RIMS Kôkyûroku Bessatsu
B19 (2010), 187–220

Stability and Arithmetic: An extract of essence

By

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Abstract

Stability plays a central role in arithmetic. In this article, we explain some basic ideas and present certain constructions for such studies. There are two aspects: namely, general Class Field Theories for Riemann surfaces using semi-stable parabolic bundles & for p-adic number fields using what we call semi-stable filtered (phi,N;omega)-modules; and non-abelian zeta functions for function fields over finite fields using semi-stable bundles & for number fields using semi-stable lattices.

Introduction

Most part of the following text is a resume of the paper [W7], based on which my talk was delivered. Due to its large size of 135 pages and to the importance of the topics treated, the author feels the necessity to provide a digest version to help potential readers. Different from [W7], where various aspects of the theories related to stability are discussed, here we only focus on the crucial roles played by stability in our studies of zeta functions and of general class field theories.

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Stability has been proved to be very fundamental in algebraic geometry and differential geometry. Comparably, this concept appears relatively new to many who are working in arithmetic geometry and number theory. Nevertheless, in the past a decade or so, importance of stability was gradually noticed by some working in arithmetic. For examples, we now have

(i) Existence theorem and reciprocity law of a non-abelian class field theory for function fields over complex numbers, based on Seshadri's work of semi-stable parabolic bundles over Riemann surfaces;

Received March 28, 2009. Reviced November 20, 2009.
2000 Mathematics Subject Classification(s): 11R40, 11R42, 11S37, 11S40, 14G10, 14G22, 22E99.
Partially supported by JSPS
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(ii) High rank zeta functions for global fields, defined as natural integrations over moduli spaces of semi-stable bundles/lattices; and
(iii) Characterization of the so-called semi-stable representations for absolute Galois groups of p-adic number fields, in terms of weakly admissible filtered (φ, N)-modules, or better, semi-stable filtered (φ, N)-modules of slope zero.

Along with this line, in this paper, we explain some basic ideas and present certain constructions on using stability to study non-abelian aspects of arithmetic. This consists of two aspects, one at a micro level and the other on large scale.

I) Micro Level

We, at this micro level, want to give a characterization for each individual Galois representation. For this, first we classify Galois representations into four types, namely, v-adic/adelic representations for local/global (number) fields.

In general, arbitrary Galois representations are too complicated to have clearer structures, certain natural restrictions should be imposed:

(i) v-adic Galois Representations for

(i.a) *Local Field* K_w: $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ involved are for Galois group G_{K_w} of a local w-adic field K_w with coefficients in a v-adic field F_v. Motivated by Monodromy Theorems, we assume
(**pST**) $\rho_{w,v}$ *is potentially semi-stable.*

(i.b) *Global Field* K: $\rho_{K,v} : G_K \to GL_n(F_v)$ involved are for Galois group G_K of a number field K with coefficients in a v-adic field F_v. Motivated by etale cohomology theory, we assume
(**pST**) *For all local completions K_w, the associated local v-adic representations $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ satisfies condition* **pST** *of (i.a);* and
(**Unr**) *For almost all w, the associated v-adic representations $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ are unramified.*

(ii) Adelic Galois Representations for

(ii.a) *Local Field* K_w: $\rho_{w,\mathbb{A}_F} : G_{K_w} \to GL_n(\mathbb{A}_F)$ involved are for Galois group G_{K_w} of a w-adic field K_w with coefficients in the adelic space $\mathbb{A}_F$ associated to a number field F. Continuity of $\rho_{w,\mathbb{A}_F}$ proves to be too loose. Motivated by etale cohomology theory, and Deligne's solution to the Weil conjecture when $v \nparallel w$, together with Katz-Messing's modification when $v \| w$, we assume that
(**Unr**) *For almost all v (in coefficients), the associated v-adic representation $\rho_{w,v} : G_{K_w} \to GL_n(F_v)$ are unramified;* and
(**Inv**) *For all v, i.e., for v satisfying either $v \| w$ or $v \nparallel w$, the associated characteristic polynomials of the Frobenius induced from $\rho_{w,v}$ are the same, particularly, independent*

of v.

We call such a representation a *thick one*, as the invariants do not depend on the coefficients chosen.

Remark. The compatibility conditions stated here are standard. However, from our point of view, the **Inv** condition appears to be to practical. In other words, it would be much better if the Inv condition can be replaced by other principles, e.g., certain compatibility from class field theory. (See e.g. [Kh1,2,3])

(ii.b) *Global Field K:* $\rho_{K,\mathbb{A}_F} : G_K \to GL_n(\mathbb{A}_F)$ involved are for Galois group G_K of a number field K with coefficients in the adelic space $\mathbb{A}_F$ associated to a number field F. As above, only continuity of $\rho_{w,\mathbb{A}_F}$ appears to be too weak to get a good theory. Certainly, there are two different directions to be considered, namely, the horizontal one consisting of places w of K, and the vertical one consisting of places v of coefficients field F. From ii.a), we assume that

(**Comp**) *For every fixed place w of K, the induced representation $\rho_{w,\mathbb{A}_F} : G_{K_w} \to \mathrm{GL}_n(\mathbb{A}_F)$ forms a compatible system.*

As such, the corresponding theory is a thick one. Hence, by **Inv**, we are able to select good representatives for $\rho_{w,\mathbb{A}_F}$, e.g., the induced $\rho_{w,v} : G_{K_w} \to \mathrm{GL}_n(F_v)$ where $v\|w$. In this language, we then further assume that the admissible conditions for the other direction v can be read from these selected $\rho_{w,v}$, $v\|w$. More precisely, we assume that

(**dR**) *All $\rho_{w,v}$, $v\|w$, are of de Rham type;*

(**Crys**) *For almost all w and v, $\rho_{v,w}$ are crystalline.*

For this reason, we may form what we call the *anleric ring*

$$\mathbb{B}_{\mathbb{A}} := \prod{}' \Big(\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{crys}}^{+}\Big),$$

where $\mathbb{B}_{\mathrm{dR}}$ denotes the ring of de Rham periods, and $\mathbb{B}_{\mathrm{crys}}^{+}$ the ring of crystalline periods, and $\prod'$ means the restricted product. As such, the final global condition we assume is the following:

(**Adm**) *$\{\rho_{w,v}\}_{v\|w}$ are $\mathbb{B}_{\mathbb{A}}$-admissible.*

Even this admissibility is not clearly stated due to 'the lack of space', one may sense it say via determinant formalism from abelian CFT, (see e.g., the reformulation by Serre for rank one case ([Se2]) and the conjecture of Fontaine-Mazur on geometric representations ([FM]). For the obvious reason, we will call such a representation a *thin* one.

With the restrictions on Galois representations stated, let us next turn our attention to their characterizations. Here by a characterization, we mean a certain totally independent but intrinsic structure from which the original Galois representation can be reconstructed. There are two different approaches, analytic one and algebraic one.

• **Analytic One** Here the objects seeking are supposed to be equipped with analytic structures such as connections and residues (at least for v-adic representations).

• **Algebraic One** Here the structures involved are supposed to be purely alebraic. We will leave the details to the main text. Instead, let me point out that for local theories, when $l \neq p$, we should equally have l-adic analogues $\mathbb{B}_{\mathrm{total}}$, $\mathbb{B}_{\mathrm{pFM\&N}}$, $\mathbb{B}_{\mathrm{ur}}$ of Fontaine's p-adic ring of de Rham, semi-stable, crystalline periods, namely, $\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{st}}$, $\mathbb{B}_{\mathrm{crys}}$, respectively.

To uniform the notation, denote the corresponding rings of periods in both l-adic theory and p-adic theory by $\mathbb{B}_{\mathrm{dR}}$, $\mathbb{B}_{\mathrm{st}}$, $\mathbb{B}_{\mathrm{ur}}^+$. Accordingly, for adelic representations of local fields, we then can formulate a huge *anleric ring* $\mathbb{B}_{\mathbb{A}} := \prod{}'\Big(\mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{ur}}^+\Big)$, *of adelic periods*, namely, the restricted product of $\mathbb{B}_{\mathrm{dR}}$ with respect to $\mathbb{B}_{\mathrm{ur}}^+$. In this language, the algebraic condition for thin adelic Galois representations of global fields along with the vertical direction may also be stated as:

(**Adm**) *It is* $\mathbb{B}_{\mathbb{A}}$*-admissible.*

II) Large Scale

A characterization of each individual Galois representation in terms of pure algebraic structures may be called a Micro Reciprocity Law, MRL for short, as it exposes an intrinsic connection between Galois representations and certain algebraic aspects of the base fields. Assuming such a MRL, we then are in a position to understand the mathematics involved in a global way. There are also two different approaches, at least when the coefficients are local. Namely, the categorical theoretic one, based on the fact that Galois representations selected automatically form a Tannakian category, and the moduli theoretic one, based on the fact that the associated algebraic structures admit GIT stability interpretations. (In the case when the coefficients are global adelic spaces, the existing standard Tannakian category theory and GIT should be extended.)

• **Tannakian Categories** The main aim here is to offer a general Class Field Theory, CFT for short, for the associated base field. Roughly speaking, this goes as follows, at least when the coefficients are local fields. With the Micro Reciprocity Law, we then can get a clone Tannakian category, consisting of certain intrinsically defined pure algebraic objects associated to the base fields, for the Tannakian category consisting of selected Galois representations. As a direct consequence of potentially semi-stability, using the so-called finitely generated sub-Tannakian categories and automorphism groups of the associated restrictions of the fiber functors, one then can establish an existence theorem and a global reciprocity law for all finite (non-abelian) extensions of the base fields so as to obtain a general CFT for them. As one may expect here, much refined results can be obtained.

• **Moduli Spaces** From the MRL, Galois representations selected can be characterized by intrinsically defined algebraic structures associated to based fields. These algebraic structures are further expected to be able to put together to form well-controlled moduli spaces. Accordingly, we have certain geometric objects to work with. The importance of such geometric spaces can hardly be overestimated since, with such spaces, we can introduce intrinsic (non-abelian) invariants for base fields.

To achieve this, we clearly need to have a good control of objects selected. As usual, this is quite delicate: If the selection is too restrictive, then there might not be enough information involved; on the other hand, it should not be too loose, as otherwise, it is too complicated to see structures in a neat manner, even we know many things are definitely there. (The reader can sense this from our current studies of the Langlands Program.) It is for the purpose of overcoming such difficulties that we introduce the following

Key: **Stability** This is supposed to be a condition which helps us to make *good selections* and hence to get nice portions among all possibilities. Particularly, for the algebraic objects selected, we then expect to establish a general MRL (using them) so that the Tannakian category formalism can be applied and a general CFT can be established; and to construct moduli spaces (for them) so that intrinsic invariants can be introduced naturally. This condition is *Stability.* In accordance with what said above, as a general principle of selection, the condition of stability then should be (a) algebraic, (b) intrinsic, and (c) rigid.

This paper consists of four chapters. They are: 1. Guidances from Geometry, 2. High Rank Zetas and Stability, 3. General CFT and Stability, and 4. Two Approaches to Conjectural Micro Reciprocity Law.

Acknowledgement. This paper was written when I visited UCLA. I would like to thank Hida for his keen interests and kind invitation(s). Special thanks also due to Ichikawa for the invitation and to Nakamura for the comments.

