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Purity of Stratifications of Shimura Varieties in Positive Characteristic

By

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Abstract

Let p be a prime number. We describe purity results for some stratifications of reductions of Shimura varieties in characteristic p, developed in the previous decade by various people. We also prove a new purity result for reductions of Hilbert modular varieties at places dividing the discriminant.

§ 1. Part I: Survey.

§ 1.1. General introduction

This paper is composed of two parts. In the first part, we give a survey of purity results for stratifications of special fibers of integral models of Shimura varieties, focusing on stratifications arising from Barsotti–Tate groups. In the second part, building on works of Manin and others, we prove a new purity result for special fibers of integral models of Hilbert modular varieties at places dividing the discriminant.

The two main examples of Shimura varieties arising in this note are Siegel modular varieties and Hilbert modular varieties. Their reductions are much simpler to describe than for general Shimura varieties, while illustrating the gist of purity in the context of higher dimensional arithmetic geometry in characteristic $p > 0$.

This paper was complemented by notes for a lecture series given in Bordeaux in March 2008 at the invitation of A. Cadoret. In particular, we have tried to fill in references for some standard algebraic geometry implicitly used in the literature. We

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point out other survey papers describing stratifications of Shimura varieties: [Rap3], [Ha], and [Va2]. In contrast, we focus unabashedly on purity.
N.B. We allowed ourselves the liberty to use footnotes, for which we ask the reader's indulgence if they happen to be distracting.

§ 1.2. Siegel and Hilbert modular varieties

1.2.1. Curves

A modular curve stratifies in two strata: a closed zero-dimensional stratum and its open complement. For example, the j-line $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$ parametrizing isomorphism classes of elliptic curves in characteristic p can be decomposed in two parts: the ordinary and the supersingular loci. Recall that an elliptic curve E defined over $\overline{\mathbb{F}}_p$ (an algebraic closure of $\mathbb{F}_p$) is called supersingular if $E[p](\overline{\mathbb{F}}_p) = 0$; otherwise, E is called ordinary and then $|E[p](\overline{\mathbb{F}}_p)| = p$.
As a non-proper (integral, separated) smooth curve is always affine by the Riemann-Roch theorem, and the zero-dimensional stratum is trivially affine, we see that in the case of curves, any non-trivial stratification is affine. A naive question is then: what happens in dimension > 1? The classical Hasse invariant allows a more fruitful point of view on modular curves: it displays the zero-dimensional locus as the zero locus of a global section (i.e., the Hasse invariant) of an *ample* invertible sheaf given by a suitable power of the Hodge bundle. This technique, in some lucky cases, allows proving results in higher dimensions also (see [Itō1], [Itō2]). Besides a trick suggested in Section 2, this seems the only known method to prove (absolute) affineness results.
We introduce two classical families of Shimura varieties which include modular curves: the Siegel modular varieties and the Hilbert modular varieties. These varieties parameterize families of abelian varieties equipped with some additional structures. Recall that modular curves are moduli spaces of elliptic curves i.e., of abelian varieties of dimension one.
A Shimura variety is a quasi-projective algebraic variety defined over a number field K, called the reflex field. When it is possible, describing a Shimura variety via a moduli problem is an efficient recipe to define a unique integral model i.e., over $\mathcal{O}_K[\frac{1}{N}] \subset K$, for $\mathcal{O}_K$ the ring of integers of K, and $N \in \mathbb{N}$. Henceforth, we may study the reduction of this integral model modulo a prime $\mathfrak{p}$ of $\mathcal{O}_K$, $(\mathfrak{p}, N) = 1$. In the examples that follows, the reflex field K will be equal to $\mathbb{Q}$ and thus $\mathcal{O}_K = \mathbb{Z}$.

1.2.2. Basic example I: Siegel modular varieties

Siegel modular varieties $\mathcal{A}_{g,1}$ are a very important, yet relatively easy to define, family of higher dimensional Shimura varieties. Besides modular curves, the Siegel modular threefold $\mathcal{A}_{2,1}$ (not $\mathcal{A}_{3,1}$!) associated to the group $\mathrm{GSp}_4/\mathbb{Q}$ has been most

studied, in particular in connection with Galois representations associated to Siegel modular forms of genus two.
Roughly, Siegel modular varieties $\mathcal{A}_{g,1,N}$ parametrize principally polarized abelian varieties of dimension g equipped with a certain level N structure. The "$g, 1$" subscript denotes the dimension g and the degree 1 of the polarisation.
Let S be a scheme over which N is invertible.

Definition 1.1. *Let $\mathbb{N} \ni N \geq 3$. The Siegel moduli space $\mathcal{A}_{g,1,N}$ of symplectic similitude level structure N is the fine moduli scheme defined over $\mathbb{Z}[1/N]$ which associates to an S-scheme T the triple $\underline{A} := (A, \lambda, \epsilon)/_{\cong}$, where:*

1. *A is an abelian scheme $A \longrightarrow T$ of relative dimension g;*
2. *$\lambda : A \xrightarrow{\cong} A^t$ is a principal polarization;*
3. *a symplectic similitude level N structure ϵ.*

N.B. The representability of this moduli problem is due to Mumford (see [Mu]). As level structures play a peripheral rôle in this part, we refer to [FC] or [Ko2] for further details.

Theorem 1.2 ([Mu])**.** *The Siegel moduli space $\mathcal{A}_{g,1,N}$ is smooth over $\mathbb{Z}[1/N]$ and of relative dimension $\frac{g(g+1)}{2}$.*

1.2.3. Basic example II: Hilbert modular varieties

We refer to [Vo] and the references therein for details on the moduli interpretation of Hilbert modular varieties. We fix some notation. Let L be a totally real number field of degree $g = [L : \mathbb{Q}]$, with $\mathcal{O}_L$ its ring of integers. The inverse different $\mathcal{D}_{L/\mathbb{Q}}^{-1}$ is defined by $\{x \in L | \mathrm{Tr}_{L/\mathbb{Q}}(x\mathcal{O}_L) \subset \mathbb{Z}\}$. The discriminant d_L of L is given by $\mathrm{Norm}(\mathcal{D}_{L/\mathbb{Q}})$.
Roughly, Hilbert modular varieties parametrize abelian varieties of dimension g equipped with an action of $\mathcal{O}_L$, a rigid level structure and polarization data.

Definition 1.3. *Let $\mathbb{N} \ni N \geq 3$. The Hilbert modular space $\mathfrak{M}(S, N) \longrightarrow S$ of level structure N is the fine moduli scheme defined over $\mathbb{Z}[1/N]$, which associates to an S-scheme T the triple $\underline{A} := (A, \iota, \epsilon)/_{\cong}$, where:*

1. *(A, ι) is an abelian scheme $A \longrightarrow T$ of relative dimension g equipped with real multiplication:*
$$\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_T(A);$$
2. *a level N structure*
$$\epsilon : \Big(\mathcal{O}_L/N\mathcal{O}_L)^2 \cong_{\mathcal{O}_L} A[N].$$

3. The Deligne-Pappas condition is satisfied:

$$A \otimes_{\mathcal{O}_L} \mathrm{Hom}_{\mathcal{O}_L}(A, A^t)^{sym} \xrightarrow{\cong} A^t.$$

The $\mathcal{O}_L$-module $\mathrm{Hom}_{\mathcal{O}_L}(A, A^t)^{sym}$ is defined as: $\{\lambda : A \longrightarrow A^t | \lambda = \lambda^t \text{ and } \iota^t \circ \lambda = \lambda \circ \iota\}$.

Theorem 1.4 ([Rap1], [DP]). *The morphism $\mathfrak{M}(S, N) \longrightarrow S$ is flat of relative dimension g and is locally of complete intersection. Moreover, it is smooth over $\mathbb{Z}[1/Nd_L]$. If $p|d_L$, the geometric fibers in characteristic p are normal varieties; the singular locus has codimension two.*

§ 1.3. Stratifications via Barsotti–Tate groups

Let k be an algebraically closed field of characteristic $p > 0$. Let A be an abelian variety over k. We can associate to A its Barsotti–Tate group $A[p^\infty] := \varinjlim A[p^n]$, where $A[p^n]$ denotes the kernel of the multiplication-by-p^n map. It is also called a p-divisible group. There exist numerous discrete invariants of $A[p^\infty]$. For example, we can consider the isogeny class of $A[p^\infty]$ i.e., the Newton polygon. The Dieudonné-Manin classification of p-divisible groups up to isogeny (see Thm 1.9 below) extends to F-isocrystals. As a consequence, stratifications of moduli spaces of K3 surfaces can also be defined and studied (as they fall outside the realm of Barsotti–Tate groups, we refer the reader directly to works of: Ogus ([Og]), van der Geer-Katsura ([vdGK1]) and references therein). Alternatively, we may also restrict our attention to the finite flat group scheme $A[p^m]$, for some level $m \in \mathbb{N}$. Other interesting invariants are provided by the a-number (see Example 1.17), the p-rank (see Definition 1.13), etc.

1.3.1. Examples of stratifications

We first define the concept of stratification.

Definition 1.5. *Let I be an index set. A stratification of a reduced k-scheme S is a decomposition (of the geometric, closed points):*

$$S(k) = \bigsqcup_{i \in I} S_i(k),$$

by subschemes S_i which are disjoint, reduced, locally closed.

It would be desirable to know whether: $\overline{S_i} = \bigsqcup_{j \in J_i} S_j$ i.e., the Zariski closure of a stratum is given by disjoint union of strata but this is only conjectured in general for (good reductions of) Shimura varieties (see [Rap3]). Note that this can easily be disproved for arbitrary $\overline{\mathbb{F}}_p$-schemes.

We do not build in more properties in the definition, as e.g., quasi-affineness does not hold for so-called the p-rank stratification. Moreover, Newton polygon strata are not smooth in general (see below for definitions).

We focus on stratifications that can be defined via Barsotti–Tate groups. We recall this notion.

Definition 1.6. *A Barsotti–Tate group (BT-group for short) is an fppf*[1] *sheaf of abelian groups G/S such that (denoting $G(n) := \mathrm{Ker}(p^n : G \longrightarrow G)$) the following three axioms hold:*

1. *G is p-divisible i.e., $p : G \longrightarrow G$ is surjective;*
2. *G is p-torsion i.e., $G = \varinjlim G(n)$;*
3. *$G(1)$ is representable by a finite, flat group scheme over S.*

The height of a BT-group G is the integer $h \geq 0$ such that $rk(G(1)) = p^h$.

We are now sufficiently prepared to introduce examples of stratifications. In parallel to the examples, we provide some brief observations pertaining e.g., to smooth families of abelian varieties. In general, we construct stratifications by defining invariants of Barsotti–Tate groups, and then study how a given invariant varies in a family e.g., of abelian varieties with additional structures. We will leave formulating the precise definitions of the stratifications in complete generality to the reader (or consult [Va2]) and we shall focus instead on the invariants themselves (but see Examples 1.19 and 1.21).

Example I. The Newton stratification

The Newton polygon is a handy device to encode the isogeny class of a BT-group. When the base scheme is an algebraically closed field, it is a complete invariant for the isogeny class.

Definition 1.7 (Newton polygon)**.**

Let $h, d \in \mathbb{N} \cup \{0\}$ and $d \leq h$. A Newton polygon is a piecewise linear, continuous function $\gamma : [0, h] \longrightarrow [0, d]$ such that:

- *γ starts at $(0, 0)$ and finishes at (h, d);*
- *γ is lower convex;*
- *each slope β of γ is a rational number comprised between 0 and 1 i.e., $\beta \in [0, 1] \cap \mathbb{Q}$;*

We customarily order the slopes of a Newton polygon in increasing order.

[1] **f**idèlement **p**lat, **p**résentation **f**inie

Example 1.8. *The Newton polygon of constant slope* $1/2$ *is called supersingular. For height* 2 *and dimension* 1*, there is only one other Newton polygon: the ordinary polygon of slopes* 0 *and* 1.

