RIMS Kôkyûroku Bessatsu
B19 (2010), 249–254

Factorization of Shintani's ray class invariant for totally real fields

By

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Abstract

In this article, we first announce the results of the paper [7], in which some formulas are given for L-functions of totally real number fields and Shintani's ray class invariant $X(\mathfrak{C})$. For detailed proofs, please refer to [7]. Moreover, in §3, we give a new formula for the Shintani invariant in the case of real quadratic fields.

§ 1. Partial zeta function and cone decomposition

Let F be a totally real number field of degree n. For an integral ideal $\mathfrak{f}$ of F, we denote by $Cl_F(\mathfrak{f})$ the narrow ray class group modulo $\mathfrak{f}$. We are interested in the *partial zeta function*

$$\zeta(s,\mathfrak{C}) := \sum_{\mathfrak{a}\in\mathfrak{C},\, \mathfrak{a}\subset O_F} \mathbf{N}(\mathfrak{a})^{-s}$$

attached to each class $\mathfrak{C} \in Cl_F(\mathfrak{f})$ (O_F denotes the ring of integers in F and $\mathbf{N}$ the norm for $F/\mathbb{Q}$). In this section, we recall a method to analyze these functions given by T. Shintani [2].

First, let us choose an integral representative $\mathfrak{a} \in \mathfrak{C}$ and an element $z \in F_+$ (here and in the following, the notation A_+ means the subset of totally positive elements of A). Then, if we put $\mathfrak{b} = z\mathfrak{a}^{-1}\mathfrak{f}$, we have

$$\zeta(s,\mathfrak{C}) = \mathbf{N}(\mathfrak{b}\mathfrak{f}^{-1})^s \sum_{\beta\in(z+\mathfrak{b})_+/E_\mathfrak{f}} \mathbf{N}(\beta)^{-s},$$

Received March 31, 2009. Revised February 24, 2010.
2000 Mathematics Subject Classification(s): 11M20, 11R42, 11R80
Supported by Grant-in-Aid for JSPS Fellows, No. 21·5093
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where $E_{\mathfrak{f}}$ denotes the group of totally positive units which are congruent to 1 modulo $\mathfrak{f}$. Hence we may study the function

$$\zeta_{\mathfrak{f}}(s, z+\mathfrak{b}) := \sum_{\beta \in (z+\mathfrak{b})_+/E_{\mathfrak{f}}} \mathbf{N}(\beta)^{-s}$$

instead of the original partial zeta function $\zeta(s, \mathfrak{C})$.

Next, we further rewrite the above function by using a cone decomposition. A *cone* σ is a subset of $F \otimes \mathbb{R} \cong \mathbb{R}^n$ which can be written as $\sigma = \mathbb{R}_+\omega_1 + \cdots + \mathbb{R}_+\omega_d$ with linearly independent vectors $\omega_1, \ldots, \omega_d$. It is called *rational* if the generators $\omega_1, \ldots, \omega_d$ can be taken from F.

We take a finite set Φ of rational cones in $(F \otimes \mathbb{R})_+$ satisfying

$$(F \otimes \mathbb{R})_+ = \coprod_{\varepsilon \in E_{\mathfrak{f}}} \coprod_{\sigma \in \Phi} \varepsilon\sigma,$$

i.e. the (disjoint) union of cones in Φ forms a fundamental domain for $(F \otimes \mathbb{R})_+/E_{\mathfrak{f}}$ (the existence of such Φ was shown by Shintani [2, Proposition 4]). Then we have

$$\zeta_{\mathfrak{f}}(s, z+\mathfrak{b}) = \sum_{\sigma \in \Phi} \zeta(s, z+\mathfrak{b}, \sigma), \tag{1.1}$$

where

$$\zeta(s, z+\mathfrak{b}, \sigma) := \sum_{\beta \in (z+\mathfrak{b}) \cap \sigma} \mathbf{N}(\beta)^{-s}. \tag{1.2}$$

The identity (1.1) says that the study of the partial zeta function can be reduced to the combinatorics on the set of cones Φ and the analysis of each function $\zeta(s, z+\mathfrak{b}, \sigma)$. Our results, explained in the next section, is obtained in this way.