Chapter 1. Guidances from Geometry

§ 1. Micro Reciprocity Law

Let $\rho : \pi_1(M^o, *) \to \mathrm{GL}(V)$ be an irreducible *unitary representation* of the fundamental group $\pi_1(M^o, *)$ of an open Riemann surface M^o. Then ρ satisfies the *finite monodromy* property at all punctures P_i's. Hence there exists a finite Galois covering $\pi' : M' \to M$ of compact Riemann surfaces ramified possibly at P_i's such that ρ naturally induces a unitary representation $\rho' : \pi_1(M', *) \to \mathrm{GL}(V)$ of the fundamental group of the *compact* Riemann surface M' on V. As such, by the uniformization theorem, we obtain a *unitary flat bundle* over M' equipped with a natural action of the Galois group $\mathrm{Gal}(\pi')$, namely, the four-tuple

$$\Big(M', E_{\rho'} := \big(\pi_1(M',*),\rho'\big)\backslash\big(\mathfrak{H}^{(+)}\times V\big), \nabla_{\rho'}; \mathrm{Gal}(\pi')\Big).$$

One checks that

$$\Big(\pi'_*\big(E_{\rho'}\otimes\Omega^1_{M'}\big)\Big)^{\mathrm{Gal}(\pi')} = E_\rho\otimes\Omega^1_M(\log Z)$$

where $Z = P_1+P_2+\cdots+P_N$ denotes the reduced branch divisor on M. Consequently, we then obtain a *logarithmic unitary flat bundle* $\big(E_\rho, \nabla_\rho(\log Z)\big)$ on the compact Riemann surface M. Thus by using $\mathrm{Res}_{P_i}\nabla_\rho(\log Z)$, we then obtain Seshadri's *parabolic structures* on the fibers of E_ρ at punctures P_i's. As such, an important discovery of Seshadri is that the parabolic bundle obtained then is stable of degree zero. More strikingly, the converse is correct as well. Namely, any stable parabolic bundle of degree zero can be constructed in this manner.

- **Micro Reciprocity Law** ((Weil, Mumford, Narasimhan-Seshadri,) Seshadri)
  There exists a natural one-to-one and onto correspondence
$$\Big\{\, irreducible\ unitary\ representations\ of\ \pi_1(M^o,*)\,\Big\}$$
$$\Updownarrow$$
$$\Big\{\, stable\ parabolic\ bundles\ of\ degree\ zero\ on\ (M^o,M)\,\Big\};$$

- **Ramifications versus Parabolic Structures** ((Grothendieck), Seshadri)
  There exists a natural one-to-one and onto correspondence
$$\Big\{\, vector\ bundles\ W/M'\ with\ compatible\ action\ Gal(M'/M)\,\Big\}$$
$$\Updownarrow$$
$$\Big\{\, parabolic\ bundles\ E_*/(M^o,M)\ with\ compatible\ parabolic\ weights\,\Big\}.$$

§ 2. Arithmetic CFT: Class Field Theory

Let us consider the category consisting of semi-stable parabolic bundles of (parabolic) degree zero over (M^o, M). This category is in fact Tannakian. Denote it by $\Big(\mathbb{PV}^{\mathrm{ss};0}_{M^o,M};\mathbb{F}\Big)$.

Main Theorem of Arithmetic CFT ([W1])

- (**Existence**) *There exists a canonical one-to-one and onto correspondence*
$$\Big\{Finitely\ Generated\ Sub-Tannakian\ Cats\ \big(\Sigma,\mathbb{F}|_\Sigma\big)\ of\ \Big(\mathbb{PV}^{\mathrm{ss};0}_{M^o,M};\mathbb{F}\Big)\Big\}$$
$$\Big\Updownarrow\ \Pi$$
$$\Big\{\, Finite\ Galois\ Coverings\ M'\to(M^o,M)\,\Big\}$$

which induces naturally an isomorphism

• (**Reciprocity Law**)

$$\mathrm{Aut}^{\otimes}\Big(\Sigma, \mathbb{F}|_{\Sigma}\Big) \simeq \mathrm{Gal}\Big(\Pi(\Sigma, \mathbb{F}|_{\Sigma})\Big).$$

§ 3. Geometric CFT: Conformal Field Theory

For a fixed compact Riemann surface M, denote by $\mathcal{M}_M(r,0)$ the moduli spaces of rank r semi-stable bundles of degree zero on M. Over such moduli spaces, we can construct many global invariants. Analytically we may expect that a still ill-defined Feymann integral would give us something interesting. We will not pursue this line further, instead, let us start with an algebraic construction.

Since each moduli point corresponds to a semi-stable vector bundle, it makes sense to talk about the associated cohomology groups. As such, then we may form the so-called Grothendieck-Mumford determinant line of cohomologies. Consequently, if we move our moduli points over all moduli spaces, we obtain the so-called Grothendieck-Mumford determinant line bundles λ_M on $\mathcal{M}_M(r,0)$. Note that the Picard group of $\mathcal{M}_M(r,0)$ is isomorphic to $\mathbb{Z}$, we see that a suitable multiple of λ_M is indeed very ample. (For all this, we in fact need to restrict ourselves only to the stable part.) It then makes sense to talk about the $\mathbb{C}$-vector space $H^0\big(\mathcal{M}_M(r,0), \lambda_M^{\otimes n}\big)$ (for n sufficiently away from 0).

The most interesting and certain a very deep point is somehow we expect that the space itself $H^0\big(\mathcal{M}_M(r,0), \lambda_M^{\otimes n}\big)$, also called *conformal blocks*, does not really very much related with the complex structure on M used. More precisely, let us now move M in $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}_g}$, the moduli space of compact Riemann surfaces of genus $g = g(M)$ and its stable compactification. Denote by Δ_{bdy} the boundary of $\mathcal{M}_g$. Then the conformal blocks form a natural vector bundle $\Pi_*\Big(\lambda_M^{\otimes n}\Big)\Big|_{\mathcal{M}_g}$ on $(\mathcal{M}_g \hookrightarrow)\overline{\mathcal{M}_g}$.

Main Theorem in Geometric CFT: (Tsuchiya-Ueno-Yamada) *There exists a projectively flat logarithmic connection on the bundle* $\Pi_*\Big(\lambda_M^{\otimes n}\Big)\Big|_{\mathcal{M}_g}$ *over* $(\mathcal{M}_g, \Delta_{\mathrm{bdy}})$.

Chapter 2. High Rank Zetas and Stability

§ 4. High Rank Zetas for Function Fields

Let C be a regular, geometrically connected projective curve of genus g defined over $\mathbb{F}_q$, the finite field with q elements, and $\mathcal{M}_{C,r}$ the moduli space of semi-stable bundles of rank r over C. These spaces are projective varieties. So following Weil, we may try to attach them with the standard Artin-Weil zeta functions. However, there is another more intrinsic way. Namely, instead of simply viewing these moduli spaces as algebraic varieties, we here want to fully use the moduli aspect by viewing rational points of these varieties as rational bundles: This is possible at least for the stable part by a work of Harder-Narasimhan on Brauer groups ([HN]). Accordingly, for each rational moduli point, we can have a very natural weighted count. All this then leads to the following

Definition. (Weng) *The rank r zeta function for $C/\mathbb{F}_q$ is defined by*

$$\zeta_{C,\mathbb{F}_q;r}(s) := \sum_{V\in[V]\in\mathcal{M}_{C,r}} \frac{q^{h^0(C,V)}-1}{\#\mathrm{Aut}(V)}\cdot\Big(q^{-s}\Big)^{\deg(V)}, \qquad \mathrm{Re}(s)>1.$$

Here as usual, $[V]$ denotes the Seshadri class of (a rational) semi-stable bundle V, and $\mathrm{Aut}(V)$ *denotes the automorphism group of V.*

By semi-stable condition, the summation above is only taken over the part of moduli space whose points have non-negative degrees. Thus by the duality, Riemann-Roch and a Clifford type lemma for semi-stable bundles, we then can expose the following basic properties for our zeta functions of curves.

Zeta Facts. (Weng) (0) *Rank one $\zeta_{C,1,\mathbf{F}_q}(s)$ coincides with the classical Artin zeta function $\zeta_C(s)$ for curve C;*
(1) *$\zeta_{C,r,\mathbf{F}_q}(s)$ is well-defined for* $\mathrm{Re}(s)>1$, *and admits a meromorphic continuation to the whole complex s-plane;*
(2) (**Rationality**) *Set $t:=q^{-s}$ and $\zeta_{C,r,\mathbf{F}_q}(s)=:Z_{C,r,\mathbf{F}_q}(t)$, $|t|<1$. Then there exists a polynomial $P_{C,r,\mathbf{F}_q}(s)\in\mathbf{Q}[t]$ such that*

$$Z_{C,r,\mathbf{F}_q}(t)=\frac{P_{C,r,\mathbf{F}_q}(t)}{(1-t^r)(1-q^rt^r)};$$

(3) (**Functional Equation**) *Set*

$$\xi_{C,r,\mathbf{F}_q}(s):=\zeta_{C,r,\mathbf{F}_q}(s)\cdot(q^s)^{r(g-1)}.$$

Then

$$\xi_{C,r,\mathbf{F}_q}(s)=\xi_{C,r,\mathbf{F}_q}(1-s).$$

Remarks. (1) (**Count in Different Ways**) The above weighted count is designed for all rational semi-stable bundles, motivated by Harder-Narasimhan's interpretation on Siegel's work about Tamagawa numbers ([HN]). For this reason, modifications for the definition of high rank zetas can be given, say, count only one within a fixed Seshadri class, or count only what are called strongly semi-stable bundles, etc...

(2) (**Stratifications and Cohomological Interpretations**) Deninger once asked whether there was a cohomological interpretation for our zeta functions. There is a high possibility for it: We expect that our earlier works on refined Brill-Noether loci would play a key role here, since refined Brill-Noether loci induce natural stratifications on moduli spaces.

§ 5. High Rank Zetas for Number Fields

Let F be a number field with usual $\mathcal{O}_F$, Δ_F, r_1 and r_2 etc... By definition, an $\mathcal{O}_F$-*lattice* $\Lambda = (P, \rho)$ of rank r consisting of a rank r projective $\mathcal{O}_F$-module P and a metric ρ on the space $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r$. Recall that, being projective, there exists a fractional idea $\mathfrak{a}$ of F such that $P \simeq \mathcal{O}_F^{r-1} \oplus \mathfrak{a}$. Particularly, the natural inclusion $O_F^{r-1} \oplus \mathfrak{a} \hookrightarrow F^r$ induces a natural embedding of P into $\big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\big)^r$ via the compositions

$$P \simeq O_F^{r-1} \oplus \mathfrak{a} \hookrightarrow F^r \hookrightarrow \Big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\Big)^r \simeq \big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}.$$

As such, then the image of P naturally offers us a lattice Λ in the metrized space $\Big(\big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}, \rho\Big)$.

An $\mathcal{O}_F$-lattice is called *semi-stable* if for all sub-$\mathcal{O}_F$-lattice Λ_1 of Λ, we have

$$\mathrm{Vol}(\Lambda_1)^{\mathrm{rank}(\Lambda)} \geq \mathrm{Vol}(\Lambda)^{\mathrm{rank}(\Lambda_1)},$$

where the volume $\mathrm{Vol}(\Lambda)$ of Λ is usually called the covolume of Λ, namely,

$$\mathrm{Vol}(\Lambda) := \mathrm{Vol}\Big(\Big(\big(\mathbb{R}^r\big)^{r_1} \times \big(\mathbb{C}^r\big)^{r_2}, \rho\Big)\Big/\Lambda\Big).$$

Denote by $\mathcal{M}_{F,r}$ the moduli space of semi-stable $\mathcal{O}_F$ lattices of rank r, i.e., the space of isomorphism classes of semi-stable $\mathcal{O}_F$ lattices of rank r.

