INVERSE SCATTERING PROBLEMS FOR DISPERSIVE EQUATIONS WITH A NON-LOCAL NONLINEARITY

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ABSTRACT. We discuss inverse scattering problems for nonlinear dispersive equations. The nonlinearity of the equations is a non-local term with unknown interaction potentials. We determine the unknown interaction potentials and reconstruct the nonlinearity by the knowledge of the scattering states.

1. INTRODUCTION

In this note, we review the author’s works [21, 22, 23] of inverse scattering problems for dispersive equations with a non-local nonlinearity. In particular, we treat the nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u = V_0 u + F(u) \quad \text{in } \mathbb{R} \times \mathbb{R}^n. \]  

(NLS)

Here, \( u = u(t, x) \) is a complex-valued unknown function, \( i = \sqrt{-1}, \partial_t = \partial/\partial t, \Delta \) is the Laplacian in \( \mathbb{R}^n \), \( F(u) \) is a cubic convolution defined by

\[ F(u) = \lambda(V_1 * |u|^2)u, \]

(1.1)

\( V_0(x), V_1(x) \) and \( \lambda(x) \) are real-valued measurable functions and \( * \) is the convolution in \( \mathbb{R}^n \). The equation (NLS), which is initially studied by Chadam–Glassey [7], is approximately derived by the multi body Schrödinger equation (for the detailed derivation, see, e.g., Watanabe [31]). If \( \lambda \equiv 1 \), then the nonlinearity \( F(u) \) means a non-local interaction whose interaction potential is \( V_1 \).

1.1. Definition of the scattering operator. The inverse scattering problem for the equation (NLS) means to identify unknown \( V_0(x), V_1(x) \) and \( \lambda(x) \) by using the knowledge of the scattering operator \( S \) for (NLS). To treat inverse scattering problems, we first consider the direct scattering problem. For this purpose, we

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give some definitions. Let $X$ be a Hilbert space. For $\delta > 0$, we denote a closed ball \( \{ \phi \in X; \| \phi \|_X \leq \delta \} \) by \( B(\delta; X) \). Let \( H \) be a linear operator defined by \( H = -\Delta + V_0 \). We now assume that \( H \) is self-adjoint on \( X \). We remark that for any \( \phi \in X \), \( e^{-itH} \phi \) solves the linear Schrödinger equation \((\text{NLS})\) with \( F(u) \equiv 0 \) whose initial data is \( \phi \). The scattering operator for \((\text{NLS})\)

\[ S : B(\delta; X) \ni \phi_- \mapsto \phi_+ \in X \]

is defined if the following property holds for some function space \( Z \subset C(\mathbb{R}; X) \) and for some positive \( \delta \):

For \( \phi_- \in B(\delta; X) \), there exist a function \( u \in Z \) and a data \( \phi_+ \in X \) such that \( u(t) \) is a time-global solution to the integral form of \((\text{NLS})\)

\[ u(t) = e^{-itH} \phi_- - i \int_{-\infty}^{t} e^{-i(t-\tau)H} F(u(\tau))d\tau \] \hspace{1cm} (1.2)

and \( u(t) \) satisfies

\[ \lim_{t \to \pm \infty} \| u(t) - e^{it\Delta} \phi_\pm \|_X = 0, \]

respectively.

In Section 2 below, we see that if \( V_0, V_1 \) and \( \lambda \) satisfy suitable conditions, then the scattering operator for \((\text{NLS})\) is well-defined in some sense.

1.2. Known results. Before reviewing the works [21, 22, 23], we introduce some known results of inverse scattering problems for nonlinear Schrödinger equations. Strauss [25] considered the Schrödinger equation with a power nonlinearity

\[ i\partial_t u + \Delta u = V(x)|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \]

Suppose that \( p \) is an integer satisfying some conditions and \( V(x) \) is real-valued continuous and bounded, whose derivatives up to order \( l > 3n/4 \) are bounded. Then the scattering operator \( S \) is well-defined. It was shown that \( V(x) \) is recovered from the scattering operator by the following way: For \( s \in \mathbb{R} \), let \( H^s(\mathbb{R}^n) \) be the Sobolev space \((1 - \Delta)^{-s/2}L^2(\mathbb{R}^n)\). For any \( \phi \in H^1(\mathbb{R}^n) \cap L^{1+1/p}(\mathbb{R}^n) \), we have

\[ V(x_0) = \lim_{\alpha \to 0} \frac{\alpha^{-(n+2)}I[\phi_{\alpha,x_0}]}{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |e^{it\Delta} \phi(x)|^{p+1}dxdt}, \] \hspace{1cm} (1.3)

where \( \phi_{\alpha,x_0}(x) = \phi(\alpha^{-1}(x-x_0)), \alpha > 0, x, x_0 \in \mathbb{R}^n \),

\[ I[\phi] = \lim_{\varepsilon \to 0} \frac{i}{\varepsilon^p} \langle (S - id)(\varepsilon \phi), \phi \rangle, \]

\( id \) is the identity mapping and \( \langle \cdot, \cdot \rangle \) denotes the \( L^2(\mathbb{R}^n) \) inner product. The above limit \( I[\phi] \) is called the small amplitude limit. Later, Weder [32, 34, 35, 36, 38] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover,
a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [25].

We next review known results of the inverse scattering problem for (NLS). In particular, we focus on the reconstruction theorem (for the uniqueness theorem, see Watanabe [28] and Sasaki–Watanabe [24]). Remark that even if $V_0 \equiv 0$ and $\lambda \equiv 1$, we cannot determine $V_1$ by the reconstruction formula (1.3). Indeed, we formally conclude

$$\lim_{\alpha \to 0} \alpha^{-(n+2)} I[\phi, x_0] = \int_{\mathbb{R}} \int_{\mathbb{R}^n} V_1(0) |e^{it\Delta} \phi(x)|^2 \ast |e^{it\Delta} \phi(x)|^2 \, dx \, dt$$

whose right hand side ordinarily diverges. Therefore, in order to determine $V_1$, we have to make some modified version of (1.3) or to find other methods. Watanabe [27] studied the inverse scattering problem for (NLS) with $V_1(x) = |x|^{-\sigma}$. It was proved that if $\sigma$ is a given number, then we can reconstruct $V_0$ and $\lambda$ by the knowledge of the scattering operator. Furthermore, Watanabe [29, 30] determined unknown $\sigma$ and $\lambda$ of the term $F(u) = \lambda(|\cdot|^{-\sigma} \ast |u|^2)u$ if $V_0 \equiv 0$ and $\lambda$ is a non-zero constant. It was shown by [24] that if $V_0 \equiv 0$, $\lambda \equiv 1$ and $V_1$ is integrable on $\mathbb{R}^n$, then $\hat{V}_1(0)$ can be uniquely determined. Here, $\hat{V}_1$ denotes the Fourier transform of $V_1$. For the time dependent Hartree–Fock system, Watanabe [31] gave the reconstruction formula for determining interaction potentials.

1.3. Outline of the author’s works. As we mention before, this note is devoted to introducing the author’s works concerned with inverse scattering problems for dispersive equations with a non-local term. Let us focus on the equation (NLS) again. An aim of the works [21, 22, 23] is to make reconstruct formulas for determining $V_0$, $V_1$ and $\lambda$ in the case where we can not determine only by using the above known results. In the rest of this section, we review an outline of such the author’s works (the detailed statements are given in Sections 4–6 below).

In [21], it is shown that if $V_0 \equiv 0$, $\lambda \equiv 1$ and unknown $V_1(x)$ is expressed by some radial and smooth function $v(r)$ with $r = |x|$ vanishing at infinity, then we can determine the exact value of $\partial_r^{M} \hat{v}(0)$ for any $M = 0, 1, 2, \cdots$. It is also proved by [21] that if $V_0 \equiv 0$, $\lambda \equiv 1$ and $V_1(x)$ is equal to $\sum_{k=1}^{N} \lambda_k |x|^{-\sigma_k}$ with unknown

$$1 < N < \infty, \quad \{\lambda_k\}_{k=1}^{N} \subset \mathbb{R}, \quad 2 \leq \sigma_N < \sigma_{N-1} < \cdots < \sigma_2 < \sigma_1 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5, \end{cases}$$

then we can completely reconstruct $V_1$.

From [22], assuming that $V_0 \equiv 0$, $\lambda \in (C^1 \cap W^1_\infty)(\mathbb{R}^n)$ and $V_1(x)$ equals to $|x|^{-\sigma}$ with unknown $\sigma \in [2, 4] \cap [2, n)$, we can determine $\lambda$ and $\sigma$. Remark that the result [22] is applicable to the case where both $\lambda$ and $\sigma$ are unknown.

In [23], it is proved that if we suppose that $n = 3$, $\lambda \equiv 1$ and that $V_j$, $j = 0, 1, \ldots$ are expressed by the Yukawa potentials $V_j(x) = Q_j \exp(-\mu_j |x|)/|x|$ with unknown numbers $Q_j \in \mathbb{R}$ and $\mu_j > 0$, respectively, then we can determine $V_0$ and $V_1$. 

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Furthermore, in [22, 23], the author studied inverse scattering problems for some relativistic equations with the nonlinearity \( F(u) \) (for the detail, see Section 7).

The key of the proof of such author's results is to calculate the parameterized small amplitude limit. For example, assuming that \( V_1 \) is expressed by the radial function \( v \) and by defining one-parameter function \( \varphi(\alpha) \) by

\[
\varphi(\alpha) = \alpha^{n+2} \lim_{\epsilon \to 0} \frac{i}{\epsilon^{3}} \langle (S-id)(\epsilon \phi^{\alpha}), \phi \rangle, \quad \phi^{\alpha}(x) := \phi(\alpha x),
\]

we have for any \( M \geq 0 \),

\[
\partial^M \varphi(0) = C_0 \lim_{\beta \to 0} \frac{d^M \varphi(\beta)}{d\alpha^M},
\]

where the constant \( C_0 \) is given only by \( M \) and \( \phi \).

Introducing the contents of the rest of this note, we close this section. In Section 2, it is shown that if \( V_0, V_1 \) and \( \lambda \) satisfy suitable conditions, then the scattering operator \( S \) is well-defined on some 0-neighborhood of the Sobolev space \( H^s(\mathbb{R}^n) \). The proof of such existence results is based on Stricharz estimates for linear Schrödinger equations. In Section 3, using properties proved in Section 2, we show the small amplitude limit of the scattering operator for (NLS). In Sections 4, 5 and 6, we mention detailed statements concerned with the works [21], [22] and [23], respectively. In Section 7, inverse scattering problems for relativistic equations with \( F(u) \) are studied.

2. DIRECT SCATTERING PROBLEM

In this section, we introduce existence results of scattering operators for (NLS). We first define some notation. For \( 1 \leq p \leq \infty \), \( W^1_p(\mathbb{R}^n) \) is the Sobolev space defined by \( W^1_p(\mathbb{R}^n) = \{ \phi \in L^p(\mathbb{R}^n); \nabla \phi \in L^p(\mathbb{R}^n) \} \). Put \( X_1 = H^1(\mathbb{R}^n), X_2 = L^2(\mathbb{R}^3), Z_1 = C(\mathbb{R}; X_1) \cap L^{3/2}(\mathbb{R}; W^{1,6n/(3n-4)}(\mathbb{R}^n)) \) and \( Z_2 = C(\mathbb{R}; X_2) \cap L^{3/2}(\mathbb{R}; L^{18/5}(\mathbb{R}^3)) \). We again define \( H \) by \( H = -\Delta + V_0 \).

