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Lifting Galois representations over arbitrary number fields: A resume

By

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Let $\mathbf{k}$ be a finite field of characteristic $p > 5$. Let K be a number field of finite degree over $\mathbb{Q}$ and G_K its absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. Let $\bar{\rho}\colon G_K \to \mathrm{GL}_2(\mathbf{k})$ be a continuous representation and let $W(\mathbf{k})$ be the ring of Witt vectors of $\mathbf{k}$. We consider the following question:

Question 0.1. Is there a continuous representation $\rho\colon G_K \to \mathrm{GL}_2(W(\mathbf{k}))$ satisfying $\bar{\rho} = \rho \mod p$?

This question has been motivated by a conjecture of Serre ([S]) saying that all odd absolutely irreducible continuous representations $\bar{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbf{k})$ are modular. This implies the existence of a lift to characteristic zero. This conjecture was proved by Khare and Wintenberger in [KW1, KW2].

In [K], Khare proved the existence of a lift to $W(\mathbf{k})$ for any $\bar{\rho}\colon G_K \to \mathrm{GL}_2(\mathbf{k})$ which is reducible. Thus we may assume that $\bar{\rho}$ is irreducible.

For a place v of K, let K_v be the completion of K at v, and let G_v be its absolute Galois group $\mathrm{Gal}(\bar{K_v}/K_v)$. Let $\mathrm{Ad}^0\bar{\rho}$ be the $\mathbf{k}$-vector space of all trace zero two-by-two matrices over $\mathbf{k}$ on which G_K acts by conjugation. Our main result is the following:

Theorem 0.2. *Assume that* $H^2(G_v, \mathrm{Ad}^0\bar{\rho}) = 0$ *for each place* $v \mid p$. *Then* $\bar{\rho}$ *lifts to a continuous representation* $\rho\colon G_K \to \mathrm{GL}_2(W(\mathbf{k}))$ *which is unramified outside a finite set of places of* K.

For $K = \mathbb{Q}$, Ramakrishna proved under very general conditions on $\bar{\rho}$ that there exist lifts to $W(\mathbf{k})$ in [R1, R2]. Gee ([G]) and Manoharmayum ([M]) proved, independently, that there exist lifts to $W(\mathbf{k})$ for K satisfying $[K(\mu_p) : K] > 2$, where μ_p is the group of p-th roots of unity.

Our method used in the proof of the Theorem is essentially that of Ramakrishna [R1, R2], but we follow the more axiomatic treatment presented in [Ta]. We denote by S

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a finite set of places of K containing the places above p, the infinite places and the places at which $\bar{\rho}$ is ramified. Let K_S denote the maximal algebraic extension of K unramified outside S and put $G_{K,S} = \mathrm{Gal}(K_S/K)$. Thus $\bar{\rho}$ factors through $G_{K,S}$. The existence of a lifting of $\bar{\rho}$ follows from the triviality of a certain class of $H^2(G_{K,S}, \mathrm{Ad}^0\bar{\rho})$. Since it is difficult to calculate it directly, we reduce the calculations of global obstructions to those of local obstructions, which are much better understood. By Taylor [Ta], it boils down to showing the vanishing of a certain Selmer group. We can prove the triviality of this group by extending S by a suitably chosen finite set Q. With the choice of Q of Ramakrishna and Gee, they could not prove the triviality of the Selmer group for an arbitrary number field. Our choice of Q is different from Ramakrishna's and Gee's. For more details, see our preprint [To].

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