For an $\mathcal{O}_F$-lattice Λ, define its *geo-arithmetical cohomology groups* by

$$H^0(F, \Lambda) := \Lambda, \qquad \text{and} \qquad H^0(F, \Lambda) := \Big(\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\Big)^r\Big/\Lambda.$$

Unlike in algebraic geometry and/or in arithmetic geometry, cohomological groups H^i are not vector spaces, but locally compact topological groups. Accordingly, we have a topological rooted duality, and an geo-ari Riemann Roch from Fourier analysis.

Definition. (Weng) *The rank r zeta function of F is defined by*

$$\xi_{F,r}(s) := \Big(\Big|\Delta_F\Big|^s\Big)^{\frac{r}{2}} \cdot \int_{\mathcal{M}_{F,r}} \Big(e^{h^0(F,\Lambda)} - 1\Big) \cdot \Big(e^{-s}\Big)^{\deg(\Lambda)} d\mu(\Lambda), \ \mathrm{Re}(s) > 1.$$

Tautologically, from the duality and the geo-arithmetical Riemann-Roch, we obtain

Zeta Facts. (Weng) (0) (Iwasawa) $\xi_{F,1}(s) \doteq \xi_F(s)$, *the completed Dedekind zeta for* F*;*
(1) (**Meromorphic Extension**) *Non-abelian zeta function*

$$\xi_{F,r}(s) := \left(\Delta_F^{\frac{r}{2}}\right)^s \int_{\Lambda\in\mathcal{M}_{F,r}} \left(e^{h^0(F,\Lambda)}-1\right)\left(e^{-s}\right)^{\deg(\Lambda)}\cdot d\mu$$

converges absolutely and uniformly when $\mathrm{Re}(s)\geq 1+\delta$ *for any* $\delta>0$*. Moreover,* $\xi_{F,r}(s)$ *admits a unique meromorphic continuation to the whole complex* s*-plane;*
(2) (**Functional Equation**) *The extended* $\xi_{F,r}(s)$ *satisfies the functional equation*

$$\xi_{F,r}(s)=\xi_{F,r}(1-s);$$

(3) (**Singularities**) *The extended* $\xi_{F,r}(s)$ *has two singularities, all simple poles, at* $s=0\,1$*, with*

$$\mathrm{Res}_{s=0}\,\xi_{F,r}(s) \;=\; \mathrm{Res}_{s=0}\,\xi_{F,r}(s) \;=\; \mathrm{Vol}\Big(\mathcal{M}_{F,r}[\Delta_F^{\frac{r}{2}}]\Big).$$

§ 6. Geometric Characterization of Stability

Minkowski embeddings of F into $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$ induces a class of natural moduli points $\tau_\Lambda\in\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$ associated to a rank two $\mathcal{O}_F$-lattice $\Lambda=\big(\mathcal{O}_F\oplus\mathfrak{a};\rho\big)$. Here, as usual, denote by

$$\mathcal{H}:=\{z=x+iy\in\mathbb{C}:x\in\mathbb{R},y\in\mathbb{R}_+^*\},$$

the upper half plane, and

$$\begin{aligned}\mathbb{H}:=&\mathbb{C}\times]0,\infty[=\Big\{(z,r):z=x+iy\in\mathbb{C},r\in\mathbb{R}_+^*\Big\}\\ =&\Big\{(x,y,r):x,y\in\mathbb{R},r\in\mathbb{R}_+^*\Big\}\end{aligned}$$

the 3-dimensional hyperbolic space.

The natural embedding $SL(\mathcal{O}_F\oplus\mathfrak{a})\hookrightarrow SL(2,\mathbb{R})^{r_1}\times SL(2,\mathbb{C})^{r_2}$ induces a natural action of $SL(\mathcal{O}_F\oplus\mathfrak{a})$ on $\mathbb{P}^1(F)$, viewed as a part of the boundary $\mathbb{P}^1(\mathbb{R})^{r_1}\times\mathbb{P}^1(\mathbb{C})^{r_2}$ of the upper half space $\mathcal{H}^{r_1}\times\mathbb{H}^{r_2}$.

Recall that for a cusp $\eta=\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\in\mathbb{P}^1(F)$, by the Cusp-Ideal Class Correspondence of Maaβ, we obtain a natural ideal class associated to the fractional ideal $\mathfrak{b}:=\mathcal{O}_F\cdot\alpha+\mathfrak{a}\cdot\beta$. Moreover, by assuming that α,β are all contained in $\mathcal{O}_F$, as we may, we know that the corresponding stablizer group Γ_η is given by

$$A^{-1}\cdot\Gamma_\eta\cdot A=\left\{\gamma=\begin{pmatrix}u & z\\ 0 & u^{-1}\end{pmatrix}\in\Gamma:u\in U_F,z\in\mathfrak{a}\mathfrak{b}^{-2}\right\},$$

where $A \in SL(2,F)$ satisfying $A\infty = \eta$ which may be further chosen in the form $A = \begin{pmatrix} \alpha\,\alpha^* \\ \beta\,\beta^* \end{pmatrix} \in SL(2,F)$ so that $\mathcal{O}_F\beta^* + \mathfrak{a}^{-1}\alpha^* = \mathfrak{b}^{-1}$.

Now for $\tau = (z_1,\dots,z_{r_1}; P_1,\cdots,P_{r_2}) \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$, set

$$N(\tau) := N\Big(\mathrm{ImJ}(\tau)\Big) = \prod_{i=1}^{r_1} \Im(z_i) \cdot \prod_{j=1}^{r_2} J(P_j)^2 = \Big(y_1\cdot\ldots\cdot y_{r_1}\Big)\cdot\Big(v_1\cdot\ldots\cdot v_{r_2}\Big)^2.$$

Then for all $\gamma = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix} \in SL(2,F)$, $N\Big(\mathrm{ImJ}(\gamma\cdot\tau)\Big) = \frac{N(\mathrm{ImJ}(\tau))}{\|N(c\tau+d)\|^2}$. Moreover, following [Sie] and [W5], define the *reciprocal distance* $\mu(\eta,\tau)$ *from the point* $\tau \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ *to the cusp* $\eta = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ *in* $\mathbb{P}^1(F)$ by

$$\begin{aligned}\mu(\eta,\tau) :=& N\Big(\mathfrak{a}^{-1}\cdot(\mathcal{O}_F\alpha + \mathfrak{a}\beta)^2\Big)\\ &\times \frac{\Im(z_1)\cdots\Im(z_{r_1})\cdot J(P_1)^2\cdots J(P_{r_2})^2}{\prod_{i=1}^{r_1}|(-\beta^{(i)}z_i+\alpha^{(i)})|^2\prod_{j=1}^{r_2}\|(-\beta^{(j)}P_j+\alpha^{(j)})\|^2}\\ =&\frac{1}{N(\mathfrak{a}\mathfrak{b}^{-2})}\cdot\frac{N(\mathrm{ImJ}(\tau))}{\|N(-\beta\tau+\alpha)\|^2},\end{aligned}$$

and the *distance of* τ *to the cusp* η by

$$d(\eta,\tau_\Lambda) := \frac{1}{\mu(\eta,\tau_\Lambda)} \geq 1.$$

Then, with the use of a crucial result of Tsukasa Hayashi [Ha], we are ready to state the following fundamental result, which exposes a beautiful intrinsic relation between stability and the distance to cusps.

Theorem. (Weng) *The lattice* Λ *is semi-stable if and only if the distances of corresponding moduli point* $\tau_\Lambda \in \mathcal{H}^{r_1} \times \mathbb{H}^{r_2}$ *to all cusps are all bigger or equal to 1.*

§ 7. Algebraic Characterization of Stability

Let $\Lambda = \Lambda^g$ be a rank r lattice associated to $g \in \mathrm{GL}_r(\mathbb{A})$ and P a parabolic subgroup. Denote the sublattices filtration associated to P by

$$0 = \Lambda_0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_{|P|} = \Lambda.$$

Assume that P corresponds to the partition $I = (d_1, d_2, \cdots, d_{n=:|P|})$. Consequently, we have

$$\mathrm{rk}(\Lambda_i) = r_i := d_1 + d_2 + \cdots + d_i, \qquad \text{for } i = 1,2,\cdots,|P|.$$

Let $p,q:[0,r]\to\mathbb{R}$ be two polygons such that $p(0)=q(0)=p(r)=q(r)=0$. Then following Lafforgue, we say q is *bigger than* p *with respect to* P and denote it by $q>_P p$, if $q(r_i)-p(r_i)>0$ for all $i=1,\cdots,|P|-1$. Moreover, using Harder-Narasimhan filtration type consideration, we can associate a canonical polygon $\bar{p}_\Lambda$ to an $\mathcal{O}_F$-lattice; and for a parabolic subgroup P, p_P^g denotes the polygon induced by P for (the lattice corresponding to) the element $g\in G(\mathbb{A})$. Accordingly, introduce also the characteristic function $\mathbf{1}(\overline{p}^*\le p)$ by

$$\mathbf{1}(\overline{p}^g\le p)=\begin{cases}1, & \text{if } \overline{p}^g\le p;\\ 0, & \text{otherwise.}\end{cases}$$

Fundamental Relation. (Lafforgue, Weng) *Let* $p:[0,r]\to\mathbb{R}$ *be a fixed convex polygon such that* $p(0)=p(r)=0$. *Then we have*

$$\mathbf{1}(\overline{p}^g\le p)=\sum_{P:\,\text{stand parabolic}}(-1)^{|P|-1}\sum_{\delta\in P(F)\backslash G(F)}\mathbf{1}(p_P^{\delta g}>_P p)\qquad\forall g\in G(\mathbb{A}).$$

§ 8. Analytic Characterization of Stability

Let G be a reductive group defined over a number field F with usual $B,P,\mathfrak{a}_0,\hat{\tau}_P$ etc...

Definition. (Arthur) *Fix a suitably regular point* $T\in\mathfrak{a}_0^+$. *If* ϕ *is a continuous function on* $G(F)\backslash G(\mathbb{A})^1$, *define Arthur's analytic trunction* $\Big(\Lambda^T\phi\Big)(x)$ *to be the function*

$$\Big(\Lambda^T\phi\Big)(x):=\sum_P(-1)^{\dim(A/Z)}\sum_{\delta\in P(F)\backslash G(F)}\phi_P(\delta x)\cdot\hat{\tau}_P\Big(H(\delta x)-T\Big),$$

where

$$\phi_P(x):=\int_{N(F)\backslash N(\mathbb{A})}\phi(nx)\,dn$$

denotes the constant term of ϕ *along* P, *and the sum is over all (standard) parabolic subgroups.*

The main purpose for introducing analytic truncation is to give a natural way to construct integrable functions: even from the example of GL_2, we know that automorphic forms are generally not integrable over the total fundamental domain $G(F)\backslash G(\mathbb{A})^1$ mainly due to the fact that in the Fourier expansions of such functions, constant terms are only of moderate growth (hence not integrable). Thus in order to naturally obtain integrable functions, we should truncate the original function along the cuspidal regions by removing constant terms. Simply put, Arthur's analytic truncation is a well-designed device in which constant terms are tackled in such a way that different levels of parabolic subgroups are suitably counted at the corresponding cuspidal region so that the whole truncation will not be overdone while there will be no parabolic subgroups left untackled.