Theorem 1.9 (Dieudonné-Manin). *Let* $k = \bar{k}$. *There is a bijection between isogeny classes of BT-groups of height* h *and dimension* d *over* k *and Newton polygons.*

In particular, the Newton stratification has finitely many strata. We refer to [Rap3], [Ha], [Va2] for basic properties of the Newton stratification. In particular, the Newton strata are not necessarily smooth.

Examples II. Truncations and their stratifications

We have seen that the association $A \mapsto A[p^\infty]$ simplifies greatly when the BT-group is considered up to isogeny. Another natural idea, given the data $A[p^\infty] = \varinjlim A[p^n]$ is to truncate in level m i.e., to forget about the $A[p^i]$'s for $i > m$, for some $m \in \mathbb{N}$. In its most drastic guise, we may consider the association $A \mapsto A[p^m]$ up to isomorphism, for $m = 1$. This level 1 case was originally studied by Ekedahl and Oort; the corresponding level 1 stratification has finitely many strata. In level $m > 1$, in general we may encounter infinitely many strata.

Manin proved in [Ma] that a BT-group G over an algebraically closed field k is determined by its truncation $G(m)$ for $m >> 0$. One may ask more precise questions: Traverso publicized the following two conjectures in 1979 ([Tr, §40, Conj. 4, 5]). Let $c, d \in \mathbb{N}$ such that $c + d > 0$. Let H be a p-divisible group over k of codimension c and dimension d. Recall that the codimension is the dimension of the dual p-divisible group. Let $n \in \mathbb{N}$ be the smallest number such that H is uniquely determined up to isogeny by $H[p^n]$. It is called the isogeny cutoff of H (terminology introduced in [NVI]).

Conjecture 1.10 (Traverso Isogeny Conjecture). *The number* n *is bounded from above by* $\lceil \frac{cd}{c+d} \rceil$ *i.e., we have* $n \leq \lceil \frac{cd}{c+d} \rceil$.

Theorem 1.11 ([NVI]). *The Traverso isogeny conjecture is true.*

D. Eriksson remarked to the author [2] that this theorem improves a result of Tate on the zeta functions of abelian varieties defined over finite fields (cf. [Ta]). For abelian varieties $c = d = \dim(A)$, and the result says that the truncation of level $\lceil g/2 \rceil$ is enough to determine uniquely the zeta function. Tate's result required a priori to compute up to level g.

Similarly, let $m \in \mathbb{N}$ be the smallest number such that H is uniquely determined up to isomorphism by $H[p^m]$. It is called the isomorphism number of H (terminology introduced in [Va1, Def. 3.1.4]).

[2] in a private conversation at Tōdai's Itatoma in Komaba.

Conjecture 1.12 (Traverso Isomorphism Conjecture). *We have $m \leq \min\{c, d\}$.*

As of December 2008[3], this conjecture was known to hold only in some very particular cases e.g., for the slope 1/2 case (see [NVI, Thm. 1.2., Ex. 3.3] and references therein), and for quasi-special p-divisible groups over k (see [Va3, Thm. 1.5.2]). We point out that in the slope 1/2 case, a variant with principal polarization holds with the same bound ([NVI, Thm. 1.3]).
It would be interesting to investigate the properties of stratifications coming from isogeny cutoffs n or isomorphism numbers m.

Example III. Other stratifications

Definition 1.13. *The p-rank of an abelian variety A over k is defined as $\log_p |A[p](k)| \in \{0, \ldots, \dim(A)\}$.*

The Newton stratification and the level 1 stratification are both refinements of the p-rank stratification, but in general none is a refinement of the other.

In Part II of this paper, we introduce the so-called Manin stratification, a stratification into finitely many strata which applies to some reductions of Shimura varieties at ramified primes. It can be seen as a finite version (in the case of bad reduction) of the level m stratification for $m = \infty$.

1.3.2. Purity

Suppose that we have a stratification $S = \coprod_{i \in I} S_i$. A given stratum S_i may satisfy various purity properties:

$$\begin{array}{ccc}
 & & \text{Principal purity} \\
 & & \Downarrow \\
\text{Affineness} & \underset{(S \text{ separated})}{\Longrightarrow} & \text{Purity i.e., } S_i \overset{\text{affine}}{\hookrightarrow} S \\
 & & \Downarrow \quad {\scriptstyle (S \text{ locally noetherian})} \\
 & & \text{Codimension 1 purity}
\end{array}$$

Other variants are readily available, but it is still unclear to us at this point how useful they are in the context of Shimura varieties. We point out that a typical stratum S_i is not necessarily closed, but only locally closed. A closed stratum is automatically pure, as a closed immersion is affine. The absolute purity conjecture of Grothendieck, which concerns *closed* immersions of regular, noetherian schemes, has been proven by O. Gabber (see [Fu] for a statement and complete proof).
We define the above notions, for which general results are known to us.

[3] See [LNV] for a counterexample to the original conjecture and optimal positive results.

Principal purity is a formalization of the local existence of generalized Hasse–Witt invariants:

Definition 1.14. Principal purity ([NVW, Def. 1.7]). *Let $T \to S$ be a quasi-compact immersion and let $\bar{T}$ be the scheme-theoretic closure of T in S. Then T is called* Zariski locally principally pure in S *if locally for the Zariski topology of $\bar{T}$, there exists a function $f \in \Gamma(\bar{T}, \mathcal{O}_{\bar{T}})$ such that we have*

$$T = \bar{T}_f,$$

where $\bar{T}_f$ is the largest open subscheme of $\bar{T}$ over which f is invertible.

Supposing $T \hookrightarrow S$ quasi-compact is a sufficient condition to insure the existence of $\bar{T}$ as a scheme ([GD1, Prop 9.5.10]). The underlying topological (Zariski) closure (also written $\bar{T}$) of course always exists.
In the definition of principal purity, the Zariski topology may be replaced by another Grothendieck topology $\mathcal{T}$. If $\mathcal{T}$ is coarser than the fpqc [4] topology, then principal purity for $\mathcal{T}$ implies purity, as affineness is then a local property for $\mathcal{T}$. We refer to [Del] for a quick introduction to Grothendieck topologies sufficient for our purpose.

The property we call 'purity' [5] is the relative analogue of affineness (see [GD2, §1.2], where the implication affineness $\implies$ relative affineness is explained, for S separated):

Definition 1.15. Purity ([Va1, Sec. 2.1.1], [NVW, Def. 1.1]). *A subscheme T of a scheme S is called* pure in S *if the immersion $T \hookrightarrow S$ is affine.*

The codimension 1 purity is defined as follows:

Definition 1.16. Codimension 1 purity. *Let S be locally noetherian. Let T be a subscheme of S. If Y is an irreducible component of the Zariski closure $\bar{T}$ of T in S, then the complement of $Y \cap T$ in Y is either empty or of pure codimension* 1.

It follows from [GD4-4, 21.12.7] that purity implies codimension 1 purity when S is locally noetherian.

Example 1.17. **Impurity.** *Let A be an abelian variety defined over $k = \overline{\mathbb{F}}_p$. The a-number of A is defined as: $a(A) := \dim_k(\alpha_p, A) \in \{0, \ldots, \dim(A)\}$, where α_p is*

[4] **f**idèlement **p**lat, **q**uasi-**c**ompact

[5] The term "purity" takes its origin in the Zariski-Nagata purity theorem. In my talk in Kyōto, I put forward a *mnemotechnical* association between Tōkkaidō shinkansen's names and the various purity properties, suggested by the relative qualities of the trains (speed, frequency, convenience, etc.): principal purity is 'light' [hikari], purity is 'echo' [kodama] and affineness is 'hope' [nozomi], etc. Relative affineness is typically proved as a consequence of affineness results e.g., for orbit spaces of Barsotti–Tate groups over fields, so maybe the word [kodama] conveys this as well.

the local-local group scheme of order p i.e., $\alpha_p = \mathrm{Spec}(k[T]/(T^p))$. Let $N \in \mathbb{N}$ such that $(N,p) = 1$. Let T_a denote the locus of geometric, closed points A of $\mathcal{A}_{g,1,N} \otimes \overline{\mathbb{F}}_p$ where $a(A) \geq a$. Consider the stratification of $\mathcal{A}_{g,1,N} \otimes \overline{\mathbb{F}}_p$ defined by $S_a := T_a \backslash T_{a+1}$. An easy calculation shows that the strata S_a are smooth (see [Ni1, Thm. 2.4.14]) *for $a > 0$. On the other hand, it is well-known* ([vdGK2, §6]) *that $\mathrm{codim}(S_a) = a(a+1)/2$, and therefore pure codimension 1 cannot hold.*

We address the natural question: does purity always hold?

We present briefly what is known about the following four stratifications:

★	Newton	Level m	p-rank	Manin
Purity	[Va1]	[NVW]	Unknown	[N., Part II]
Codim. 1 purity	[dJO]	OK from above	[Zi]	OK from above

Recall that

$$\text{principal purity} \implies \text{purity} \implies \text{codimension 1 purity.}$$

We describe in some detail the most recent developments concerning purity, principal or not. The main theorem of [NVW] goes beyond level m stratifications per se (which have been developed for good reductions of Shimura varieties of Hodge type in [Va4]):

Theorem 1.18 (N.-Vasiu-Wedhorn)**.** *Let k be a field of characteristic $p > 0$. Let D_m be a BT_m over k (i.e., an m-truncated Barsotti–Tate group over k). Let S be a k-scheme and let X be a BT_m over S. Let $S_{D_m}(X)$ be the subscheme of S which describes the locus where X is locally for the fppf topology isomorphic to D_m. If $p \geq 5$, then $S_{D_m}(X)$ is pure in S i.e., the immersion $S_{D_m}(X) \hookrightarrow S$ is affine.*

For reasons of space, it is quite difficult to describe Shimura varieties in a degree of generality matching the applicability of this result. Hence we confine ourselves to the classical examples introduced earlier.

Example 1.19. *Let $N \geq 3$ be an integer prime to p. Let $\mathcal{A}_{g,1,N}$ be the Siegel moduli scheme parametrizing the usual objects over $\mathbb{F}_p$-schemes. Let $(\mathcal{U}, \Lambda)$ be the principally quasi-polarized p-divisible group of the universal principally polarized abelian scheme over $\mathcal{A}_{g,1,N}$. If k is algebraically closed and if (D, λ) is a principally quasi-polarized p-divisible group of height $2g$ over k, let $\mathfrak{s}_{D,\lambda}(m)$ be the unique reduced locally closed subscheme of $\mathcal{A}_{g,1,N,k}$ that satisfies the following identity of sets*

$$\mathfrak{s}_{D,\lambda}(m)(k) = \{y \in \mathcal{A}_{g,1,N}(k) | y^*(\mathcal{U}, \Lambda)[p^m] \cong (D, \lambda)[p^m]\}.$$

Then $\mathfrak{s}_{D,\lambda}(m)$ is smooth and equidimensional, and we have:

Proposition 1.20 ([NVW, Thm. 1.6]). *If either $p = 3$ and $g \leq 6$ or $p \geq 5$, then the subscheme $\mathfrak{s}_{D,\lambda}(m)$ is pure in $\mathcal{A}_{g,1,N}$.*

Example 1.21. *Let $N \geq 3$ be an integer prime to p. Let $\mathfrak{M}(S,N)$ be the Hilbert moduli scheme parametrizing the usual objects over $\mathbb{F}_p$-schemes. Let $(\mathcal{U},\Lambda)$ be the quasi-polarized p-divisible group of the universal polarized abelian scheme with real multiplication by $\mathcal{O}_L$ over $\mathfrak{M}(S,N)$. If k is algebraically closed and if (D,λ) is a quasi-polarized p-divisible group of height $2g$ over k with real multiplication by $\mathcal{O}_L$, let $\mathfrak{s}_{D,\lambda}(m)$ be the unique reduced locally closed subscheme of $\mathfrak{M}(S,N)_k$ that satisfies the following identity of sets*

$$\mathfrak{s}_{D,\lambda}(m)(k) = \{y \in \mathfrak{M}(S,N)(k) | y^*(\mathcal{U},\Lambda)[p^m] \cong_{/\mathcal{O}_L} (D,\lambda)[p^m]\}.$$

Then $\mathfrak{s}_{D,\lambda}(m)$ is smooth and equidimensional, and purity holds as above.

