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Kisin conjecture on the moduli spaces of finite flat models

By

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Abstract

We explain a relationship between a local universal deformation ring and a moduli space of finite flat models. We also give an outline of a proof of the Kisin conjecture on the connected components of the moduli space of finite flat models.

Introduction

Let $K$ be a $p$-adic field for $p > 2$. We consider a two-dimensional continuous representation $V_{K}$ of the absolute Galois group $G_{K}$ over a finite field $\mathbb{F}$ of characteristic $p$. By a finite flat model of $V_{K}$, we mean a finite flat group scheme $G$ over $\mathcal{O}_{K}$, equipped with an action of $\mathbb{F}$, and an isomorphism $V_{K} \sim G(\overline{K})$ that respects the action of $G_{K}$ and $\mathbb{F}$. Then there exists a moduli space of finite flat models of $V_{K}$, which is projective scheme over $\mathbb{F}$, and we denoted it by $\mathcal{G}_{V_{K},0}$. Let $\mathcal{G}_{V_{K},0}^{v}$ be the closed subscheme of $\mathcal{G}_{V_{K},0}$ determined by the condition that the $p$-adic Hodge type is (1).

It is important to study the connected components of $\mathcal{G}_{V_{K},0}^{v}$, since it gives us information of a deformation ring. The ordinary component of $\mathcal{G}_{V_{K},0}^{v}$ was determined in [Kis], and Kisin conjectured that the non-ordinary component is connected. In this survey paper, we explain the relationship between $\mathcal{G}_{V_{K},0}^{v}$ and a deformation ring. We also give an outline of a proof of the Kisin conjecture. The theory of the moduli space of finite flat models was established in [Kis], and the Kisin conjecture was proved by Kisin in [Kis] if $K$ is totally ramified over $\mathbb{Q}_{p}$, by Gee in [Gee] if $V_{K}$ is the trivial representation, and by the author in [Ima] for general $K$ and $V_{K}$.
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Notation

Throughout this paper, we use the following notation. Let $p > 2$ be a prime number. For a positive number $m$, the finite field of cardinality $p^m$ is denoted by $\mathbb{F}_{p^m}$. Let $k$ be the finite extension of $\mathbb{F}_p$ of cardinality $q = p^n$. For a ring $R$, the ring of Witt vectors over $R$ with respect to $p$ is denoted by $W(R)$. We put $K_0 = W(k)[1/p]$. Let $K$ be a totally ramified extension of $K_0$ of degree $e$. The ring of integers of $K$ is denoted by $\mathcal{O}_K$, and the absolute Galois group of $K$ is denoted by $G_K$. Let $\mathbb{F}$ be a finite field of characteristic $p$. The formal power series ring of $u$ over $\mathbb{F}$ is denoted by $\mathbb{F}[[u]]$, and its quotient field is denoted by $\mathbb{F}((u))$. Let $v_u$ be the valuation of $\mathbb{F}((u))$ normalized by $v_u(u) = 1$. For a local ring $A$, the maximal ideal of $A$ is denoted by $\mathfrak{m}_A$. For a topological space $X$, the set of connected components of $X$ is denoted by $\pi_0(X)$.

§ 1. Deformation ring and moduli space of finite flat models

In this section, we explain the relationship between a deformation ring and a moduli space of finite flat models.

First, we are going to introduce a deformation ring. Let $V_{\mathbb{F}}$ be a two-dimensional continuous $G_K$-representation over $\mathbb{F}$ with a fixed ordered basis. A $G_K$-representation over a finite ring is said to be flat if and only if it is isomorphic to the generic fiber of a finite flat group scheme over $\mathcal{O}_K$ as a $G_K$-module. We assume that $V_{\mathbb{F}}$ is flat. Let $\mathfrak{A}\mathfrak{R}_{W(\mathbb{F})}$ be the category of Artin local finite $W(\mathbb{F})$-algebra $A$ whose residue field is isomorphic to $\mathbb{F}$ as a $W(\mathbb{F})$-algebra. To define a deformation, we use a notion of groupoids. For the notion of groupoids, please consult [Kis, Appendix on groupoids]. The framed flat deformation $D_{V_{\mathbb{F}}}^{\mathrm{fl}\square}$ of $V_{\mathbb{F}}$ over $\mathfrak{A}\mathfrak{R}_{W(\mathbb{F})}$ is a groupoid $D_{V_{\mathbb{F}}}^{\mathrm{fl}\square}$ over $\mathfrak{A}\mathfrak{R}_{W(\mathbb{F})}$ determined as in the followings:

- For an object $A$ in $\mathfrak{A}\mathfrak{R}_{W(\mathbb{F})}$, an object of $D_{V_{\mathbb{F}}}^{\mathrm{fl}\square}(A)$ is a triple $(V_A, \psi, \beta)$, where $V_A$ is a flat continuous $G_K$-representation that is a free $A$-module of rank 2 with an ordered basis $\beta$ over $A$, and $\psi : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$ is an $\mathbb{F}$-linear $G_K$-isomorphism sending $\beta$ to the fixed ordered basis of $V_{\mathbb{F}}$.

- A morphism $(V_A, \psi, \beta) \to (V_{A'}, \psi', \beta')$ covering a given morphism $A \to A'$ in $\mathfrak{A}\mathfrak{R}_{W(\mathbb{F})}$ is an equivalence class $[\alpha]$, where $\alpha : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$ is an $A'$-linear $G_K$-isomorphism that is compatible with the morphisms $\psi$, $\psi'$ and sending $\beta$ to $\beta'$, and two morphisms are equivalent if they differ by an element of $A'^\times$. 
Then the framed flat deformation \( D_{V_{\xi}}^{\square} \) is pro-represented by a complete local \( W(\mathbb{F}) \)-algebra \( R_{V_{\xi}}^{\square} \).

We are going to define a deformation ring with the condition that the \( p \)-adic Hodge type \( \mathbf{v} = (1) \), which is denoted by \( R_{V_{\xi}}^{\square, \mathbf{v}} \). Let \( (R_{V_{\xi}}^{\square, [1/p]})^{\mathbf{v}} \) be the quotient of \( R_{V_{\xi}}^{\square, [1/p]} \) corresponding to the connected components of \( \text{Spec } R_{V_{\xi}}^{\square, [1/p]} \) whose closed points \( \xi \) satisfy the following:

If \( V_{\xi} \) is the deformation corresponding to \( \xi \), then \( \text{Fil}^0 \text{D}_{\text{crys}}(V_{\xi}[1/p])_K \) is free of rank 1 over \( k(\xi) \otimes_{\mathbb{Q}_p} K \). Here, \( k(\xi) \) is the residue field of \( \xi \).

We note that \( V_{\xi}[1/p] \) is Barsotti-Tate representation, since we are considering a flat deformation. Then we define \( R_{V_{\xi}}^{\square, \mathbf{v}} \) by the image of \( R_{V_{\xi}}^{\square} \) in \( (R_{V_{\xi}}^{\square, [1/p]})^{\mathbf{v}} \).

The information of the connected components of \( \text{Spec } R_{V_{\xi}}^{\square, \mathbf{v}, [1/p]} \) is very important for an application to a theorem comparing a deformation ring and a Hecke ring ([Kis, Theorem 3.4.11], [Ima, Theorem 3.1]). So we want to know \( \pi_0(\text{Spec } R_{V_{\xi}}^{\square, \mathbf{v}, [1/p]}) \).

Next, we are going to explain the Kisin module and the moduli space of finite flat models of \( V_{\xi} \). By a finite flat model of \( V_{\xi} \), we mean a finite flat group scheme \( \mathcal{G} \) over \( \mathcal{O}_K \), equipped with an action of \( \mathbb{F} \), and an isomorphism \( V_{\xi} \sim \mathcal{G}(\overline{K}) \) that respects the action of \( G_K \) and \( \mathbb{F} \).

Let \( \mathcal{G} = W(k)[[u]] \), and \( \mathcal{O}_\mathcal{E} \) be the \( p \)-adic completion of \( \mathcal{G}[1/u] \). We consider the action of \( \phi \) on \( \mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F} \cong k((u)) \otimes_{\mathbb{F}_p} \mathbb{F} \) defined by \( p \)-th power on \( k((u)) \). Let \( \Phi M_{\mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F}} \) be the category of finite \( \mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F} \)-modules \( M \) with \( \phi \)-semi-linear map \( \phi : M \rightarrow M \) such that the induced linear map \( \phi^* M \rightarrow M \) is bijective.