As an example, we may consider Authur's analytic truncation for the constant function $\mathbf{1}$. For a sufficiently regular $T \in \mathfrak{a}_0$, introduce the truncated subset $\Sigma(T) := \Big(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})\Big)_T$ of the space $G(F)\backslash G(\mathbb{A})^1$ by

$$\Sigma(T) := \Big(Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A})\Big)_T := \Big\{g \in Z_{G(\mathbb{A})}G(F)\backslash G(\mathbb{A}) : \Lambda^T\mathbf{1}(g) = 1\Big\}.$$

Lemma. (Arthur) *For sufficiently regular $T \in \mathfrak{a}_0^+$, $\Sigma(T)$ is compact.*

Moreover, we have the following:

Global Bridge. (Lafforgue, Weng) *For a fixed normalized convex polygon $p : [0,r] \to \mathbb{R}$, let $T(p) \in \mathfrak{a}_0$ be the associated vector defined by*

$$\Big(p(1), p(2)-p(1), \cdots, p(i)-p(i-1), \cdots, p(r-1)-p(r-2), -p(r-1)\Big).$$

If $T(p)$ is sufficiently positive, then

$$\mathbf{1}(\bar{p}^g \leq p) = \Big(\Lambda^{T(p)}\mathbf{1}\Big)(g).$$

§ 9. Non-Abelian L-Functions

Recall that the rank r non-abelian zeta function $\xi_{\mathbb{Q},r}(s)$ of $\mathbb{Q}$ is given by

$$\xi_{\mathbb{Q},r}(s) = \int_{\mathcal{M}_{\mathbb{Q},r}} \Big(e^{h^0(\mathbb{Q},\Lambda)} - 1\Big) \cdot \big(e^{-s}\big)^{\deg(\Lambda)}\, d\mu(\Lambda), \qquad \mathrm{Re}(s) > 1,$$

with $e^{h^0(\mathbb{Q},\Lambda)} := \sum_{x\in\Lambda} \exp\big(-\pi|x|^2\big)$ and $\deg(\Lambda) = -\log \mathrm{Vol}\big(\mathbb{R}^r/\Lambda\big)$.

Introduce the completed Epstein zeta function for Λ by

$$\hat{E}(\Lambda; s) := \pi^{-s}\Gamma(s) \cdot \sum_{x\in\Lambda\backslash\{0\}} |x|^{-2s}.$$

One checks that

Proposition. (Weng) (**Eisenstein Series and High Rank Zetas**)

$$\xi_{\mathbb{Q},r}(s) = \frac{r}{2}\int_{\mathcal{M}_{\mathbb{Q},r}[1]} \hat{E}(\Lambda, \frac{r}{2}s)\, d\mu_1(\Lambda).$$

Motivated by this, we next introduce general non-abelian L-functions as follows: For a fixed convex polygon $p : [0,r] \to \mathbb{R}$, we obtain compact moduli spaces

$$\mathcal{M}_{F,r}^{\leq p}[\Delta_F^{\frac{r}{2}}] := \Big\{g \in GL_r(F)\backslash GL_r(\mathbb{A}) : \deg g = 0, \bar{p}^g \leq p\Big\}.$$

For example, $\mathcal{M}_{\mathbb{Q},r}^{\leq 0}[1] = \mathcal{M}_{\mathbb{Q},r}[1]$, (the adelic inverse image of) the moduli space of rank r semi-stable $\mathbb{Z}$-lattices of volume 1.

As usual, we fix the minimal parabolic subgroup P_0 corresponding to the partition $(1, \cdots, 1)$ with M_0 consisting of diagonal matrices. Then $P = P_I = U_I M_I$ corresponds to a certain partition $I = (r_1, \cdots, r_{|P|})$ of r with M_I the standard Levi and U_I the unipotent radical. Fix also an irreducible automorphic representation π of $M_I(\mathbb{A})$.

Definition. (Weng) *The rank r non-abelian L-function $L_{F,r}^{\leq p}(\phi, \pi)$ associated to the L^2-automorphic form $\phi \in A^2(U_I(\mathbb{A})M_I(F)\backslash G(\mathbb{A}))_\pi$ for the number field F is defined by the following integration*

$$L_{F,r}^{\leq p}(\phi, \pi) := \int_{\mathcal{M}_{F,r}^{\leq p}[\Delta_F^{\frac{r}{2}}]} E(\phi, \pi)(g)\, dg, \qquad \mathrm{Re}\,\pi \in \mathcal{C}.$$

For $w \in W$ the Weyl group of $G = GL_r$, fix once and for all representative $w \in G(F)$ of w. Set $M' := wMw^{-1}$ and denote the associated parabolic subgroup by $P' = U'M'$. As usual, define the associated *intertwining operator* $M(w, \pi)$ by

$$(M(w, \pi)\phi)(g) := \int_{U'(F)\cap wU(F)w^{-1}\backslash U'(\mathbb{A})} \phi(w^{-1}n'g)dn', \qquad \forall g \in G(\mathbb{A}).$$

Basic Facts of Non-Abelian L-Functions. (Langlands, Weng)
• (**Meromorphic Continuation**) *$L_{F,r}^{\leq p}(\phi, \pi)$ for* $\mathrm{Re}\pi \in \mathcal{C}$ *is well-defined and admits a unique meromorphic continuation to the whole space $\mathfrak{P}$;*
• (**Functional Equation**) *As meromorphic functions on $\mathfrak{P}$,*

$$L_{F,r}^{\leq p}(\phi, \pi) = L_{F,r}^{\leq p}(M(w, \pi)\phi, w\pi), \qquad \forall w \in W.$$

• (**Holomorphicity**) (i) *When* $\mathrm{Re}\pi \in \mathcal{C}$, *$L_{F,r}^{\leq p}(\phi, \pi)$ is holomorphic;*
(ii) *$L_{F,r}^{\leq p}(\phi, \pi)$ is holomorphic at π where* $\mathrm{Re}\pi = 0$*;*
• (**Singularities**) *Assume further that ϕ is a cusp form. Then*
(i) *There is a locally finite set of root hyperplanes D such that the singularities of $L_{F,r}^{\leq p}(\phi, \pi)$ are supported by D;*
(ii) *Singularities of $L_{F,r}^{\leq p}(\phi, \pi)$ are without multiplicities at π if $\langle \mathrm{Re}\pi, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta_M^G$;*
(iii) *There are only finitely many of singular hyperplanes of $L_{F,r}^{\leq p}(\phi, \pi)$ which intersect $\{\pi \in \mathfrak{P} : \langle \mathrm{Re}\pi, \alpha^\vee \rangle \geq 0, \forall \alpha \in \Delta_M\}$.*

§ 10. Symmetries and the Riemann Hypothesis

Characterizations of stability in terms of geometric, algebraic and analytic structures open new dimensions for the study of our high rank zeta functions. Here, we briefly recall how the analytic one enables us to use powerful techniques from trace formulas to expose the abelian zeta functions associated to pairs of reductive groups and their maximal parabolic subgroups.

For simplicity, assume that we are now working only over the field of rational. Then high rank zetas can be written as integrations of Epstein zetas over moduli spaces of semi-stable lattices. But Epstein zetas are special kinds of Eisenstein series, namely, the Eisenstein series $E^{SL_r/P_{r-1,1}}(\mathbf{1};s;g_\Lambda)$ induced from the constant function $\mathbf{1}$ over the maximal parabolic subgroup $P_{r-1,1}$ of SL_r corresponding to the partition $r=(r-1)+1$. Thus, via the analytic and algebraic approaches, we can further express our non-abelian zetas as integrations of truncated Eisenstein series $\Lambda^0 E^{SL_r/P_{r-1,1}}(\mathbf{1};s;g_\Lambda)$ over the total fundamental domain of $SL(r,\mathbb{Z})$. In this way, we deduce our non-abelian zetas to what we call Eisenstein periods.

In general, Eisenstein periods associated to L^2-automorphic forms such as these associated to our high rank zetas, are very difficult to be calculated. However, using techniques from trace formula, particularly, what we call an advanced version of Rankin-Selberg & Zagier method, see e.g. [JLR]/[W4], we know that Eisenstein periods associated to cusp forms can be evaluated. This, together with our earlier down-to-earth works on SL_3, then leads to calculations of Eisenstein periods associated to constant function $\mathbf{1}$ over the Borels, since, following Siegel and Langlands, (details are given by Diehl [D]), Epstein zetas can be realized as residues of these Eisenstein series coming from Borels. All in all, the up-shot is the following:

Definition. (Weng) *Let G be a reductive group and B a fixed Borel, both defined over a number field F. Denote by Δ_0 the corresponding set of simple roots, and W the associated Weyl group. The period of G over F is defined by*

$$\omega_F^G(\lambda) := \sum_{w\in W}\left(\frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle}\cdot\prod_{\alpha>0,w\alpha<0}\frac{\xi_F(\langle\lambda,\alpha^\vee\rangle)}{\xi_F(\langle\lambda,\alpha^\vee\rangle+1)}\right)$$

for λ in a suitable positive chamber of the root space. Here, as usual, $\alpha^\vee$ denotes the co-root corresponding to α, $\rho:=\frac{1}{2}\sum_{\alpha>0}\alpha$, and $\xi_F(s)$ denotes the completed Dedekind zeta function of F.

The above periods for G are several variables. To get a single variable one, say these corresponding to, but not coinciding with, our non-abelian zetas, as said above, we need to first properly choose singular hyper-planes and then take residues along them. In the cases of SL and Sp associated to high rank zetas, all this can be completed with the work of Diehl; but general cases are still quite complicated. To see structures more clearly, we decided to choose G_2 to test. This proves to be very crucial as it singles out the crucial role played by maximal parabolic subgroups. As a result, we have the following:

Definition. (Weng) *Let (G,P) be a pair of reductive group G and its maximal parabolic*

subgroup defined over a number field F. Denote by α_P the single element of Δ_0 corresponding to P and $s := \langle \lambda - \rho, \alpha_P^\vee \rangle$. Then we define
(i) the period of (G, P) over F by

$$\omega_F^{G/P}(s) := \mathrm{Res}_{\langle \lambda-\rho,\alpha^\vee\rangle=0,\alpha\in\Delta_0\setminus\{\alpha_P\}}\Big(\omega_F^G(\lambda)\Big).$$

(ii) the abelian zeta function associated to $(G, P)/F$ to be the function obtained from the period of (G, P) over F by making the following normalizations: clearing up Dedekind zetas appeared in the denominators and making a possible parallel shift of s:

$$\xi_{\mathbb{Q}}^{G/P}(s) := \mathrm{Norm}\Big[\mathrm{Res}_{\langle \lambda-\rho,\alpha^\vee\rangle=0,\alpha\in\Delta_0\setminus\{\alpha_P\}}\Big(\omega_F^G(\lambda)\Big)\Big]$$

where as above,

$$\omega_F^G(\lambda) := \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta_0}\langle w\lambda-\rho,\alpha^\vee\rangle} \cdot \prod_{\alpha>0,w\alpha<0} \frac{\xi_F(\langle\lambda,\alpha^\vee\rangle)}{\xi_F(\langle\lambda,\alpha^\vee\rangle+1)}.$$

As such, then easily, $\xi_{\mathbb{Q}}^{G/P}(s)$ is a well-defined meromorphic function on the whole complex s-plane. And strikingly, the structures of all this zetas can be summarized by the following

Main Conjecture. (i) (**Functional Equation**) $\xi_{\mathbb{Q}}^{G/P}(1-s) = \xi_{\mathbb{Q}}^{G/P}(s)$;
(ii) (**The Riemann Hypothesis**)

$$\xi_{\mathbb{Q}}^{G/P}(s) = 0 \text{ implies that } \mathrm{Re}(s) = \tfrac{1}{2}.$$

Remarks. (i) Even when $(G, P) = \big(SL_r, P_{r-1,1}\big)$, these new abelian zetas are not rank r zetas. In fact, abelain zetas here are related with the so-called constant terms of Eisenstein series $E^{SL_r/B}(\mathbf{1}, \lambda, g)$ only, while non-abelian high rank zetas are related to all parts;
(ii) Functional equation is first checked in [W7] for 10 examples listed in the appendix there, namely for the groups $SL(2, 3, 4, 5), Sp(4)$ and G_2; then by Kim-Weng for $(SL_r, P_{r-1,1})$. Recently, Komori [Ko] is able to establish the following basic

Functional Equation: $\xi_{\mathbb{Q}}^{G/P}(1-s) = \xi_{\mathbb{Q}}^{G/P}(s)$.