The (expected) failure of principal purity may be seen as yet another manifestation of Murphy's Law[6] in higher dimensional algebraic geometry (for a somewhat analogous result in the context of Zariski-Nagata purity, see [Gr, Ex. 3.13]). More precisely, for $g \geq 4$, the Siegel modular variety $\mathcal{A}_{g,1,N} \otimes \overline{\mathbb{F}}_p$ does not admit generalized Hasse–Witt invariants for the p-rank stratification (see [NVW, Prop. 1.8]).

As for positive results regarding principal purity, we mention the construction of (global) generalized Hasse–Witt invariants by T. Itō (cf. [Itō1]) for $U(n,1)$ Shimura varieties, exploiting the fact that the underlying p-divisible group has dimension one. We refer to [Itō2] for similar results about the (minimal compactification of the) Siegel modular 3-fold $\mathcal{A}_{2,1}$, in which case the underlying p-divisible group has dimension two. The 6-fold $\mathcal{A}_{3,1}$ remains somewhat mysterious.

We also point out that the usual stratification of moduli spaces of $K3$ surfaces by the height satisfies principal purity, and moreover, it is expected that its strata are affine (see [vdGK1]).

1.3.3. Applications of purity

We try illustrating some typical applications of purity, picking out examples where a lesser variant will not yield the desired outcome.

Pure codimension one: Classical codimension one purity allows to get lower bounds on dimensions of strata.

Purity: What extra information can we derive from purity that is not already a consequence of codimension one purity? An elementary, direct application of purity is as follows: suppose that $S_i, T_i, i = 1,2$ are locally closed subschemes in some scheme X.

[6] "Anything that can possibly go wrong, will."

If S_1, T_1 are pure, then it follows from the definition that $S_1 \cap T_1$ is also pure. On the other hand, if we only assume that S_2 has pure codimension one and that T_2 is pure, we cannot logically deduce even the weaker statement that $S_2 \cap T_2$ has pure codimension 1, due to the existence of counterexamples.
Another classical application of purity is as follows: affine morphisms f behave well under natural operations e.g., [AGV, Thm. 3.1] or the perverse sheaf theoretic variant that $Rf_!$ is t-exact (see [BBD, 4.1.10]). For example, P. Boyer uses mere purity (and not the stronger affineness results which are nonetheless available in his setting) in this latter form to prove the monodromy-weight conjecture for $U(n,1)$ Shimura varieties (see [Bo, Partie III, Prop. 6.2]).
Affineness: This is used in geometric proofs of existence of companion forms (special cases of Serre-type conjectures) e.g., proof of Gross's theorem via the modular curves $X_1(N)$ by Faltings-Jordan ([FJ]), work in progress of F. Herzig and J. Tilouine (see [HT]), etc.

§ 1.4. Open questions

We gather in this section the open questions which are more or less explicit in the main text.

Question 1.22. *What are the relationships between the various stratifications? E.g., describe the intersection $S_i \cap T_j$ of various strata S_i, T_j.*

Question 1.23. *Is the stratification by the p-rank pure?*

Question 1.24. *Can global Hasse–Witt invariants be constructed for the (minimal compactification of the) Siegel modular variety $\mathcal{A}_{3,1,N} \otimes \overline{\mathbb{F}}_p$ of dimension 6? The most mysterious part of this variety seems the Newton stratum of slopes $\{1/3, 2/3\}$.*

§ 1.5. Appendix of Part I: affineness criteria

For the convenience of the reader, we collect classical affineness criteria.

Affineness as cohomological purity

Theorem 1.25 ([GD3, Thm. 1.3.1]). *Let X be an affine scheme. For all quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we have $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Moreover, we have the criterion of Serre in the noetherian case:

Theorem 1.26 ([GD2, Thm. 5.2.1]). *Let X be a noetherian scheme. The following are equivalent:*

1. *X is an affine scheme;*
2. *there exists a family $f_\alpha \in A = \Gamma(X, \mathcal{O}_X)$ such that X_{f_α} are affine and the ideal generated by the f_α's in A is equal to A;*
3. *the fonctor $\Gamma(X, \mathcal{F})$ is exact in the term $\mathcal{F}$ in the category of quasi-coherent $\mathcal{O}_X$-modules;*
4. *$H^1(X, \mathcal{F}) = 0$ for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$;*
5. *$H^1(X, \mathcal{J}) = 0$ for all quasi-coherent sheaves of ideals $\mathcal{J}$ in $\mathcal{O}_X$.*

Among quasi-affine subsets, affine subsets are characterized by the following vanishing property:

Corollary 1.27. *Let $X = \mathrm{Spec}(A)$ be a scheme of finite type over a field, $U \subset X$ a quasi-compact Zariski open subset. Then U is affine iff $H^i(U, \mathcal{O}_U) = 0$ for all $i > 0$.*

Proof. (Sketch) Since X is a scheme of finite type over a field, the structure sheaf $\mathcal{O}_X$ is ample, and so is $\mathcal{O}_U$. Also, every coherent sheaf on U admits a resolution by direct sums of $\mathcal{O}_X$. As the cohomological dimension is finite, we are done. □

A theorem of Chevalley

Theorem 1.28 ([GD2, Prop. 6.7.1])**.** *Let X be a separated, affine scheme, Y a noetherian scheme, and $f : X \longrightarrow Y$ a finite, surjective morphism. Then Y is an affine scheme.*

In particular, Chevalley's theorem implies that affineness is a local property in the fpqc topology.
N.B. This result also holds for varieties i.e., schemes of finite type over a field.

A theorem of Cline-Parshall-Scott

Let $\mathfrak{G}$ be an affine, smooth connected group over a field k. A smooth subgroup scheme $\mathfrak{F}$ of $\mathfrak{G}$ is called *exact* if the induction of rational $\mathfrak{F}$-modules to rational $\mathfrak{G}$-modules preserves short exact sequences.

Theorem 1.29 ([CPS, Thm. 4.3])**.** *A smooth subgroup scheme $\mathfrak{F}$ of $\mathfrak{G}$ is exact if and only if the quotient variety $\mathfrak{G}/\mathfrak{F}$ is affine.*

This affineness criterion was used crucially in [NVW, Thm. 5.1] to prove purity for level m stratifications associated to Shimura varieties of Hodge type.

§ 2. Part II. The Manin Stratification of Hilbert Modular Varieties

In Part II, we introduce the Manin stratification, using [Ma] heavily. Our main goal is to illustrate purity explicitly with Hilbert modular varieties at a place of bad reduction. We also point out a connection between the singular superspecial locus (a zero-dimensional stratum) and base change for GL_2. We postpone discussing other Shimura varieties over ramified primes to future work, as additional techniques come into play.

§ 2.1. Introduction to Part II

The Manin stratification is expected to give finitely many reduced, locally closed strata, which are smooth, equidimensional, quasi-affine, relatively affine and such that the Zariski closure of a stratum is given by the union of strata. We identify the Manin stratification of Hilbert modular varieties with the slope stratification previously studied in [AG1] and [AG2], where all properties but purity were previously verified. We prove purity i.e., relative affiness of strata, via explicit computations. These computations are tractable because the slope stratification is defined using modules of rank two. We note that purity of the supersingular strata was first proved in [Ni1].

The main virtue of the Manin stratification is to provide a finite stratification adapted to the study of reductions of some classical modular varieties at ramified primes. Our motivation for studying such singular Shimura varieties is the possibility of establishing a geometric version of base change via vanishing cycles. We give partial evidence for GL_2.

We describe the contents of each section. Section 2 classifies Dieudonné modules with real multiplication (RM) by Manin's method. Section 3 applies this classification to the study of Hilbert modular varieties. We show that the slope stratification coincides with the Manin stratification. The explicit nature of our computations exhibits algebraic varieties which are easily seen to be affine, as in Manin's original work. Section 4 describes the quaternion orders appearing as endomorphism orders of superspecial abelian varieties with RM in this ramified setting and revisits the Eichler Basis Problem. In section 5, we recall the definition of ℓ-adic character groups using vanishing cycles, and we use it to draw a tentative connection between Hilbert modular surfaces associated to a real quadratic field $\mathbb{Q}(\sqrt{D})$, modular curves $X_1(p)$ and the classical Doi-Naganuma lifting. This paper extends slightly [Ni2], where p is assumed to be unramified.
N.B. We assume throughout Part II that p is a totally ramified prime in $\mathcal{O}_L$ i.e., $p\mathcal{O}_L = \mathfrak{p}^g$ for a prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$.

§ 2.2. Classification of Dieudonné modules up to isomorphism over totally ramified Witt vectors

Dieudonné modules arise in geometry e.g., as the first crystalline cohomology group $H^1_{cris}(A/W(k))$ of an abelian variety A defined over a perfect field k. Since cohomology is functorial, additional structures (such as real multiplication) carry over from A to the Dieudonné module $\mathbb{D}(A)$, and we may incorporate the additional structure and try to classify the resulting objects. In this section, we establish the classification of Dieudonné modules up to isomorphism over *totally ramified* Witt vectors i.e., over a totally ramified extension of the Witt vectors; as a very particular case, we recover all instances of such enhanced Dieudonné modules arising from superspecial points on Hilbert modular varieties over a totally ramified prime. Recall that an abelian variety A defined over k is superspecial if $A \cong_{\overline{k}} E^g$, for E a supersingular elliptic curve.
N.B. This whole section rests on the observation that the proofs in [Ma] carry over mutatis mutandis to totally ramified extensions of $W(k)$; indeed, all ideas in this section are due to Manin. We see no point in reproducing his proofs word by word. Instead, we sketch the proofs, pointing out the minor, required modifications and hopefully we describe the results in sufficient detail to make the geometric application in Subsection 2.3 intelligible.

Let k be an algebraically closed field, and let $\mathfrak{F}$ be a finite, totally ramified extension of $\mathbb{Q}_p$. As k is perfect, we can form the ring of Witt vectors $W(k)$. The Witt vectors $W(k)$ are a complete discrete valuation ring in characteristic zero with residue field k i.e., $W(k)/pW(k) = k$. Let K be the fraction field of $W(k)$. The Frobenius automorphism σ of the residue field k induces an automorphism of K that we note also σ by a slight abuse of notation. Denote by $K_{\mathfrak{F}} := K \cdot \mathfrak{F}$ the compositum of K and $\mathfrak{F}$, with ring of integers $W_{\mathfrak{F}}$. The ring $W_{\mathfrak{F}}(k)$ is a totally ramified extension of $W(k)$ of degree $[\mathfrak{F} : \mathbb{Q}_p] = g$. We sometimes use the shorthand notation $W_{\mathfrak{F}}$, omitting the mention of the residue field k. Since k is algebraically closed, we may fix an uniformizer $T \in W_{\mathfrak{F}}$ such that $T^\sigma = T$.
The main tools that appear in Manin's classification are two finiteness theorems and some algebro-geometric classifying spaces that we shall call Manin spaces, for short. The key idea behind Manin's finiteness theorems is the concept of a *special* module; a crucial fact is that every Dieudonné module has a unique maximal special submodule, of finite colength. We go over the definitions and describe these results.

Definition 2.1. *A Dieudonné module $\mathbb{D}$ is a left $W_{\mathfrak{F}}[F, V]$-module free of finite rank over $W_{\mathfrak{F}}$ with the condition that $\mathbb{D}/F\mathbb{D}$ has finite length.*

Recall that the ring $W_{\mathfrak{F}}[F, V]$ is non-commutative (except if $k = \mathbb{F}_p$) : F is σ-linear i.e., $Fx = x^\sigma F$, V is σ^{-1}-linear i.e., $Vx^\sigma = xV$, and moreover $FV = VF = p$.

