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Kisin conjecture on the moduli spaces of finite flat models

By

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Abstract

We explain a relationship between a local universal deformation ring and a moduli space of finite flat models. We also give an outline of a proof of the Kisin conjecture on the connected components of the moduli space of finite flat models.

Introduction

Let $K$ be a $p$-adic field for $p > 2$. We consider a two-dimensional continuous representation $V_{F}$ of the absolute Galois group $G_K$ over a finite field $F$ of characteristic $p$. By a finite flat model of $V_{F}$, we mean a finite flat group scheme $G$ over $O_K$, equipped with an action of $F$, and an isomorphism $V_{F} \sim G(K)$ that respects the action of $G_K$ and $F$. Then there exists a moduli space of finite flat models of $V_{F}$, which is projective scheme over $F$, and we denoted it by $\mathscr{G} \mathscr{R}_{V_{F},0}$. Let $\mathscr{G} \mathscr{R}_{V_{F},0}^{\vee}$ be the closed subscheme of $\mathscr{G} \mathscr{R}_{V_{F},0}$ determined by the condition that the $p$-adic Hodge type is (1).

It is important to study the connected components of $\mathscr{G} \mathscr{R}_{V_{F},0}^{\vee}$, since it gives us information of a deformation ring. The ordinary component of $\mathscr{G} \mathscr{R}_{V_{F},0}^{\vee}$ was determined in [Kis], and Kisin conjectured that the non-ordinary component is connected. In this survey paper, we explain the relationship between $\mathscr{G} \mathscr{R}_{V_{F},0}^{\vee}$ and a deformation ring. We also give an outline of a proof of the Kisin conjecture. The theory of the moduli space of finite flat models was established in [Kis], and the Kisin conjecture was proved by Kisin in [Kis] if $K$ is totally ramified over $\mathbb{Q}_p$, by Gee in [Gee] if $V_{F}$ is the trivial representation, and by the author in [Ima] for general $K$ and $V_{F}$. 
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Notation

Throughout this paper, we use the following notation. Let \( p > 2 \) be a prime number. For a positive number \( m \), the finite field of cardinality \( p^m \) is denoted by \( \mathbb{F}_{p^m} \). Let \( k \) be the finite extension of \( \mathbb{F}_p \) of cardinality \( q = p^m \). For a ring \( R \), the ring of Witt vectors over \( R \) with respect to \( p \) is denoted by \( W(R) \). Let \( K \) be a totally ramified extension of \( K_0 \) of degree \( e \). The ring of integers of \( K \) is denoted by \( \mathcal{O}_K \), and the absolute Galois group of \( K \) is denoted by \( G_K \). Let \( \mathbb{F} \) be a finite field of characteristic \( p \). The formal power series ring of \( u \) over \( \mathbb{F} \) is denoted by \( \mathbb{F}[[u]] \), and its quotient field is denoted by \( \mathbb{F}(u) \). Let \( v_u \) be the valuation of \( \mathbb{F}(u) \) normalized by \( v_u(u) = 1 \). For a local ring \( A \), the maximal ideal of \( A \) is denoted by \( \mathfrak{m}_A \). For a topological space \( X \), the set of connected components of \( X \) is denoted by \( \pi_0(X) \).

§ 1. Deformation ring and moduli space of finite flat models

In this section, we explain the relationship between a deformation ring and a moduli space of finite flat models.

First, we are going to introduce a deformation ring. Let \( V_{\mathbb{F}} \) be a two-dimensional continuous \( G_K \)-representation over \( \mathbb{F} \) with a fixed ordered basis. A \( G_K \)-representation over a finite ring is said to be flat if and only if it is isomorphic to the generic fiber of a finite flat group scheme over \( \mathcal{O}_K \) as a \( G_K \)-module. We assume that \( V_{\mathbb{F}} \) is flat. Let \( \mathfrak{AR}_{W(\mathbb{F})} \) be the category of Artin local finite \( W(\mathbb{F}) \)-algebra \( A \) whose residue field is isomorphic to \( \mathbb{F} \) as a \( W(\mathbb{F}) \)-algebra. To define a deformation, we use a notion of groupoids. For the notion of groupoids, please consult [Kis, Appendix on groupoids]. The framed flat deformation \( D_{V_{\mathbb{F}}}^{\mathfrak{AR}} \) of \( V_{\mathbb{F}} \) over \( \mathfrak{AR}_{W(\mathbb{F})} \) is a groupoid \( D_{V_{\mathbb{F}}}^{\mathfrak{AR}} \) over \( \mathfrak{AR}_{W(\mathbb{F})} \) determined as in the followings:

- For an object \( A \) in \( \mathfrak{AR}_{W(\mathbb{F})} \), an object of \( D_{V_{\mathbb{F}}}^{\mathfrak{AR}}(A) \) is a triple \( (V_A, \psi, \beta) \), where \( V_A \) is a flat continuous \( G_K \)-representation that is a free \( A \)-module of rank 2 with an ordered basis \( \beta \) over \( A \), and \( \psi : V_A \otimes_A \mathbb{F} \longrightarrow V_{\mathbb{F}} \) is an \( \mathbb{F} \)-linear \( G_K \)-isomorphism sending \( \beta \) to the fixed ordered basis of \( V_{\mathbb{F}} \).

- A morphism \( (V_A, \psi, \beta) \rightarrow (V_{A'}, \psi', \beta') \) covering a given morphism \( A \rightarrow A' \) in \( \mathfrak{AR}_{W(\mathbb{F})} \) is an equivalence class \( [\alpha] \), where \( \alpha : V_A \otimes_A A' \longrightarrow V_{A'} \) is an \( A' \)-linear \( G_K \)-isomorphism that is compatible with the morphisms \( \psi, \psi' \) and sending \( \beta \) to \( \beta' \), and two morphisms are equivalent if they differ by an element of \( A'^\times \).
Then the framed flat deformation $D^\square_{V_{\xi}}$ is pro-represented by a complete local $W(F)$-algebra $R^\square_{V_{\xi}}$.

We are going to define a deformation ring with the condition that the $p$-adic Hodge type $\nu = (1)$, which is denoted by $R^\square_{V_{\xi}}$. Let $(R^\square_{V_{\xi}}[1/p])^\nu$ be the quotient of $R^\square_{V_{\xi}}[1/p]$ corresponding to the connected components of Spec $R^\square_{V_{\xi}}[1/p]$ whose closed points $\xi$ satisfy the following:

If $V_{\xi}$ is the deformation corresponding to $\xi$, then $\Fil^0 D_{\text{crys}}(V_{\xi}[1/p])_K$ is free of rank 1 over $k(\xi) \otimes_{\mathbb{Q}_p} K$. Here, $k(\xi)$ is the residue field of $\xi$.

We note that $V_{\xi}[1/p]$ is Barsotti-Tate representation, since we are considering a flat deformation. Then we define $R^\square_{V_{\xi}}$ by the image of $R^\square_{V_{\xi}}$ in $(R^\square_{V_{\xi}}[1/p])^\nu$.

The information of the connected components of Spec $R^\square_{V_{\xi}}[1/p]$ is very important for an application to a theorem comparing a deformation ring and a Hecke ring ([Kis, Theorem 3.4.11], [Ima, Theorem 3.1]). So we want to know $\pi_0(\text{Spec } R^\square_{V_{\xi}}[1/p])$.

Next, we are going to explain the Kisin module and the moduli space of finite flat models of $V_{\xi}$. By a finite flat model of $V_{\xi}$, we mean a finite flat group scheme $G$ over $\mathcal{O}_K$, equipped with an action of $F$, and an isomorphism $V_F \sim G(K)$ that respects the action of $G_K$ and $F$.

