Global DIV-CURL Lemma in 3D bounded domains

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1 Introduction.

Let $\Omega$ be an open set in $\mathbb{R}^3$. It is well-known that if $u_j \rightharpoonup u$, $v_j \rightharpoonup v$ weakly in $L^2(\Omega)$ and if $\{\text{div } u_j\}_{j=1}^\infty$ and $\{\text{rot } v_j\}_{j=1}^\infty$ are bounded in $L^2(\Omega)$, then it holds that $u_j \cdot v_j \rightharpoonup u \cdot v$ in the sense of distributions in $\Omega$. This is the original Div-Curl lemma. For instance, we refer to Tartar [5]. The purpose of this article is to deal with a similar lemma to bounded domains where the convergence $u_j \cdot v_j \rightharpoonup u \cdot v$ holds in the sense that

\begin{equation}
\int_{\Omega} u_j \cdot v_j \ dx \rightarrow \int_{\Omega} u \cdot v \ dx \quad \text{as } j \rightarrow \infty.
\end{equation}

Our result may be regarded as a global version of the Div-Curl lemma, which includes the previous one. To obtain such a global version, we need to pay an attention to the behaviour of $\{u_j\}_{j=1}^\infty$ and $\{v_j\}_{j=1}^\infty$ on the boundary $\partial \Omega$ of $\Omega$. Indeed, an additional bound of $\{u_j \cdot \nu|_{\partial \Omega}\}_{j=1}^\infty$ or that of $\{v_j \times \nu|_{\partial \Omega}\}_{j=1}^\infty$ in $H^{1/2}(\partial \Omega)$ on the boundary $\partial \Omega$ plays an essential role for our convergence, where $\nu$ denotes the unit outward normal to $\partial \Omega$.

In what follows, we impose the following assumption on the domain $\Omega$:

Assumption. $\Omega$ is a bounded domain in $\mathbb{R}^3$ with $C^\infty$-boundary $\partial \Omega$.

Before stating our result, we first recall the generalized trace theorem for $u \cdot \nu$ and $u \times \nu$ on $\partial \Omega$ defined on the Banach spaces $E_{\text{div}}^q(\Omega)$ and $E_{\text{rot}}^q(\Omega)$ for $1 < q < \infty$, where

\begin{align*}
E_{\text{div}}^q(\Omega) & \equiv \{ u \in L^q(\Omega); \text{div } u \in L^q(\Omega) \} \text{ with the norm } ||u||_{E_{\text{div}}^q} = ||u||_q + ||\text{div } u||_q, \\
E_{\text{rot}}^q(\Omega) & \equiv \{ u \in L^q(\Omega); \text{rot } u \in L^q(\Omega) \} \text{ with the norm } ||u||_{E_{\text{rot}}^q} = ||u||_q + ||\text{rot } u||_q.
\end{align*}

Here and in what follows, $|| \cdot ||_q$ denotes the usual $L^q$-norm over $\Omega$. It is known that there are bounded operators $\gamma_{\nu}$ and $\tau_{\nu}$ on $E_{\text{div}}^q(\Omega)$ and $E_{\text{rot}}^q(\Omega)$ with properties that

\begin{align*}
\gamma_{\nu} : u & \in E_{\text{div}}^q(\Omega) \mapsto \gamma_{\nu} u \in W^{1-1/q',q'}(\partial \Omega)^d, \quad \gamma_{\nu} u = u \cdot \nu|_{\partial \Omega} \text{ if } u \in C^1(\bar{\Omega}), \\
\tau_{\nu} : u & \in E_{\text{rot}}^q(\Omega) \mapsto \tau_{\nu} u \in W^{1-1/q',q'}(\partial \Omega)^d, \quad \tau_{\nu} u = u \times \nu|_{\partial \Omega} \text{ if } u \in C^1(\bar{\Omega}).
\end{align*}
respectively, where $1/q + 1/q' = 1$. The range $W^{1-1/q',q'}(\partial \Omega)^*$ of $\gamma_{\nu}$ and $\tau_{\nu}$ is the dual space of $W^{1-1/q',q'}(\partial \Omega)$ which is the image of the trace on $\partial \Omega$ of functions in $W^{1,q'}(\Omega)$. Indeed, the following generalized Stokes formula holds

\begin{equation}
(u, \nabla p) + (\text{div} u, p) = \langle \gamma_{\nu} u, \gamma_{0} p \rangle_{\partial \Omega}
\end{equation}

for all $u \in E_{\text{div}}^{q}(\Omega)$ and all $p \in W^{1,q'}(\Omega)$,

\begin{equation}
(u, \text{rot} \phi) = (\text{rot} u, \phi) + \langle \tau_{\nu} u, \gamma_{0} \phi \rangle_{\partial \Omega}
\end{equation}

for all $u \in E_{\text{rot}}^{q}(\Omega)$ and all $\phi \in W^{1,q'}(\Omega)$,

where $\gamma_{0}$ denotes the usual trace operator from $W^{1,q'}(\Omega)$ onto $W^{1-1/q',q'}(\partial \Omega)$, and $\langle \cdot, \cdot \rangle_{\partial \Omega}$ is the duality paring between $W^{1-1/q',q'}(\partial \Omega)^*$ and $W^{1-1/q',q'}(\partial \Omega)$. Here and in what follows, $(\cdot, \cdot)$ denotes the duality paring between $L^{q}(\Omega)$ and $L^{q'}(\Omega)$.