We choose a system \( (\pi_m)_{m \geq 1} \) of elements in \( \overline{K} \) such that \( \pi_m^p = \pi \) and \( \pi_{m+1}^p = \pi_m \) for \( m \geq 1 \), and put \( K_\infty = \bigcup_{m \geq 1} K(\pi_m) \). Let \( \text{Rep}_\mathbb{F}(G_{K_\infty}) \) be the category of finite-dimensional continuous \( G_{K_\infty} \)-representations over \( \mathbb{F} \).

Then the functor

\[
T : \Phi M_{\mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F}} \rightarrow \text{Rep}_\mathbb{F}(G_{K_\infty}) : M \mapsto \left( k((u)) \otimes k((u)) M \right)^{\phi=1}
\]

is an equivalence of abelian categories. We take \( M_{\mathbb{F}} \in \Phi M_{\mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F}} \) such that \( T(M_{\mathbb{F}}) \) is isomorphic to \( V_{\xi}(-1)|_{G_{K_\infty}} \). Here \((-1)\) denotes the inverse of the Tate twist. Then \( M_{\mathbb{F}} \) is a free \( (\mathcal{O}_\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{F}) \)-module of rank 2.

We put \( \mathcal{S}_\mathbb{F} = \mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{F} \). Let \( \text{(Mod } / \mathcal{S}_\mathbb{F}) \) be the category of finite free \( \mathcal{S}_\mathbb{F} \)-modules \( \mathfrak{M} \) with \( \phi \)-semi-linear map \( \phi : \mathfrak{M} \rightarrow \mathfrak{M} \) such that the cokernel of the induced linear map \( \phi^* \mathfrak{M} \rightarrow \mathfrak{M} \) is killed by \( u^e \). An object of \( \text{(Mod } / \mathcal{S}_\mathbb{F}) \) is called a Kisin module with coefficients in \( \mathbb{F} \). Let \( \text{(}\mathbb{F} \text{-Gr}/\mathcal{O}_K\text{)} \) be the category of finite flat group schemes over \( \mathcal{O}_K \) with a structure of an \( \mathbb{F} \)-vector space.

**Theorem 1.1.** There exists an equivalence of categories

\[
\text{Gr} : \text{(Mod } / \mathcal{S}_\mathbb{F}) \rightarrow (\mathbb{F} \text{-Gr}/\mathcal{O}_K).
\]
Proof. This follows from [Br, Théorème 4.2.1.6] and [Kis, Lemma 1.2.5]. \( \square \)

**Proposition 1.2** ([Kis, Proposition 1.1.13]). For an object \( \mathcal{M} \) of \( \text{Mod}/\mathfrak{G}_{\varphi} \), there exists a canonical isomorphism

\[
T(\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}} \mathcal{M})(1) \simeq \text{Gr}(\mathcal{M})(\overline{K})|_{G_{\mathcal{K}_{\infty}}}
\]

as \( G_{\mathcal{K}_{\infty}} \)-representations. Here (1) denotes the Tate twist.

By this proposition, we see that a Kisin module which is a sublattice of \( M_{\mathcal{E}} \) corresponds to a finite flat model of \( V_{\mathcal{E}} \). Here and in the sequel, a sublattices means a finite free \( \mathfrak{G}_{\varphi} \)-submodule of \( M_{\mathcal{E}} \) that spans \( M_{\mathcal{E}} \) over \( \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}} \mathcal{F} \). In the above, we have defined a Kisin module with coefficients in \( \mathbb{F} \). More generally, we can define a Kisin module with coefficients in a \( \mathbb{Z}_{p} \)-algebra (cf. [Kis, (1.2)]). Using this general Kisin module, we can construct a moduli space of Kisin modules, which is denoted by \( \mathfrak{S}_{\mathcal{E}} \) and projective over \( \text{Spec} \mathcal{R}^{\mathfrak{F}, \square}_{\mathcal{E}} \) (cf. [Kis, (2.1)]). The closed fiber of \( \mathfrak{S}_{\mathcal{E}} \) over \( \text{Spec} \mathcal{R}^{\mathfrak{F}, \square}_{\mathcal{E}} \) is denoted by \( \mathcal{R}_{\mathcal{E}_{0}, 0} \). The scheme \( \mathcal{R}_{\mathcal{E}_{0}, 0} \) is a moduli space of finite flat models of \( V_{\mathcal{E}} \) in the sense of the following proposition.