Let $\mathcal{G} = W(k)[[u]]$, and $\mathcal{O}_E$ be the $p$-adic completion of $\mathcal{G}[1/u]$. We consider the action of $\phi$ on $\mathcal{O}_E \otimes_{\mathbb{Z}_p} F \cong k((u)) \otimes_{\mathbb{F}_p} F$ defined by $p$-th power on $k((u))$. Let $\Phi M_{\mathcal{O}_E \otimes_{\mathbb{Z}_p} F}$ be the category of finite $\mathcal{O}_E \otimes_{\mathbb{Z}_p} F$-modules $M$ with $\phi$-semi-linear map $\phi : M \rightarrow M$ such that the induced linear map $\phi^* M \rightarrow M$ is bijective.

We choose a system $(\pi_m)_{m \geq 1}$ of elements in $K$ such that $\pi_m^p = \pi$ and $\pi_{m+1}^p = \pi_m$ for $m \geq 1$, and put $K_f = \bigcup_{m \geq 1} K(\pi_m)$. Let $\text{Rep}_F(G_{K_f})$ be the category of finite-dimensional continuous $G_{K_f}$-representations over $F$.

Then the functor

$$T : \Phi M_{\mathcal{O}_E \otimes_{\mathbb{Z}_p} F} \rightarrow \text{Rep}_F(G_{K_f}); M \mapsto \left(\overline{k((u))} \otimes k((u))\right) M^{\phi = 1}$$

is an equivalence of abelian categories. We take $M_F \in \Phi M_{\mathcal{O}_E \otimes_{\mathbb{Z}_p} F}$ such that $T(M_F)$ is isomorphic to $V_F(-1)_{G_{K_f}}$. Here $(-1)$ denotes the inverse of the Tate twist. Then $M_F$ is a free $(\mathcal{O}_E \otimes_{\mathbb{Z}_p} F)$-module of rank 2.

We put $\mathcal{G}_F = \mathcal{G} \otimes_{\mathbb{Z}_p} F$. Let $(\text{Mod }/\mathcal{G}_F)$ be the category of finite free $\mathcal{G}_F$-modules $\mathfrak{M}$ with $\phi$-semi-linear map $\phi : \mathfrak{M} \rightarrow \mathfrak{M}$ such that the cokernel of the induced linear map $\phi^* \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $u^e$. An object of $(\text{Mod }/\mathcal{G}_F)$ is called a Kisin module with coefficients in $F$. Let $(\text{F-Gr}/\mathcal{O}_K)$ be the category of finite flat group schemes over $\mathcal{O}_K$ with a structure of an $F$-vector space.

**Theorem 1.1.** There exists an equivalence of categories

$$\text{Gr} : (\text{Mod }/\mathcal{G}_F) \rightarrow (\text{F-Gr}/\mathcal{O}_K).$$
\textbf{Proof.} This follows from \cite[Théorème 4.2.1.6]{Br} and \cite[Lemma 1.2.5]{Kis}. \hfill $\square$

\textbf{Proposition 1.2 ([Kis, Proposition 1.1.13])}. For an object $\mathcal{M}$ of $(\text{Mod} / \mathcal{G}_F)$, there exists a canonical isomorphism

$$T(\mathcal{O}_E \otimes \mathcal{M})(1) \sim \text{Gr}(\mathcal{M})(\overline{K})|_{G_{K_\infty}}$$

as $G_{K_\infty}$-representations. Here (1) denotes the Tate twist.

By this proposition, we see that a Kisin module which is a sublattice of $M_F$ corresponds to a finite flat model of $V_F$. Here and in the sequel, a sublattices means a finite free $\mathfrak{S}_F$-submodule of $M_F$ that spans $M_F$ over $\mathcal{O}_E \otimes \mathbb{Z}_p F$. In the above, we have defined a Kisin module with coefficients in $\mathbb{F}$. More generally, we can define a Kisin module with coefficients in a $\mathbb{Z}_p$-algebra (cf. \cite[(1.2)]{Kis}). Using this general Kisin module, we can construct a moduli space of Kisin modules, which is denoted by $\mathscr{G}_F V_F$ and projective over Spec $R_{\mathfrak{F}}^{\mathfrak{F}}$ (cf. \cite[(2.1)]{Kis}). The closed fiber of $\mathscr{G}_F V_F$ over Spec $R_{\mathfrak{F}}^{\mathfrak{F}}$ is denoted by $\mathscr{G}_F V_{\mathfrak{F},0}$. The scheme $\mathscr{G}_F V_{\mathfrak{F},0}$ is a moduli space of finite flat models of $V_{\mathfrak{F}}$ in the sense of the following proposition.