**Theorem 2.1.** Let \( n \geq 3 \). We assume that \( V_0 \equiv 0, \lambda \in (C^1 \cap W^1_\infty(\mathbb{R}^n)) \) and

\[
|V_1(x)| \leq C (|x|^{-\sigma_1} + |x|^{-\sigma_2}), \quad 2 \leq \sigma_1, \sigma_2 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5 \end{cases}
\]

for some positive constant \( C \) independent of \( x \). Then there exists a positive number \( \delta \) such that for any \( \phi_- \in B(\delta; X_1) \), the integral equation (1.2) with \( H = -\Delta \) has a unique solution \( u \in Z_1 \) satisfying

\[
\|u\|_{Z_1} \leq C\|\phi_-\|_{X_1}, \quad (2.1)
\]

\[
\|u - e^{it\Delta} \phi_-\|_{Z_1} \leq C\|\phi_-\|^3_{X_1}, \quad (2.2)
\]
and
\[
\lim_{t \to -\infty} \| u(t) - e^{it\Delta}\phi_- \|_{X_1} = 0.
\]
Furthermore, there exists a data \( \phi_+ \in X_1 \) such that
\[
\lim_{t \to +\infty} \| u(t) - e^{it\Delta}\phi_+ \|_{X_1} = 0.
\]
Therefore, the scattering operator \( S_1 : B(\delta; X_1) \ni \phi_- \mapsto \phi_+ \in X_1 \) is well-defined.

**Theorem 2.2.** ([23]) We assume that \( \lambda \equiv 1, \ V_0 \in X_2 \),
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V_0(x)V_0(y)|}{|x-y|^2} \, dy \, dx < (4\pi)^2, \quad \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V_0(y)|}{|x-y|} \, dy < 4\pi
\]
and \( V_1 \in L^{3/2}(\mathbb{R}^3) \). Then \( H \) is absolutely continuous self-adjoint and there exists a positive number \( \delta \) such that for any \( \phi_- \in B(\delta; X_2) \), the integral equation (1.2) has a unique solution \( u \in Z_2 \) satisfying
\[
\| u \|_{Z_2} \leq C \| \phi_- \|_{X_2},
\]
\[
\| u - e^{-itH}\phi_- \|_{Z_2} \leq C \| \phi_- \|_{X_2}^3
\]
and
\[
\lim_{t \to -\infty} \| u(t) - e^{it\Delta}\phi_- \|_{X_2} = 0.
\]
Furthermore, there exists a data \( \phi_+ \in X_2 \) such that
\[
\lim_{t \to +\infty} \| u(t) - e^{it\Delta}\phi_+ \|_{X_2} = 0.
\]
Therefore, the scattering operator \( S_2 : B(\delta; X_2) \ni \phi_- \mapsto \phi_+ \in X_2 \) is well-defined.

Since the first theorem directly follows from Mochizuki [12], we prove only the second one. For this purpose, we introduce the following Strichartz type estimate:

**Proposition 2.3.** Suppose that \( V_0 \) satisfies (2.3). Then \( H \) is an absolutely continuous self-adjoint operator and we have for any \( \phi \in X_2 \) and any \( f \in (C \cap L^\infty)(\mathbb{R}; X_2) \),
\[
\| e^{-itH}\phi \|_{Z_2} \leq C \| \phi \|_{X_2}
\]
and
\[
\left\| \int_{-\infty}^{t} e^{i(t-\tau)H} f(\tau) \, d\tau \right\|_{Z_2} \leq C \| f \|_{L^1(\mathbb{R}; X_2)}.
\]

**Proof.** By Rodionianski–Schlag [20], it follows from the condition (2.3) that for any \( t \neq 0 \),
\[
\| e^{itH}\phi \|_{L^\infty(\mathbb{R}^3)} \leq C |t|^{-3/2} \| \phi \|_{L^1(\mathbb{R}^3)}.
\]
On the other hand, by the standard argument, we immediately see that $H$ is an absolutely continuous self-adjoint operator (for the detail, see, e.g., [17, 19]). Therefore, for any $t \in \mathbb{R}$, we have

$$\|e^{itH}\phi\|_{X_2} = \|\phi\|_{X_2}. \quad (2.11)$$

From the Riesz-Thorin interpolation theorem, we obtain a decay estimate

$$\|e^{itH}\phi\|_{L^p(\mathbb{R}^3)} \leq C|t|^{-3/2 + 3/p}\|\phi\|_{L^{p'}(\mathbb{R}^3)}. \quad (2.12)$$

Here, $2 \leq p \leq \infty$ and $p'$ is the Hölder conjugate defined by $1/p' = 1 - 1/p$. Furthermore, the $T^*T$ argument given by Ginibre-Velo [9] implies

$$\|e^{-itH}\phi\|_{L^{q_{1}}(\mathbb{R};L^{r_{1}}(\mathbb{R}^3))} \leq C\|\phi\|_{X_2},$$

$$\left\|\int_{-\infty}^{t}e^{i(t-\tau)H}f(\tau)d\tau\right\|_{L^{q_{2}}(\mathbb{R};L^{r_{2}}(\mathbb{R}^3))} \leq C\|f\|_{L^{q_{3}'}(\mathbb{R};L^{r_{3}'}(\mathbb{R}^3))}$$

for any $2 \leq q_j \leq \infty$, $2 \leq r_j \leq 6$ ($j = 1, 2, 3$) with $2/q_j = 3(1/2 - 1/r_j)$. Therefore, we have (2.8) and (2.9).

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 2.3, we have already seen that $H$ is an absolutely continuous self-adjoint operator. Put $\|\phi_-\|_{X_2} \leq \delta$ and

$$G[u](t) = e^{-itH}\phi_- - i\int_{-\infty}^{t}e^{-i(t-\tau)H}F(u(\tau))d\tau.$$ 

From (2.8) and (2.9), we obtain

$$\|G[u](t)\|_{Z_2} \leq C(\|\phi_-\|_{X_2} + \|F(u)\|_{L^1(\mathbb{R};X_2)})$$

We see that

$$\|F(u(t))\|_{X_2} \leq \|V_1 \ast |u(t)|^2\|_{L^{2/3}(\mathbb{R}^3)} \|u(t)\|_{L^{18/5}(\mathbb{R}^3)} \leq C\|u(t)\|_{L^{2/3}(\mathbb{R}^3)}^2 \|u(t)\|_{L^{18/5}(\mathbb{R}^3)}^2,$$

where we have used the Hölder-Young inequality in the second inequality. We hence see that

$$\|G[u](t)\|_{Z_2} \leq C(\|\phi_-\|_{X_2} + \|u(t)\|_{Z_2}^3).$$

Similarly, we obtain

$$\|G[u](t) - G[\tilde{u}](t)\|_{Z_2} \leq C\|u - \tilde{u}\|_{Z_2}(\|u\|_{Z_2} + \|\tilde{u}\|_{Z_2})^2.$$

It is clear that $G[u] \in C(\mathbb{R}; X_2)$. Therefore, we see that there uniquely exists $u \in Z_2$ such that $G[u] = u$ for sufficiently small $\delta > 0$. We can immediately find that the fixed point $u$ solves the equation (1.2). Furthermore, we obtain (2.4) and (2.5). It follows from $u \in L^3(\mathbb{R}; L^{18/5}(\mathbb{R}^3))$ that

$$\|u(t) - e^{-itH}\phi_-\|_{X_2} \leq \int_{-\infty}^{t}\|F(u(t))\|_{X_2}dt \leq \int_{-\infty}^{t}\|u(t)\|_{L^{18/5}(\mathbb{R}^3)}^3dt \rightarrow 0 \text{ as } t \rightarrow \infty.$$
Furthermore, if we put
\[ S_F(\phi_-) = \phi_- + \frac{1}{i} \int_\mathbb{R} e^{itH} F(u(t)) dt, \] (2.13)
we have
\[ \lim_{t \to \pm \infty} \| u(t) - e^{-itH} S_F(\phi_-) \|_{X_2} = 0. \]

In order to prove the existence of the scattering operator for (1.2), we denote wave operators \( \Omega_- \) and \( \Omega_+ \) by
\[ \Omega_\pm = s - \lim_{t \to \pm \infty} e^{-it\Delta} e^{-itH}, \]
respectively. Since \( \Omega_\pm \) are unitary operators in \( X_2 \), there exists some \( \delta_0 > 0 \) such that if \( \phi_- \in B(\delta_0, X_2) \), then there uniquely exists \( v \in Z_2 \) satisfying \( G[v] = v \) and
\[ \lim_{t \to -\infty} \| v(t) - e^{-itH} \Omega_-(\phi_-) \|_{X_2} = 0. \] (2.14)
Moreover, we have
\[ \lim_{t \to \infty} \| v(t) - e^{-itH} S_F \Omega_-(\phi_-) \|_{X_2} = 0. \] (2.15)
By (2.11), we see from (2.14) and (2.15) that
\[ \| v(t) - e^{it\Delta} \phi_- \|_{X_2} \leq \| v(t) - e^{-itH} \Omega_-(\phi_-) \|_{X_2} + \| e^{-itH} \Omega_-(\phi_-) - e^{it\Delta} \phi_- \|_{X_2} \]
\[ = \| v(t) - e^{-itH} \Omega_-(\phi_-) \|_{X_2} + \| \Omega_- (\phi_-) - e^{itH} e^{it\Delta} \phi_- \|_{X_2} \]
\[ \to 0 \quad \text{as } t \to -\infty \]
and
\[ \| v(t) - e^{it\Delta} \Omega_+^{-1} S_F \Omega_-(\phi_-) \|_{X_2} \leq \| v(t) - e^{-itH} S_F \Omega_-(\phi_-) \|_{X_2} + \| e^{-itH} S_F \Omega_-(\phi_-) - e^{it\Delta} \Omega_+^{-1} S_F \Omega_-(\phi_-) \|_{X_2} \]
\[ = \| v(t) - e^{-itH} S_F \Omega_-(\phi_-) \|_{X_2} + \| S_F \Omega_-(\phi_-) - e^{itH} e^{it\Delta} \Omega_+^{-1} S_F \Omega_-(\phi_-) \|_{X_2} \]
\[ \to 0 \quad \text{as } t \to +\infty, \]
respectively. Thus, (2.6) and (2.7) hold if we define \( S_2 = \Omega_+^{-1} S_F \Omega_- \). This completes the proof. \( \square \)

3. Small amplitude limit

In this section, we focus on the small amplitude limit of the scattering operator for \( S_1 \) derived by Theorems 2.1. The following proposition will be used in Sections 4 and 5 below:
Proposition 3.1. Let $n \geq 3$. We assume that $V_0 \equiv 0$, $\lambda \in (C^1 \cap W^{1}_{\infty})(\mathbb{R}^n)$ and

$$|V_1(x)| \leq C \left(|x|^{-\sigma_1} + |x|^{-\sigma_2}\right), \quad 2 \leq \sigma_1, \sigma_2 \begin{cases} < n & \text{if } n = 3, 4, \\ \leq 4 & \text{if } n \geq 5 \end{cases}$$

for some positive constant $C$ independent of $x$. Then for any $\phi \in X_1$, we have

$$\lim_{\epsilon \to 0} \frac{i}{\epsilon^3} \langle (S_1 - \text{id})(\epsilon \phi), \phi \rangle = \int_{\mathbb{R}} \langle F(e^{it\Delta} \phi), e^{it\Delta} \phi \rangle \, dt. \quad (3.1)$$

Remark 3.1. From Theorem 2.1, $S_1$ is well-defined on $B(\delta; X_1)$ with sufficiently small positive $\delta$. Thus, the left hand side of limit (3.1) makes sense.