(iii) Based on symmetries, the RH for the above 10 examples is solved partially by Lagarias-Suzuki, particularly by Suzuki, and fully by Ki. Ki's method is expected to have more applications. For details, please go to ([LS], [Su1,2], [SW], [Ki1,2]).

Chapter 3. General CFT and Stability

The study of the so-called Hodge-Tate, de Rham, semi-stable and crystalline representations plays a central role in Fontaine's theory of p-adic Galois representations.

These representations are closely related with p-adic Hodge theory. Main results here are
(i) A characterization of semi-stable representations in terms of weakly admissible filtered (φ, N)-modules ([CF]);
(ii) Monodromy Theorem for p-adic Galois representations ([B]); and
(iii) Semi-stable conjecture in p-adic Hodge ([Tsu], [Ni], [Fal1,2]).

In this chapter, based on (i) and (ii), we will first introduce what we call ω-structures to tackle ramifications, then formulate a conjectural Micro Reciprocity Law characterizing de Rham representations in terms of semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero, and finally establish a general CFT for p-adic number fields using Tannakian category theory.

§ 11. Filtered (φ, N)-Modules & Semi-Stable Reps

Let K be a p-adic number field with k the residue field and $K_0 := \mathrm{Fr}\, W(k)$. Denote by $\mathbb{B}_{\mathrm{HT}}, \mathbb{B}_{\mathrm{dR}}, \mathbb{B}_{\mathrm{st}}, \mathbb{B}_{\mathrm{crys}}$ Fontaine's rings of Hodge-Tate, de Rham, semi-stable, crystalline periods, respectively.

Let $\rho : G_K \to \mathrm{GL}(V)$ be a p-adic Galois representation. Following Fontaine, define the associated spaces of periods by

$$\mathbb{D}_{\mathrm{HT}}(V) := \Big(\mathbb{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V\Big)^{G_K}, \qquad \mathbb{D}_{\mathrm{dR}}(V) := \Big(\mathbb{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V\Big)^{G_K},$$

$$\mathbb{D}_{\mathrm{st}}(V) := \Big(\mathbb{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V\Big)^{G_K}, \qquad \mathbb{D}_{\mathrm{crys}}(V) := \Big(\mathbb{B}_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V\Big)^{G_K}.$$

In particular, if V is semi-stable, then $\mathbb{D}(V) := \Big(\mathbb{D}_{\mathrm{st}}(V), \mathbb{D}_{\mathrm{dR}}(V)\Big)$ admits a natural filtered (φ, N)-module structure. Hence it makes sense to talk about the corresponding Hodge-Tate slope μ_{HT} and Newton slope μ_N. Along with this line, an important discovery of Fontaine is the following basic:

Theorem. (Fontaine) *Let $\rho : G_K \to \mathrm{GL}(V)$ be a semi-stable p-adic representation of G_K and set $\mathbb{D} := \big(D_0, D\big)$ with*

$$D := \mathbb{D}_{\mathrm{dR}}(V) \quad \textit{and} \quad D_0 := \mathbb{D}_{\mathrm{st}}(V).$$

Then
(i) $\mu_{\mathrm{HT}}(\mathbb{D}) = \mu_{\mathrm{N}}(\mathbb{D})$; and
(ii) $\mu_{\mathrm{HT}}(\mathbb{D}') \leq \mu_{\mathrm{N}}(\mathbb{D}')$ for any saturated filtered (φ, N)-submodule $\mathbb{D}' = (D_0', D')$ of $\mathbb{D} = (D_0, D)$.

If a filtered (φ, N)-module (D_0, D) satisfies the above two conditions (i) and (ii), we, following Fontaine, call it a *weakly admissible filtered (φ, N)-module.* So the above result then simply says that for a semi-stable V, its associated periods $\mathbb{D} := \Big(\mathbb{D}_{\rm st}(V), \mathbb{D}_{\rm dR}(V)\Big)$ is weakly admissible. More surprisingly, the converse holds correctly. That is to say, we have also the following

Theorem. (Fontaine||Colmez-Fontaine) *If (D_0, D) is a weakly admissible filtered (φ, N)-module. Then there exists a semi-stable p-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$ such that*

$$D = \mathbb{D}_{\rm dR}(V) \quad \text{and} \quad D_0 = \mathbb{D}_{\rm st}(V).$$

Remark. (A||B), for contributors, means that the assertion is on one hand conjectured by A and on the other proved by B.

§ 12. Monodromy Theorem for p-adic Galois Reps

We have already explained one of three fundamental results for p-adic Galois representations. Here we introduce another one, the so-called Monodromy Theorem for p-adic Galois Representations.

To explain this, let us recall that a p-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$ is called *potentially semi-stable*, if there exists a finite Galois extension L/K such that the induced Galois representation $\rho|_{G_L} : G_L(\hookrightarrow G_K) \to \mathrm{GL}(V)$ is semi-stable. One checks easily that every potentially semi-stable representation is de Rham. As a p-adic analogue of the Monodromy Theorem for l-adic Galois Representations, we have then the following fundamental:

Monodromy Theorem for p-adic Galois Reps. (Fontaine||Berger)
All de Rham representations are potentially semi-stable.

Started with Sen's theory for $\mathbb{B}_{\rm dR}$ of Fontaine, bridged by overconvergence of p-adic representations due to (Cherbonnier||Cherbonnier-Colmez), Berger's proof is based on the so-called p-adic monodromy theorem (for p-adic differentials equations) of (Crew, Tsuzuki||Crew, Tsuzuki, Andre, Kedelaya, Mebkhout). For more details, please refer to Ch. 4.

§ 13. Semi-Stability of Filtered $(\varphi, N; \omega)$-Modules

§ 13.1. Weak Admissibility = Stability & of Slope Zero

With the geometric picture in mind, particularly the works of Weil, Grothendieck, Mumford, Narasimhan-Seshadri and Seshadri, we then notice that weakly admissible condition for filtered (φ, N)-module $\mathbb{D} = (D_0, D)$ is an arithmetic analogue of the condition on semi-stable bundles of slope zero. Indeed, if we set

$$\mu_{\rm total}(\mathbb{D}) := \mu_{\rm HT}(D) - \mu_{\rm Dieu}(D_0)$$

then the first condition of weak admissibility, namely,
(i) $\mu_{\mathrm{HT}}(D) = \mu_{\mathrm{Dieu}}(D_0)$
is *equivalent* to the slope zero condition that
(i)′ $\mu_{\mathrm{total}}(\mathbb{D}) = 0$;
and the second condition
(ii) $\mu_{\mathrm{HT}}(D') = \mu_{\mathrm{Dieu}}(D'_0)$ for any saturated filtered (φ, N)-submodule (D'_0, D') of (D_0, D),
is *equivalent* to the semi-stability condition that
(ii)′ $\mu_{\mathrm{total}}(\mathbb{D}') \leq \mu_{\mathrm{total}}(\mathbb{D}) = 0$ for all saturated filtered (φ, N)-submodule $\mathbb{D}'$ of $\mathbb{D}$.
As such, then the above correspondence between semi-stable Galois representations and weakly admissible filtered (φ, N)-modules may be understood as an arithmetic analogue of the Narasimhan-Seshadri correspondence between (irreducible) unitary representations and stable bundles of degree zero over compact Riemann surfaces.

Accordingly, in order to establish a general class field theory for p-adic number fields, motivated by what we saw in algebraic geometry explained in Ch. 1, we need to introduce some new structures to tackle ramifications. Recall that in algebraic geometry, there are two parallel theories for this purpose, namely, the π-bundle one on the covering space at the top using Galois groups; and the parabolic bundle one on the base space at the bottom using parabolic structures. Hence, in arithmetic setting now, we would like to develop corresponding theories. The π-bundle analogue is easy based on Monodromy theorem for p-adic Galois Representations. In fact, we have the following orbifold version:

Theorem. (Fontaine||Fontaine, Colmez-Fontaine, Berger)
There exists a natural one-to-one and onto correspondence

$$\Big\{ \textit{de Rham Galois representations of } G_K \Big\}$$
$$\Updownarrow$$
$$\Big\{ \textit{semi-stable filtered } (\varphi, N; G_{L/K})\textit{-modules of slope zero: } \exists\ L/K \textit{ finite Galois} \Big\}.$$

§ 13.2. Ramifications

In geometry, parabolic structures take care of ramifications. Recall that if $M^0 \hookrightarrow M$ is a punctured Riemann surface, then around the punctures $P_i \in M \backslash M^0, i = 1, 2, \dots, N$, the associated monodromy groups generated by parabolic elements S_i are isomorphic to $\mathbb{Z}$, an abelian group. Thus for a unitary representation $\rho : \pi_1(M^0; *) \to \mathrm{GL}(V)$, the images of $\rho(S_i)$ are given by diagonal matrices with diagonal entries $\exp(2\pi\sqrt{-1}\alpha_{i;k})$, that is to say, they are determined by unitary characters $\exp(2\pi\sqrt{-1}\alpha)$, $\alpha \in \mathbb{Q}$. As such, to see the corresponding ramifications, one usually chooses a certain cyclic covering with ramifications around P_i's such that the orbifold semi-stable bundles can be characterized by semi-stable parabolic bundles on (M^0, M).

However, in arithmetic, the picture is much more complicated since there is no simple way to make each step abelian. By contrast, the good news is that there is a well-established theory in number theory to measure ramifications, namely, the theory of high ramification groups.