\textbf{Proposition 1.3 ([Kis, Corollary 2.1.13])}. For any finite extension $\mathbb{F}'$ of $\mathbb{F}$, there is a natural bijection between the set of isomorphism classes of finite flat models of $V_{\mathfrak{F}'} = V_{\mathfrak{F}} \otimes \mathbb{F}'$ and $\mathscr{G}_F V_{\mathfrak{F},0}(\mathbb{F}')$.

From now on, we assume $\mathbb{F}_q \subset \mathbb{F}$ and fix an embedding $k \hookrightarrow \mathbb{F}$. This assumption does not matter, since we may extend $\mathbb{F}$ to prove the main theorem. We consider the isomorphism

$$\mathcal{O}_E \otimes \mathbb{Z}_p \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \sim \prod_{\sigma \in \text{Gal}(k/\mathbb{F}_p)} \mathbb{F}((u)); \quad \left( \sum_i a_i u^i \right) \otimes b \mapsto \left( \sum_i \sigma(a_i) bu^i \right)$$

and let $\epsilon_\sigma \in k((u)) \otimes_{\mathbb{F}_p} \mathbb{F}$ be the primitive idempotent corresponding to $\sigma$. Take $\sigma_1, \ldots, \sigma_n \in \text{Gal}(k/\mathbb{F}_p)$ such that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$. Here we regard $\phi$ as the $p$-th power Frobenius, and use the convention that $\sigma_{n+i} = \sigma_i$. In the sequel, we often use such conventions. Then we have $\phi(\epsilon_{\sigma_i}) = \epsilon_{\sigma_{i+1}}$, and $\phi : M_F \to M_F$ determines $\phi : \epsilon_{\sigma_i} M_F \to \epsilon_{\sigma_{i+1}} M_F$. For $(A_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}((u)))^n$, we write

$$M_F \sim (A_1, A_2, \ldots, A_n) = (A_i)$$

if there is a basis $\{e_1^i, e_2^i\}$ of $\epsilon_{\sigma_i} M_F$ over $\mathbb{F}((u))$ such that $\phi \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} = A_i \begin{pmatrix} e_1^{i+1} \\ e_2^{i+1} \end{pmatrix}$. We use the same notation for any sublattice $M_{\mathfrak{M}} \subset M_F$ similarly.

Finally, for any sublattice $M_{\mathfrak{M}} \subset M_F$ with a chosen basis $\{e_1^i, e_2^i\}_{1 \leq i \leq n}$ and $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}((u)))^n$, the module generated by the entries of $\langle B_i \begin{pmatrix} e_1^i \\ e_2^i \end{pmatrix} \rangle$ with
the basis given by these entries is denoted by $B \cdot \mathcal{M}_F$. Note that $B \cdot \mathcal{M}_F$ depends on the choice of the basis of $\mathcal{M}_F$.

A closed subscheme $\mathcal{A}^{\nabla}_{V_F} \subset \mathcal{A}^{\nabla}_{V_F}$ is defined by the condition that $p$-adic Hodge type $\nu = (1)$ as in [Kis, (2.4.2)]. The closed fiber of $\mathcal{A}^{\nabla}_{V_F}$ over $\text{Spec } R_{V_F}^\nabla$ is denoted by $\mathcal{A}^{\nabla}_{V_F,0}$. The rational points of $\mathcal{A}^{\nabla}_{V_F,0}$ is characterized by the following Lemma.