Proof of Proposition 3.1. Let $\epsilon$ be a sufficiently small positive number. From the proof of Theorem 2.1, we see that $S_1$ is expressed by

$$S_1(\epsilon \phi) = \epsilon \varphi - i \int_{\mathbb{R}} e^{-it\Delta} F(u_{\epsilon}(t)) \, dt,$$

where $u_{\epsilon}(t)$ is the solution to (1.2) with $\phi_- = \epsilon \phi$. Therefore, we obtain

$$(S_1 - \text{id})(\epsilon \phi) = -i \int_{\mathbb{R}} e^{-it\Delta} F(u_{\epsilon}(t)) \, dt.$$ 

Set $F(v_1, v_2, v_3) = (V_1 * v_1 \overline{v_2})v_3$. Then it follows that

$$\|F(v_1, v_2, v_3)\|_{Z_1} \leq C \prod_{j=1}^{3} \|v_j\|_{Z_1} \quad (3.2)$$

and we see that

$$i \langle (S_1 - \text{id})(\epsilon \phi), \phi \rangle = \int_{\mathbb{R}} \langle F(u_{\epsilon}(t)), e^{it\Delta} \phi \rangle \, dt$$

$$= \epsilon^3 \int_{\mathbb{R}} \langle F(e^{it\Delta} \phi), e^{it\Delta} \phi \rangle \, dt$$

$$+ \int_{\mathbb{R}} \langle F\left(u_{\epsilon}(t) - e^{it\Delta}(\epsilon \phi), u_{\epsilon}(t), u_{\epsilon}(t)\right), e^{it\Delta} \phi \rangle \, dt$$

$$+ \int_{\mathbb{R}} \langle F\left(e^{it\Delta}(\epsilon \phi), u_{\epsilon}(t) - e^{it\Delta}(\epsilon \phi), u_{\epsilon}(t)\right), e^{it\Delta} \phi \rangle \, dt$$

$$+ \int_{\mathbb{R}} \langle F\left(e^{it\Delta}(\epsilon \phi), e^{it\Delta}(\epsilon \phi), u_{\epsilon}(t) - e^{it\Delta}(\epsilon \phi)\right), e^{it\Delta} \phi \rangle \, dt$$

$$=: (I) + (II)_1 + (II)_2 + (II)_3.$$

By (3.2), (2.1) and (2.2), we obtain for any $j = 1, 2, 3$,

$$|\langle (II)_j \rangle| \leq C \|u_{\epsilon}(t) - e^{it\Delta}(\epsilon \phi)\|_{Z_1} \|u_{\epsilon}(t)\|_{Z_1}^2 \|e^{it\Delta} \phi\|_{X_1} \leq C \|\epsilon \phi\|_{X_1}^5 \|\phi\|_{X_1}.$$

Therefore, we have (3.1).
4. NLS (Case I)

In this section, we review the work [21] concerned with the inverse scattering problem for (NLS). Let $n \geq 3$ and $S(\mathbb{R}^n)$ be the Schwartz class. Suppose that, $V_0 \equiv 0$ and $\lambda \equiv 1$. Moreover, assume that $V_1$ satisfies either the following two conditions:

(A) $V_1$ is equal to $\sum_{k=1}^{N} \lambda_k |x|^{-\sigma_k}$ with

$$1 < N < \infty, \quad \{\lambda_k\}_{k=1}^{N} \subset \mathbb{R}, \quad 2 \leq \sigma_N < \cdots < \sigma_2 < \sigma_1 \left\{ \begin{array}{l} < n \quad \text{if } n = 3, 4, \\ \leq 4 \quad \text{if } n \geq 5. \end{array} \right.$$

(B) $V_1$ is radial function belonging to $S(\mathbb{R}^3)$.

From Theorem 2.1, we see that the scattering operator $S_1 : B(\delta; X_1) \ni \phi_\perp \mapsto \phi_\perp \in X_1$ for (NLS) is well-defined. The goal of [21] is to identify unknown $V_1$ under the assumption (A) or (B). In order to state main results of [21], we list some notation. For $\phi \in X_1$, let $\mathcal{I}[\phi]$ be the small amplitude limit

$$\mathcal{I}[\phi] = \lim_{\epsilon \to 0} \frac{i}{\epsilon^3} \langle (S_1 - id)(\epsilon \phi), \phi \rangle. \quad (4.1)$$

For $j = 1, 2, \cdots, N$, we put

$$I_j = \int_{\mathbb{R}} \langle |\cdot|^{-\sigma_j} * |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \rangle dt.$$

For $\phi \in X_1$, $\alpha > 0$ and $x \in \mathbb{R}^n$, put $\phi_\alpha(x) = \phi(\alpha^{-1}x)$. We define $\psi(\alpha)$ and $\varphi(\alpha)$ by

$$\psi(\alpha) = \alpha^{2n-2} \mathcal{I}[\phi_\alpha]$$

and

$$\varphi(\alpha) = \alpha^{n+2} \mathcal{I}[\phi_{\alpha^{-1}}],$$

respectively. The Fourier transform on $S(\mathbb{R}^n)$ is given by

$$(\mathcal{F}\phi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \phi(x) dx$$

for all $\phi \in S(\mathbb{R}^n)$. It is well-known that the Fourier transform on $S'(\mathbb{R}^n)$ is defined uniquely. We also define the inverse Fourier transform on $S'(\mathbb{R}^n)$ by

$$(\mathcal{F}^{-1}\phi)(\xi) = (\mathcal{F}\tilde{\phi})(\xi),$$

where we set $\tilde{\phi}(x) = \phi(-x)$. We are ready to state the main results.

**Theorem 4.1.** Assume that $V_0 \equiv 0$ and $\lambda \equiv 1$ and that $V_1$ satisfies the condition (A). Put $\phi \in X_1 \setminus \{0\}$. Then we can determine $N$, $\{\lambda_j\}_{j=1}^{N}$ and $\{\sigma_j\}_{j=1}^{N}$ by the following steps:

(Step I.) We have

$$\sigma_1 = - \lim_{\alpha \to 0} \ln \frac{\psi(e\alpha)}{|\psi(\alpha)| + 1}. \quad (4.2)$$
Furthermore, using the given \( \sigma_1 \), we obtain
\[
\lambda_1 = I_1^{-1} \lim_{\alpha \to 0} (\alpha^{\sigma_1} \psi(\alpha)).
\] (4.3)

(Step II-1.) Suppose that we have already determined \( \{\lambda_j\}_{j=1}^k \) and \( \{\sigma_j\}_{j=1}^k \). Put
\[
\psi^k(\alpha) = \psi(\alpha) - \sum_{j=1}^k \alpha^{-\sigma_j} \lambda_j I_j.
\]

If \( \lim_{\alpha \to 0} |\psi^k(\alpha)| = 0 \), then \( N = k \).

(Step II-2.) Suppose that we have already determined \( \{\lambda_j\}_{j=1}^k \) and \( \{\sigma_j\}_{j=1}^k \). If
\[
\lim_{\alpha \to 0} |\psi^k(\alpha)| = \infty,
\]
then we have \( N > k \) and
\[
\sigma_{k+1} = -\lim_{\alpha \to 0} \ln \frac{|\psi^{k}(e\alpha)|}{|\psi^{k}(\alpha)| + 1}.
\]
Furthermore, using the given \( \sigma_{k+1} \), we obtain
\[
\lambda_{k+1} = I_{k+1}^{-1} \lim_{\alpha \to 0} (\alpha^{\sigma_{k+1}} \psi^k(\alpha)).
\]

Remark 4.1. The value of \( \lim_{\alpha \to 0} |\psi^k(\alpha)| \) must be either 0 or \( \infty \).

**Theorem 4.2.** Let \( n \geq 3 \). Assume that \( V_0 \equiv 0 \) and \( \lambda \equiv 1 \) and that \( V_1 \) satisfies the condition (B). If we define \( V(\rho), \rho \geq 0, \) by \( V(|\xi|) = FV_1(|\xi|) \) and we set \( d = d/d\rho \), then we have for any \( M = 0, 1, \ldots \) and any \( \phi \in H^{M/2+n/4-1/2}(\mathbb{R}^n) \setminus \{0\},
\]
\[
(d^M V)(0) = \frac{\lim_{\alpha \to 0} (d^M \varphi)(\alpha)}{\int_{\mathbb{R}} \| \mathcal{F}(e^{\imath \Delta} \phi) \|^2_{L^2(\mathbb{R}^n)} dt}.
\] (4.4)

Remark 4.2. Suppose that \( V_1 \) satisfy (B) and that we have already seen that \( \mathcal{F}V_1 \) is analytic. Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\} \). If we obtain \( \limsup_{M \to \infty} |(d^M V)(0)|^{1/M} = 0 \), then it follows from the Maclaurin theorem that
\[
V(\rho) = \sum_{M=0}^{\infty} \frac{(d^M V)(0)}{M!} \rho^M.
\]
Therefore, for any \( x \in \mathbb{R}^n \), we can reconstruct \( V_1 \) by
\[
V_1(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \sum_{M=0}^{\infty} \frac{(d^M V)(0)}{M!} e^{ix \cdot \xi} |\xi|^M d\xi.
\]

**Proof of Theorem 4.1.** Suppose that \( V_1 \) satisfies (A) and \( 0 < \alpha < 1 \). Since
\[
(V_1)_{\alpha^{-1}}(x) = \sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j |x|^{-\sigma_j},
\]
we obtain
\[ \psi(\alpha) = \int_{\mathbb{R}} \left( \sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j \cdot |\cdot|^{-\sigma_j} \right) * |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \right) dt. \]
Hence we see that
\[ \psi(\alpha) = \sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j \int_{\mathbb{R}} \left( |\cdot|^{-\sigma_j} * |e^{it\Delta} \phi|^2, |e^{it\Delta} \phi|^2 \right) dt = \sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j I_j. \] 
(4.5)
Therefore, we obtain
\[ \frac{|\psi(e^\alpha)|}{|\psi(\alpha)| + 1} = \frac{\sum_{j=1}^{N} e^{-\sigma_j} \alpha^{-\sigma_j} \lambda_j I_j}{\sum_{j=1}^{N} \alpha^{-\sigma_j} \lambda_j I_j + 1} = \frac{\sum_{j=1}^{N} e^{-\sigma_j} \alpha^{\sigma_1-\sigma_j} \lambda_j I_j}{\sum_{j=1}^{N} \alpha^{\sigma_1-\sigma_j} \lambda_j I_j + \alpha^{\sigma_1}}. \]
By \( \sigma_1 > \sigma_j \) for any \( j > 1 \), it follows from the boundedness of \( I_j \) that
\[ \lim_{\alpha \to 0} \frac{|\psi(e^\alpha)|}{|\psi(\alpha)| + 1} = e^{-\sigma_1}. \]
Thus, (4.2) holds. From (4.5), we immediately obtain (4.3).