Accordingly, let then $G_K^{(r)}$ be the upper-indexed high ramification groups of G_K, parametrized by non-negative reals $r \in \mathbb{R}_{\geq 0}$. (See e.g., [Se2].) Denote then by $V^{(r)} := V^{G_K^{(r)}}$ the invariant subspace of V under $G_K^{(r)}$, and $K^{(r)} := \overline{K}^{G_K^{(r)}}$. For a p-adic Galois representation V, define the associated r-th graded piece by

$$\mathrm{Gr}^{(r)}V := \bigcap_{s:s\geq r} V^{(s)} / \bigcup_{s:s<r} V^{(s)},$$

and its *Swan conductor* by

$$\mathrm{Sw}(\rho) := \sum_{r\in\mathbb{R}_{\geq 0}} r \cdot \dim_{\mathrm{Q_p}} \mathrm{Gr}^{(r)}V.$$

Proposition. *Let $\rho : G_K \to \mathrm{GL}(V)$ be a de Rham representation.*
*(i) (**Hasse-Arf Lemma**) All jumps of $\mathrm{Gr}^{(r)}V$ are rational;*
*(ii) (**Artin, Fontaine**) There exists a Swan representation $\rho_{\mathrm{Sw}} : G_K \to \mathrm{GL}(V_{\mathrm{Sw}})$ such that*

$$\langle \rho_{\mathrm{Sw}}, \rho \rangle = \mathrm{Sw}(\rho).$$

In particular, $\mathrm{Sw}(\rho) \in \mathbb{Z}_{\geq 0}$.

In fact, Monodromy Theorem for p-adic Galois representations can be refined as follows:

Theorem$'$. *A p-adic Galois representation $\rho : G_K \to \mathrm{GL}(V)$ is de Rham if and only if there exists a finite Galois extension L/K such that for all $r \in \mathbb{R}_{\geq 0}$, $\rho|_{G_{L\cap K^{(r)}}} : G_{L\cap K^{(r)}} \to \mathrm{GL}\Big(V^{(r)}\Big)$ is semi-stable.*

§ 13.3. ω-structures

Recall that in geometry ([MY]), parabolic structures, taking care of ramifications, can also be characterized via an $\mathbb{R}$-index filtration

$$E_t := \Big(p_*\Big(W \otimes \mathcal{O}_Y\big(-[\#\Gamma \cdot t]D\big)\Big)\Big)^{\Gamma},$$

and its associated parabolic degree is measured by

$$\sum_i \alpha_i \cdot \dim_{\mathbb{C}} \mathrm{Gr}^i V.$$

Moreover, it is known that the filtration E_t is

(i) left continuous;
(ii) has jumps only at $t = \alpha_i - \alpha_{i-1} \in \mathbb{Q}$; and
(iii) with parabolic degree in $\mathbb{Z}_{\geq 0}$.

Even we have not yet checked with geometers whether their ramification filtration constructions are motivated by the arithmetic one related to the filtration of upper indexed high ramification groups, the similarities between both constructions are quite apparent. Indeed, it is well-known that, for the filtrations on Galois groups G and on representations V induced from that of high ramification groups $G_K^{(r)}$,
(i) by definition, $G_K^{(r)}$ and hence $V^{(r)}$ are left continuous;
(ii) from the Hasse-Arf Lemma, all jumps of $G_K^{(r)}$ and hence of $V^{(r)}$ are rational; and
(iii) according to essentially a result of Artin, the Artin/Swan conductors are non-negative integers.

Motivated by this, we make the following
Definition. *Let D be a finite dimensional K-vector space. Then, an ω-filtration* $\mathrm{Fil}_\omega^r D$ *on D is by definition an $\mathbb{R}_{\geq 0}$-indexed increasing but exhaustive filtration of finite dimensional K-vector subspaces of D satisfying the following properties:*
*(i) (***Continuity***) it is left continuous;*
*(ii) (***Hasse-Arf's Rationality***) it has all jumps at rationals;*
Define then the associated r-th graded piece by

$$\mathrm{Gr}_\omega^{(r)} D := \bigcap_{s:s\geq r} \mathrm{Fil}_\omega^{(s)} D \Big/ \bigcup_{s:s<r} \mathrm{Fil}_\omega^{(s)} D,$$

and its ω-*slope* by

$$\mu_\omega(D) := \frac{1}{\mathrm{dim}_K D} \cdot \sum_{r\in\mathbb{R}_{\geq 0}} r \cdot \mathrm{dim}_{\mathrm{Q_p}} \mathrm{Gr}_\omega^{(r)} D.$$

(iii) (**Artin's Integrality**) *The ω-degree*

$$\mathrm{deg}_\omega(D) := \sum_{r\in\mathbb{R}_{\geq 0}} r \cdot \mathrm{dim}_K \mathrm{Gr}_\omega^{(r)} D = \mathrm{dim}_K D \cdot \mu_\omega(D)$$

is a non-negative integer.

§ 13.4. Semi-Stability of Filtered $(\varphi, N; \omega)$-Modules

By the monodromy theorem of p-adic Galois representations, for a de Rham representation V of G_K, there exists a finite Galois extension L/K such that V, as a representation of G_L, is semi-stable. As such, then, over the extension field L, the weakly admissible filtered (φ, N)-structure on $\Big(\mathbb{D}_{\mathrm{st},L}(V), \mathbb{D}_{\mathrm{dR},L}(V)\Big)$ is equipped with a compatible Galois action of $G_{L/K}$. On the other hand, instead of working over L, from the original base field K, we, motivated by algebraic geometry, expect that the ω-structure would play the role of parabolic structures. Accordingly, we make the following

Definition. *(i) A filtered $(\varphi, N; \omega)$-module $\mathbf{D} := \Big(D_0, D; \mathrm{Fil}^r_\omega D\Big)$ is by definition a filtered (φ, N)-module (D_0, D) equipped with a compatible ω-structure on D;*
(ii) Tautologically, we have the notion of saturated filtered $(\varphi, N; \omega)$-submodule $\mathbf{D}' := \Big(D_0', D'; \mathrm{Fil}^r_\omega D'\Big)$ of $\mathbf{D} = \Big(D_0, D; \mathrm{Fil}^r_\omega D\Big)$;
(iii) Define the total slope of a filtered $(\varphi, N; \omega)$-module $\mathbf{D} := \Big(D_0, D; \mathrm{Fil}^r_\omega D\Big)$ by

$$\mu_{\mathrm{total}}(\mathbf{D}) := \mu_{\mathrm{HT}}(D) - \mu_{\mathrm{Dieu}}(D_0) - \mu_\omega(D).$$

(iv) A filtered $(\varphi, N; \omega)$-module $\mathbf{D} = \Big(D_0, D; \mathrm{Fil}^r_\omega D\Big)$ is called semi-stable of slope zero if
*(a) (**Slope 0**) it is of total slope zero, i.e.,*

$$\mu_{\mathrm{total}}(\mathbf{D}) = 0;$$

*(b) (**Semi-Stability**) For every saturated filtered $(\varphi, N; \omega)$-module $\mathbf{D}'$ of $\mathbf{D}$, we have*

$$\mu_{\mathrm{total}}(\mathbf{D}') \leq \mu_{\mathrm{total}}(\mathbf{D}).$$

§ 14. General CFT for p-adic Number Fields

With all these preparations, we are now ready to make the following:

Conjectural Micro Reciprocity Law. *There exists a canonical one-to-one and onto correspondence*

$$\Big\{\text{de Rham representations of } G_K\Big\}$$
$$\Updownarrow$$
$$\Big\{\text{semi-stable filtered } (\varphi, N; \omega)\text{-modules of slope zero over } K\Big\}.$$

Denote the category of semi-stable filtered $(\varphi, N; \omega)$-modules of slope zero over K by $\mathrm{FM}_K^{\mathrm{ss};0}(\varphi, N; \omega)$. Assuming the MRL, i.e., the micro reciprocity law, then we can show easily that, with respect to natural structures, $\mathrm{FM}^{\mathrm{ss};0}(\varphi, N; \omega)$ becomes a Tannakian category. Denote by $\mathbb{F}$ the natural fiber functor to the category of finite $\mathbb{Q}_p$-vector spaces. Then, from the standard Tannakian category theory, we obtain the following

General CFT for p-adic Number Fields.

• **Existence Theorem** *There exists a canonical one-to-one and onto correspondence*

$$\Big\{\text{Finitely Generated Sub-Tannakian Categories } \Big(\Sigma, \mathbb{F}|_\Sigma\Big) \text{ of } \mathrm{FM}_K^{\mathrm{ss};0}(\varphi, N; \omega)\ \Big\}$$

$$\Updownarrow \Pi$$

$$\Big\{\text{Finite Galois Extensions L/K}\Big\};$$

Moreover,

• **Reciprocity Law** *The above canonical correspondence induces a natural isomorphism*

$$\mathrm{Aut}^{\otimes}\Big(\Sigma, \mathbb{F}|_\Sigma\Big) \simeq \mathrm{Gal}\Big(\Pi\Big(\Sigma, \mathbb{F}|_\Sigma\Big)\Big).$$

In fact much refined result remains correct: By using ω-filtration, for all $r \in \mathbb{R}_{\geq 0}$, we may form sub-Tannakian category $(\Sigma^{(r)}, \mathbb{F}|_{\Sigma^{(r)}})$ of $(\Sigma, \mathbb{F}|_\Sigma)$, consisting of objects admitting trivial $\mathrm{Fil}_\omega^{r'}$ for all $r' \geq r$.

• **Refined Reciprocity Law** *The natural correspondence* Π *induces, for all* $r \in \mathbb{R}_{\geq 0}$, *canonical isomorphisms*

$$\mathrm{Aut}^{\otimes}\Big(\Sigma^{(r)}, \mathbb{F}|_{\Sigma^{(r)}}\Big) \simeq \mathrm{Gal}\Big(\Pi\Big(\Sigma, \mathbb{F}|_\Sigma\Big)\Big)\Big/\mathrm{Gal}^{(r)}\Big(\Pi\Big(\Sigma, \mathbb{F}|_\Sigma\Big)\Big).$$

§ 15. Moduli Spaces and Polarizations

Let $\mathbf{D} := \Big(D_0, D; \mathrm{Fil}_\omega^r(D)\Big)$ be a filtered $(\varphi, N; \omega)$-module of rank d over K, $P(\kappa_{\mathrm{Dieu}})$ and $P(\kappa_{\mathrm{HT}})$ be the corresponding parabolic subgroups of $\mathrm{GL}(D_0)$ and of $GL(D)$. Define the character $L_{\kappa_{\mathrm{HT}}}$ of $P(\kappa_{\mathrm{HT}})$ by

$$L_{\kappa_{\mathrm{HT}}} := \bigotimes_{i \in \mathbb{Z}} \Big(\det \mathrm{Gr}_{\mathrm{HT}}^i(D)\Big)^{\otimes -i}.$$

Similarly, define the (rational) character $L_{\kappa_{\mathrm{Dieu}}}$ of $P(\kappa_{\mathrm{Dieu}})$ by

$$L_{\kappa_{\mathrm{Dieu}}} := \bigotimes_{l \in \mathbb{Q}} \Big(\det \mathrm{Gr}_{\mathrm{Dieu}}^l(D_0)\Big)^{\otimes -l}.$$

(Unlike $L_{\kappa_{\mathrm{HT}}}$, which is an element of the group $X^*(P_{\kappa_{\mathrm{HT}}})$ of characters of $P_{\kappa_{\mathrm{HT}}}$, being rationally indexed, $L_{\kappa_{\mathrm{Dieu}}}$ is in general not an element of $X^*(P_{\kappa_{\mathrm{Dieu}}})$, but a rational character, i.e., it belongs to $X^*(P_{\kappa_{\mathrm{Dieu}}}) \otimes \mathbb{Q}$.)

