leading term of p-adic L-function and p-adic height pairing

We observe that the knowledge of the non-torsion class z_λ induces a trivialization of $\det_S \mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda)$, and hence allows a formulation of a variant of the ETNC, even though the p-adic L-function vanishes and even though we do not assume that the p-adic height pairing is not degenerate.

When the p-adic height pairing is known to be non-degenerate and z_λ is not torsion, then the convoluted construction of $\mathscr{X}_f(T_\lambda,S)$ can be replaced by the product of the determinants of the following complexes:

$$\mathrm{Cone}(S[-1]\xrightarrow{z_\lambda}\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda)),$$
$$\mathrm{Cone}(S[-1]\xrightarrow{h(z_\lambda,\cdot)}\mathrm{R}\,\mathrm{Hom}_S(\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda),S)).$$

Interpreting $\mathscr{X}_f(T_\lambda,S)$ as the determinant of the cone of the morphism

$$S[-1]\xrightarrow{h(z,z)}S[-1]$$

associates $h(z,z)$ with an element of S which we naturally conjecture to be the algebraic part of $\lambda(L_p(\mathcal{T}_{\mathrm{Iw}}))$. Here again, the author would like to express his scepticism about the literal truth of the previous statement as he feels the issue of whether it is the improved or standard p-adic L-function which appears has not been enough explored in what precedes.

§ 4.2. Totally real field F and quaternion algebras

Theorems 2.8 and 2.9 can be generalized fairly well to nearly ordinary automorphic representations of the multiplicative group of a quaternion algebra over a totally real field such that at most one infinite place does not ramify, though several hurdles appear in this more general setting.

In the indefinite case, we refer the reader to [5] and explain here the most serious difficulties. First, one replaces the tower of modular curves $X_1(Np^s)$ by a tower of compact Shimura curves $X(s)$. However, as the q-expansion principle is then lacking, the freeness of $e_{\mathfrak{m}}^{ord}H^1_{et}(X(1)\times_F\bar{F},\mathcal{O})$ over the Hecke algebra is in general not known (it is typically known under some conditions on $\bar{\rho}_f$, in which case it follows from the Taylor-Wiles machinery). Considering general quaternion algebra allows for more supple choices of N and K, as the construction of CM points does not require that all primes dividing N split in K, as in [10, 4]. But under these more general conditions, the class z_λ is not known to belong to $H^1_f(G_{K_v},T_\lambda)$ at places v dividing N and inert in K. Consequently, we are only able to prove that $\mathscr{I}_f(T_\lambda\otimes_S\Lambda_a,T_\lambda\otimes_S\Lambda_a)$ is in $\Lambda_a[1/p]$.

Multiplying z_λ by the product of the Tamagawa numbers at prime dividing N and inert in K allows for a proof of $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ along the same lines as in this text. This is in agreement with the remark of V.Vatsal that the μ-invariant of anticyclotomic p-adic L-functions should not always be zero.

Recent and forthcoming works of M.Longo and S.Vigni treat simultaneously the definite and indefinite case over $\mathbb{Q}$.

Appendix

In the proof of proposition 3.2, we use results from [9]. However, this article makes the assumption that the image of $\bar{\rho}_{|G_K}$ contains all homotheties. In this appendix, we explain how to modify the proof of [9, Theorem 1.6.1] in order to replace the hypothesis that the image of $\bar{\rho}_{|G_K}$ contains all homotheties with our standing assumption that $\bar{\rho}_{|G_K}$ is absolutely irreducible. The author thanks the referee for insisting that the proof of proposition 3.2 be firmly linked with the axiomatic method of [9]. All references are to [9].

Let G be the image of $\bar{\rho}_{|G_K}$. The hypothesis that G contains all homotheties is used in [9] in precisely one instance: in the proof of theorem 1.6.5, it is used to show that $H^1(G, \overline{T})$ vanishes. This vanishing implies that $\bar{\rho}_{|G_K}$ satisfies Hypothesis H.2. Hypothesis H.2 is part of the axiomatic setting used to prove theorem 1.6.1, which we wish to use. It is thus enough to prove that $H^1(G, \overline{T})$ vanishes. The proof that $\bar{\rho}$ satisfies Hypothesis H.2 will then be identical to that given in theorem 1.6.5.

Let k be the field of coefficients of $\bar{\rho}$. Assume first that $p \nmid |G|$. Because $\overline{T}$ is a k-vector space and G is a finite group, the group $H^1(G, \overline{T})$ then vanishes. Hence, we can assume that $p||G|$. Because $p \geq 5$, Dickson's classification of finite subgroups of $GL_2(\bar{\mathbb{F}}_p)$ implies that the elements of order divisible by p are included inside a Borel B or generate $\mathrm{SL}_2(\mathbb{F}_q)$ for some power q of p. Assume the first possibility holds. Then all elements of order p stabilize a line D in k^2. Because conjugation does not change the order of an element, the group G fixes D so G is included inside B. This contradicts the fact that G acts irreducibly. Hence, we are in the second case. Then G contains the non-trivial homothety $\mu = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Because μ is in the center of G and $\mu - 1$ is invertible, Sah's lemma imply that $H^1(G, \overline{T})$ vanishes.

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