Since
\[ \psi^k(\alpha) = \sum_{j=k+1}^{N} \alpha^{-\sigma_j} \lambda_j I_j, \]
we can determine \( N, \{\lambda_j\}_{j=1}^{N} \) and \( \{\sigma_j\}_{j=1}^{N} \) by using the steps (Step II-1) and (Step II-2). This completes the proof. \( \square \)

For the proof of Theorem 4.2, we prepare the following lemma:

**Lemma 4.3.** Let \( n \geq 2 \). For any \( s > 0 \) and any \( \phi \in H^{s+n/4-1/2}(\mathbb{R}^n) \), we have
\[ \int_{\mathbb{R}} \left\| |\cdot|^s \mathcal{F} \left( |e^{it\Delta} \phi|^2 \right) \right\|^2_{L^2(\mathbb{R}^n)} dt \leq C \|\phi\|^4_{H^{s+n/4-1/2}(\mathbb{R}^n)}. \] 
(4.6)

**Proof of Theorem 4.2.** Suppose that \( V_1 \) satisfies the condition (B) and \( 0 < \alpha < 1 \). By
\[ \mathcal{F}(V_1)_{\alpha} = \alpha^{-n}(\mathcal{F}V_1)_{\alpha^{-1}}, \]
it follows from (3.1) and the Plancherel theorem that
\[ \varphi(\alpha) = \alpha^{n+2} T[\varphi_{\alpha^{-1}}] = \int_{\mathbb{R}} \left\langle \left( (\mathcal{F}V_1)_{\alpha^{-1}} \mathcal{F} \left( |e^{it\Delta} \phi|^2 \right) \right), \mathcal{F} \left( |e^{it\Delta} \phi|^2 \right) \right\rangle dt. \]
By (4.6) and \( (\mathcal{F}V_1)_{\alpha^{-1}}(\xi) = \mathcal{V}(\alpha|\xi|) \), we obtain
\[ \frac{d^M}{d\alpha^M} \varphi(\alpha) = \int_{\mathbb{R}} \left\langle |\cdot|^M (d^M \mathcal{V})(\alpha|\cdot|), \mathcal{F} \left( |e^{it\Delta} \phi|^2 \right), \mathcal{F} \left( |e^{it\Delta} \phi|^2 \right) \right\rangle dt. \]
for any $M = 0, 1, \cdots$. The Lebesgue dominated theorem implies that

$$\lim_{\alpha \to 0} (d^M \varphi)(\alpha) = (d^M \mathcal{V})(0) \int_{\mathbb{R}} \left\langle |\cdot|^M \mathcal{F}(|e^{it\Delta} \phi|^2), \mathcal{F}(|e^{it\Delta} \phi|^2) \right\rangle dt.$$ 

Thus, we have (4.4). This completes the proof. \[\square\]

5. NLS (Case II)

In this section, we review a part of the work [22] which treats the inverse scattering problem (NLS). Suppose that $n \geq 3$ and that $V_0$, $V_1$ and $\lambda$ satisfy the following condition:

(C) $V_0$ and $\lambda$ satisfy $V_0 \equiv 0$ and $\lambda \in (C^1 \cap W_{\infty}^1)(\mathbb{R}^n)$, respectively. Furthermore, $\lambda(0) \neq 0$ and $V_1(x)$ is expressed by $V_1(x) = |x|^{-\sigma}$ for some unknown $\sigma \in [2, 4] \cap [2, n]$.

From Theorem 2.1, we see that the scattering operator $S_1 : B(\delta; X_1) \ni \phi_{-} \mapsto \phi_{-} \in X_1$ for (NLS) is well-defined. The goal of [22] for (NLS) is to identify unknown $\lambda$ and $\sigma$. In order to state the main result of [22] for (NLS), we list some notation. Let $e$ be the base of the natural logarithm. For $\phi \in X_1$, the small amplitude limit $\mathcal{I}[\phi]$ is defined by 4.1. For $\phi \in X_1$, $\alpha > 0$ and $x, x_0 \in \mathbb{R}^n$, we again put $\phi_{\alpha, x_0}(x) = \phi(\alpha^{-1}(x - x_0))$. In particular, we denote $\phi_{\alpha, 0}$ by $\phi_\alpha$. We are ready to state the main result.

**Theorem 5.1.** Let $n \geq 3$. Assume that $V_0$, $V_1$ and $\lambda$ satisfy the condition (C). Then for any $\phi \in X_1$, the unknown $\sigma$ is given by

$$\sigma = 2n + 2 - \lim_{\alpha \to 0} \ln \frac{[\mathcal{I}[\phi_{\alpha}]]}{[\mathcal{I}[\phi_\alpha]] + \alpha^{2n+2}}. \quad (5.1)$$

Furthermore, using the given $\sigma$, we have for any $x_0 \in \mathbb{R}^n$,

$$\lambda(x_0) = \frac{\lim_{\alpha \to 0} \alpha^{-(2n+2-\sigma)} [\mathcal{I}[\phi_{\alpha, x_0}] - \mathcal{I}[\phi_{\alpha}]}{\int |y|^{-\sigma} |e^{it\Delta} \phi(x-y)|^2 |e^{it\Delta} \phi(x)|^2 d(t, x, y)}. \quad (5.2)$$

**Proof.** By (3.1), it follows that

$$\mathcal{I}[\phi] = \int_{\mathbb{R}^{1+n+n}(t, x, y)} \lambda(x) |y|^{-\sigma} |e^{it\Delta} \phi(x-y)|^2 |e^{it\Delta} \phi(x)|^2 d(t, x, y). \quad (5.3)$$

Having in mind that

$$e^{it\Delta} \phi_{\alpha} = (e^{it\alpha^{-2} \Delta} \phi)_{\alpha},$$

we see that

$$\mathcal{I}[\phi_\alpha] = \alpha^{2n+2-\sigma} \int_{\mathbb{R}^{1+n+n}(t, x, y)} \lambda(\alpha x) R(\phi)(t, x, y) d(t, x, y),$$

where

$$R(\phi)(t, x, y) = |y|^{-\sigma} |e^{it\Delta} \phi(x-y)|^2 |e^{it\Delta} \phi(x)|^2.$$
We see that $R(\phi)$ is integrable. Hence it follows that
\[
\int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(\beta x)R(\phi)(t, x, y)d(t, x, y)
\rightarrow \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(0)R(\phi)(t, x, y)d(t, x, y) \neq 0 \text{ as } \beta \to 0
\]
if $\phi \neq 0$. Hence we obtain
\[
\frac{|\mathcal{I}[\phi_{e\alpha}]|}{|\mathcal{I}[\phi_\alpha]| + \alpha^{2n+2}} = \frac{e^{2n+2-\sigma}}{|\int \lambda(\alpha x)R(\phi)d(t, x, y)| + \alpha^{\sigma}} \to e^{2n+2-\sigma} \text{ as } \alpha \to 0.
\]
Thus, we have (5.1). We immediately see that the remainder (5.2) follows from (3.1). This completes the proof. \[\square\]

6. NLS (Case III)

In this section, we review a part of the work [23] which is concerned with the inverse scattering problem for (NLS). Suppose that $n = 3$ and that $V_0$, $V_1$ and $\lambda$ satisfy the following condition:

**(D).** $\lambda$ satisfy $\lambda \equiv 1$. Furthermore, $V_j$, $j = 0, 1$, are expressed by the Yukawa potentials
\[
V_j(x) = \frac{Q_j \exp(-\mu_j |x|)}{|x|}, \quad x \in \mathbb{R}^3.
\]

Here, $Q_j \in \mathbb{R}$ and $\mu_j > 0$ are unknown parameters with
\[
|Q_0| < \mu_0.
\]

Since the condition (6.2) implies (2.3), we see from Theorem 2.2 that the scattering operator $S_2 : B(\delta; X_2) \ni \phi_+ \mapsto \phi_- \in X_2$ for (NLS) is well-defined. The goal of [23] is to identify unknown $Q_j$ and $\mu_j$. In order to state the main result of [23] for (NLS), we list some notation and proposition. We set $r(x) = |x|$. we again put wave operators $\Omega_\pm = s - \lim_{t \to \pm \infty} e^{-it\Delta} e^{-itH}$ and remark that $\Omega_\pm$ are bijective functions from $X_2$ into itself. We define a mapping $S_{V_0}$ by
\[
S_{V_0} = \Omega_+^{-1}\Omega_- : X_2 \to X_2.
\]
The operator $S_{V_0}$ is the scattering operator for the linear Schrödinger equation ((NLS) with $F(u) \equiv 0$). Using the method of [27, 32, 34, 35, 36], we see that $S_{V_0}$ can be determined from the knowledge of $S_1$.

**Theorem 6.1.** ([27, 32, 34, 35, 36]) Assume that (6.1) and (6.2) hold. For any $\phi \in X_2 \setminus \{0\}$, we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} S_2(\varepsilon \phi) = S_{V_0}(\phi) \text{ in } X_2.
\]
Once we have determined $S_{V_{0}}$, we can reconstruct $V_{0}$, $e^{\pm itH}$, $\Omega_{\pm}$, $\Omega_{\pm}^{-1}$ by Enss–Weder [6]. The main result of [23] concerned with (NLS) is to determine the remainder $Q_{1}$ and $\mu_{1}$.

**Theorem 6.2.** Assume that (6.1) and (6.2) hold and that $\phi \in C_{c}^{\infty}(\mathbb{R}^{3}\setminus\{0\})$ satisfies $\phi \neq 0$ and $(\Delta^{2} + 1)^{-1}\phi \in C_{c}^{\infty}(\mathbb{R}^{3}\setminus\{0\})$.