**Proposition 1.3** ([Kis, Corollary 2.1.13]). For any finite extension \( \mathcal{F}' \) of \( \mathbb{F} \), there is a natural bijectination between the set of isomorphism classes of finite flat models of \( V_{\mathcal{E}'} = V_{\mathcal{E}} \otimes_{\mathcal{F}} \mathcal{F}' \) and \( \mathcal{R}_{\mathcal{E}_{0}, 0}(\mathcal{F}') \).

From now on, we assume \( \mathcal{F}_{q} \subset \mathbb{F} \) and fix an embedding \( k \hookrightarrow \mathbb{F} \). This assumption does not matter, since we may extend \( \mathbb{F} \) to prove the main theorem. We consider the isomorphism

\[
\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{F} = k((u)) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{F} \simeq \prod_{\sigma \in \text{Gal}(k/\mathcal{F}_{p})} \mathbb{F}((u)) \otimes \left( \sum_{i} a_{i} u^{i} \right) \otimes b \mapsto \left( \sum_{i} \sigma(a_{i})bu^{i} \right)_{\sigma}
\]

and let \( \epsilon_{\sigma} \in k((u)) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{F} \) be the primitive idempotent corresponding to \( \sigma \). Take \( \sigma_{1}, \ldots, \sigma_{n} \in \text{Gal}(k/\mathcal{F}_{p}) \) such that \( \sigma_{i+1} = \sigma_{i} \circ \phi^{-1} \). Here we regard \( \phi \) as the \( p \)-th power Frobenius, and use the convention that \( \sigma_{n+i} = \sigma_{i} \). In the sequel, we often use such conventions. Then we have \( \phi(\epsilon_{\sigma_{i}}) = \epsilon_{\sigma_{i+1}} \), and \( \phi : M_{\mathcal{E}} \rightarrow M_{\mathcal{E}} \) determines \( \phi : \epsilon_{\sigma_{i}}M_{\mathcal{E}} \rightarrow \epsilon_{\sigma_{i+1}}M_{\mathcal{E}} \). For \( (A_{i})_{1 \leq i \leq n} \in GL_{2}(\mathcal{F}((u)))^{n} \), we write

\[
M_{\mathcal{E}} \sim (A_{1}, A_{2}, \ldots, A_{n}) = (A_{i})_{i}
\]

if there is a basis \( \{e_{1}^{i}, e_{2}^{i}\} \) of \( \epsilon_{\sigma_{i}}M_{\mathcal{E}} \) over \( \mathbb{F}((u)) \) such that \( \phi \begin{pmatrix} e_{1}^{i} \\ e_{2}^{i} \end{pmatrix} = A_{i} \begin{pmatrix} e_{1}^{i+1} \\ e_{2}^{i+1} \end{pmatrix} \). We use the same notation for any sublattice \( \mathcal{M}_{\mathcal{E}} \subset M_{\mathcal{E}} \) similarly.

Finally, for any sublattice \( \mathcal{M}_{\mathcal{E}} \subset M_{\mathcal{E}} \) with a chosen basis \( \{e_{1}^{i}, e_{2}^{i}\}_{1 \leq i \leq n} \) and \( B = (B_{i})_{1 \leq i \leq n} \in GL_{2}(\mathcal{F}((u)))^{n} \), the module generated by the entries of \( \left\langle B_{i} \begin{pmatrix} e_{1}^{i} \\ e_{2}^{i} \end{pmatrix} \right\rangle \) with
the basis given by these entries is denoted by \( B \cdot \mathfrak{M}_F \). Note that \( B \cdot \mathfrak{M}_F \) depends on the choice of the basis of \( \mathfrak{M}_F \).

A closed subscheme \( \mathcal{R}_V^V \subset \mathcal{R}_V \) is defined by the condition that \( p \)-adic Hodge type \( v = (1) \) as in [Kis, (2.4.2)]. The closed fiber of \( \mathcal{R}_V^V \) over \( \text{Spec} \, R_{V_{F}}^{\square, 1} \) is denoted by \( \mathcal{R}_{V_{F},0}^V \). The rational points of \( \mathcal{R}_{V_{F},0}^V \) is characterized by the following Lemma.