Moreover, since all jumps of an ω-structure are rationals, it makes sense to define the associated parabolic subgroup $P(\kappa_\omega)$ and a (rational) character L_{κ_ω} of $P(\kappa_\omega)$ by

$$L_{\kappa_\omega} := \bigotimes_{r \in \mathbb{R}_{\geq 0}} \Big(\det \mathrm{Gr}_\omega^r(D)\Big)^{\otimes -r}.$$

As usual, identify $L_{\kappa_{\mathrm{HT}}}$ with an element of $\mathrm{Pic}^{\mathrm{GL}(D)}\Big(\mathrm{Flag}(\kappa_{\mathrm{HT}})\Big)$, where $\mathrm{Flag}(\kappa_{\mathrm{HT}})$ denotes the partial flag variety consisting of all filtrations of D with the same graded piece dimensions $\dim_K \mathrm{Gr}^k_{\mathrm{HT}}(D)$. (We have identified $\mathrm{Flag}(\kappa_{\mathrm{HT}})$ with $\mathrm{GL}(D)/P_{\kappa_{\mathrm{HT}}}$.) Similarly, we get an element L_{κ_ω} of $\mathrm{Pic}^{\mathrm{GL}(D)}\Big(\mathrm{Flag}(\kappa_\omega)\Big)\otimes\mathbb{Q}$, with $\mathrm{Flag}(\kappa_\omega)$ the partial flag variety consisting of all filtrations of D with the same $\dim_K \mathrm{Gr}^r_\omega(D)$. Thus, it makes sense to talk about the rational line bundle $\Big(L_{\kappa_{\mathrm{HT}}}\boxtimes L_{\kappa_\omega}\Big)\otimes L_{\kappa_{\mathrm{Dieu}}}$ on the product variety $\mathrm{Flag}(\kappa_{\mathrm{HT}})\times\mathrm{Flag}(\kappa_\omega)$. Moreover, define $J=J_K$ be an algebraic group whose $\mathbb{Q}_p$-rational points consist of automorphisms of the filtered $(\varphi,N;\omega)$-module $\mathbf{D}$ over K. As such, then essentially following Langton, Mehta-Seshadri, Rapoport-Zink, and particularly, Totaro, we can manage to have the following:

Proposition. *Assume k is algebrically closed. Then $\Big(D_0,D;\mathrm{Fil}^r_\omega(D)\Big)$ is semi-stable of slope zero if and only if the corresponding point*

$$\Big(\mathrm{Fil}^i_{\mathrm{HT}}(D),\mathrm{Fil}^r_\omega(D)\Big)\in\mathrm{Flag}(\kappa_{\mathrm{HT}})\times\mathrm{Flag}(\kappa_\omega)$$

is semi-stable with respect to all one-parameter subgroups $\mathbb{G}_m\to J$ defined over $\mathbb{Q}_p$ and the rational J-line bundle

$$\Big(L_{\kappa_{\mathrm{HT}}}\boxtimes L_{\kappa_\omega}\Big)\otimes L_{\kappa_{\mathrm{Dieu}}}$$

on $\mathrm{Flag}(\kappa_{\mathrm{HT}})\times\mathrm{Flag}(\kappa_\omega)$.

As a direct consequence, following Mumford's Geometric Invariant Theory ([M]), we then obtain the moduli space $\mathfrak{M}^{\varphi,N;\omega}_{K;d,0}$ of rank d semi-stable filtered $(\varphi,N;\omega)$-modules of slope zero over K. In particular, when there is no ω-structure involved, we denote the corresponding moduli space simply by $\mathfrak{M}^{\varphi,N}_{K;d,0}$.

Remark. The notion of semi-stable filtered $(\varphi,N;\omega)$-modules of slope s and the associated moduli space $\mathfrak{M}^{\varphi,N;\omega}_{K;r,s}$ for arbitrary s can also be introduced similarly. We leave the details to the reader.

With moduli spaces of semi-stable filtered $(\varphi,N;\omega)$-modules built, next we want to introduce various invariants (using these spaces). Recall that in (algebraic) geometry for semi-stable vector bundles, this process is divided into two: First we construct natural polarizations via the so-called Mumford-Grothendieck determinant line bundles of cohomologies; then we study the cohomologies of these polarizations.

Moduli spaces of semi-stable filtered $(\varphi,N;\omega)$-modules, being projective, admit natural geometrized polarizations as well. However, such geometric polarizations, in general, are quite hard to be used arithmetically, due to the fact that it is difficult to reinterpret them in terms of arithmetic structures involved. To overcome this difficulty, we here want to use Galois cohomologies of p-adic representations, motivated by the $(\mathfrak{g},K)$-modules interpretations of cohomology of (certain types of) vector bundles over homogeneous spaces.

On the other hand, as said, such polarizations, or better, determinant line bundles,

if exist, should be understood as arithmetic analogues of Grothendieck-Mumford determinant line bundles constructed using cohomologies of vector bundles. Accordingly, if we were seeking a perfect theory, we should first develop an analogue of sheaf cohomology for filtered $(\varphi, N; \omega)$-modules. We will discuss this elsewhere, but merely point out here the follows:
(i) a good cohomology theory in the simplest abelian case of $r = 1$ is already very interesting since it would naturally lead to a true arithmetic analogue of the theory of Picard varieties, an understanding of which is expected to play a key role in our intersectional approach to the Riemann Hypothesis proposed in our Program paper [W2];
(ii) the yet to be developed cohomology theory would help us to build up p-adic L-functions algebrically. This algebraically defined L-function for filtered $(\varphi, N; \omega)$-modules then should be compared to p-adic L-functions for p-adic representations defined using Galois cohomology ([PR]). We expect that these two different types of L's correspond to each other in a canonical way and further can be globalized within the framework of the thin theory of adelic Galois representations proposed in the introduction.

Chapter 4. Two Approaches to Conjectural MRL

§ 16. Algebraic Method

There are two different approaches to establish the conjectural Micro Reciprocity Law. Namely, algebraic one and geometric one.

Let us start with algebraic approach. Here, we want to establish a correspondence between filtered $\big(\varphi, N; G\big)$-modules M and filtered $\big(\varphi, N; \omega\big)$-modules D. Obviously, this is an arithmetic analogue of Seshadri's correspondence between π-bundles and parabolic bundles over Riemann surfaces. Therefore, we expect further that our correspondence satisfies the following two compatibility conditions:
(i) it induces a natural correspondence between saturated subobjects M' and D' of M and D; and
(ii) it scales the slopes by a constant multiple of $\#G$. Namely,

$$\mu_{\text{total}}(M') = \#G \cdot \mu_{\text{total}}(D').$$

Assume the existence of such a correspondence. Then, as a direct consequence of the compatibility conditions, semi-stable filtered $\big(\varphi, N; G\big)$-modules M of slope zero correspond naturally to semi-stable filtered $\big(\varphi, N; \omega\big)$-modules D of slope zero.

In this way, via the MRL with limited ramifications and the Monodromy Theorem for p-adic Galois Representations, we are able to establish the conjectural MRL.

So the problem is what is this correspondence? For this, we propose the follows:

Let then $\mathbf{D}_L := (D_0, D)$ be a filtered $(\varphi, N; G_{L/K})$-module. So D_0 is defined over L_0 and D is over L. By the compactness of the Galois groups, there exists a lattice version of (D_0, D) which we denote by (Λ_0, Λ). In particular, Λ_0 is an $\mathcal{O}_{L_0}$-lattice with a group action G_{L_0/K_0}. Consider then the finite covering map $\pi_0 : \operatorname{Spec}\mathcal{O}_{L_0} \to \operatorname{Spec}\mathcal{O}_{K_0}$. We identify Λ_0 with its associated coherent sheaf on $\operatorname{Spec}\mathcal{O}_{L_0}$. Set

$$\Lambda_{0,K} := \Big((\pi_0)_*\Lambda_0\Big)^{\mathrm{Gal}(L_0/K_0)}.$$

Clearly, there is a natural (φ, N)-structure on $\Lambda_{0,K}$.

Moreover, for the natural covering map $\pi : \operatorname{Spec}\mathcal{O}_L \to \operatorname{Spec}\mathcal{O}_K$, view Λ as a coherent sheaf on $\operatorname{Spec}\mathcal{O}_L$ and form the coherent sheaf $\mathcal{O}_L\Big(-[\deg(\pi)\cdot t]\mathfrak{m}_L\Big)$, where $t \in \mathbb{R}_{\geq 0}$ and $\mathfrak{m}_L$ denotes the maximal idea of $\mathcal{O}_L$. Consequently, it makes sense to talk about

$$\Lambda_K(t) := \bigg(\pi_*\Big(\Lambda \otimes \mathcal{O}_L\Big(-\big[\deg(\pi)\cdot t\big]\mathfrak{m}_L\Big)\Big)\bigg)^{\mathrm{Gal}(L/K)}.$$

Or equivalently, in pure algebaric language,

$$\Lambda_K(t) := \Big(\Lambda \otimes \mathfrak{m}_L^{\left[t\cdot\#G_{L/K}\right]}\Big)^{\mathrm{Gal}(L/K)}.$$

Even we can read ramification information involved from this decreasing filtration consisting of invariant $\mathcal{O}_K$-lattices, unfortunately, we have not yet been able to obtain its relation with ω-structure wanted.

§ 17. Infinitesimal, Local and Global

In this section, we briefly recall how micro arithmetic objects of Galois representations are naturally related with global geometric objects of the so-called overconvergent F-isocrystals.

From Arithmetic to Geometry: The shift from arithmetic to geometry is carried out via Fontaine-Winterberger's fields of norms.

Let then K be a p-adic number field with $\overline{K}$ a fixed separable closure and $K_\infty = \cup_n K_n$ with $K_n := K(\mu_{p^n})$ the cyclotomic extension of K by adding p^n-th root of unity. Denote by $k := k_K$ its residue field, and $K_0 := \operatorname{Fr}W(k)$ the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Set $\varepsilon := (\varepsilon^{(n)})$ with $\varepsilon^{(n)} \in \mu_{p^n}$ satisfying $\varepsilon^{(1)} \neq 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$, and introduce the base field $E_{K_0} := k_K((\varepsilon - 1))$. Then, from the theory of fields of norms, associated to K, there exists a finite extension E_K of E_{K_0} in a fixed separated closure $E_{K_0}^{\mathrm{sep}}$ such that we have a canonical isomorphism

$$H_K := \mathrm{Gal}\Big(\overline{K}/K_\infty\Big) \simeq \mathrm{Gal}\Big(E_K^{\mathrm{sep}}/E_K\Big),$$

where, in particular, $E_K^{\mathrm{sep}} := \bigcup_{L/K:\text{finite Galois}} E_L$ is then a separable closure of E_K. In this way, the arithmetically defined Galois group H_K for p-adic field K_∞ is tranformed into the geometrically defined Galois group $\mathrm{Gal}\Big(E_K^{\mathrm{sep}}/E_K\Big)$ for the field E_K of power series defined over finite field.

From Infinitesimal to Global: Let $\rho : G_K \to GL(V)$ be a p-adic representation of G_K. Then, following Fontaine, we obtain an etale (φ, Γ)-module $\mathbb{D}(V)$. Moreover, by a result of Cherbonnier-Colmez, $\mathbb{D}(V)$ is overconvergent. Note that now Γ_K, being the Galois group of K_∞/K, is abelian which may be viewed as an open subgroup of $\mathbb{Z}_p$ via cyclotomic character. Hence, following Sen, we can realize the action of Γ_K by using a certain natural operator, or better, a connection. In this way, we are able to transform our initial arithmetic objects of Galois representations into the corresponding structures in geometry, namely, that of p-adic differential equations with Frobenius structure, following Berger. However, despite of this successful transformation, we now face a new challenge – In general, the p-adic differential equations obtained are singular. It is for the purpose to remove these singularities that we are naturally led to the category of de Rham representations, thanks to the works of Fontaine and Berger.