(i) We have

$$
\frac{Q_{1}}{\mu_{1}^{2}} = \frac{\lim_{\alpha \to -\infty} i\alpha^{4} \langle (\Omega_{+}S_{2}\Omega_{-}^{-1} - id)(\alpha^{-3}\phi_{\alpha}), \phi_{\alpha} \rangle}{4\pi \|e^{it\Delta}\phi\|_{L^{4}(\mathbb{R},L^{4}(\mathbb{R}^{3}))}^{4}}. \tag{6.4}
$$

(ii) Suppose that $Q_{1} \neq 0$. Put

$$
b = \left| \frac{Q_{1}}{\mu_{1}^{2}} \right|^{1/2}, \quad H(b) = -\Delta - bQ_{0} \exp(-b\mu_{0}r),
$$

$$
\Psi_{1}(\alpha) = \int_{\mathbb{R}} \frac{\alpha \exp(-\sqrt{|\alpha|}r)}{r} * |e^{-itH(b)}\phi|^{2}, |e^{-itH(b)}\phi|^{2} \, dt, \quad \alpha \in \mathbb{R},
$$

$$
a = \lim_{\epsilon \to 0} i\epsilon^{-3}b^{-7} \langle (\Omega_{+}S_{2}\Omega_{-}^{-1} - id)(\epsilon\phi_{\epsilon}), \phi_{\epsilon} \rangle,
$$

$$
m_{0} = \max \{ m \in \mathbb{Z}_{\geq 0}; \Psi_{1}(m) \leq |a| \},
$$

$$
q_{1} = \max \left\{ q = 0, 1; \Psi_{1} \left( m_{0} + \frac{q}{2} \right) \leq |a| \right\},
$$

$$
q_{j+1} = \max \left\{ q = 0, 1; \Psi_{1} \left( m_{0} + \sum_{k=1}^{j} \frac{q_{k}}{2^{k}} + \frac{q}{2^{j+1}} \right) \leq |a| \right\}, \quad j = 1, 2, \cdots.
$$

Then we have

$$
Q_{1} = \text{sign} \left( \frac{Q_{1}}{\mu_{1}^{2}} \right) \left( m_{0} + \sum_{j=1}^{\infty} \frac{q_{j}}{2^{j}} \right). \tag{6.5}
$$

6.1. **Proof of Theorem 6.2.** In the rest of this section, we give the proof of Theorem 6.2. For this purpose, we first list some notation and propositions. For $\alpha > 0$ and $y \in \mathbb{R}^{3}$, let $H(\alpha, y)$ be a linear operator on $X_{2}$ defined by

$$
D(H(\alpha, y)) = D(-\Delta), \quad H(\alpha, y) = -\Delta + \alpha^{2}V_{0}(\alpha x - y).
$$

Furthermore, we put

$$
H(\alpha) = H(\alpha, 0).
$$

The operator $H(\alpha, y)$ becomes a self-adjoint operator on $X_{2}$. We denote norms $\| \cdot \|_{X_{2}}$, $\| \cdot \|_{L^{q}(\mathbb{R}^{3})}$ ($q \in [1, \infty]$) and $\| \cdot \|_{L^{4}(\mathbb{R},L^{4}(\mathbb{R}^{3}))}$ by $\| \cdot \|$, $\| \cdot \|_{q}$ and $\| \cdot \|_{(4,4)}$, respectively. For $E \subset \mathbb{R}^{3}$, let $C_{c}^{\infty}(E)$ be the set of all smooth functions with compact support in $E$.

**Proposition 6.3.** Let $\alpha > 0$ and $\phi \in X_{2}$. Then we have the following identities:
(i) \[
(e^{-itH}\phi_\alpha)(x) = \left(e^{-i\alpha^{-2}tH(\alpha)}\phi\right)(\alpha^{-1}x). \tag{6.6}
\]

(ii) \[
(e^{-itH(\alpha)}\phi)(x - \alpha^{-1}y) = \left(e^{-i\alpha^{-2}tH(\alpha,y)}\tau_\alpha^{-1}y\phi\right)(x), \tag{6.7}
\]

where \(\tau_z\phi(x) := \phi(x - z), z \in \mathbb{R}^3\).

**Proposition 6.4.** If \(\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})\), then we have
\[
\lim_{\alpha \to \infty} \| (e^{-itH(\alpha,y)} - e^{it\Delta}) ((-\Delta + i)(-\Delta - i)\phi) \| = 0. \tag{6.8}
\]

**Proposition 6.5.** For any \(\alpha > 0\) and for any \(\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})\), we have
\[
\| e^{-itH(\alpha)}\phi \|_6 \leq C(\phi)(1 + |t|)^{-1/2}. \tag{6.9}
\]

Here, the constant \(C(\phi)\) is independent of \(\alpha\) and \(t\).

We omit the proof of Proposition 6.3. We now show Proposition 6.4. The proof of is essentially similar to that of Theorem VIII.20 in [16].

**Proof of Proposition 6.4.** We give the proof dividing two steps.

(Step I.) For \(\alpha \in \mathbb{R}\) and \(m \in \mathbb{Z}_{\geq 0}\), let \(f(\alpha) = e^{-i\alpha}\) and \(g_m(\alpha) = e^{-\alpha^2/m^2}\). Let \(\psi \in X_2\) and \(\epsilon > 0\). Then there exists some \(m_0 \in \mathbb{Z}_{\geq 0}\) such that
\[
\| g_m(-\Delta)\psi - \psi \| \leq \epsilon \tag{6.10}
\]

for any \(m \geq m_0\). Henceforth, we assume that \(m = m_0\). Since \(g_m\) is a continuous function vanishing at infinity, it follows from the Stone-Weierstrass theorem (see, e.g., [16]) that there exists some two-parameter polynomial \(P(\alpha, \beta)\) such that
\[
\sup_{\alpha \in \mathbb{R}} |g_m(\alpha) - P((\alpha + i)^{-1}, (\alpha - i)^{-1})| \leq \epsilon.
\]

Therefore, for any self-adjoint operator \(A\), we have
\[
\| g_m(A) - P((A + i)^{-1}, (A - i)^{-1}) \| \leq \epsilon. \tag{6.11}
\]

Thus, we obtain
\[
\|[g_m(H(\alpha, y)) - g_m(-\Delta)]\psi\| \\
\leq \|[g_m(H(\alpha, y)) - P((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1})]\| \|\psi\| \\
+ \|[g_m(-\Delta) - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1})]\| \|\psi\| \\
+ \|[P((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1}) \psi - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \psi]\| \\
\leq 2\epsilon\|\psi\| + \|[P((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1}) \psi - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \psi]\|. \tag{6.12}
\]
Here, we have used the property (6.11) in the last inequality. Since it follows that
\[ P(\alpha_1, \beta_1) - P(\alpha_2, \beta_2) = \sum_{k,l} C_{k,l} \left\{ (\alpha_1 - \alpha_2) \tilde{P}^{k-1}(\alpha_1, \alpha_2) \beta_1^l + (\beta_1 - \beta_2) \tilde{P}^{l-1}(\beta_1, \beta_2) \alpha_2^k \right\} \]
for some two-parameter polynomial \( \tilde{P}^k, k = -1, 0, 1, 2, \cdots \), we obtain
\[
\| P ((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1}) \psi - P ((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \psi \|
\leq C \left\{ \| ((H(\alpha, y) + i)^{-1} - (-\Delta + i)^{-1}) \psi \| + \| ((H(\alpha, y) - i)^{-1} - (-\Delta - i)^{-1}) \psi \| \right\}.
\]
(6.13)
Henceforth, we put \( \psi = (\Delta + i)(\Delta - i)\phi, \phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\}) \). Then we see that
\[
\| ((H(\alpha, y) \pm i)^{-1} - (-\Delta \pm i)^{-1}) \psi \|
\leq \| ((H(\alpha, y) \pm i)^{-1} - (-\Delta \pm i)^{-1}) (\Delta \mp i) \phi \|
\leq \alpha^2 V_0(\cdot \frac{-\alpha^{-1} y}{\eta})(-\Delta \mp i) \phi \|
\leq \alpha^2 V_0(\cdot \frac{-\alpha^{-1} y}{\eta})(\pm i) \phi \|
\]
We set \( \eta = \text{dist(supp}\phi, 0) \). If \( \alpha > 0 \) is sufficiently large, then we have
\[
\| ((H(\alpha, y) \pm i)^{-1} - (-\Delta \pm i)^{-1}) \psi \| \leq C \| Q_0 \| \alpha^2 \frac{\exp(-\mu_0|\alpha \eta - y|)}{|\alpha \eta - y|} \|(-\Delta \mp i) \phi \|.
\]
(6.14)
We see from (6.12)–(6.14) that
\[
\lim_{\alpha \to \infty} \| g_m(H(\alpha, y)) \psi - g_m(-\Delta) \psi \| = 0.
\]
(6.15)
(Step II.) Since \( fg_m \) is a continuous function vanishing at infinity, it follows from the same argument of the proof of (6.12) that
\[
\| (f(H(\alpha, y)) - f(-\Delta)) \psi \|
\leq \| (fg_m)(H(\alpha, y)) \psi - f(H(\alpha, y)) \psi \| + \| (fg_m)(-\Delta) \psi - f(-\Delta) \psi \|
+ \| (fg_m)(H(\alpha, y)) \psi - (fg_m)(-\Delta) \psi \|
\leq \| f(H(\alpha, y)) \| \| g_m(H(\alpha, y)) \psi - \psi \| + \| f(-\Delta) \| \| g_m(-\Delta) \psi - \psi \| + 2\varepsilon \| \psi \|
+ \| \hat{P} ((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1}) \psi - \hat{P} ((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \psi \|
\leq \| g_m(H(\alpha, y)) \psi - g_m(-\Delta) \psi \| + 2\| g_m(-\Delta) \psi - \psi \| + 2\varepsilon \| \psi \|
+ \| \hat{P} ((H(\alpha, y) + i)^{-1}, (H(\alpha, y) - i)^{-1}) \psi - \hat{P} ((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \psi \|
\]
(6.16)
for some two-parameter polynomial \( \hat{P} \). Here, we have used the unitarity of \( f(A) \) in the last inequality. By (6.10) and (6.13)–(6.15), (6.8) holds. □

We next show Proposition 6.5.
Proof of Proposition 6.5. Let $C_b$ be the best constant for the embedding $\dot{W}^1_2(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$:

$$C_b = \sup_{\psi \in H^1 \setminus \{0\}} \frac{\|\psi\|_6}{\|
abla \psi\|}.$$ 

Then we obtain

$$\|e^{-itH(\alpha)}\phi\|^2_6 \leq C_b^2 \|\nabla e^{-itH(\alpha)}\phi\|^2_6$$
$$\leq C_b^2 \left\langle \nabla e^{-itH(\alpha)}\phi, \nabla e^{-itH(\alpha)}\phi \right\rangle$$
$$\leq C_b^2 \left\langle (-\Delta)e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle$$
$$\leq C_b^2 \left\langle H(\alpha)e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle - C_b^2 \left\langle \alpha^2(\alpha^{-1})^2e^{-itH(\alpha)}\phi, e^{-itH(\alpha)}\phi \right\rangle$$
$$\leq C_b^2 \|\alpha^2(\alpha^{-1})\|_{3/2} \|e^{-itH(\alpha)}\phi\|^2_6$$
$$\leq C_b^2 \|\nabla \phi\|^2 + C_b^2 \left\langle \alpha^2(\alpha^{-1})\phi, \phi \right\rangle + C_b^2 \|\alpha(\alpha^{-1})\|_{3/2} \|e^{-itH(\alpha)}\phi\|^2_6.$$

It follows that

$$\left| \left\langle \alpha^2(\alpha^{-1})\phi, \phi \right\rangle \right| \leq |Q_0|\alpha \frac{\exp(-\mu_0 \eta \alpha)}{\eta} \|\phi\|^2 \leq \frac{|Q_0|\|\phi\|^2}{e^{\mu_0 \eta^2}},$$

where $\eta = \text{dist}(\text{supp}\phi, 0)$. By Aubin and Talenti [1, 26], the best constant $C_b$ is explicitly given by $C_b = 3^{-1}2^{1/3}\pi^{-4/3}$. Therefore, we have

$$C_b^2 \left\| \frac{e^{-r}}{r} \right\|_{3/2} < 1. \quad (6.17)$$

We hence see that

$$\|e^{-itH(\alpha)}\phi\|^2_6 \leq \frac{\|\nabla \phi\|^2 + \frac{|Q_0|}{e\mu_0 \eta^2} \|\phi\|^2}{C_b^{-2} - \frac{|Q_0|}{\mu_0} \left\| \frac{e^{-r}}{r} \right\|_{3/2}}. \quad (6.18)$$

Furthermore, by [20] and (6.6), we have

$$\|e^{-itH(\alpha)}\phi\|^2_6 = \left\| \left( e^{-ita^2H(\alpha)}\phi \right)_{\alpha^{-1}} \right\|_6$$
$$= \alpha^{-1/2} \left\| e^{-ita^2H(\alpha)}\phi \right\|_6$$
$$\leq C\alpha^{-1/2} \|a^{2/3}t^{-3(1/2-1/6)}\|_{6/5}$$
$$= C\alpha^{-1/2} - 2 + 5/2 \|t^{-1}\|_{6/5}$$
$$= Ct^{-1} \|\phi\|_{6/5}. \quad (6.19)$$

From (6.18) and (6.19), we have (6.9).