Remark. In fact, for our purpose, it is sufficient to consider a finite set Φ of rational cones and a collection of real numbers $(\alpha_\sigma)_{\sigma \in \Phi}$ such that

$$\mathbf{1}_{(F \otimes \mathbb{R})_+} = \sum_{\varepsilon \in E_{\mathfrak{f}}} \sum_{\sigma \in \Phi} \alpha_\sigma \mathbf{1}_{\varepsilon\sigma},$$

where $\mathbf{1}_A$ denotes the defining function of A. Obviously, this leads to the identity

$$\zeta_{\mathfrak{f}}(s, z+\mathfrak{b}) = \sum_{\sigma \in \Phi} \alpha_\sigma \zeta(s, z+\mathfrak{b}, \sigma).$$

Such a 'weighted' cone decomposition might be convenient from the computational point of view (cf. [1, Section 5]).

§ 2. Main theorem

Here we state the main results obtained in [7], about the *Shintani invariant*

$$X(\mathfrak{C}) := \exp\bigl(-\zeta'(0,\mathfrak{C}) + (-1)^n \zeta'(0,\mu\mathfrak{C})\bigr). \tag{2.1}$$

Here μ denotes an element of O_F which is totally negative and congruent to 1 modulo $\mathfrak{f}$ (the ray class of the principal ideal generated by such μ is independent of the choice of μ). This is a direct generalization of the invariant defined by Shintani [5] for real quadratic fields, up to inversion.

To state the precise theorem, we number n embeddings of F into $\mathbb{R}$ and denote them by $x \longmapsto x^{(i)}$ $(i = 1, \ldots, n)$. Then, for each i, we put

$$\zeta_i(s, z+\mathfrak{b}, \sigma) := \sum_{\beta \in (z+\mathfrak{b})\cap\sigma} \bigl(\beta^{(i)}\bigr)^{-s} \tag{2.2}$$

(compare with (1.2)).

Theorem 2.1.

(1) We have a factorization $X(\mathfrak{C}) = X_1(\mathfrak{C}) \cdots X_n(\mathfrak{C})$, *where*

$$X_i(\mathfrak{C}) := \prod_{\sigma\in\Phi} \exp\bigl\{-\zeta_i'(0, z+\mathfrak{b}, \sigma) + (-1)^{\dim\sigma} \zeta_i'(0, -z+\mathfrak{b}, \sigma)\bigr\}. \tag{2.3}$$

(2) $X_i(\mathfrak{C})$ *is independent of the choices of* $\mathfrak{a}$, z *and* Φ *(and hence of* $\mathfrak{b}$*).*

(3) If μ_i *is an element of* O_F *satisfying*

$$\mu_i \equiv 1 \bmod \mathfrak{f}, \quad \mu_i^{(i)} < 0 \ \ \textit{and} \ \ \mu_i^{(j)} > 0 \ (j \neq i),$$

then we have

$$X_i(\mu_i\mathfrak{C}) = X_i(\mathfrak{C}) \ \ \textit{and} \ \ X_i(\mu_j\mathfrak{C}) = X_i(\mathfrak{C})^{-1} \ (j \neq i). \tag{2.4}$$

We call $X_i(\mathfrak{C})$ the *i-th factor* of the Shintani invariant. It may be regarded as the contribution of the i-th real place of F to the Shintani invariant $X(\mathfrak{C})$. The following formula, which is a direct consequence of (2.4), indicates an intrinsic meaning of each factor $X_i(\mathfrak{C})$:

Corollary 2.2. *Let* $\chi\colon Cl_F(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$ *be a Dirichlet character modulo* $\mathfrak{f}$ *and* $L(s,\chi) = \sum_{\mathfrak{C}} \chi(\mathfrak{C})\zeta(s,\mathfrak{C})$ *the associated L-function. Assume that there is an index* $i \in \{1, \ldots, n\}$ *such that*

$$\chi(\mu_i) = +1 \ \ \textit{and} \ \ \chi(\mu_j) = -1 \ \ (j \neq i).$$

Then we have

$$L'(0,\chi) = -\frac{1}{2}\sum_{\mathfrak{C}\in Cl_F(\mathfrak{f})} \chi(\mathfrak{C}) \log X_i(\mathfrak{C}). \tag{2.5}$$

Note that, if χ is a primitive Dirichlet character, the order of $L(s,\chi)$ at $s=0$ is equal to the number of $+1$ in $\chi(\mu_1),\dots,\chi(\mu_n)$. When the order is 1, the above corollary says that the leading Taylor coefficient $L'(0,\chi)$ can be expressed by the contribution of the only real place at which $\chi(\mu_i)=+1$ holds. Similar formulas in the case of higher order have not been found so far.