On the other hand, contrast to this local, or better, infinitesimal theory, thanks to the works of Levelt and Katz ([Le], [Ka2]), we are led to a corresponding global theory, the framework of which was first built up by Crew based on Berthelot's overconvergent isocrystals ([Ber], [BO]). For more details, see the discussion below. Simply put, the up-shot is the follows: If $X^0 \hookrightarrow X$ is a marked regular algebraic curve defined over $\mathbb{F}_q$, then, Crew (for rank one) and Tsuzuki (in general) show that there exists a canonical one-to-one and onto correspondence between p-adic representations of $\pi_1(X^0, *)$ with finite monodromy along $Z = X\backslash X^0$ and the so-called unit-root F-isocrystals on X^0 overconvergent around Z. This result is an arithmetic-geometric analogue of the result of Weil on correspondence between complex representations of fundamental groups and flat bundles over compact Riemann surfaces, at least when Z is trivial.

Conversely, to go from global overconvergent isocrystals to micro p-adic Galois representations, aiming at establishing the conjectural MRL relating de Rham representations to semi-stable filtered $(\varphi, N; \omega)$-modules, additional works should be done. To sense it, we suggest the reader to go to the papers [Ts2] and [Mar].

§ 18. Convergent F-isocrystals and Rigid Stable F-Bundles

Recall that the p-adic Monodromy Theorem is built up on Crew and Tsuzuki's works about overconvergent unit-root F-isocrystals. To understand it, in this section, we make some preparations following [Cre]. Along with this same line, we also offer a notion called semi-stable rigid F-bundles of slope zero in rigid analytic geometry, which is the key to our algebraic characterization of p-adic representations of fundamental groups of complete, regular, geometrically connected curves defined over finite fields.

Assume the language of formal, rigid analytic geometry, particularly, usual notations $X, \mathfrak{X}, \mathfrak{X}^{\mathrm{an}},]X[$ etc...

Theorem. (Crew) *Let X/k be a smooth k-scheme and suppose that $\mathbb{F}_q \subset k$. Then there exists a natural equivalence of categories $\mathbb{G} : \mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big) \simeq \mathrm{Isoc}^{F;\mathrm{ur}}(X/K)$ where $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)$ denotes the category of K-representations of the fundamental group $\pi_1(X)$ of X, and $\mathrm{Isoc}^{F;\mathrm{ur}}(X/K)$ denotes the category of unit-root F-isocrystals on X/K.*

This result is based on Katz's work on the correspondence between K-representations of $\pi_1(X)$ and the so-called unit-root F-lattices on $\mathfrak{X}/R$ ([Ka1]).

The above result of Crew may be viewed as an arithmetic analogue of Weil's result on the correspondence between representations of fundamental groups and flat bundles over compact Riemann surfaces. However now the context is changed to curves defined over finite fields of characteristic p, the representations are p-adic, and, accordingly the flat bundles are replaced by unit-root F-isocrystals. In fact, the arithmetic result is a bit more refined: since the associated fundamental group is pro-finite, the actural analogue in geometry is better to be understood as the one for unitary representations and unitary flat bundles.

With this picture in mind, it is then very naturally to ask whether an arithmetic structure in parallel with Narasimhan-Seshadri correspondence between unitary representations and semi-stable bundles of slope zero can be established in the current setting. This is our next topic.

With the same notationa as above, assume in addition that X is completed. Then it makes sense to talk about locally free F-sheaves $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules. If $X = \mathrm{Spec}(k)$, then $\mathcal{E}$ is nothing but a finite-dimensional K-vector space V endowed with a σ automorphism $\Phi : \sigma^* V \simeq V$. Similarly, we can talk about its associated Dieudonne slope. Consequently, for general X, if $\mathcal{E}$ is a locally free F-sheaves $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules, then we can talk above its fibers at points of X with values in a perfect field. By definition, a locally free F-sheaves $\mathcal{E}$ of $\mathcal{O}_{]X[}$-modules is called of *slope* $s \in \mathbb{Q}$, denoted by $\mu(\mathcal{E}) = s$, if all its fibers have slope s; and $\mathcal{E}$ is called *semi-stable* if for all saturated F-submodules $\mathcal{E}'$, we have all slopes of the fibers of $\mathcal{E}'$ is at most $\mu(\mathcal{E})$. As usual, if the slopes satisfy the strict inequalities, then we call $\mathcal{E}$ *stable.* For simplicity, we call such locally free objects semi-stable (resp. stable) rigid F-bundles on X/K of slope s.

Conjectural MRL in Rigid Analytic Geometry. *Let X be a regular projective curve defined over k. There is a natural one-to-one and onto correspondence between absolutely irreducible K-representations of $\pi_1(X)$ and stable rigid F-bundles on X/K of slope zero.*

Remark. It is better to rename the above as a Working Hypothesis: Unlike previous a few conjectures, there are certain points here which have not yet been completed understood due to lack of time. (For example, in terms of intersection, the so-called Hodge polygon is better than Newton polygon adopted here, etc...) See however [Ked].

§ 19. Overconvergent F-Isocrystals, Log Geometry & Stability

From now on assume that X/k is a regular geometrically connected curve with regular compatification $\bar{X}$. Let $Z := \bar{X} - X$. By definition, a p-adic representation $\rho : \pi_1(X) \to GL(V)$ is called having *finite (local) monodromy around Z* if for each $x \in Z$, the image under ρ of the inertia group at x is finite. Denote by $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^{\mathrm{fin}}$ the associated Tannakian category and by $\mathrm{OIsoc}^{F;\mathrm{ur}}(X/K)$ the category of the so-called unit-root overconvergent F-isocrystals on X/K.

Theorem. (Crew‖Crew for rank one, Tsuzuki in general) *The restriction of the Crew equivalence $\mathbb{G}$ induces a natural equivalence*

$$\mathbb{G}^\dagger : \mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^{\mathrm{fin}} \to \mathrm{OIsoc}^{F;\mathrm{ur}}(X/K).$$

Remarks. (i) More generally, instead of unit-root condition, there is a notion of quasi-unipotency. In this language, then the p-adic Monodromy Theorem is nothing but the following

p-adic Monodromy Theorem. (Crew, Tsuzuki‖Crew, Tsuzuki, Andre, Kedlaya, Mebkhout) *Every overconvergent F-isocrystal is quasi-unipotent.*
(ii) Quasi-unipotent overconvergent F-isocrystal has been beautifully classified by Matsuda ([Mat]). Simply put, we now have the following structural

Theorem. (Crew, Tsuzuki, MA(C)K, Matsuda) *Every overconvergent F-isocrystal is Matsudian, i.e., admits a natural decomposition to the so-called Matsuda blocks defined by tensor products of etale and unipotent objects.*
In a certain sense, while unit-root objects are coming from representations of fundamental groups, quasi-unipotent objects are related with representations of central extension of fundamental groups.
(iii) Finally, we would like to recall that overconvergent isocrystals have been used by Shiho to define crystalline fundamental groups for high dimensional varieties ([Sh1,2]).

The above result of Crew & Tsuzuki is built up from the open part X of $\bar{X}$, a kind of arithmetic analogue of local constant systems over $\mathbb{C}$. As we have already seen, in Ch. 1, to have a complete theory, it is even better if such a theory can be studied over the whole $\bar{X}$: After all, for representation side, $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^{\mathrm{fin}}$ really means $\mathbb{R}\mathrm{ep}_K\big(\pi_1(X)\big)^Z$, that is, p-adic representations of $\pi_1(X)$ with finite local monodromy around every mark $P \in Z$. For doing so, we propose two different approaches, namely, analytic one and algebraic one.

Let us start with the analytic approach. Recall that the analytic condition of unit-root F-isocrystals on X overconvergent around Z is defined over (infinitesmal neighborhood of) X. We need to extend it to the total space $\bar{X}$. As usual, this can be done if we are willing to pay the price, i.e., allowing singularities along the boundary. Certainly, in general terms, singularities are very hard to deal with. However, with our experience over $\mathbf{C}$, particularly, the work of Deligne on local constant systems ([De1]), for the case at hands, fortunately, we expect that singularities involved are very mild – There are only logarithmic singularities appeared. This then leads to the notion of logarithmic convergent F-isocrystals $\mathcal{E}$ over $(\bar{X}, Z)$: Simply put, it is an overconvergent F-isocrystal that can be extended and hence realized as a locally free sheaf of $\mathcal{O}_{]\bar{X}[}$-module $\mathcal{E}$, endowed with an integral connection ∇ with logarithmic singularities along Z

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{]\bar{X}[}(\log Z),$$

not only defined over the first infinitesimal neighborhood but over all levels of infinitesimal neighborhoods.

Let us next turn to algebraic approach. With the notion of semi-stable rigid F-bundles introduced previously, it is not too difficult to introduce the notion of what should be called semi-stable parabolic rigid F-bundles.

Even we understand that additional work has to be done here using what should be called logarithmic formal, rigid analytic geometry, but with current level of understanding of mathematics involved, we decide to leave the details to the ambitious reader. Nevertheless, we would like to single out the following

Correspondence I. *There is a natural one-to-one and onto correspondence between unit-root F-isocrystals on X overconvergent around $Z := \bar{X} - X$ and what should be called unit-root logarithmic overconvergent F-isocrystals on $(X, Z)/K$.*

Correspondence II. *There is a natural one-to-one and onto correspondence between unit-root F-isocrystals on X overconvergent around $Z := \bar{X} - X$ and what should be called semi-stable parabolic rigid F-bundles of slope zero on $\big(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\big)$. Here $(\mathfrak{X}, \mathfrak{Z})$ denotes a logarithmic formal scheme associated to (X, Z).*

Moreover, by comparing the theory to be developed here with that for π-bundles of algebraic geometry for Riemann surfaces recalled in Ch. 1, for a fixed finite Galois covering $\pi : Y \to X$ ramified at Z, branched at $W := \pi^{-1}(Z)$, it is also natural for us to expect the following

Correspondence III. *There is a natural one-to-one and onto correspondence between orbifold rigid F-bundles on* $\big(\mathfrak{Y}^{\mathrm{an}}, \mathfrak{W}^{\mathrm{an}}\big)$ *and rigid parabolic F-bundles on* $\big(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\big)$ *satisfying the following compatibility conditions:*
(i) it induces a natural correspondences among saturated sub-objects; and
(ii) it scales the slopes by a constant multiple $\deg(\pi)$.

Assuming all this, then we can obtain the following
Micro Reciprocity Law in Log Rigid Analytic Geometry. *There is a natural one-to-one and onto correspondence*

$$\Big\{\text{irreducible } p\text{-adic representations of } \pi_1(X,*) \\ \text{with finite monodromy along } Z := \bar{X}\backslash X\Big\} \\ \Updownarrow \\ \Big\{\text{stable parabolic rigid } F\text{-bundles of slope 0 on } \big(\mathfrak{X}^{\mathrm{an}}, \mathfrak{Z}^{\mathrm{an}}\big)\Big\}.$$

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