We are now ready to state the proof of Theorem 6.2.
Proof of Theorem 6.2. We first show (6.4). Remark that for any $\phi \in X_2 \setminus \{0\}$ and any $\alpha > \|\phi\|^{3/2} \delta^{-1}$, $(\Omega_+ S_2 \Omega_-^{-1} - id)(\alpha^{-3} \phi_\alpha)$ is well-defined because we have

$$\|\alpha^{-3} \phi_\alpha\| \leq \alpha^{-3/2} \|\phi\|.$$ 

Let $u_\alpha$ be the time-global solution to (1.2) with $\phi_- = \alpha^{-3} \phi_\alpha$. Put

$$u_\alpha^0 = e^{-itH}(\alpha^{-3} \phi_\alpha), \quad \widetilde{u}_\alpha^0 = e^{-itH}(\phi_\alpha), \quad u_\alpha^1 = u_\alpha - u_\alpha^0.$$ 

Then we obtain

$$i\alpha^4 \langle (\Omega_+ S_2 \Omega_-^{-1} - id)(\alpha^{-3} \phi_\alpha), \phi_\alpha \rangle = (I)_\alpha + (II)_\alpha^1 + (II)_\alpha^2 + (II)_\alpha^3,$$

where

$$(I)_\alpha = \alpha^4 \int_{\mathbb{R}} \langle (V_1 * u_\alpha^0 \overline{u}_\alpha^0) u_\alpha^0, \overline{u}_\alpha^0 \rangle dt,$$

$$(II)_\alpha^1 = \alpha^4 \int_{\mathbb{R}} \langle (V_1 * u_\alpha^1 \overline{u}_\alpha^0) u_\alpha^0, \overline{u}_\alpha^0 \rangle dt,$$

$$(II)_\alpha^2 = \alpha^4 \int_{\mathbb{R}} \langle (V_1 * u_\alpha \overline{u}_\alpha^1) u_\alpha^0, \overline{u}_\alpha^0 \rangle dt,$$

$$(II)_\alpha^3 = \alpha^4 \int_{\mathbb{R}} \langle (V_1 * u_\alpha \overline{u}_\alpha) u_\alpha^1, \overline{u}_\alpha^0 \rangle dt.$$ 

Following the proof of Proposition 3.1, we see that for $j = 1, 2, 3$,

$$|(II)_\alpha^j| \leq C \alpha^{-7/2} \|\phi\|^6 \to 0 \quad \text{as } \alpha \to \infty.$$ 

By proposition (6.3), we obtain

$$(I)_\alpha = \alpha^{-5} \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1 |y|)}{|y|} e^{-i\alpha^{-2} t H(\alpha)} \phi(\alpha^{-1} x - \alpha^{-1} y) \overline{e^{-i\alpha^{-2} t H(\alpha)} \phi(\alpha^{-1} x)} d(t, x, y)$$

$$= \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1 |y|)}{|y|} |e^{-itH(\alpha, y)} \phi(x)|^2 |e^{-itH(\alpha)} \phi(x)|^2 d(t, x, y)$$

$$= \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1 |y|)}{|y|} \Phi(\alpha, y) dy,$$

where

$$\Phi(\alpha, y) = \int_{\mathbb{R}^{1+3}} |e^{-i\alpha^{-2} t H(\alpha, y)} \phi(x)|^2 |e^{-i t H(\alpha)} \phi(x)|^2 d(t, x).$$

For the function $\Phi$, we have the following property:

**Lemma 6.6.** Assume that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1} \phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.

(i) For any $y \in \mathbb{R}^3$, we have

$$\lim_{\alpha \to \infty} \Phi(\alpha, y) = \|e^{it\Delta} \phi\|^4_{(4,4)}.$$
For any $\alpha > 0$ and $y \in \mathbb{R}^3$, we have

$$|\Phi(\alpha, y)| \leq C(\phi),$$

where the constant $C(\phi)$ is independent of $\alpha$ and $y$.

**Proof of Lemma 6.6.** It follows from the Hölder inequality that

\[
|\Phi(\alpha, y)| - \|e^{it\Delta} \phi\|_{(4,4)}^4 \leq \int_{\mathbb{R}^{1+3}} \left|e^{-itH(\alpha, y)}\tau_{\alpha^{-1}y} \phi(x)\right|^2 \left|e^{-itH(\alpha)}\phi(x)\right|^2 dt, \quad \text{for}$

where we have used the equality

\[
\|e^{-itH(\alpha, y)}\tau_{\alpha^{-1}y} \phi\|_6 = \|e^{-itH(\alpha)}\phi\|_6,
\]

which is given by (6.7), in the last inequality. We can easily see that

$$\|e^{it\Delta} \phi\|_6 \leq (1 + |t|)^{-1} (\|\nabla \phi\| + \|\phi\|/4).$$

Therefore, by Propositions 6.4 and 6.5 and by applying the Lebesgue dominated theorem with respect to the variable $t$, we obtain (i). Similarly, we have (ii). \qed

Let us go back to the proof of Theorem 6.2. Henceforth, we suppose that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1} \phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Using the above Lemma 6.6, we see from the Lebesgue dominated theorem with respect to the
variable $y$ that
\[
\lim_{\alpha \to \infty} (I)_{\alpha} = \|e^{\mu \Delta} \phi\|_{(4,4)}^{4} \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1 |y|)}{|y|} dy
\]
\[
= 4\pi \frac{Q_1}{\mu_1^2} \|e^{it\Delta} \phi\|_{(4,4)}^{4},
\]
which implies (6.4).

We next show (6.5). Suppose that $Q_1 \neq 0$. Recall the definition of $b$, $\Psi_1$, $m_0$ and $q_j$, $j = 1, 2, \cdots$. It follows from Propositions 3.1 and 6.3, that
\[
a = b^{-7} K[\phi_b] = b^{-7} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} \ast |e^{-itH} \phi_b|^2, |e^{-itH} \phi_b|^2 \right\rangle dt
\]
\[
= b^{-7} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} \ast \left( e^{-it(b^{-2}H(b))} \phi \right)_b, \left( e^{-it(b^{-2}H(b))} \phi \right)_b \right\rangle dt
\]
\[
= b^{-7+2+3+3} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\sqrt{|Q_1|} r)}{br} \ast |e^{-itH(b)} \phi|^2, |e^{-itH(b)} \phi|^2 \right\rangle dt
\]
\[
= Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\sqrt{|Q_1|} r)}{r} \ast |e^{-itH(b)} \phi|^2, |e^{-itH(b)} \phi|^2 \right\rangle dt
\]
\[
= \Psi_1(Q_1).
\]

By the Plancherel theorem, we have
\[
\Psi_1(\alpha) = 4\pi \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{\alpha}{|\alpha| + |\xi|^2} \left| \mathcal{F} \left( e^{-itH(b)} \phi \right)(\xi) \right|^2 d\xi dt.
\]

Therefore, $\Psi_1 : \mathbb{R} \to \Psi_1(\mathbb{R})$ is odd, continuous, bijective and monotonically increasing. Thus, we obtain
\[
\Psi_1 \left( m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} \right) \leq \Psi_1(|Q_1|) < \Psi_1 \left( m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} + \frac{1}{2^j} \right)
\]
and
\[
m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} \leq |Q_1| < m_0 + \sum_{k=1}^{j} \frac{q_k}{2^k} + \frac{1}{2^j}.
\]
Hence (6.5) holds.

7. APPLICATIONS

In this section, dividing two subsections, we review a part of the works [22] and [23] which is concerned with the inverse scattering problem for relativistic equations with
a non-local term. In the first subsection, we consider the nonlinear Klein-Gordon equation
\[ \partial_t^2 u - \Delta u + u = \lambda \left( V_1 \ast |u|^2 \right) u, \quad \text{in } \mathbb{R} \times \mathbb{R}^n. \] (NLKG)

Here, $V_1$ and $\lambda$ satisfy the condition (C) appeared in Section 5. The existence of the scattering operator for (NLKG) is directly proved by [12]. We introduce the result of [22], which is concerned with determining unknown $\sigma$ and $\lambda$ by using the scattering states.

On the other hand, in the second subsection, we consider the semi-relativistic Hartree equation
\[ i \partial_t u - (1 - \Delta)^{1/2} u = \left( \frac{Q_1 \exp(-\mu_1 |x|)}{|x|} \ast |u|^2 \right) u, \quad \text{in } \mathbb{R} \times \mathbb{R}^3. \] (SRH)

Here, the real number $Q_1$ and the positive number $\mu_1$ are unknown. The existence of the scattering operator for (SRH) is directly proved by Cho–Ozawa [5]. We introduce a part of the result of [23], which is concerned with determining unknown $Q_1$ and $\mu_1$ by using the scattering states.

Before treating the inverse scattering problem for the above two relativistic equations, we first review known results for nonlinear Klein-Gordon equations. Morawetz–Strauss [11] initially studied the inverse scattering problem for the Klein-Gordon equation with power nonlinearity. Later, Bachelot [2] considered more general cases. Weder [33, 37] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [11, 2]. The inverse scattering problem for the Klein-Gordon equation with a cubic convolution
\[ \partial_t^2 w - \Delta w + w = (V \ast |w|^2)w, \quad \text{in } (t, x) \in \mathbb{R}^{1+n} \]
was initially studied by Sasaki–Watanabe [24]. As far as the author knows, there is no result except [23] of the inverse scattering problem for semi-relativistic Hartree equation.

7.1. Nonlinear Klein-Gordon equation. Let us focus on the inverse scattering problem for the nonlinear Klein-Gordon equation (NLKG). We here suppose that $V_1$ and $\lambda$ satisfy the condition (C) appeared in Section 5. We first mention the existence theorem for the scattering operator for (NLKG).