Remark. The equation (2.5) leads to a relation between the invariants $X_i(\mathfrak{C})$ and certain Stark units. In fact, $X_i(\mathfrak{C})$ is equal to an (absolute value of) Stark unit for the class field corresponding to the group $Cl_F(\mathfrak{f})/\{1,\mu_i\}$, under the assumption of the existence of such a unit (the precise statement is given in [7, §2]). This explains an arithmetic meaning of the factors $X_i(\mathfrak{C})$.

§ 3. Another expression of $X_i(\mathfrak{C})$ in the quadratic case

By theorem 2.1 (3), we may define the invariant $X_i(\mathfrak{C})$ by $X_i(\mathfrak{C}) = \big(X(\mathfrak{C})X(\mu_i\mathfrak{C})\big)^{\frac{1}{2}}$. Then the equation (2.3) will be an analytic expression of $X_i(\mathfrak{C})$. Indeed, the right hand side of (2.3) is a finite product of special values of multiple sine functions. Hence, if the Stark conjecture is true, this formula gives a partial answer to Hilbert's twelfth problem (generating the class fields of a number field by special values of certain analytic functions).

In this section, we give another expression for $X_i(\mathfrak{C})$ in the case that F is a real quadratic field, by transforming the finite product (2.3). Although we explain this result by an example, we can easily generalize it to general real quadratic fields by using the theory of continued fractions (cf. [6]).

We use the following example from [6] (this is essentially an example given in [3]): For $K=\mathbb{Q}(\sqrt{21})$, $\mathfrak{f}=(3)$ and $\mathfrak{C}=[O_K]$, we have

$$\begin{aligned} X_1(\mathfrak{C}) = X_2(\mathfrak{C}) &= \sqrt{\frac{1}{2}\left(\frac{1+\sqrt{21}}{2} - \sqrt{\frac{3+\sqrt{21}}{2}}\right)} \\ &= \mathcal{S}_2\left(\frac{\varepsilon}{3},\varepsilon\right)\mathcal{S}_2\left(\frac{2\varepsilon+2}{3},\varepsilon\right)\mathcal{S}_2\left(\frac{3\varepsilon+1}{3},\varepsilon\right), \end{aligned} \tag{3.1}$$

where $\varepsilon = \frac{5+\sqrt{21}}{2}$ and $\mathcal{S}_2$ is the double sine function defined by

$$\mathcal{S}_2(z,\omega) = \exp\big\{-\zeta_2'(0,z,\omega) + \zeta_2'(0,1+\omega-z,\omega)\big\},$$

$$\zeta_2(s,z,\omega) = \sum_{m,n=0}^{\infty} (z+m\omega+n)^{-s}.$$

Now let us recall the following formula obtained by Shintani [3]: if $\operatorname{Im}\tau > 0$,

(3.2)

$$\mathcal{S}_2(z,\tau) = \sqrt{i}\exp\frac{\pi i}{12}\Big(\tau+\frac{1}{\tau}\Big)\frac{\prod_{m=0}^{\infty}\Big(1-\exp 2\pi i(m\tau+z)\Big)}{\prod_{m=1}^{\infty}\Big(1-\exp 2\pi i\big(\frac{-m+z}{\tau}\big)\Big)}\exp\frac{\pi i}{2}\Big\{\frac{z^2}{\tau}-\Big(1+\frac{1}{\tau}\Big)z\Big\}.$$

In particular, if we put

$$f(x,y,\tau) = \prod_{m=0}^{\infty}\{1-\exp 2\pi i(m\tau+x\tau+y)\},$$

we have

$$\mathcal{S}_2(x\omega+y,\omega) = \lim_{\tau\to\omega}\big|\mathcal{S}_2(x\tau+y,\tau)\big| = \lim_{\tau\to\omega}\left|\frac{f(x,y,\tau)}{f(1-y,x,-1/\tau)}\right| \tag{3.3}$$

for $\omega > 0$ and $x, y \in \mathbb{R}$ (note that the left hand side is a positive real number).