**Lemma 1.4** ([Gee, Lemma 2.2]). If \( F' \) is a finite extension of \( F \), the elements of \( \mathcal{R}_{V_{F},0}(F') \) naturally correspond to free \( k[[u]] \otimes_{\mathbb{F}_p} F' \)-submodules \( \mathfrak{M}_{F'} \subset M_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} F' \) of rank 2 that satisfy the following:

1. \( \mathfrak{M}_{F'} \) is \( \phi \)-stable.
2. For some (so any) choice of \( k[[u]] \otimes_{\mathbb{F}_p} F' \)-basis for \( \mathfrak{M}_{F'} \), and for each \( \sigma \in \text{Gal}(k/F'_p) \),
   
   \[
   \phi : \epsilon_\sigma \mathfrak{M}_{F'} \to \epsilon_{\sigma \circ \phi^{-1}} \mathfrak{M}_{F'}
   \]
   
   has determinant \( \alpha u^e \) for some \( \alpha \in F'[u]^\times \).

Then there is the following relation between the deformation ring \( R_{V_{F}}^{\square, v} \) and the moduli space \( \mathcal{R}_{V_{F},0}^V \).

**Proposition 1.5.** There exists a natural bijection

\[
\pi_0(\text{Spec} \, R_{V_{F}}^{\square, v}[1/p]) \cong \pi_0(\mathcal{R}_{V_{F},0}^V).
\]

**Proof.** This follows from [Kis, Corollary 2.4.10], since \( \mathcal{R}_{V_{F},0}^{\text{loc}} = \mathcal{R}_{V_{F},0}^V \) by [Kis, Proposition 2.4.6] if the \( p \)-adic Hodge type \( v = (1) \). \( \square \)

So the problem has been reduced to study \( \pi_0(\mathcal{R}_{V_{F},0}^V) \). The connected components \( \mathcal{R}_{V_{F},0}^{\text{ord}} \subset \mathcal{R}_{V_{F},0}^V \) is defined by the points corresponding to the ordinary finite flat group schemes. We can easily determine the set \( \pi_0(\mathcal{R}_{V_{F},0}^{\text{ord}}) \) as in the following:

**Proposition 1.6** ([Kis, Proposition 2.5.15]). If \( \mathcal{R}_{V_{F},0}^{\text{ord}} \) is non-empty, then it consist of a single point, unless \( V_{F} \sim \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} \) where \( \chi_1 \) and \( \chi_2 \) are unramified characters of \( G_K \). In the latter case, we have the followings:

1. If \( \chi_1 \neq \chi_2 \), then \( \mathcal{R}_{V_{F},0}^{\text{ord}} \) consists of two points.
2. If \( \chi_1 = \chi_2 \), then \( \mathcal{R}_{V_{F},0}^{\text{ord}} \cong \mathbb{P}^1_{\mathbb{F}_p} \).

Next, we consider the non-ordinary part. We put

\[
\mathcal{R}_{V_{F},0}^{\text{non-ord}} = \mathcal{R}_{V_{F},0}^V \setminus \mathcal{R}_{V_{F},0}^{\text{ord}}.
\]
Then Kisin conjectured that $\mathcal{G}^{\text{non-ord}}_{V_{F},0}$ is connected.

§ 2. Proof of Kisin conjecture

We use the following Lemma on the structure of $M_{F}$.

**Lemma 2.1** ([Ima, Lemma 1.2]). Suppose $V_{F}$ is absolutely irreducible and $\mathbb{F}_{q^{2}} \subset \mathbb{F}$. If $\mathbb{F}'$ is the quadratic extension of $\mathbb{F}$, then

$$M_{F} \otimes_{\mathbb{F}} \mathbb{F}' \sim \left(\begin{array}{ccc}\alpha_{1} & 0 \\ \alpha_{1}u^{s} & 0 & \alpha_{2} \\ & \alpha_{2} & \alpha_{n} \end{array}\right)_{1 \leq i \leq n}$$

for some $\alpha_{i} \in (\mathbb{F}')^{\times}$ and a positive integer $s$ such that $(q + 1) \nmid s$.

In fact, we prove that $\mathcal{G}^{\text{non-ord}}_{V_{F},0}$ is rationally connected. To join two points by $\mathbb{P}^{1}_{\mathbb{F}}$, we use the following two Lemmas.