Theorem 7.1. Let $n \geq 3$, $2 \leq \sigma \leq 4$ and $\sigma < n$. Put $X_3 = H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ and $Z_3 = C(\mathbb{R}; H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^3(\mathbb{R}; H^1_{q_3})$ with $1/q_3 = 1/2 - 2/3(n - 1 + \theta)$, $\rho_3 = (n + 1 + \theta)/(n - 1 + \theta)$ and $\theta = 2 - \sigma/2$. There exists some $\delta > 0$ such that for any $\psi_- = (\psi_{-1}, \psi_{-2}) \in B(\delta; X_3)$, we uniquely have $w \in Z_3$ and $\psi_+ \in X_3$ such
that

\[ W = \mathcal{L}(w, \partial_t w) \in C^1(\mathbb{R}; H^1(\mathbb{R}^n)) \oplus C(\mathbb{R}; L_2(\mathbb{R}^n)), \]

\[ W(t) = U(t)\phi_- + \frac{1}{i} \int_{-\infty}^{t} U(t - \tau) \begin{pmatrix} 0 \\ iF(w(\tau)) \end{pmatrix} d\tau, \]

\[ \|W(t) - U(t)\phi_\pm\|_{X_3} \to 0 \quad \text{as} \; t \to \pm \infty, \]

\[ \|w\|_{Z_3} \leq C\|\psi_-\|_{H^1(\mathbb{R}^n)}, \quad (7.1) \]

\[ \left\| w - \left( \cos(t\omega)\psi^1_- + \omega^{-1}\sin(t\omega)\psi^2_- \right) \right\|_{Z_3} \leq C\|\psi_-\|^3_{X_3}, \quad (7.2) \]

\[ \left\| \int_{\mathbb{R}} e^{it\omega} (V_1 \ast (w_1 \overline{w_2})) w_3 dt \right\|_{L^2(\mathbb{R}^n)} \leq C\Pi_{j=1}^{3} \|w_j\|_{Z_3}. \quad (7.3) \]

Moreover, we can define the scattering operator for (NLKG)

\[ S : B(\delta; X_3) \ni \psi_- \mapsto \psi_+ = \psi_- + \frac{1}{i} \int_{\mathbb{R}} U(-t) \begin{pmatrix} 0 \\ iF(w(t)) \end{pmatrix} dt \in X_3. \]

The goal of [22] is to identify unknown \( \lambda \) and \( \sigma \). In order to state the main result of [22], we list some notation. For \( \phi \in L^2(\mathbb{R}^n) \), the small amplitude \( \mathcal{K}[\phi] \) denotes

\[ \mathcal{K}[\phi] = \lim_{\epsilon \to 0} \frac{i}{\epsilon^3} \langle (S - I)(\epsilon^{t}(\phi, 0))^{t}(0, \phi) \rangle_{X_3}. \]

For \( s \in \mathbb{R} \) and \( q \in [1, \infty] \), we denote by \( H^s_q(\mathbb{R}^3) = (1 - \Delta)^{-s/2}L^q(\mathbb{R}^n) \). For \( q \in [1, \infty] \), \( q' \) means the Hölder conjugate of \( q \). We are ready to state the main result.

**Theorem 7.2.** Suppose that \( n \geq 3, 2 \leq \sigma \leq 4, \sigma < n, \)

\[ \phi \in H^1 \cap \bigcap_{r \in \ell} H^{(n+1)(1/2-1/r)}_r \]

and \( \phi \neq 0 \). Let \( I = (6(n-1)/(3n-5), 2n/(n-2)) \). Then we have the formula for determining \( \sigma \)

\[ \sigma = 2n + 1 - \lim_{\alpha \to 0} \ln \frac{|\mathcal{K}[\phi_{\alpha些}|}{|\mathcal{K}[\phi_{0些}|} + \alpha^{2n+1}. \quad (7.4) \]

Furthermore, the reconstruction formula for \( \lambda \) can be also given by

\[ \lambda(x_0) = \frac{\lim_{\alpha \to 0} \alpha^{-2n+1-\sigma} \mathcal{K}[\phi_{\alpha x_0}]}{\int |y|^{-\sigma} |w_0(t, x - y)|^2 |w_0(t, x)|^2 d(t, x, y)}, \quad (7.5) \]

where \( w_0 = \cos(t\sqrt{-\Delta})\phi \).
Now we prove Theorem 7.2. To derive (7.4), we follow the line of the proof of Theorem 5.1. Let $\sigma \in [2, 4] \cap [2, n)$.

We here assume that $$\phi \in \Lambda := H^1 \cap \bigcap_{r \in I} H_r^{(n+1)(1/2-1/r)}$$
and $\phi \neq 0$, where $I = (6(n-1)/(3n-5), 2n/(n-2)]$. By Theorem 7.1, and $\Lambda \subset H^1$, $K[\phi]$ is well-defined. By (7.1)–(7.3), it follows from the proof of (3.1) that

$$K[\phi] = \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(x)|y|^{-\sigma}[\Gamma^1(t, x-y)]^2[\Gamma^1(t, x)]^2 d(t, x, y),$$

where $\Gamma^m(t) = \cos(t\sqrt{m^2-\Delta})\phi$.

Having in mind that $e^{it\sqrt{\alpha^2-\Delta}}\phi = (e^{i\alpha^{-1}\sqrt{\alpha^2-\Delta}}\phi)\alpha$, we also have

$$K[\phi_{\alpha}] = \alpha^{2n+1-\sigma}\int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(\alpha x)Q^\alpha(\phi)(t, x, y)d(t, x, y),$$

where

$$Q^m(\phi)(t, x, y) = |y|^{-\sigma}|\Gamma^m(t, x-y)|^2|\Gamma^m(t, x)|^2.$$

In order to see the convergence of $|K[\phi_{\alpha}]/(|K[\phi_{\alpha}]+\alpha^{2n+1})$, we give the following lemma:

**Lemma 7.3.** Let $I = (6(n-1)/(3n-5), 2n/(n-2)]$ and $0 \leq \beta \leq 1$. For $n/(n-1) < \sigma \leq 4$, $\sigma < n$, and $\phi \in H^1 \cap \bigcap_{r \in I} H_r^{(n+1)(1/2-1/r)}$, then $Q^\beta(\phi)(t, x, y)$ is integrable on $\mathbb{R}_{(t,x,y)}^{1+n+n}$. Moreover, we have

$$\int Q^\beta(\phi)d(t, x, y) \to \int Q^0(\phi)d(t, x, y) \quad \text{as } \beta \to 0.$$  

(7.6)

**Proof.** We first state $L^p - L^\beta$ estimates (see, e.g., [14, 15])

$$\|e^{it\sqrt{-\Delta}}\phi\|_r \leq C|t|^{-(n-1)(1/2-1/r)}\|\hat{\phi}\|_{\dot{H}_p^{(n+1)(1/2-1/r)}}$$  

(7.7)

and

$$\|e^{it\sqrt{-\Delta}}\phi\|_r \leq C|t|^{-(n-1)(1/2-1/r)}\|\hat{\phi}\|_{\dot{H}_p^{(n+1)(1/2-1/r)}},$$  

(7.8)

where $2 < r < \infty$ and $\dot{H}_p^{s}$ is the homogeneous Sobolev space (for the definition, see, e.g., [3, 8]).
By the embedding \( H^s_{r'} \hookrightarrow \dot{H}^s_r \) and \( H^{n(1/2-1/r)} \hookrightarrow L^r(\mathbb{R}^n) \), it follows from (7.7) that
\[
\| e^{it\sqrt{\Delta}} \phi \|_r \leq C(1 + |t|)^{-(n-1)(1/2-1/r)} N_r(\phi),
\]
where
\[
N_r(\phi) = \max \{ \| \phi \|_{H^{n(1/2-1/r)}}, \| \phi \|_{H^{n(1/1)(1/2-1/r)}} \}.
\]

On the other hand, Using the identity
\[
e^{it\sqrt{m^2-\Delta}} \phi = (e^{itm\sqrt{1-\triangle}} \phi_m)_{m-1},
\]
and (7.8), we obtain
\[
\| e^{it\sqrt{m^2-\Delta}} \phi \|_r = m^{-n/r} \| e^{itm\sqrt{1-\Delta}} \phi_m \|_r
\leq C m^{-n/r} |tm|^{-(n-1)(1/2-1/r)} \| \phi_m \|_{H^{n(1/2-1/r)}}
\leq C m^{(n+1)(1/2-1/r)} (1 + m^{-(n+1)(1/2-1/r)})
\times |t|^{-(n-1)(1/2-1/r)} \| \phi \|_{H^{n(1/2-1/r)}}
\leq C |t|^{-(n-1)(1/2-1/r)} \| \phi \|_{H^{n+1}(1/2-1/r)}
\]
if \( 0 < m \leq 1 \). By the embedding \( H^{n(1/2-1/r)} \hookrightarrow L^r \), it follows from (7.9) that
\[
\| e^{it\sqrt{\beta^2-\Delta}} \phi \|_r \leq C(1 + |t|)^{-(n-1)(1/2-1/r)} N_r(\phi).
\]

Let \( s \in [0, 1] \). By the Hardy-Littlewood-Sobolev inequality, the Hölder inequality and the embedding \( L^{(1/2+s/n)^{-1}}(\mathbb{R}^n) \hookrightarrow H^s \), we have
\[
\| Q^m(\phi)-Q^0(\phi) \|_{L^1(\mathbb{R}_{x,y}^{2n})}
\leq \| (| \cdot |^{-\sigma} * (\Gamma^m - \Gamma^0)) \Gamma^m \|_{L^1(\mathbb{R}^n)}
+ \| (| \cdot |^{-\sigma} * \Gamma^0(\Gamma^m - \Gamma^0)) \Gamma^m \|_{L^1(\mathbb{R}^n)}
+ \| (| \cdot |^{-\sigma} * \Gamma^0(\Gamma^m - \Gamma^0)) \Gamma^m \|_{L^1(\mathbb{R}^n)}
+ \| (| \cdot |^{-\sigma} * \Gamma^0(\Gamma^m - \Gamma^0)) \Gamma^m \|_{L^1(\mathbb{R}^n)}
\leq \sum_{\beta_1, \beta_2 \in \{0,1\}} \| (| \cdot |^{-\sigma} * |\Gamma^{\beta_1}|^2) |\Gamma^{\beta_2}| \|_{H^{-s}}
\leq C \| \Gamma^m - \Gamma^0 \|_{H^s}
\sum_{\beta_1, \beta_2 \in \{0,1\}} \| (| \cdot |^{-\sigma} * |\Gamma^{\beta_1}|^2) |\Gamma^{\beta_2}| \|_{H^{-s}}
\leq C \| \Gamma^m - \Gamma^0 \|_{H^1}
\sum_{\beta_1, \beta_2 \in \{0,1\}} \| (| \cdot |^{-\sigma} * |\Gamma^{\beta_1}|^2) |\Gamma^{\beta_2}| \|_{(1/2, s)^{-1}}
\leq C \| \Gamma^m - \Gamma^0 \|_{H^1} \{ \| \Gamma^m \|_{r}^2 + \| \Gamma^0 \|_{r}^2 \},
where we have used the equality
\[ \int (|\cdot|^{-\sigma} * v_1)v_2 = \int (|\cdot|^{-\sigma} * v_2)v_1 \]
for the second inequality, and \( r \) satisfies
\[ \frac{3}{2} + \frac{s}{n} = \frac{\sigma}{n} + \frac{3}{r}, \quad 2 < r < \infty. \]
Using (7.10), we see that
\[ \|Q^m(\phi) - Q^0(\phi)\|_{L^1(\mathbb{R}_{(t,x,y)}^{2n})} \leq C(1 + |t|)^{-3(n-1)(1/2-1/r)}(N_r(\phi))^3\|\Gamma^m - \Gamma^0\|. \quad (7.11) \]
Since \( n/(n-1) < \sigma \leq 4 \) and \( \sigma < n \), we can put \( r \in I \). Thus, we have \( n(1/2-1/r) \leq 1 \) and \( -3(n-1)(1/2-1/r) < -1 \). Hence, \( Q^0(\phi) \) is integrable, and the right hand side of (7.11) is bounded by some integrable function which is independent of \( m \). For all \( t \in \mathbb{R} \), we can easily show that
\[ \|\Gamma^m(t) - \Gamma^0(t)\|_{H^s} \rightarrow 0 \quad \text{as} \quad m \rightarrow 0. \]
Thus, applying the Lebesgue dominate theorem on \( \mathbb{R}_t \), we have (7.6). This completes the proof. \( \square \)

Let us go back to the proof of Theorem 7.2. We again assume \( \phi \in \Lambda \). From Lemma 7.3, we see that
\[ \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(\alpha x)Q^\alpha(\phi)(t, x, y)d(t, x, y) \rightarrow \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(0)Q^0(\phi)(t, x, y)d(t, x, y) \neq 0 \quad \text{as} \quad \alpha \rightarrow 0. \]
So we obtain
\[ \frac{\|\mathcal{K}[\phi_{e\alpha}]\|}{|\mathcal{K}[\phi_\alpha]| + \alpha^{2n+1}} = \frac{e^{2n+1-\sigma}|\int \lambda(e\alpha x)Q^0(\phi)(t, x, y)|}{\|\int \lambda(\alpha x)Q^0(\phi)(t, x, y)d(t, x, y)\|} \rightarrow e^{2n+1-\sigma} \quad \text{as} \quad \alpha \rightarrow 0. \]
Thus, we have (7.4).