Definition 2.2. *Two Dieudonné modules $\mathbb{D}_1, \mathbb{D}_2$ are isogenous if there is an* *injective $W_{\mathfrak{F}}[F,V]$-homomorphism $\phi : \mathbb{D}_1 \hookrightarrow \mathbb{D}_2$ such that $\mathbb{D}_2/\phi(\mathbb{D}_1)$ has finite length. If $\mathbb{D}_1$ is isogenous to $\mathbb{D}_2$, we write: $\mathbb{D}_1 \sim \mathbb{D}_2$.*

The isogeny relation $\sim$ is indeed symmetric. Since k is algebraically closed, the associated F-isocrystal uniquely determines the isogeny class of the Dieudonné module.

Definition 2.3. *An F-isocrystal (V,Φ) is a finite dimensional space V over $K_{\mathfrak{F}}$ equipped with a σ-linear bijection Φ.*

Theorem 2.4 (Dieudonné-Manin)**.** *Let k be an algebraically closed field. The category of F-isocrystals over $K_{\mathfrak{F}}$ is semisimple and with simple objects parametrized by $\mathbb{Q}$. To $\lambda \in \mathbb{Q}$ correspond the simple object E_λ, defined as follows. If $\lambda = \frac{r}{s}$, with $r,s \in \mathbb{Z}, s > 0, (r,s) = 1$, then*

$$E_\lambda = K_{\mathfrak{F}}(k)[F]/(F^s - T^r),$$

where T is a uniformizer of $K_{\mathfrak{F}}$, and $Fx = x^\sigma F$.

Proof. [Ma, Chap. 2], cf. [Ko1, Chap. 3]. □

The rational numbers $\{\lambda_i\}$ associated to a semisimple object $\oplus_i E_{\lambda_i}$ are called the slopes. By a standard argument (cf. [La, Chap. VI 5.7-5.8, p.180]), any Dieudonné module $\mathbb{D}$ is decomposable uniquely in a direct sum: $\mathbb{D} = \mathbb{D}_{\text{étale}} \oplus \mathbb{D}_{\text{local}}$, where F is an isomorphism on $\mathbb{D}_{\text{étale}}$, and F is topologically nilpotent on $\mathbb{D}_{\text{local}}$ i.e., $\cap_{i\geq 0} F^i \mathbb{D}_{\text{local}} = 0$. Repeating this decomposition with V in lieu of F, we get:

$$\mathbb{D} = \mathbb{D}_{\text{étale,étale}} \oplus \mathbb{D}_{\text{étale,local}} \oplus \mathbb{D}_{\text{local,étale}} \oplus \mathbb{D}_{\text{local,local}}.$$

Since $\mathbb{D}_{\text{étale,étale}}$ is easily seen to be zero, it suffices to classify local Dieudonné modules by the standard dévissage via Cartier duality. Manin's proofs are written in this setting.

Recall that a Dieudonné module $\mathbb{D}$ is *isoclinic* if the set of slopes of $\mathbb{D}$ is a singleton.

Definition 2.5 (Special module)**.**

- *An isoclinic Dieudonné module of slope $\{\frac{r}{s}\}$ is special if $F^r\mathbb{D} = T^s\mathbb{D}$.*
- *An (arbitrary) Dieudonné module $\mathbb{D}$ is special if $\mathbb{D} \cong \oplus_i \mathbb{D}_i$, where $\mathbb{D}_i$ are maximal isoclinic special submodules of $\mathbb{D}$.*

We denote by $K_{\mathfrak{F}}(\mathbb{F}_{p^r})$ the subfield of $K_{\mathfrak{F}}$ fixed under σ^r (e.g., $K_{\mathfrak{F}}(\mathbb{F}_p) = \mathfrak{F}$).

Definition 2.6 (Cyclic local algebra)**.** *Let $E_{r,s}$ be the associative $W_{\mathfrak{F}}(\mathbb{F}_{p^r})$-algebra (with unit) generated by θ such that:*

$$\theta^r = T, \quad \theta\alpha = \alpha^{\sigma^{-b}}\theta, \quad \alpha \in W_{\mathfrak{F}}(\mathbb{F}_{p^r}),$$

where b is such that $-bs = 1 \mod r$.

Let $K_{r,s}$ denote the division algebra $E_{r,s} \otimes \mathbb{Q}$. Define the isosimple module $M_{r,s} := W_{\mathfrak{F}}(k) \otimes_{W_{\mathfrak{F}}(\mathbb{F}_{p^r})} K_{r,s}$. The Dieudonné module structure is defined as follows: $F\theta^i := \theta^{i+s}, T\theta^i := \theta^{i+r}$, and the action of V is uniquely determined by the relation $FV = p$. As the notation suggests, the slope of $M_{r,s}$ is $\frac{r}{s}$. It follows from [Ma, Lem. 3.6] that every element $x \in M_{r,s}$ can be uniquely expressed as: $x = \sum_{i>-\infty} \epsilon_i \theta^i, \epsilon_i \in \mathfrak{T}$, for any multiplicative system $\mathfrak{T} \subset W_{\mathfrak{F}}(k)$ of representatives of k. In particular, for any element $x \in \mathbb{D}$ of a module $\mathbb{D}$ of slope $\frac{r}{s}$, we can define $\nu(x)$ as the minimal integer i such that $\epsilon_i \neq 0$. By picking a suitable embedding (cf. [Ma, Section 2, p.47]), we can view $\mathbb{D}$ as included in the submodule $W_{\mathfrak{F}}(k) \otimes E_{r,s}$, and containing an element congruent to 1 modulo $W_{\mathfrak{F}}(k) \otimes E_{r,s}\theta$.
We can thus define $J = J(\mathbb{D}) := \{\nu(x) | x \in \mathbb{D}\}$. It is invariant under translation by $r, s, gr - s$, and $\mathbb{N} \backslash J$ is finite.

Lemma 2.7. *Let $\mathbb{D}$ be an isosimple module of slope $\frac{r}{s}$.*

- *The finite set $\overline{J} := \mathbb{N} \backslash J$ does not depend on the choice of the embedding (if we restrict ourselves to embeddings that satisfy $\min\{\nu(x) | x \in \mathbb{D}\} = 0$) i.e., $\overline{J}$ is an invariant of $\mathbb{D}$.*
- *For the given embedding $\mathbb{D} \hookrightarrow W_{\mathfrak{F}}(k) \otimes E_{r,s}$, the module $\mathbb{D}$ contains a system of elements of the form:*

$$z_j = \theta^j + \sum_{\ell \in \overline{J}, \ell > j} \epsilon_\ell \theta^\ell, \quad \epsilon_\ell \in \mathfrak{T},$$

where $\mathfrak{T}$ is a multiplicative system of representative for k, and j runs over all elements of J such that the translates belong to $\overline{J}$. The system $\{z_j\}$ is uniquely determined and coincides with a minimal generating set of the Dieudonné module $\mathbb{D}$.

Proof. See [Ma, Lem. 3.9]. □

Corollary 2.8 (First Finiteness Theorem for isosimple modules)**.** *There exists only a finite number of non-isomorphic special modules isogenous to a fixed isosimple module.*

Proof. See [Ma, Cor.1,p.48]. □

Proposition 2.9 (First Finiteness Theorem for isoclinic modules)**.** *There exists only a finite number of non-isomorphic special modules isogenous to a fixed isoclinic module.*

Proof. See [Ma, Lem. 3.10,p.51]. □

Corollary 2.10 (First Finiteness Theorem (general case)). *Let $\mathbb{D}$ be a Dieudonné module. There exists only a finite number of non-isomorphic special modules isogenous to $\mathbb{D}$.*

Proof. This follows from the definition of a special module and the First Finiteness Theorem for isoclinic modules. □

Theorem 2.11 (Second Finiteness Theorem). *Let $\mathbb{D}$ be a Dieudonné module. The module $\mathbb{D}$ has a unique maximal special submodule $\mathbb{D}_0$. The length $[\mathbb{D} : \mathbb{D}_0]$ is bounded uniformly in the isogeny class of $\mathbb{D}$.*

Proof. See [Ma, Thm. 3.1, p.39] for the first assertion, and [Ma, Section 6, Thm 3.8] for the second assertion. □

Theorem 2.12 (Classification Theorem). *Let k be an algebraically closed field. A Dieudonné module $\mathbb{D}$ is determined uniquely up to (non-unique) isomorphism by the following collection of invariants:*

- *the slopes of $\mathbb{D}$;*
- *the maximal special submodule $\mathbb{D}_0 \subset \mathbb{D}$ (parametrized by discrete invariants);*
- *a $\Gamma(\mathbb{D}_0, h)$-orbit of a point corresponding to $\mathbb{D}$ in a constructible algebraic set $A(\mathbb{D}_0, h)$, where h is a nonnegative integer that depends only on the slopes; $A(\mathbb{D}_0, h)$ and $\Gamma(\mathbb{D}_0, h)$ depend only on $\mathbb{D}_0$ and h, and $\Gamma(\mathbb{D}_0, h)$ is a finite group.*

Proof. See [Ma, Chap. 3, Section 3, Thm. 3.2]. □

We need to explain a few elementary facts concerning special modules before presenting an easy illustration of the general classification.

Definition 2.13 (Special element).

- *Let M be an isoclinic module of slope $\frac{r}{s}$. An element $x \in M$ is special if $F^r x = T^s x$.*
- *Let M be an arbitrary Dieudonné module. An element $x \in M$ is special if the projections to all maximal isoclinic special submodules of M are special.*

Lemma 2.14. *An isoclinic module M of slope $\frac{r}{s}$ is special if and only if as a $W_{\mathfrak{F}}(k)$-module, it has a basis consisting of special elements (i.e., a so-called special basis).*

Proof. (See [Ma, Lem. 3.3]). This is a consequence of the so-called Fitting Lemma: a non-trivial σ-linear additive bijection of a finite dimensional k-vector space admits a basis of eigenvectors with eigenvalue 1. Fitting's Lemma shows that there is a special basis modulo p, and a standard bootstrap argument finishes the proof. □

We illustrate the above theory in a typical computation. Recall that a supersingular Dieudonné module is an isoclinic Dieudonné module of (Newton polygon) slope $\frac{1}{2}$.

Remark 2.15. *A supersingular Dieudonné module is superspecial if and only if it is special.*

Corollary 2.16. *The number of isomorphism classes of superspecial Dieudonné modules with RM by $\mathcal{O}_L$ of rank 2 over a totally ramified prime $p = \mathfrak{p}^g$ is: $\left[\frac{g}{2}\right] + 1$.*

Proof. The supersingular isocrystal has slope $\frac{1}{2}$. We are classifying rank 2 modules over $W_{\mathfrak{F}}(k)$. If g is odd, the supersingular isocrystal is given by the isosimple module $W_{\mathfrak{F}}(k)[F,V]/(F-V)$. We can count the number of special modules isogenous to $W_{\mathfrak{F}}(k)[F,V]/(F-V)$ by looking at the discrete invariants of Lemma 2.7. The triplet $\{r, s, gr - s\}$ boils down to $\{g, 2\}$, hence for $g = 2k+1$, the sets $\overline{J}$ have the shape $(1, 3, \ldots, 2c-1)$, where $0 \leq c \leq k$ ($\overline{J}$ is empty if $c = 0$). The complement of such a set is: $J_c := \{2a+(2k+1)b\} \cup \{2c+1+2a+(2k+1)b\}, a, b, c \geq 0$. Recall that by Lemma 2.7, the distinguished special submodule M containing 1 is generated by elements $\{1, \theta^{2c+1}\}$ if the set $J(\mathbb{D})$ coincides with J_c, and all the corresponding modules are non-isomorphic. This gives precisely $k+1 = [g/2]+1$ modules, hence finishes the proof for g odd. If g is even, the isogeny class is given by the non-simple module $2 \cdot W_{\mathfrak{F}}(k)[F,V]/(F - T^{g/2})$. By changing variables, the generating system of the special module $\mathbb{D}$ can be chosen to be $\{1, \theta^c\}$ for $0 \leq c \leq g/2$, since it depends only on the valuations of the generators. This gives also $[g/2]+1$ modules, so the proof is finished. □

Remark 2.17. *C.-F. Yu gave in [Yu1, Lem. 4.5, 4.6] an ad hoc classification of superspecial crystals.*

§ 2.3. Application to Hilbert moduli spaces over totally ramified primes

In this section, we show that the slope stratification of the Hilbert moduli space over a totally ramified prime $p\mathcal{O}_L = \mathfrak{p}^g$ introduced by Andreatta-Goren in [AG1] coincides with the stratification suggested by the Manin classification.