**Lemma 1.4** ([Gee, Lemma 2.2]). If $F'$ is a finite extension of $F$, the elements of $\mathcal{A}^{\nabla}_{V_{F'},0}(F')$ naturally correspond to free $k[[u]] \otimes_{\mathbb{F}_p} F'$-submodules $\mathcal{M}_{F'} \subset M_F \otimes_F F'$ of rank $2$ that satisfy the following:

1. $\mathcal{M}_{F'}$ is $\phi$-stable.
2. For some (so any) choice of $k[[u]] \otimes_{\mathbb{F}_p} F'$-basis for $\mathcal{M}_{F'}$, and for each $\sigma \in \text{Gal}(k/F_p)$, the map
   \[
   \phi : \epsilon_\sigma \mathcal{M}_{F'} \rightarrow \epsilon_{\sigma \circ \phi^{-1}} \mathcal{M}_{F'}
   \]
   has determinant $\alpha u^e$ for some $\alpha \in F'[\{u\}]^\times$.

Then there is the following relation between the deformation ring $R_{V_F}^{\nabla,\nu}$ and the moduli space $\mathcal{A}^{\nabla}_{V_{F'},0}$.

**Proposition 1.5.** There exists a natural bijection
   \[\pi_0(\text{Spec } R_{V_F}^{\nabla,\nu}[1/p]) \sim \pi_0(\mathcal{A}^{\nabla}_{V_{F'},0}).\]

**Proof.** This follows from [Kis, Corollary 2.4.10], since $\mathcal{A}^{\nabla,\text{loc}}_{V_{F'},0} = \mathcal{A}^{\nabla}_{V_{F'},0}$ by [Kis, Proposition 2.4.6] if the $p$-adic Hodge type $\nu = (1)$. \qed

So the problem has been reduced to study $\pi_0(\mathcal{A}^{\nabla}_{V_{F'},0})$. The connected components $\mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0} \subset \mathcal{A}^{\nabla}_{V_{F'},0}$ is defined by the points corresponding to the ordinary finite flat group schemes. We can easily determine the set $\pi_0(\mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0})$ as in the following:

**Proposition 1.6** ([Kis, Proposition 2.5.15]). If $\mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0}$ is non-empty, then it consist of a single point, unless $V_F \sim \left( \begin{array}{cc} \chi_1 & 0 \\ 0 & \chi_2 \end{array} \right)$ where $\chi_1$ and $\chi_2$ are unramified characters of $G_K$. In the latter case, we have the followings:

1. If $\chi_1 \neq \chi_2$, then $\mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0}$ consists of two points.
2. If $\chi_1 = \chi_2$, then $\mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0} \cong \mathbb{P}^1_{\mathbb{F}_p}$.

Next, we consider the non-ordinary part. We put
   \[\mathcal{A}^{\nabla,\text{non-ord}}_{V_{F'},0} = \mathcal{A}^{\nabla}_{V_{F'},0} \setminus \mathcal{A}^{\nabla,\text{ord}}_{V_{F'},0}.\]
Then Kisin conjectured that $\mathcal{R}_{V_{F},0}^{\text{non-ord}}$ is connected.

§ 2. Proof of Kisin conjecture

We use the following Lemma on the structure of $M_{F}$.

**Lemma 2.1** ([Ima, Lemma 1.2]). Suppose $V_{F}$ is absolutely irreducible and $\mathbb{F}_{q^{2}} \subset F$. If $\mathbb{F}'$ is the quadratic extension of $F$, then

$$M_{F} \otimes_{F} \mathbb{F}' \sim \left(\begin{array}{cccc}
\alpha_{1} & 0 & \cdots & 0 \\
0 & \alpha_{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_{n}
\end{array}\right)$$

for some $\alpha_{i} \in (\mathbb{F}')^{\times}$ and a positive integer $s$ such that $(q + 1) \mid s$.

In fact, we prove that $\mathcal{R}_{V_{F},0}^{\text{non-ord}}$ is rationally connected. To join two points by $\mathbb{P}_{F}^{1}$, we use the following two Lemmas.