It remains to show the reconstruction formula (7.5). We put \( \phi \in \Lambda \). Then we have
\[ \mathcal{K}[\phi_{x_0}] = \alpha^{2n+1-\sigma} \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(\alpha x + x_0)Q^\alpha(\phi)(t, x, y)d(t, x, y). \]
Here, \( \sigma \) is a known number which is determined in the above. Since \( \lambda \) is bounded and continuous, it follows from Lemma 7.3 that
\[ \lim_{\alpha \rightarrow 0} \alpha^{-(2n+1-\sigma)}\mathcal{K}[\phi_{x_0}] = \int_{\mathbb{R}_{(t,x,y)}^{1+n+n}} \lambda(x_0)Q^0(\phi)(t, x, y)d(t, x, y). \]
Thus, we have (7.5). This completes the proof of Theorem 7.2.
7.2. **Semi-relativistic Hartree equation.** Let us focus on the inverse scattering problem for the semi-relativistic Hartree equation (SRH). We first mention the existence theorem for the scattering operator for (SRH).

**Theorem 7.4.** Let \( s \geq 5/6 \). Put \( U_2(t) = e^{-it\sqrt{1-\Delta}} \), \( X_4 = H^s(\mathbb{R}^3) \) and \( Z_4 = C(\mathbb{R}; X_4) \cap L^2(\mathbb{R}; H^{s-5/6}_6(\mathbb{R}^3)) \). Then there exists some \( \delta > 0 \) satisfying the following properties:

If \( \phi_- \in B(\delta; X_4) \), then there uniquely exist \( w \in Z_4 \) and \( \phi_+ \in X_4 \) such that

\[
\begin{align*}
w(t) &= U_2(t)\phi_- + \frac{1}{i} \int_{-\infty}^{t} U_2(t - \tau)F(w(\tau))d\tau, \\
\phi_+ &= \phi_- + \frac{1}{i} \int_{\mathbb{R}} U_2(-t)F(w(t))dt,
\end{align*}
\]

\[
\|w\|_{Z_4} \leq C\|\phi_-\|_{X_4},
\]

\[
\|w - U_2(t)\phi_-\|_{Z_4} \leq C\|\phi_-\|_{X_4}^3,
\]

\[
\lim_{t \to \pm \infty} \|w(t) - U_2(t)\phi_\pm\|_{X_4} = 0.
\]

Therefore, we can define the scattering operator for (SRH)

\[
S_4 : B(\delta; X_4) \ni \phi_- \mapsto \phi_+ \in X_4.
\]

The goal of [23] for (SRH) is to identify unknown \( \lambda \) and \( \sigma \). The main result of [23] is the following state:

**Theorem 7.5.** Let \( s \) be a positive number given by Theorem 7.4. Let \( 1 \leq p < 12/7 \) and \( k > 11/12 \). Assume that

\[
\phi \in (H^s(\mathbb{R}^3) \cap H^k_p(\mathbb{R}^3)) \setminus \{0\}.
\]

(i) we have

\[
\frac{Q_1}{\mu_1^2} = \lim_{\alpha \to \infty} \frac{i\alpha^4 \langle (S_4 - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha \rangle}{4\pi \|e^{i\frac{3}{2}\Delta}\phi\|_{(4,4)}^4}.
\]
(ii) Put
\[ d = \left| \frac{Q_1}{\mu_1^2} \right|^{1/2}, \]
\[ \Psi_2(\alpha) = \int_{\mathbb{R}} \left\{ \frac{\alpha \exp(-\sqrt{|\alpha|}r)}{r} \ast |e^{-it\sqrt{d^2-\Delta}}\phi|^2, |e^{-it\sqrt{d^2-\Delta}}\varphi'|^2 \right\} dt, \quad \alpha \in \mathbb{R}, \]
\[ h = \lim_{\epsilon \to 0} i\epsilon^{-3}d^{-6}\langle (S_4-id)(\varepsilon \phi_d), \phi_d \rangle, \]
\[ l_0 = \max \{ l \in \mathbb{Z}_{\geq 0}; \Psi_2(l) \leq |h| \}, \]
\[ p_1 = \max \{ p = 0, 1; \Psi_2 \left( l_0 + \frac{p}{2} \right) \leq |h| \}, \]
\[ p_{j+1} = \max \{ p = 0, 1; \Psi_2 \left( l_0 + \sum_{k=1}^{j} \frac{p_k}{2^k} + \frac{p}{2^{j+1}} \right) \leq |h| \}, \quad j = 1, 2, \cdots. \]

Then we have
\[ Q_1 = \sign \left( \frac{Q_1}{\mu_1^2} \right) \left( l_0 + \sum_{j=1}^{\infty} \frac{p_j}{2^j} \right). \quad (7.13) \]

In order to show Theorem 7.2, we first prepare the following lemma:

**Lemma 7.6.** For \( \alpha > 0 \), let
\[ U^\alpha(t) = e^{it\alpha^2 - it\alpha\sqrt{\alpha^2 - \Delta}}, \quad U^\infty(t) = e^{i\frac{t}{2}\Delta}. \]
Assume that \( 1 \leq p < 12/7 \) and \( k > 11/12 \). If \( \phi \in H^s(\mathbb{R}^3) \cap H_p^k(\mathbb{R}^3) \), then we have
\[ \lim_{\alpha \to \infty} \| U^\alpha(t)\phi - U^\infty(t)\phi \|_{(4,4)} = 0. \quad (7.14) \]

**Proof.** We again denote the norm \( \| \cdot \|_{L^q(\mathbb{R}^3)} \) (\( q \in [1, \infty] \)) by \( \| \cdot \|_{q} \). From the embedding \( H^{3/4}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3) \) and the Plancherel theorem we obtain
\[ \| U^\alpha(t)\phi - U^\infty(t)\phi \|_4 \leq C \| U^\alpha(t)\phi - U^\infty(t)\phi \|_{H^{3/4}(\mathbb{R}^3)} \]
\[ \leq C \left\| \left( e^{it\alpha^2 - it\alpha\sqrt{\alpha^2 + |\xi|^2}} - e^{-it|\xi|^2/2} \right)(1 + |\xi|)^{3/4}d\phi \right\|_{L^2(\mathbb{R}^3_{\xi})}. \]

Since \( (\xi)^{3/4} \mathcal{F} \phi \in X_2 \) and
\[ \alpha^2 - \alpha\sqrt{\alpha^2 + |\xi|^2} = \frac{-|\xi|^2}{1 + \sqrt{1 + |\xi|^2/\alpha^2}} \to \frac{-|\xi|^2}{2} \quad \text{as} \quad \alpha \to \infty \]
for any \( \xi \in \mathbb{R}^3 \), it follows from the Lebesgue dominated theorem that
\[ \lim_{\alpha \to \infty} \| U^\alpha(t)\phi - U^\infty(t)\phi \|_4 = 0 \quad \text{for any} \quad t \in \mathbb{R}. \quad (7.15) \]
Now we put $\alpha > 1$. By the $L^p - L^q$ estimate for the free Klein-Gordon equation in [4], we obtain

$$
\Vert U^\alpha(t)\phi \Vert_4 = \left\| (e^{-it\alpha^2\sqrt{1-\Delta}}\phi_\alpha)_{\alpha^{-1}} \right\|_4 \\
= \alpha^{-3/4}\|e^{-it\alpha^2\sqrt{1-\Delta}}\phi_\alpha\|_4 \\
\leq C\alpha^{-3/4}|t\alpha^2|^{-3/4}\|\phi\|_{H^{5/4}} \\
\leq C|t|^{-3/4}\|\phi\|_{H^{5/4}}. 
$$

(7.16)

Using the complex interpolation method for the linear operator $U^\alpha(t)$, we see from

$$
\Vert U^\alpha(t)\phi \Vert_4 \leq C\|U^\alpha(t)\phi\|_{H^{3/4}} \leq C\|\phi\|_{H^{3/4}}
$$

and (7.16) that

$$
\Vert U^\alpha(t)\phi \Vert_4 \leq C(1 + |t|)^{-3\theta/4}\|\phi\|_{A_\theta},
$$

(7.17)

where

$$
A_\theta = H^{3/4}(\mathbb{R}^3) \cap H^{k_\theta}_{p_\theta}((\mathbb{R}^3), \quad k_\theta = \frac{3}{4} + \frac{\theta}{2}, \quad p_\theta = \frac{1}{2} + \frac{\theta}{4}, \quad 0 \leq \theta \leq 1.
$$

Thus, we can easily see that the left hand side of (7.17) belongs $L^4(\mathbb{R})$ if $1/3 < \theta \leq 1$. Therefore, we obtain

$$
\Vert U^\alpha(t)\phi - U^\infty(t)\phi \Vert_4 \leq Cg(t),
$$

where $g \in L^4(\mathbb{R})$ is some suitable function independent of $\alpha$. By (7.15), it follows from the Lebesgue dominated theorem with respect to time $t$ that (7.14) holds. \(\square\)

We are ready to prove Theorem 7.2.

**Proof of Theorem 7.2.** Following the line of the proof of (6.4), we obtain

$$
\lim_{\alpha \to \infty} i\alpha^4 \left\langle (S_4 - id)(\alpha^{-3}\phi_\alpha), \phi_\alpha \right\rangle \\
= \lim_{\alpha \to \infty} i\alpha^{-5} \int_\mathbb{R} \left\langle F(U_2(t)\phi_\alpha), U_2(t)\phi_\alpha \right\rangle dt \\
= \lim_{\alpha \to \infty} \int_{\mathbb{R}^{1+2}} Q_1 \exp(-\mu_1|y|)|U^\alpha(t)\phi(x - \alpha^{-1} y)|^2 |U^\alpha(t)\phi(x)|^2 d(t, x, y).
$$

By (7.14), we have (7.12). The remaining formula (7.13) can be shown by the same argument as the proof of (7.12). \(\square\)

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