We recall briefly the definition of the slope stratification of [AG1]. Recall that L is a totally real field of degree g over $\mathbb{Q}$, with ring of integers $\mathcal{O}_L$, and different $\mathcal{D}^{-1}_{L/\mathbb{Q}}$. We fix a set of fractional ideals $\{\mathfrak{L}_1, \cdots, \mathfrak{L}_{h^+}\}$ that form a complete set of representatives of $Cl(L)^+$, the narrow class group of L, $h^+ = |Cl(L)^+|$.

Definition 2.18. *Let S be a scheme. Let N be a positive integer. If T is a scheme over S, the objets of the Hilbert moduli space $\mathfrak{M}(S,\mu_N) \longrightarrow S$ are quadruples $(A,\iota,\lambda,\epsilon)/_{\cong}$ consisting of:*

(a) an abelian scheme $A \longrightarrow T$ of relative dimension g;

(b) an $\mathcal{O}_L$-action i.e., a ring homomorphism $\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_T(A)$;

(c) a polarisation

$$\lambda : (\mathcal{P}_A, \mathcal{P}_A^+) \xrightarrow{\cong} (\mathfrak{L}, \mathfrak{L}^+),$$

i.e., an $\mathcal{O}_L$-linear isomorphism on the étale site of S between the invertible $\mathcal{O}_L$-module $\mathcal{P}_A := \mathrm{Hom}_{\mathcal{O}_L}(A, A^t)^{sym}$ and one of the fixed $\mathfrak{L}_i$'s, $1 \leq i \leq h^+$, identifying the positive cone of polarisations $\mathcal{P}_A^+$ with $\mathfrak{L}^+$. Moreover, we require that the morphism $A \otimes_{\mathcal{O}_L} \mathcal{P}_A \longrightarrow A^t$ be an isomorphism.

(d) an $\mathcal{O}_L$-linear injective homomorphism

$$\epsilon : \mu_N \otimes_{\mathbb{Z}} \mathcal{D}_L^{-1} \hookrightarrow A,$$

where for any scheme T' over T, $(\mu_N \otimes_{\mathbb{Z}} \mathcal{D}_L^{-1})(T') := \mu_N(T') \otimes_{\mathbb{Z}} \mathcal{D}_L^{-1}$.

The stack $\mathfrak{M}(S,\mu_N)$ decomposes as disjoint union $\bigsqcup_{\mathfrak{L}_i} \mathfrak{M}(S,\mu_N,\mathfrak{L})$, according to the polarisation modules. We point out that the Deligne-Pappas condition for A is equivalent to the existence, locally for the étale topology, of an $\mathcal{O}_L$-linear p-principal polarisation on A ([AG1, Prop. 3.1], [Vo, Prop. 2.2]).
Let A/k be a polarized abelian variety with RM, defined over a field k of characteristic p. Fix an isomorphism $\mathcal{O}_L \otimes_{\mathbb{Z}} k \cong k[T]/(T^g)$. Recall the short exact sequence:

$$0 \longrightarrow H^0(A,\Omega_A^1) \longrightarrow H^1_{dR}(A) \longrightarrow H^1(A,\mathcal{O}_A) \longrightarrow 0.$$

These modules are Dieudonné modules of group schemes, and we may rewrite this exact sequence in terms of Frobenius and Verschiebung as:

$$0 \longrightarrow (k, \mathrm{Fr}^{-1}) \otimes_k \mathbb{D}(\mathrm{Ker}(\mathrm{Fr})) \longrightarrow \mathbb{D}(A[p]) \longrightarrow \mathbb{D}(\mathrm{Ker}(\mathrm{Ver})) \longrightarrow 0.$$

Since $H^1_{dR}(A)$ is a free $k[T]/(T^g)$-module of rank 2, there are two generators α and β such that:

$$H^1(A,O_A) = (T^i)\alpha + (T^j)\beta, i \geq j, i+j = g.$$

The index $j = j(A)$ is called the singularity index. The slope $n = n(A)$ is defined by $j(A) + n(A) = a(A)$, where $a(A) := \mathrm{Hom}(\alpha_p, A)$ is the a-number of the abelian variety. The subschemes $\mathfrak{W}_{(j,n)}$ parameterizing abelian varieties with singularity index j and slope n are quasi-affine, locally closed and thus form a stratification (see [AG1, Thm. 10.1], [AG2, §6.1]). Note that for any Dieudonné module $\mathbb{D}$ with RM of rank 2, we can define abstractly $j(\mathbb{D})$ and $n(\mathbb{D})$ without any reference to abelian varieties i.e., $j(\mathbb{D}) = j$ is the integer such that

$$(T^i)\alpha + (T^j)\beta = \mathrm{Ker}(V : \mathbb{D}/p\mathbb{D} \longrightarrow \mathbb{D}/p\mathbb{D}), i \geq j,$$

for α, β some generators of $\mathbb{D}$. The slope is $n(\mathbb{D}) := a(\mathbb{D}) - j(\mathbb{D})$.

For a fixed special module $\mathbb{D}_c$, denote by $\{\mathcal{M}_c^d\}_{d\geq 0}$ the finitely many irreducible algebraic varieties classifying modules over the special module $\mathbb{D}_c$, ordered in increasing dimension.

N.B. The index d is justified a posteriori by the fact that d is precisely the dimension of $\mathcal{M}_c^d$ (or, equivalently, its e-index) i.e., there are no gaps, or missing dimensions, starting from dimension zero.

We use this set-theoretic decomposition to define a stratification of the Hilbert moduli space.

Definition 2.19. *Define $\mathfrak{N}_c^d$ as the locus on the Hilbert moduli space such that for $\underline{A} \in \mathfrak{N}_c^d$, the Dieudonné module $\mathbb{D}(\underline{A})$ belongs to $\mathcal{M}_c^d$. We call such a decomposition the Manin stratification.*

The main theorem of this section justifies our terminology:

Theorem 2.20. *The Manin stratification $\{\mathfrak{N}_c^d\}_{c,d}$ coincides with the slope stratification $\{\mathfrak{W}_{(j,n)}\}_{j,n}$ of the Hilbert moduli space $\mathfrak{M}(\overline{\mathbb{F}}_p, \mu_N), (N,p) = 1$.*

Let $\mathbb{D}$ be a Dieudonné module, and let $\mathbb{D}_c$ be its maximal special submodule. We shall see that the slope $n(\mathbb{D})$ of $\mathbb{D}$ depends only on the maximal special submodule $\mathbb{D}_c$. Moreover, a-number of $\mathbb{D}$ depends only on the e-index $(0, d)$ of $\mathbb{D}$ over its superspecial module i.e., $a(\mathbb{D}) = a(\mathbb{D}_c) - d$.

We prove Theorem 2.20 by giving an explicit description of Manin spaces for all possible Newton polygons, following the terminology and the very similar computations of [Ma, Chap. 3, Thm. 3.12, Lem. 3.14, Thm. 3.15]). This is done for the supersingular Newton polygon stratum in Subsection 2.3.1, and for the non-supersingular Newton polygon strata in Subsection 2.3.2.

2.3.1. The supersingular Newton polygon stratum

Definition 2.21. *We define the superspecial Dieudonné module $\mathbb{D}_c$ as follows, for $c \in \{0, \dots, [g/2]\}$: $\mathbb{D}_c$ is generated by $\{1, \theta^{2c+1}\}$ if g is odd; $\{1, \theta^c\}$ if g is even.*

For short, if a module $\mathbb{D}$ is generated by $\{a, b\}$, we write $\mathbb{D} =< a, b >$.

Definition 2.22 (e-index)**.** *Let $\mathbb{D}_0 \subset \mathbb{D} \subset T^{-h}\mathbb{D}_0$ be a $W_{\mathfrak{F}}(k)$-module, for $h >> 0$. There exists a $W_{\mathfrak{F}}(k)$-basis $(x_1, \dots, x_N)$ of $\mathbb{D}_0$ such that:*

$$(T^{-e_1}x_1, \dots, T^{-e_N}x_N), \quad 0 \leq e_1 \leq e_2 \leq \cdots \leq e_N \leq h$$

is a $W_{\mathfrak{F}}(k)$-basis for $\mathbb{D}$. We call the string of integers $e(\mathbb{D}_0, \mathbb{D}) = (e_1, \ldots, e_N)$, the e-index of $\mathbb{D}$.

- g odd. As we have already seen in the proof of Corollary 2.16, any supersingular module is isogenous to the isosimple module $W_{\mathfrak{F}}(k)[F,V]/(F-V)$. By Lemma 2.7, any module $\mathbb{D}$ over $\mathbb{D}_c$ has two standard generators:

$$z_1 = 1 + \sum_{\ell=1}^{h} \epsilon_{2\ell-1}\theta^{2\ell-1}, \quad z_2 = \theta^{2h+1},$$

where $\epsilon_{2\ell-1} \in \mathfrak{T}$ are determined by $\mathbb{D}$. We define a number d, $0 \leq d \leq h$ by the conditions:

$$\epsilon_{2\ell-1} \in W_{\mathfrak{F}}(\mathbb{F}_{p^2}), \ell \leq h-d,$$

$$\epsilon_{2(\ell-d)+1} \notin W_{\mathfrak{F}}(\mathbb{F}_{p^2}).$$

Proposition 2.23. *Let $\mathbb{D}$ be a Dieudonné having $\mathbb{D}_c$ as its maximal special submodule.*

1. *The T-height of $\mathbb{D}$ is at most $[g/2]+1$.*
2. *The factor module $\mathbb{D}/\mathbb{D}_c$ is generated by the coset of one element z, where*

$$z = 1 + \sum_{\ell=1}^{d} \epsilon_{2\ell-1}\theta^{-(2\ell-1)}.$$

3. *The e-index of $\mathbb{D}$ is $(0,d)$, for some $d \leq c \leq [g/2]$.*
4. *The space $\mathcal{M}_c^d$ of modules M of e-index $(0,d)$ belonging to a fixed special module $\mathbb{D}_c$ has dimension d and is isomorphic to the complement of the disjoint union of p^2 hyperplanes in $\mathbb{A}^d$:*

$$\overline{\epsilon_d} = a, a \in \mathbb{F}_{p^2}.$$

Proof. Cf. [Ma, p.60, Thm. 3.12]. □

- g even. Any supersingular module is isogenous to $2 \cdot W_{\mathfrak{F}}(k)[F,V]/(F-T^{g/2})$. We label θ_i the generator of the cyclic local algebra coming from the i-th copy of $W_{\mathfrak{F}}(k)[F,V]/(F-T^{g/2})$. Without loss of generality, we restrict to primitive modules i.e., modules that do not contain θ_1^{-1} and θ_2^{-1}. We define an invariant d in the same way as in the g odd case.