**Lemma 2.2** ([Gee, Lemma 2.4]). Suppose $x_{0}, x_{1} \in \mathcal{R}_{V_{F},0}(\mathbb{F})$ correspond to objects $\mathfrak{M}_{0,F}, \mathfrak{M}_{1,F}$ of $(\text{Mod}/\mathcal{S}_{F})$ respectively. Let $N = (N_{i})_{1 \leq i \leq n}$ be a nilpotent element of $M_{2}(\mathbb{F}(u)))^{n}$ such that $\mathfrak{M}_{1,F} = (1 + N) \cdot \mathfrak{M}_{0,F}$ with a basis of $\mathfrak{M}_{0,F}$, and $A = (A_{i})_{1 \leq i \leq n}$ be an element of $GL_{2}(\mathbb{F}(u)))^{n}$ such that $\mathfrak{M}_{0,F} \sim A$ for the same basis of $\mathfrak{M}_{0,F}$. If $\phi(N_{i})A_{i}N_{i+1} \in M_{2}(\mathbb{F}[u])$ for all $i$, then there is a morphism $\mathbb{P}_{F}^{1} \rightarrow \mathcal{R}_{V_{F},0}$ sending 0 to $x_{0}$ and 1 to $x_{1}$.

**Proof.** We put $\mathfrak{M}_{t,F} = (1 + tN) \cdot \mathfrak{M}_{0,F}$. Then we have

$$\mathfrak{M}_{t,F} \sim (\phi(1 + tN_{i})A_{i}(1 + tN_{i+1})^{-1})_{i} = (A_{i} + t(\phi(N_{i})A_{i} - A_{i}N_{i+1}) - t^{2}\phi(N_{i})A_{i}N_{i+1})_{i}.$$ 

The $\phi$-stability of $\mathfrak{M}_{1,F}$ ensures that $(\phi(N_{i})A_{i} - A_{i}N_{i+1}) \in M_{2}(\mathbb{F}[u])$. So we get $(\phi(N_{i})A_{i} - A_{i}N_{i+1}) \in M_{2}(\mathbb{F}[u])$ by $\phi(N_{i})A_{i}N_{i+1} \in M_{2}(\mathbb{F}[u])$. Then $\mathfrak{M}_{t,F}$ is $\phi$-stable, and a parameterization $t \mapsto \mathfrak{M}_{t,F}$ gives a morphism $\mathbb{P}_{F}^{1} \rightarrow \mathcal{R}_{V_{F},0}$ sending 0 to $x_{0}$ and 1 to $x_{1}$. By the properness of $\mathcal{R}_{V_{F},0}$, this morphism extends to $\mathbb{P}_{F}^{1} \rightarrow \mathcal{R}_{V_{F},0}$. □

**Lemma 2.3** ([Ima, Lemma 2.3]). Suppose $n \geq 2$ and that $x \in \mathcal{R}_{V_{F},0}(\mathbb{F})$ corresponds to an object $\mathfrak{M}_{F}$ of $(\text{Mod}/\mathcal{S}_{F})$. Fix a basis of $\mathfrak{M}_{F}$ over $k[[u]] \otimes_{\mathbb{F},F} \mathbb{F}$. Consider $U^{(i)} = (U_{j}^{(i)})_{1 \leq j \leq n} \in GL_{2}(\mathbb{F}(u)))^{n}$ such that $U_{i}^{(i)} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ and $U_{j}^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $j \neq i$. If $U^{(i)} \cdot \mathfrak{M}_{F}$ is $\phi$-stable, it corresponds to a point $x' \in \mathcal{R}_{V_{F},0}(\mathbb{F})$, and $x'$ lies on the same connected component of $\mathcal{R}_{V_{F},0}$ as $x$. 
Proof. First, $U^{(i)} \cdot \mathcal{M}_F$ corresponds to a point $x' \in \mathcal{X}_{V_F,0}^\vee(\mathbb{F})$, since it satisfies the conditions of Lemma 1.4.