**Lemma 2.2** ([Gee, Lemma 2.4]). Suppose $x_{0}, x_{1} \in \mathcal{G}^{\text{V}}_{V_{F},0}(\mathbb{F})$ correspond to objects $\mathfrak{M}_{0,F}, \mathfrak{M}_{1,F}$ of $(\text{Mod} / \mathfrak{S}_{F})$ respectively. Let $N = (N_{i})_{1 \leq i \leq n}$ be a nilpotent element of $M_{2}(\mathbb{F}((u)))^{n}$ such that $\mathfrak{M}_{1,F} = (1 + N) \cdot \mathfrak{M}_{0,F}$ with a basis of $\mathfrak{M}_{0,F}$, and $A = (A_{i})_{1 \leq i \leq n}$ be an element of $\text{GL}_{2}(\mathbb{F}((u)))^{n}$ such that $\mathfrak{M}_{0,F} \sim A$ for the same basis of $\mathfrak{M}_{0,F}$. If $\phi(N_{i})A_{i}N_{i+1} \in M_{2}(\mathbb{F}[[u]])$ for all $i$, then there is a morphism $\mathbb{P}^{1}_{\mathbb{F}} \rightarrow \mathcal{G}^{\text{V}}_{V_{F},0}$ sending 0 to $x_{0}$ and 1 to $x_{1}$.

**Proof.** We put $\mathfrak{M}_{t,F} = (1 + tN) \cdot \mathfrak{M}_{0,F}$. Then we have

$$\mathfrak{M}_{t,F} \sim (\phi(1 + tN_{i})A_{i}(1 + tN_{i+1})^{-1})_{i} = A_{i} + t(\phi(N_{i})A_{i} - A_{i}N_{i+1}) - t^{2}(\phi(N_{i})A_{i}N_{i+1})_{i}.$$

The $\phi$-stability of $\mathfrak{M}_{1,F}$ ensures that $(\phi(N_{i})A_{i} - A_{i}N_{i+1} - \phi(N_{i})A_{i}N_{i+1}) \in M_{2}(\mathbb{F}[[u]])$. So we get $(\phi(N_{i})A_{i} - A_{i}N_{i+1}) \in M_{2}(\mathbb{F}[[u]])$ by $\phi(N_{i})A_{i}N_{i+1} \in M_{2}(\mathbb{F}[[u]])$. Then $\mathfrak{M}_{t,F}$ is $\phi$-stable, and a parameterization $t \mapsto \mathfrak{M}_{t,F}$ gives a morphism $\mathbb{A}^{1}_{\mathbb{F}} \rightarrow \mathcal{G}^{\text{V}}_{V_{F},0}$ sending 0 to $x_{0}$ and 1 to $x_{1}$. By the properness of $\mathcal{G}^{\text{V}}_{V_{F},0}$, this morphism extends to $\mathbb{P}^{1}_{\mathbb{F}} \rightarrow \mathcal{G}^{\text{V}}_{V_{F},0}$. \hfill $\square$

**Lemma 2.3** ([Ima, Lemma 2.3]). Suppose $n \geq 2$ and that $x \in \mathcal{G}^{\text{V}}_{V_{F},0}(\mathbb{F})$ corresponds to a object $\mathfrak{M}_{F}$ of $(\text{Mod} / \mathfrak{S}_{F})$. Fix a basis of $\mathfrak{M}_{F}$ over $k[[u]] \otimes_{\mathbb{F}, F}$. Consider $U^{(i)} = (U_{j}^{(i)})_{1 \leq j \leq n} \in \text{GL}_{2}(\mathbb{F}((u)))^{n}$ such that $U_{i}^{(i)} = \left(\begin{array}{cc}u & 0 \\ 0 & u^{-1}\end{array}\right)$ and $U_{j}^{(i)} = \left(\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right)$ for all $j \neq i$. If $U^{(i)} \cdot \mathfrak{M}_{F}$ is $\phi$-stable, it corresponds to a point $x' \in \mathcal{G}^{\text{V}}_{V_{F},0}(\mathbb{F})$, and $x'$ lies on the same connected component of $\mathcal{G}^{\text{V}}_{V_{F},0}$ as $x$. 


Proof. First, $U^{(i)} \cdot \mathfrak{M}_F$ corresponds to a point $x' \in \mathcal{Y}_{V_{F},0}(\mathbb{F})$, since it satisfies the conditions of Lemma 1.4.