Proposition 2.24. *Let $\mathbb{D}$ be a Dieudonné having $\mathbb{D}_c$ as its maximal special submodule.*

1. *There is a primitive module $\mathbb{D}'$ isomorphic to $\mathbb{D}$ with T-height $d \leq g/2$.*
2. *The factor module $\mathbb{D}'/\mathbb{D}_c$ is generated by the coset of z, where:*

$$z = \theta_1^{-d} + \sum_{\ell=1}^{d} \epsilon_\ell \theta_2^{-\ell}, \epsilon_\ell \in \mathfrak{T}, \epsilon_c \neq 0.$$

3. *The e-index of $\mathbb{D}$ is $(0, d)$, for some $d \leq c \leq [g/2]$.*
4. *The space $\mathcal{M}_c^d$ of modules $\mathbb{D}$ of index $(0, d)$ belonging to a fixed special module $\mathbb{D}_c$ has dimension d, and is isomorphic to the complement of the disjoint union of p^2 hyperplanes in $\mathbb{A}^d$:*

$$\overline{\epsilon_d} = a, \quad a \in \mathbb{F}_{p^2}.$$

Proof. Cf. [Ma, p. 66, Thm. 3.15]. □

Armed with the precise description of the Manin spaces, we can compute the invariants defining the slope stratification, and prove the assertions of Theorem 2.20 in the supersingular case.

Lemma 2.25. *The a-number of the Dieudonné module $\mathbb{D}$ over its (maximal) superspecial module $\mathbb{D}_c$ depends only on the e-index $e(\mathbb{D}_c, \mathbb{D}) = (0, d)$ over this module:*

$$a(\mathbb{D}) = g - d.$$

Proof. Let $\mathbb{D}$ be a module over its maximal special module $\mathbb{D}_c$ such that $e(\mathbb{D}_c, \mathbb{D}) = (0, d)$. The a-number of $\mathbb{D}$ is, by definition, $\dim_k \mathbb{D}/F\mathbb{D} + V\mathbb{D}$. Since $\mathbb{D}_c$ is superspecial, $\dim_k \mathbb{D}_c/F\mathbb{D}_c + V\mathbb{D}_c = g$. Thence, showing that $a(\mathbb{D}) = g - d$ is equivalent to showing that the e-index of $F\mathbb{D} + V\mathbb{D}$ over $F\mathbb{D}_c + V\mathbb{D}_c$ is $(0, d)$ i.e., $e(F\mathbb{D}_c + V\mathbb{D}_c, F\mathbb{D} + V\mathbb{D}) = (0, d)$. This is possible if and only if $d \leq c \leq [g/2]$.
Let g be odd.

$$F(1 + \sum_{\ell=1}^{d} \epsilon_{2\ell-1} \theta^{-(2\ell-1)}) = \theta^g + \sum_{\ell=1}^{d} \epsilon_{2\ell-1}^{\sigma} \theta^{-2\ell+1+g}, \quad F\theta^{2c+1} = \theta^{2c+1+g},$$

and

$$V(1 + \sum_{\ell=1}^{d} \epsilon_{2\ell-1} \theta^{-2\ell+1}) = \mu \left\{ \theta^g + \sum_{\ell=1}^{d} \epsilon_{2\ell-1}^{\sigma^{-1}} \theta^{-2\ell+1+g} \right\}, \quad V\theta^{2c+1} = \mu\theta^{2c+1+g}.$$

This implies that $F\mathbb{D} + V\mathbb{D} =$

$$< \theta^g + \sum_{\ell=1}^{d} \epsilon_{2\ell-1}^{\sigma} \theta^{-2\ell+1+g}, \theta^{2c+1+g} > + < \theta^g + \sum_{\ell=1}^{d} \epsilon_{2\ell-1}^{\sigma^{-1}} \theta^{-2\ell+1+g}, \theta^{2c+1+g} >,$$

as we can ignore the unit μ by changing the generator; note the crucial difference in the action of σ (resp. σ^{-1}) for F (resp. V). Of course,

$$F\mathbb{D}_c + V\mathbb{D}_c =< \theta^g, \theta^{2c+1+g} > .$$

Since the second generator θ^{2c+1+g} of $F\mathbb{D}$ and $V\mathbb{D}$ is the same as the second generator of $F\mathbb{D}_c + V\mathbb{D}_c$, to compute the e-index of $F\mathbb{D} + V\mathbb{D}$ over $F\mathbb{D}_c + V\mathbb{D}_c$, we only need to inspect the coefficients of the first generators. Since $\epsilon_{2d-1}^{\sigma} \neq \epsilon_{2d-1}^{\sigma^{-1}}$, the corresponding coefficient in the generator of $F\mathbb{D} + V\mathbb{D}$ is non-trivial, and this implies that the e-index of $F\mathbb{D} + V\mathbb{D}$ over $F\mathbb{D}_c + V\mathbb{D}_c$ is $(0, d)$, since for g odd, $\theta^2 = T$.
Let g be even. Similarly to the g odd case, F acts by σ on the coefficients $\epsilon_\ell \in \mathfrak{T}$ and by multiplication by $\theta^{g/2}$ in the cyclic local algebra, and V acts by σ^{-1} on the coefficients $\epsilon_\ell \in \mathfrak{T}$ and by multiplication by $\mu\theta^{g/2}$ in the cyclic local algebra, where $\theta = T$. We can ignore μ as before by making the obvious change of generator of $V\mathbb{D}$. Since $\mathbb{D}_c = \{1, \theta^c\}$, $F\mathbb{D}_c + V\mathbb{D}_c = \{\theta^{g/2}, \theta^{g/2+c}\}$. In the same way as in the g odd case, the e-index of $F\mathbb{D} + V\mathbb{D}$ over $F\mathbb{D}_c + V\mathbb{D}_c$ is $(0, d)$, and we are done.

□

Lemma 2.26. *Let $\mathbb{D}$ be a module over its maximal special submodule $\mathbb{D}_c$ with e-index $(0, d)$. Then $\mathbb{D}$ has type $(c - d, g - c)$.*

Proof. The invariants $j(\mathbb{D})$ and $i(\mathbb{D})$ are computable modulo p. In particular,

$$g - j = \min\{m | T^m H^1(A, \mathcal{O}_A) = 0 \mod p\}.$$

Recall that $H^1(A, \mathcal{O}_A) \cong H^1_{dR}(A)/H^0(A, \Omega^1_A)$, and, in terms of the contravariant version of Dieudonné theory,

$$H^1(A, \mathcal{O}_A) = \mathbb{D}(A[p])/V\mathbb{D}(A[p]) = \mathbb{D}(A[p])/\mathbb{D}(A[p])[F].$$

We can compute the singularity index $j(\mathbb{D})$ by computing $\min\{m | T^m(\mathbb{D}/\mathbb{D}[F]) = 0 \mod p\}$ for any Dieudonné module $\mathbb{D}$. We reduce the claim to the case of e-index $(0, d) = (0, 0)$. Fix an isomorphism $\mathbb{D} \cong W_{\mathfrak{F}}(k) \oplus W_{\mathfrak{F}}(k)$, such that $\overline{\mathbb{D}_c} = \mathbb{D}_c \mod p \cong k[T]/(T^g) \oplus T^d k[T]/(T^g)$. In this representation, it is obvious that $j(\mathbb{D}_c) - d = j(\mathbb{D})$. We show that

$$j(\mathbb{D}_c) = c.$$

Suppose first that g is odd. Recall that the superspecial module $\mathbb{D}_c$ is generated by $\{1, \theta^{2c+1}\}$. Therefore

$$\theta^{2g-2c-1}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) = 0, \theta^{2g-2c-2}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) \neq 0,$$

and so

$$T^{g-c}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) = 0, T^{g-c-1}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) \neq 0,$$

i.e., $j(\mathbb{D}_c) = c$. Suppose now that g is even. $\mathbb{D}_c$ is generated by $\{1, \theta^c\}$. Therefore,

$$\theta^{g-c}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) = 0, \theta^{g-c-1}(\overline{\mathbb{D}}/V\overline{\mathbb{D}}) \neq 0,$$

and since $\theta = T$, $j(\mathbb{D}_c) = c$. □

2.3.2. Non-supersingular strata

The non-supersingular strata are easier to deal with, and we are briefer. In particular, the slope n uniquely defines the Newton polygon as $\{\frac{n}{g}, \frac{g-n}{g}\}$. Moreover, for every non-supersingular Newton polygon, there is a unique special module, given by

$$W_{\mathfrak{F}}(k)[F,V]/(F - T^n) \oplus W_{\mathfrak{F}}(k)[F,V]/(F - T^{g-n}).$$

In other words, the maximal special module depends only on the slope n. Following [Ma, p.63, Lem. 3.13, p. 65, Thm. 3.14], we indeed recover all non-supersingular slope strata from Manin strata.

Proposition 2.27. *Let $\mathbb{D}$ be a Dieudonné module having $W_{\mathfrak{F}}(k)[F,V]/(F - T^n) \oplus W_{\mathfrak{F}}(k)[F,V]/(F - T^{g-n})$ as its maximal special submodule, for $n < g - n$. The Manin space of such Dieudonné modules $\mathbb{D}$ splits in $n+1$ components $\mathcal{M}_d$ of index $(0,d)$, for $0 \leq d \leq n$. The component $\mathcal{M}_d$ is isomorphic to the space of orbits of a certain finite group acting on the affine space $\mathbb{A}^d$ with hyperplanes removed.*

Of course, it is possible to generalize the Manin stratification by modifying the level (e.g., replacing $A[p^\infty]$ by $A[p^n]$). In particular, the Manin stratification of level 1 would generalize the Ekedahl-Oort stratification at primes of good reduction while remaining a finite stratification at ramified primes. We note that the Manin stratification can be readily defined for Picard modular varieties over ramified primes, but since available tools are insufficient (cf. [PR]) to describe accurately the local geometry in dimension (strictly) greater than two, we postpone a detail study of such varieties to later work.

The slope stratification is pure i.e., all its strata are relatively affine. This follows from the explicit computations of the Manin spaces: since they are easily seen to be affine, the locally closed embeddings in the ambient space are affine, hence their pullbacks i.e., the locally closed embeddings of the Manin strata in the Hilbert modular variety, are also affine. We record this observation:

Proposition 2.28. *The Manin stratification of Hilbert modular varieties over totally ramified primes is pure.*

Moreover, some supersingular strata are (absolutely) affine for trivial reasons. Indeed, in the supersingular Newton polygon stratum, there are zero-dimensional strata (consisting of superspecial points) that are of course affine. We propose the following naive method to prove affineness "near" zero-dimensional strata in some cases. The idea is to use group actions to relate strata *directly* via finite, surjective maps in order to apply Chevalley's theorem.

Let $W^{[m]}_{(j,j')}$ be the scheme-theoretic image of $\mathrm{Isom}(p^{m+2}) \longrightarrow \mathrm{Isom}(p^m)$, where $\mathrm{Isom}(p^m)$ represents the functor associating to a scheme over $W_{(j,j')}$ the group of isomorphisms $\mathrm{Isom}(A[p^m] \times_{W_{(j,j')}} T, A[p^m] \times_k T)$ as group schemes over T endowed with an $\mathcal{O}_L$-action. The scheme $\mathrm{Isom}(p^m)$ is affine and of finite type over $W_{(j,j')}$.

Proposition 2.29 ([AG2, Prop. 6.1.9]). *Let $0 \leq m \leq j \leq g/2$. Let j' be either j or $g-j$. There exists a smooth, connected, affine scheme U_m over k, of dimension m, and a finite surjective map:*

$$\psi_m : W^{[m]}_{(j,j')} \times_k U_m \longrightarrow W_{(j-m,j')}.$$

N.B. The statement of [AG2] gives more information, but we shall not need the extra properties that are proved there.

Proposition 2.30. *All strata in the supersingular locus are affine.*

Proof. Since $W^{[m]}_{(j,j')}$ and U_m are affine, it follows that $W^{[m]}_{(j,j')} \times_k U_m$ is also affine. By Prop. 2.29 and Chevalley's theorem [GD2, Thm. 6.7.1], it follows that $W_{(j-m,j')}$ is affine. It is straightforward that all (supersingular) strata are obtained in this way. $\square$

§ 2.4. Superspecial orders

In this section, we give a classical description of endomorphism orders of superspecial points on the Hilbert modular variety; in short, they can be described as locally primitive (or Bass) orders i.e., orders containing the ring of integers of a quadratic extension of the center, locally at each prime. For completeness with respect to [Ni2] (whose notation and terminology we follow), we also derive from this description a parametrization of the superspecial locus, and the Eichler Basis Problem as in the unramified case.