Next, we consider $N^{(i)} = (N_j^{(i)})_{1 \leq j \leq n} \in M_2(\mathbb{F}(u))^n$ such that

$$N_i^{(i)} = \begin{pmatrix} 1 & -u \\ u^{-1} & -1 \end{pmatrix} \text{ and } N_j^{(i)} = 0 \text{ for all } j \neq i.$$ 

Then $U^{(i)} \cdot \mathcal{M}_F = (1 + N^{(i)}) \cdot \mathcal{M}_F$, since

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2u & u^{-1} \end{pmatrix} \begin{pmatrix} 2 & -u \\ 0 & 1 \end{pmatrix}.$$ 

We have $\phi(N_j^{(i)}) A_j N_j^{(i)} = 0$ for any $A = (A_j) \in GL_2(\mathbb{F}(u))^n$, since $n \geq 2$. So we can apply Lemma 2.2. \hfill \square

Theorem 2.4 (Kisin conjecture). \quad $\mathcal{X}_{V_F,0}^{\vee, \text{non-ord}}$ is connected.

Proof. If $n = 1$, this was proved in [Kis]. Here, we give an outline of a proof in the case $n \geq 2$ and $V_F$ is absolutely irreducible.

Let $\mathbb{F}'$ be a finite extension of $\mathbb{F}$. Suppose $x_1, x_2 \in \mathcal{X}_{V_F,0}^{\vee, \text{non-ord}}(\mathbb{F}')$ correspond to objects $\mathcal{M}_{1,\mathbb{F}'}, \mathcal{M}_{2,\mathbb{F}'}$ of $(\text{Mod}/\mathfrak{S}_{\mathbb{F}'})$ respectively. We are going to show that $x_1$ and $x_2$ are joined by $\mathbb{F}'$'s.

If $e < p - 1$, then $\mathcal{X}_{V_F,0}^{\vee, \text{non-ord}}(\mathbb{F}')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $e \geq p - 1$. Furthermore, extending $\mathbb{F}'$ and replacing $V_F$ by $V_F \otimes_{\mathbb{F}} \mathbb{F}'$, we may assume $\mathbb{F}' = \mathbb{F}' \supset \mathbb{F}_{q^2}$.

We construct an explicit point of $\mathcal{X}_{V_F,0}^{\vee, \text{non-ord}}$. By using Lemma 2.1, we can prove that there exists a basis of $M_F$ such that

$$M_F \sim \begin{pmatrix} \alpha_1 & 0 & u^{s_1} \\ & u^{t_1} & 0 \end{pmatrix}, \begin{pmatrix} \alpha_2 & u^{s_2} & 0 \\ & 0 & u^{t_2} \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_n & u^{s_n} & 0 \\ & 0 & u^{t_n} \end{pmatrix}$$

after replacing the field $\mathbb{F}$ by the quadratic extension. Here $\alpha_i \in \mathbb{F}, 0 \leq s_i, t_i \leq e, s_i + t_i = e$ and $|s_i - t_i| \leq p + 1$ for all $i$. Let $\mathcal{M}_{0,\mathbb{F}}$ be the sublattice of $M_F$ defined by this basis. We take a point $x_0 \in \mathcal{X}_{V_F,0}^{\vee, \text{non-ord}}(\mathbb{F})$ corresponding to $\mathcal{M}_F$.

We are going to prove that $x_0$ and $x_1$ lie on the same connected component. We can prove that $x_0$ and $x_2$ lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n$ such that $\mathcal{M}_{1,\mathbb{F}} = B \cdot \mathcal{M}_{0,\mathbb{F}}$, and $B_i = \begin{pmatrix} u^{-a_i} & v_i \\ 0 & u^{a_i} \end{pmatrix}$ for $a_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}(u)$.

Then we put $r_i = v_u(v_i)$. Now we have

$$\phi(B_1) \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1) & u^{s_1+a_2} & u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1} \\ u^{t_1+pa_1+a_2} & -v_2u^{t_1+pa_1} \end{pmatrix},$$

$$\phi(B_i) \begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} B_{i+1}^{-1} = \begin{pmatrix} u^{s_i-pa_1+a_{i+1}} \phi(v_i)u^{t_i-a_{i+1}} - v_{i+1}u^{s_i-pa_i} \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix}$$
for $2 \leq i \leq n$. On the right-hand sides, every component of the matrices is integral since $\mathfrak{M}_{1,F}$ is $\phi$-stable.