Proposition 3.1. *The value $X_1(\mathfrak{C})$ given in* (3.1) *satisfies the equality*

$$X_1(\mathfrak{C}) = \lim_{\tau\to\varepsilon}\left|\frac{f\big(1,\frac{1}{3},\tau\big)}{f\big(1,\frac{1}{3},\gamma\tau\big)}\right|,$$

where $\gamma = \begin{pmatrix}5 & -1\\ 1 & 0\end{pmatrix}^3 \in \mathrm{SL}_2(\mathbb{Z})$ *acts on the upper half plane in the usual manner.*

Proof. Applying (3.3) to the last expression in (3.1), we obtain

$$X_1(\mathfrak{C}) = \lim_{\tau_1\to\varepsilon}\left|\frac{f\big(\frac{1}{3},0,\tau_1\big)}{f\big(1,\frac{1}{3},-\frac{1}{\tau_1}\big)}\frac{f\big(\frac{2}{3},\frac{2}{3},\tau_2\big)}{f\big(\frac{1}{3},\frac{2}{3},-\frac{1}{\tau_2}\big)}\frac{f\big(1,\frac{1}{3},\tau_3\big)}{f\big(\frac{2}{3},1,-\frac{1}{\tau_3}\big)}\right|. \tag{3.4}$$

Here we define τ_k inductively by $\tau_k = 5-\frac{1}{\tau_{k+1}}$. Since $\varepsilon = \frac{5+\sqrt{21}}{2}$ satisfies $\varepsilon = 5-\frac{1}{\varepsilon}$, all τ_k tend to ε when τ_1 tends to ε.

Using the equalities $f(x,y,\tau+1) = f(x,x+y,\tau)$ and $f(x,y+1,\tau) = f(x,y,\tau)$, the right hand side of (3.4) can be rewritten as

$$\begin{aligned} X_1(\mathfrak{C}) &= \lim_{\tau_1\to\varepsilon}\left|\frac{f\big(\frac{1}{3},0,\tau_1\big)}{f\big(1,\frac{1}{3}-5,5-\frac{1}{\tau_1}\big)}\frac{f\big(\frac{2}{3},\frac{2}{3},\tau_2\big)}{f\big(\frac{1}{3},\frac{2}{3}-\frac{5}{3},5-\frac{1}{\tau_2}\big)}\frac{f\big(1,\frac{1}{3},\tau_3\big)}{f\big(\frac{2}{3},1-\frac{10}{3},5-\frac{1}{\tau_3}\big)}\right| \\ &= \lim_{\tau_1\to\varepsilon}\left|\frac{f\big(\frac{1}{3},0,\tau_1\big)}{f\big(1,\frac{1}{3},\tau_0\big)}\frac{f\big(\frac{2}{3},\frac{2}{3},\tau_2\big)}{f\big(\frac{1}{3},0,\tau_1\big)}\frac{f\big(1,\frac{1}{3},\tau_3\big)}{f\big(\frac{2}{3},\frac{2}{3},\tau_2\big)}\right| = \lim_{\tau\to\varepsilon}\left|\frac{f\big(1,\frac{1}{3},\tau\big)}{f\big(1,\frac{1}{3},\gamma\tau\big)}\right|, \end{aligned} \tag{3.5}$$

hence the proof is complete. ∎

An advantage of this new expression is that there is only a single fraction, instead of a finite product. Indeed, the reduction in the last equality of (3.5) reflects the combinatorial structure involved in the original finite product expression, hence we may concentrate on the analytic properties of the single function $f(x, y, \tau)$.

As was mentioned earlier, it is easy to generalize Proposition 3.1 to general real quadratic fields. On the other hand, to the author's knowledge, the corresponding expression has not been found yet in the case of degree greater than 2. It could be interesting to look for such a formula, while it seems more important to find good applications of the above formula in the quadratic case.

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