Next, we consider $N^{(i)} = (N_j^{(i)})_{1 \leq j \leq n} \in M_2(\mathbb{F}(u))^n$ such that

$$N_i^{(i)} = \begin{pmatrix} 1 & -u \\ u^{-1} & 1 \end{pmatrix} \quad \text{and} \quad N_j^{(i)} = 0 \quad \text{for all} \quad j \neq i.$$  

Then $U^{(i)} \cdot \mathfrak{M}_F = (1 + N^{(i)}) \cdot \mathfrak{M}_F$, since

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2u & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & -u \\ u^{-1} & 0 \end{pmatrix}.$$  

We have $\phi(N_j^{(i)}) A_j N_j^{(i)} = 0$ for any $A = (A_j) \in GL_2(\mathbb{F}(u))^n$, since $n \geq 2$. So we can apply Lemma 2.2.  \qed

**Theorem 2.4 (Kisin conjecture).** $\mathcal{Y}_{V_{F},0}^{\text{non-ord}}$ is connected.

Proof. If $n = 1$, this was proved in [Kis]. Here, we give an outline of a proof in the case $n \geq 2$ and $V_F$ is absolutely irreducible.

Let $F'$ be a finite extension of $F$. Suppose $x_1, x_2 \in \mathcal{Y}_{V_{F},0}^{\text{non-ord}}(F')$ correspond to objects $\mathfrak{M}_{1,F'}, \mathfrak{M}_{2,F'}$ of $(\text{Mod}/\mathfrak{S}_{F'})$ respectively. We are going to show that $x_1$ and $x_2$ are joined by $\mathbb{P}_{F}^{1}$'s.

If $e < p - 1$, then $\mathcal{Y}_{V_{F},0}(F')$ is one point by [Ray, Theorem 3.3.3]. So we may assume $e \geq p - 1$. Furthermore, extending $F'$ and replacing $V_F$ by $V_F \otimes_F F'$, we may assume $F = F' \supset F_q$.

We construct some explicit point of $\mathcal{Y}_{V_{F},0}$. By using Lemma 2.1, we can prove that there exists a basis of $M_F$ such that

$$M_F \sim (\alpha_1 \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix}, \alpha_2 \begin{pmatrix} u^{s_2} & 0 \\ 0 & u^{t_2} \end{pmatrix}, \ldots, \alpha_n \begin{pmatrix} u^{s_n} & 0 \\ 0 & u^{t_n} \end{pmatrix})$$

after replacing the field $F$ by the quadratic extension. Here $\alpha_i \in F$, $0 \leq s_i, t_i \leq e$, $s_i + t_i = e$ and $|s_i - t_i| \leq p + 1$ for all $i$. Let $\mathfrak{M}_{0,F}$ be the sublattice of $M_F$ defined by this basis. We take a point $x_0 \in \mathcal{Y}_{V_{F},0}(F)$ corresponding to $\mathfrak{M}_{0,F}$.

We are going to prove that $x_0$ and $x_1$ lie on the same connected component. We can prove that $x_0$ and $x_2$ lie on the same connected component by the same argument.

By the Iwasawa decomposition and the determinant conditions, we can take $B = (B_i)_{1 \leq i \leq n} \in GL_2(\mathbb{F}(u))^n$ such that $\mathfrak{M}_{1,F} = B \cdot \mathfrak{M}_{0,F}$ and $B_1 = \begin{pmatrix} u^{-a_1} & v_i \\ 0 & u^{a_i} \end{pmatrix}$ for $a_i \in \mathbb{Z}$ and $v_i \in \mathbb{F}(u)$. Then we put $r_i = v_u(v_i)$. Now we have

$$\phi(B_1) \begin{pmatrix} 0 & u^{s_1} \\ u^{t_1} & 0 \end{pmatrix} B_2^{-1} = \begin{pmatrix} \phi(v_1) u^{t_1+a_2} u^{s_1-pa_1-a_2} - \phi(v_1)v_2u^{t_1} \\ u^{t_1+pa_1+a_2} - v_2u^{t_1+pa_1} \end{pmatrix},$$

$$\phi(B_i) \begin{pmatrix} u^{s_i} & 0 \\ 0 & u^{t_i} \end{pmatrix} B_{i+1}^{-1} = \begin{pmatrix} u^{s_i-pa_i+a_i+1} \phi(v_i)u^{t_i-a_i+1} - v_{i+1}u^{s_i-pa_i} \\ 0 \\ u^{t_i+pa_i-a_i+1} \end{pmatrix}.$$
for \(2 \leq i \leq n\). On the right-hand sides, every component of the matrices is integral since \(\mathfrak{M}_{1,F}\) is \(\phi\)-stable.