First, we consider the case $t_1 + pa_1 + a_2 \leq e$. In this case, if we put $\mathfrak{M}_{3,F} =
\begin{pmatrix}
    u^{-a_i} & 0 \\
    0 & u^{a_i}
\end{pmatrix}_i \cdot \mathfrak{M}_{0,F},$
then
\[
\mathfrak{M}_{3,F} \sim \left(\alpha_1 \begin{pmatrix} 0 & u^{s_1-pa_1-a_2} \\ u^{t_1+pa_1+a_2} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2-pa_2+a_3} & 0 \\ 0 & u^{t_2+pa_2-a_3} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n-pa_n+a_1} & 0 \\ 0 & u^{t_n+pa_n-a_1} \end{pmatrix}\right)
\]
and $\mathfrak{M}_{1,F} = \left(\begin{pmatrix} 1 & v_i \nu^{-a_i} \\ 0 & 1 \end{pmatrix}_i \cdot \mathfrak{M}_{3,F}$. Note that $\mathfrak{M}_{3,F}$ satisfies the conditions of Lemma 1.4, and let $x_3$ be the point of $\mathfrak{G}_{V_0}(\mathbb{F})$ corresponding to $\mathfrak{M}_{3,F}$. If we put $N_i =
\begin{pmatrix} 0 & 0 \\ v_i \nu^{-a_i} & 0 \end{pmatrix}$, then
\[
\phi(N_1) \begin{pmatrix} 0 & u^{s_1-pa_1-a_2} \\ u^{t_1+pa_1+a_2} & 0 \end{pmatrix} N_2 = \begin{pmatrix} 0 & 0 \\ \phi(v_1) v_2 u^{t_1} & 0 \end{pmatrix},
\]
\[
\phi(N_i) \begin{pmatrix} u^{s_i-pa_i+a_{i+1}} & 0 \\ 0 & u^{t_i+pa_i-a_{i+1}} \end{pmatrix} N_{i+1} = 0
\]
for $2 \leq i \leq n$. Here we have $v_u(\phi(v_1)v_2 u^{t_1}) \geq 0$, since $s_1 - pa_1 - a_2 \geq 0$ and $v_u(u^{s_1-pa_1-a_2} - \phi(v_1)v_2 u^{t_1}) \geq 0$. Hence $x_1$ and $x_3$ lie on the same connected component by Lemma 2.2.

Further, we can prove that $x_0$ and $x_3$ are joined by $\mathbb{P}^1$s by using Lemma 2.3. Hence $x_0$ and $x_1$ lie on the same connected component in the case $t_1 + pa_1 + a_2 \leq e$.

Next, we treat the case $t_1 + pa_1 + a_2 > e$. We consider the following operations:
\[
a_i \sim a_i - 1, \ v_i \sim uv_i,
\]
if it preserves the $\phi$-stability of $B \cdot \mathfrak{M}_{0,F}$. These operations replace $x_1$ by a point that lies on the same connected component as $x_1$ by Lemma 2.3. We prove that we can continue these operations until we get to the situation where $t_1 + pa_1 + a_2 \leq e$. In other words, we reduce the problem to the case $t_1 + pa_1 + a_2 \leq e$. If we can continue the operations endlessly, we get to the situation where $t_1 + pa_1 + a_2 \leq e$, since the conditions $s_i - pa_i + a_{i+1} \geq 0$ for $2 \leq i \leq n$ exclude that both $a_1$ and $a_2$ remain bounded below. Suppose that we cannot continue the operations and $t_1 + pa_1 + a_2 > e$. The condition that we cannot continue the operations
is equivalent to the following condition:

\[ s_n - pa_n + a_1 = 0 \text{ or } r_2 + t_1 + pa_1 \leq p - 1, \]
\[ pr_1 + t_1 + a_2 = 0 \text{ or } t_2 + pa_2 - a_3 \leq p - 1, \]
\[ s_{i-1} - pa_{i-1} + a_i = 0 \text{ or } t_i + pa_i - a_{i+1} \leq p - 1 \text{ for each } 3 \leq i \leq n. \]

From these conditions, we can make a contradiction by elementary arguments. This completes the proof. \(\square\)

References