We recall a few facts about quaternion algebra (see [Bz1] for a good compendium). Let $B_{p,\infty}$ be the quaternion algebra over $\mathbb{Q}$ ramified at p and ∞.

Definition 2.31. *The dual of an $\mathcal{O}_L$-lattice $M \subset B := B_{p,\infty} \otimes L$ is defined as:*

$$M^\sharp := \{x \in B : tr(xM) \subset \mathcal{O}_L\},$$

where tr is the reduced trace. The $\mathcal{O}_L$-ideal $\mathrm{Norm}(M^\sharp)^{-1}$ is the level of M.

Definition 2.32. *An order $\mathcal{O}$ in a quaternion algebra B over a field F is primitive if it contains the maximal order of a quadratic field extension of F or the maximal order of the split extension $F \oplus F$.*

As being primitive is not a local property, we say that an $\mathcal{O}_L$-order $\mathcal{O}$ in $B_{p,\infty} \otimes L$ is locally primitive if $\mathcal{O}_v$ is primitive at all places v of L.

Definition 2.33. *An order $\mathcal{O}$ is Gorenstein if $\mathcal{O}^\sharp$ is $\mathcal{O}$-projective as a left (resp. right) $\mathcal{O}$-lattice.*

Definition 2.34. *An order $\mathcal{O}$ is a Bass order if each order in B containing it is a Gorenstein order.*

In particular, a Bass order is Gorenstein. It follows from [Bz2, Prop. 1.11] that an order in $B_{p,\infty} \otimes L$ is Bass if and only if it is locally primitive.

Definition 2.35. *Let B be the quaternion algebra over $L_\mathfrak{p}$. Let $K = K_\mathfrak{p}$ be a quadratic extension of $L_\mathfrak{p}$ contained in B. Set*

$$R_v(K) = \mathcal{O}_K + P_B^{v-1},$$

for P_B the unique maximal ideal in $\mathcal{O}_B$ and $v = 1, 2, \dots$.

We introduce a subclass of Bass orders.

Definition 2.36. *An order $\mathcal{O}$ is* superspecial *of level $\mathcal{P}$ dividing p, $\mathcal{P} = \prod_i \mathfrak{p}_i^{\alpha_i} \cdot \prod_j \mathfrak{q}_j^{\beta_j}$, for $\mathfrak{p}_i \in Ram(B_{p,\infty} \otimes L)$, $\mathfrak{q}_j \notin Ram(B_{p,\infty} \otimes L)$, if:*

- for $\alpha_i \geq 1$, there is an unramified quadratic extension $\mathcal{O}_K$ of $\mathcal{O}_{L_\mathfrak{p}}$ such that $\mathcal{O}_{\mathfrak{p}_i} = R_{\alpha_i}(K)$;
- for $\beta_j > 1$, if $f(\mathfrak{q}_j/p)$ is even, $\mathcal{O}_{q_j}$ contains a split quadratic extension; if $f(\mathfrak{q}_j/p)$ is odd, there is an unramified quadratic extension $\mathcal{O}_K$ such that

  $$\mathcal{O}_{\mathfrak{q}_j} \cong \left\{ \begin{pmatrix} \alpha & \beta^\sigma \\ \pi_{\mathfrak{q}_j}^{\beta_j} \beta & \alpha^\sigma \end{pmatrix}, \alpha, \beta \in \mathcal{O}_K \right\},$$

  for σ the involution on K, $\pi_{\mathfrak{q}_j}$ a uniformizer in $\mathcal{O}_{L_{\mathfrak{q}_j}}$;
- for any other finite prime $\mathfrak{l}$, $\mathcal{O}_\mathfrak{l}$ contains a split extension (i.e., $\mathcal{O}_{L_\mathfrak{l}} \oplus \mathcal{O}_{L_\mathfrak{l}}$).

Recall that an abelian variety with RM is an abelian variety with an action by $\mathcal{O}_L$, satisfying the Deligne-Pappas condition i.e., the canonical morphism

$$A \otimes_{\mathcal{O}_L} \mathcal{P}_A \xrightarrow{\cong} A^t, (a, \lambda) \mapsto \lambda(a),$$

is an isomorphism (see [DP, 2.1.3, p.64]).
Recall that the type of an abelian variety A is the pair $(j(A), g - j(A))$ given by the singularity index $j(A)$, see Section 2.3.

Theorem 2.37. *Let A be a superspecial abelian variety with RM of type $(j, g-j)$. Then $\mathrm{End}_{\mathcal{O}_L}(A)$ is a superspecial order of level $\mathfrak{p}^{g-2j}$, where $j \leq [g/2]$.*

Proof. This follows from [Yu1, Lem. 4.5, 4.6] and also from the $\mathcal{O}_L$-variant of Tate's theorem on endomorphisms of abelian varieties ([Ni2, Thm 2.1]). □

2.4.1. Locally principal ideals and superspecial loci

We may parametrize superspecial abelian varieties whose endomorphism order has fixed level with left ideals of that order. In order to have a bijection, it is necessary to impose that ideals are locally principal or, equivalently, projective. This condition is non-trivial e.g., for nonsquarefree levels. For simplicity, we assume that $h^+(L) = 1$.

Proposition 2.38. *Let $h^+(L) = 1$. Let A be a superspecial abelian variety with RM satisfying the Deligne-Pappas condition, and such that $\mathcal{O} = \mathrm{End}_{\mathcal{O}_L}(A)$ has level $\mathfrak{p}^n$. The map $A \mapsto A \otimes_{\mathcal{O}} I$ induces a functorial bijection between locally principal left $\mathcal{O}$-ideals I and superspecial points whose endomorphism orders have level $\mathfrak{p}^n$.*

Proof. This follows from Tate's theorem with RM and the locally principal condition, as in the unramified case (see [Ni2, Thm. 5.6] □

Corollary 2.39. *All superspecial orders of level $\mathfrak{p}^n$ arise from geometry.*

Proof. All superspecial orders of level $\mathfrak{p}^n$ are conjugate by [Bz1, Prop. 5.3], and the rest of the proof follows as in unramified case (see [Ni2, Cor. 5.7]). □

For completeness, we recall and adapt the construction of quadratic forms (and theta series) for the projective module $\mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)$, where A_1, A_2 are two p-principally polarized supersingular abelian varieties with RM having isomorphic quasi-polarized Dieudonné modules. Let $\lambda_i : A_i \xrightarrow{\cong} A_i^t$, $i = 1, 2$, be p-principal $\mathcal{O}_L$-polarisations, and define, for $\phi \in \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)$

$$||\phi||_0 := ||\phi||_{0,\lambda_1,\lambda_2} := \lambda_1^{-1} \circ \phi^t \circ \lambda_2 \circ \phi, \qquad \begin{tikzcd} A_2^t \arrow[d, "\phi^t"'] & A_2 \arrow[l, "\lambda_2"'] \\ A_1^t \arrow[r, "\lambda_1^{-1}"'] & A_1 \arrow[u, "\phi"'] \end{tikzcd} .$$

As we recall from [Ni2, Lem. 5.9], $||\phi||_0 \in L$. We define $\deg_{\mathcal{O}_L}(\lambda_i) := ||\lambda_i||_0$ for $i = 1, 2$. Finally, the $\mathcal{O}_L$-degree is defined as:

$$|| - || : \mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2) \longrightarrow \mathcal{O}_L, \phi \mapsto \frac{\deg_{\mathcal{O}_L}(\lambda_1)}{\deg_{\mathcal{O}_L}(\lambda_2)} ||\phi||_0.$$

Note that for $h^+(L) = 1$, we retrieve the $\mathcal{O}_L$-degree defined in the principally polarized case in [Ni2]. This gives a quadratic module structure on $\mathrm{Hom}_{\mathcal{O}_L}(A_1, A_2)$, and we can thus define the theta series as in [Ni2] via representations numbers, etc.

2.4.2. The Basis Problem in the totally ramified case

It is well-known how to derive directly from the Jacquet-Langlands correspondence the Eichler Basis Problem. Details can be found in [HPS, §9], [Ge2, p.294], [Ge1, §10]) and [Hi1]. The key observation of [HPS, 9.1, §9] is that the Jacquet-Langlands correspondence implies the Basis Problem for so-called minimal forms. A form is minimal if it has minimal level among all the character twists (in particular, a minimal form is a newform). Note that there are sometimes no minimal forms e.g., for even level and dyadic fields, see [BH, Section 41.5, Lem.]. We specialize this to superspecial orders: when g is even, the quaternion algebra is split at $\mathfrak{p}$, so there is no nothing to check; when g is odd, all possible levels are also of odd exponent. Exactly as in [HPS], all newforms are minimal for odd exponent levels.

Proposition 2.40. *Let $p\mathcal{O}_L = \mathfrak{p}^g$, and let $0 \leq g - 2j \leq g$, $j \in \mathbb{N}$. The space $S_2^{new}(\Gamma_0(\mathfrak{p}^{g-2j}))$ is contained in the span of theta series coming from left ideals of a superspecial order of level $\mathfrak{p}^{g-2j}$ in the quaternion algebra $B_{p,\infty} \otimes L$.*

§ 2.5. The Doi-Naganuma lifting

We describe a tentative connection between the locus of (singular) superspecial points of Hilbert modular varieties at ramified primes, and the Doi-Naganuma lifting. Recall that an abelian variety A is called superspecial if $A \cong E^g$, for some supersingular elliptic curve E.

We consider the simplest case i.e., restricting to $L = \mathbb{Q}(\sqrt{p})$, $p \equiv 1 \mod 4$, $h^+(L) = 1$ and weight two. We wish to compare the character group (defined via vanishing cycles) of the Hilbert modular surface, with the vanishing cycles sheaf cohomology of the modular curve $X_1(p)$ of $\Gamma_1(p)$-level structure. Difficulties already arise in this case and this section contains no fundamentally new result. Our aim is to sketch a very quick route through the series of algebro-geometric coincidences leading to Question 2.43 on the existence of geometric base change.

Geometric approaches to base change were pioneered in characteristic zero by Hirzebruch and Zagier ([HZ]).