First, we consider the case \(t_1 + pa_1 + a_2 \leq e\). In this case, if we put \(\mathfrak{M}_{3,F} = (u^{-a_i}) \cdot \mathfrak{M}_{0,F} \cdot i\), then

\[
\mathfrak{M}_{3,F} \sim \left(\begin{array}{cc}
\alpha_1 \left(0 u^{s_1-pa_1-a_2} \right) & \alpha_2 \left(0 u^{s_2-pa_2+a_3} \right) \\
\nu^{t_1+pa_1+a_2} & u^{t_2+pa_2-a_3}
\end{array}\right),
\]

\[
\alpha_3 \left(0 u^{s_3-pa_3+a_4} \right), \ldots , \alpha_n \left(0 u^{s_n-pa_n+a_1} \right)
\]

and \(\mathfrak{M}_{1,F} = (v_i u^{-a_i}) \cdot \mathfrak{M}_{3,F}\). Note that \(\mathfrak{M}_{3,F}\) satisfies the conditions of Lemma 1.4, and let \(x_3\) be the point of \(\mathcal{G}\mathcal{R}_{V,F,0}(F)\) corresponding to \(\mathfrak{M}_{3,F}\). If we put \(N_i = (0 v_i u^{-a_i})\), then

\[
\phi(N_1) \left(0 u^{s_1-pa_1-a_2} \right) N_2 = \left(0 \phi(v_1) v_2 u^{t_1} \right),
\]

\[
\phi(N_i) \left(u^{s_i-pa_i+a_{i+1}} \right) N_{i+1} = 0
\]

for \(2 \leq i \leq n\). Here we have \(v_u(\phi(v_1)v_2 u^{t_1}) \geq 0\), since \(s_1 - pa_1 - a_2 \geq 0\) and \(v_u(u^{s_1-pa_1-a_2} - \phi(v_1)v_2 u^{t_1}) \geq 0\). Hence \(x_1\) and \(x_3\) lie on the same connected component.

Further, we can prove that \(x_0\) and \(x_3\) are joined by \(\mathbb{P}_{F}^1\)'s by using Lemma 2.3. Hence \(x_0\) and \(x_1\) lie on the same connected component in the case \(t_1 + pa_1 + a_2 \leq e\).

Next, we treat the case \(t_1 + pa_1 + a_2 > e\). We consider the following operations:

\[
a_i \sim a_i - 1, \ v_i \sim uv_i, \text{ if it preserves the } \phi\text{-stability of } B \cdot \mathfrak{M}_{0,F}.
\]

These operations replace \(x_1\) by a point that lies on the same connected component as \(x_1\) by Lemma 2.3. We prove that we can continue these operations until we get to the situation where \(t_1 + pa_1 + a_2 \leq e\). In other words, we reduce the problem to the case \(t_1 + pa_1 + a_2 \leq e\). If we can continue the operations endlessly, we get to the situation where \(t_1 + pa_1 + a_2 \leq e\), since the conditions \(s_i - pa_i + a_{i+1} \geq 0\) for \(2 \leq i \leq n\) exclude that both \(a_1\) and \(a_2\) remain bounded below. Suppose that we cannot continue the operations and \(t_1 + pa_1 + a_2 > e\). The condition that we cannot continue the operations
Kisin conjecture on the moduli spaces of finite flat models is equivalent to the following condition:

\[
\begin{align*}
    s_n - pa_n + a_1 &= 0 \text{ or } r_2 + t_1 + pa_1 \leq p - 1, \\
p r_1 + t_1 + a_2 &= 0 \text{ or } t_2 + pa_2 - a_3 \leq p - 1, \\
    s_{i-1} - pa_{i-1} + a_i &= 0 \text{ or } t_i + pa_i - a_{i+1} \leq p - 1 \text{ for each } 3 \leq i \leq n.
\end{align*}
\]

From these conditions, we can make a contradiction by elementary arguments. This completes the proof. \(\square\)

References