Recall that the Doi-Naganuma lift is a map of the space of modular forms of level $\Gamma_0(p)$ with non-trivial quadratic character χ_p to the space of Hilbert modular forms of parallel weight:

$$DN : S_2(\Gamma_0(p), \chi_p) \longrightarrow S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L)).$$

According to Hecke, the space of modular forms of nebentype decomposes in $\pm$-spaces:

$$S_2(\Gamma_0(p), \chi_p) = S_2^+(\Gamma_0(p), \chi_p) \oplus S_2^-(\Gamma_0(p), \chi_p).$$

The kernel of the map DN is $S_2^-(\Gamma_0(p), \chi_p)$, and thus DN maps $S_2^+(\Gamma_0(p), \chi_p)$ injectively into $S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))$. The image of the map consists of symmetric Hilbert modular forms i.e., forms F such that $F(z_1, z_2) = F(z_2, z_1)$. We denote the space of symmetric Hilbert modular forms by $S_{2,2}^{sym}(\mathrm{SL}_2(\mathcal{O}_L))$. We shall try to gain some understanding of the isomorphism $S_2^+(\Gamma_0(p), \chi_p) \xrightarrow{\cong} S_{2,2}^{sym}(\mathrm{SL}_2(\mathcal{O}_L))$ via characteristic p geometry.
We fix some extra notation. Since $p\mathcal{O}_L = \mathfrak{p}^2$, note that $B_{p,\infty} \otimes L \cong B_{\infty_1,\infty_2}$, the quaternion algebra ramified only at the infinite places of L. Let $x \in B_{\infty_1,\infty_2}$. Denote by $x \mapsto \overline{x}$ the quaternion involution on B_{∞_1,∞_2}, and by $x \mapsto x^*$ the $\mathbb{Q}$-automorphism of B_{∞_1,∞_2} extending the non-trivial Galois involution $\sigma : L \longrightarrow L$ and which is trivial on $B_{p,\infty}$. Let $V := \{x \in B_{\infty_1,\infty_2} | x^* = \overline{x}\}$. Equipped with the reduced norm, it is a quadratic space over $\mathbb{Q}$ of discriminant p (not p^2!).
Let $A = E \otimes \mathcal{O}_L$, for E a supersingular elliptic curve over $\overline{\mathbb{F}}_p$. Indeed, this corresponds to a non-singular superspecial point on the Hilbert modular surface. But there is a canonical procedure to associate to it a singular superspecial point, whose endomorphism order has level 1. Indeed, for any $g = [L : \mathbb{Q}]$, there is a canonical chain of $\mathcal{O}_L$-invariant α_p-isogenies stemming from $A = E \otimes \mathcal{O}_L$ ([AG1, Prop. 6.6, (2c) and (2d)], cf. [Ni1, p.115]):

- for g odd:

$$A = A_{0,g} \overset{\exists!\alpha_p}{\longrightarrow} A_{1,g-1} \overset{\exists!\alpha_p}{\longrightarrow} A_{2,g-2} \overset{\exists!\alpha_p}{\longrightarrow} \dots \overset{\exists!\alpha_p}{\longrightarrow} A_{[g/2],[g/2]+1};$$

- for g even:

$$A = A_{0,g} \overset{\exists!\alpha_p}{\longrightarrow} A_{1,g-1} \overset{\exists!\alpha_p}{\longrightarrow} A_{2,g-2} \overset{\exists!\alpha_p}{\longrightarrow} \dots \overset{\exists!\alpha_p}{\longrightarrow} A_{g/2,g/2},$$

  where the pair $(j, i), i + j = g$ is the type of the superspecial abelian variety.

This canonical chain provides a distinguished symmetric maximal order. Recall that an order $\mathcal{O}$ is symmetric if for any $\sigma \in \mathrm{Aut}(L)$, there exists an extension $\overline{\sigma}$ to $\mathrm{Aut}(B_{\infty_1,\infty_2})$ such that $\mathcal{O}^{\overline{\sigma}} = \mathcal{O}$. For $g = 2$, this order is $\mathrm{End}_{\mathcal{O}_L}(E \otimes \mathcal{O}_L/H)$, for H the unique $\mathcal{O}_L$-invariant α_p-group scheme of the abelian surface $E \otimes \mathcal{O}_L$. The symmetry is easily checked on $\mathrm{End}_{\mathcal{O}_L}(E \otimes \mathcal{O}_L)$, and is inherited by successive quotients by $\mathcal{O}_L$-invariant α_p-subgroup schemes.

Proposition 2.41 ([Po]). *Let $h(L) = 1$. There is a bijection between: left $\mathcal{O}$-ideal classes Λ for a symmetric maximal order $\mathcal{O}$ of B_{∞_1,∞_2} and proper similitude classes of lattices with reduced discriminant p, given by the map $\Lambda \mapsto \overline{\Lambda^*}\Lambda \cap V$.*

In general (e.g., for $h(L) \neq 1$), the cardinality of proper similitude classes of lattices is given by the type of $\mathcal{O}$.
The following diagram describes in a nutshell the link between the ideal classes arising from singular superspecial points and the Doi-Naganuma lifting:

$$\begin{tikzcd}
\{\mathcal{O}\text{-ideal classes } \Lambda\} \arrow[r, "\text{bijection}", "\Lambda\mapsto\overline{\Lambda^*}\Lambda\cap V"'] \arrow[d, "\text{theta series}"'] & \{\text{Lattices of disc. } p\} \arrow[d, "\text{theta series}"] \\
S_{k,k}(\mathrm{SL}_2(\mathcal{O}_L)) & S_k(\Gamma_0(p), \chi_p) \arrow[l, "Doi-Naganuma"]
\end{tikzcd}$$

The top arrow bijection is thus an algebraic version of the Doi-Naganuma lifting. Thanks to [Hi2], the integral version of the Eichler Basis Problem is known e.g., in level 1. On other other hand, it is well-known that the basis problem for nebentypus does not always hold (even rationally) for $k = 2$ e.g., the smallest prime for which it fails is $p = 389$. On the other hand, Waldspurger ([Wa]) has shown that the basis problem holds for $k > 2$. Since we do not have a novel geometric interpretation of the spherical polynomials thus arising, we stick to $k = 2$.

2.5.1. The Hilbert modular surface $\mathfrak{M}$

To get a more geometric statement, we recall generalized character groups.
As is well-known ([Il], [Raj]), the vanishing cycles formalism allows to define a $\mathbb{Z}_\ell$-coefficients character group $\mathcal{C}_{\mathbb{Z}_\ell}(X)$ for arbitrary weights:

$$\mathcal{C}_{\mathbb{Z}_\ell}(X) := \mathrm{Im}\Big(H^d(X \times \overline{K}, \mathfrak{F}) \longrightarrow H^d(X \times \overline{k}, R\Phi(\mathfrak{F}))\Big),$$

where X is a d-dimensional scheme over $\mathcal{O}_K$, $[K : \mathbb{Q}_p] < \infty$, and with residue field k. The vanishing cycle sheaf $R\Phi(\mathfrak{F}))$ is defined for any lisse $\mathbb{Z}_\ell$-sheaf $\mathfrak{F}$ (see the discussion in [Raj]). In what follows, we shall restrict to weight two for simplicity i.e., $\mathfrak{F} = \mathbb{Z}_\ell$. Here, ℓ denotes a prime number different than p.

We denote by $\mathfrak{M}$ the Hilbert modular surface studied in Section 2.3. Its compactification at ramified primes has been described by Deligne and Pappas ([DP]), correcting earlier work of Rapoport ([Rap1]). We describe the singularities of the Hilbert modular surface, following [DP] and [BG]. The singular points are the superspecial points which do not satisfy the Rapoport condition i.e., the Lie algebra $\mathrm{Lie}(A)$ is not locally free as an $\mathcal{O}_L \otimes k$-module. There are $p + 1$ singular points on each rational component of the supersingular locus. Any singular point is ordinary with $p+1$ branches, and the tangent cone is $z^2 = xy$.

Proposition 2.42. *Fix an identification $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. We have an isomorphism of Hecke modules:*

$$\mathcal{C}_{\mathbb{Z}_\ell}(\mathfrak{M}) \otimes_{\mathbb{Z}_\ell} \mathbb{C} \cong S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L))_{\mathbb{C}}.$$

Proof (Sketch). It is clear that $\mathcal{C}_{\mathbb{Z}_\ell}(\mathfrak{M})$ is a $\mathbb{Z}_\ell$-module supported on the singular superspecial locus. The claim essentially follows from the Jacquet-Langlands correspondence for B_{∞_1,∞_2}. □

2.5.2. Geometric base change map

We define the symmetric locus $\mathcal{C}_{\mathbb{Z}_\ell}(\mathfrak{M})^{sym}$ as the sub-Hecke-module of $\mathcal{C}_{\mathbb{Z}_\ell}(\mathfrak{M})$ supported on points of the singular locus whose endomorphism order is symmetric. These symmetric superspecial abelian varieties are parametrized by $A_H \otimes I$, for $A_H := E \otimes \mathcal{O}_L/H$, $H \cong_{\mathcal{O}_L} \alpha_p$ and where I runs over all symmetric ideals of $\mathrm{End}_{\mathcal{O}_L}(A_H)$. We suppose that the symmetric Basis Problem holds i.e., that theta series coming from symmetric ideals generate the whole space of symmetric Hilbert modular forms over $\mathbb{C}$. Recall that the curve $X_1(p)$ is the moduli space [7] of elliptic curves equipped with a point of exact order p. Denote by $X_1(p)_L := X_1(p) \times_{\mathbb{Q}} L$, and $\mathrm{Res}_{L/\mathbb{Q}} X_1(p)_L$ for the Weil restriction of scalars.

Question 2.43. *Let $k = 2$. Do we have a natural Hecke-equivariant map:*

$$\mathfrak{D}\mathfrak{N} : \mathcal{C}_{\mathbb{Z}_\ell}(\mathfrak{M})^{sym} \longrightarrow H^1_c(\mathrm{Res}_{L/\mathbb{Q}} X_1(p)_L \times \overline{\mathbb{F}}_p, R\Phi(\mathbb{Z}_\ell)),$$

giving rise to the Doi-Naganuma lifting $S_2^+(\Gamma_0(p), \chi_p) \cong S_{2,2}^{sym}(\mathrm{SL}_2(\mathcal{O}_L))$ over $\mathbb{C}$?

§ 2.6. Appendix of Part II. A digression on the condition $h^+(L) = 1$.

In [Ni2] and this paper's Sections 4 and 5, we supposed that $h^+(L) = 1$. We explain in detail issues arising for a general totally real field L. The hypothesis $h^+(L) = 1$ is a natural condition to impose a classical allure: $h(L) = 1$ implies that the group of transformations of the Hilbert modular forms is conjugate to $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathcal{O}_L)$, while $h^+(L) = 1$ allows to parametrize the whole superspecial locus with double cosets corresponding to (left) ideal classes of an order in a quaternion algebra or, equivalently, to omit mentioning the polarisation modules. In the adelic approach to Hilbert modular forms, there is no disadvantage in considering an arbitrary L. As is well-known, it is possible in the quaternionic case to parametrize the superspecial locus (for fixed level) with the adelic double cosets by removing the norm 1 condition i.e., by considering instead the algebraic group G defined as $G(R) := (\mathrm{End}_{\mathcal{O}_L}(A) \otimes R)^\times$ for any ring R; (this

[7] We choose this curve because the space of cusp forms $S_2(\Gamma_1(p))$ contains $S_2(\Gamma_0(p), \chi_p)$, but this might not be the optimal choice. A problem is to construct geometrically lattices of reduced discriminant p.

is well-known, cf. [Ca]). There is of course a discrepancy between this parametrization and the ideal class parametrization, precisely at the level of polarisations. The Hilbert moduli space $\mathcal{M}$ decomposes as: $\mathcal{M} = \coprod_{(\mathfrak{A},\mathfrak{A}^+)\in Cl^+(L)} \mathcal{M}_{(\mathfrak{A},\mathfrak{A}^+)}$, where $(\mathfrak{A},\mathfrak{A}^+)$ is an $\mathcal{O}_L$-module with a notion of positivity. Suppose that we pick a superspecial point $A \in \mathcal{M}_{\mathfrak{A},\mathfrak{A}^+}$. For $h^+(L) \neq 1$, the point $A \otimes_{\mathcal{O}} I$, for $\mathcal{O} = \mathrm{End}_{\mathcal{O}_L}(A)$ does not land in the same component. More precisely, if $A \in \mathcal{M}_{(\mathfrak{A},\mathfrak{A}^+)}$, then $A \otimes_{\mathcal{O}} I \in \mathcal{M}_{(\mathfrak{A}\cdot\mathfrak{B}^2,(\mathfrak{A}\cdot\mathfrak{B}^2)^+)}$, for $\mathfrak{B} = I \cap \mathcal{O}_L$. This is the case because $(A \otimes_{\mathcal{O}} I)^t = A^t \otimes I^{-1}$. Thus, the points $A \otimes_{\mathcal{O}} I$ do not cover the whole superspecial locus. On the other hand, it is possible to parametrize the whole superspecial locus by considering a disjoint sum of double cosets, parametrized by $h^+(L)/h(L)$. The number $h^+(L)/h(L)$ thus measures the failure of the bijection between left ideal classes of a fixed level and the corresponding superspecial locus of abelian varieties with endomorphism order of the corresponding level.

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