Our result now reads:

**Theorem 1** Let $\Omega$ be as in the Assumption. Let $1 < r < \infty$ with $1/r + 1/r' = 1$. Suppose that

\begin{equation}
\{u_{j}\}_{j=1}^{\infty} \subset L^{r}(\Omega) \quad \text{and} \quad \{v_{j}\}_{j=1}^{\infty} \subset L^{r'}(\Omega)
\end{equation}

satisfy

\begin{equation}
u_{j} \to u \quad \text{weakly in} \quad L^{r}(\Omega), \quad v_{j} \to v \quad \text{weakly in} \quad L^{r'}(\Omega)
\end{equation}

for some $u \in L^{r}(\Omega)$ and $v \in L^{r'}(\Omega)$, respectively. Assume also that

\begin{equation}
\{\text{div} u_{j}\}_{j=1}^{\infty} \quad \text{is bounded in} \quad L^{q}(\Omega) \quad \text{for some} \quad q > \max\{1, 3r/(3+r)\}
\end{equation}

and that

\begin{equation}
\{\text{rot} v_{j}\}_{j=1}^{\infty} \quad \text{is bounded in} \quad L^{s}(\Omega) \quad \text{for some} \quad s > \max\{1, 3r'/(3+r')\},
\end{equation}

respectively. If either

(i) \{\gamma_{\nu} u_{j}\}_{j=1}^{\infty} \quad \text{is bounded in} \quad W^{1-1/q,q}(\partial \Omega),

or

(ii) \{\tau_{\nu} v_{j}\}_{j=1}^{\infty} \quad \text{is bounded in} \quad W^{1-1/s,s}(\partial \Omega),

then it holds that

\begin{equation}
\int_{\Omega} u_{j} \cdot v_{j} dx \to \int_{\Omega} u \cdot v dx \quad \text{as} \quad j \to \infty.
\end{equation}

In particular, if either $\gamma_{\nu} u_{j} = 0$, or $\tau_{\nu} v_{j} = 0$ for all $j = 1, 2, \cdots$ is satisfied, then we have also (1.7).

As an immediate consequence of our theorem, we have the following Div-Curl lemma in an arbitrary open set in $\mathbb{R}^{3}$.

**Corollary 1.1** (Tartar [5]) Let $D$ be an arbitrary open set in $\mathbb{R}^{3}$. Let $1 < r < \infty$. Suppose that $\{u_{j}\}_{j=1}^{\infty} \subset L^{r}(D)$ and $\{v_{j}\}_{j=1}^{\infty} \subset L^{r'}(D)$ satisfy

\begin{equation}
u_{j} \to u \quad \text{weakly in} \quad L^{r}(D), \quad v_{j} \to v \quad \text{weakly in} \quad L^{r'}(D)
\end{equation}

for some $u \in L^{r}(D)$ and $v \in L^{r'}(D)$, respectively. Assume also that

\begin{equation}
\{\text{div} u_{j}\}_{j=1}^{\infty} \quad \text{and} \quad \{\text{rot} v_{j}\}_{j=1}^{\infty} \quad \text{are bounded in} \quad L^{r}(D) \quad \text{and} \quad L^{r'}(D),
\end{equation}

respectively. Then it holds that

\begin{equation}
\int_{D} u_{j} \cdot v_{j} dx \to \int_{D} u \cdot v dx \quad \text{in the sense of distributions in} \quad D.
\end{equation}
Remarks. (i) Since $\Omega$ is a bounded domain, we may assume that $3r/(3 + r) < q \leq r$ and $3r'/3 + r' < s \leq r'$, and hence it holds that $\{u_j\}_{j=1}^{\infty} \subset E_{\text{div}}^{q}(\Omega)$ and that $\{v_j\}_{j=1}^{\infty} \subset E_{\text{rot}}^{s}(\Omega)$. Then we have that $\{\gamma_{\nu}u_j\}_{j=1}^{\infty} \subset W^{1-1/q',q'}(\partial \Omega)^{*}$ and that $\{\tau_{\nu}v_j\}_{j=1}^{\infty} \subset W^{1-1/s',s'}(\partial \Omega)^{*}$.

(ii) In Theorem 1, it is unnecessary to assume both bounds of $\{\gamma_{\nu}u_j\}_{j=1}^{\infty}$ in $W^{1-1/r,r}(\partial \Omega)$ and $\{\tau_{\nu}v_j\}_{j=1}^{\infty}$ in $W^{1-1/r',r'}(\partial \Omega)$. Indeed, what we need is only one of these bounds.

2 $L^r$-Helmholtz-Weyl decomposition.

In this section, we recall the Helmholtz-Weyl decomposition for vector fields in $L^r(\Omega)$. For a detail, we refer [2]. According to the two types $u \cdot \nu = 0$ and $u \times \nu = 0$ of boundary conditions on $\partial \Omega$, we first define harmonic vector spaces $X_{\text{har}}(\Omega)$ and $V_{\text{har}}(\Omega)$ as

$$X_{\text{har}}(\Omega) = \{h \in C^\infty(\overline{\Omega}); \text{div} h = 0, \text{rot} h = 0 \text{ in } \Omega \text{ with } h \cdot \nu = 0 \text{ on } \partial \Omega\},$$

$$V_{\text{har}}(\Omega) = \{h \in C^\infty(\overline{\Omega}); \text{div} h = 0, \text{rot} h = 0 \text{ in } \Omega \text{ with } h \times \nu = 0 \text{ on } \partial \Omega\}.$$

Moreover, for $1 < r < \infty$ let us define divergence-free vector fields $X^r_{\sigma}(\Omega)$ and $V^r_{\sigma}(\Omega)$ by

$$X^r_{\sigma}(\Omega) = \{u \in W^{1,r}(\Omega); \text{div} u = 0, \gamma_{\nu}u = 0\},$$

$$V^r_{\sigma}(\Omega) = \{u \in W^{1,r}(\Omega); \text{div} u = 0, \tau_{\nu}u = 0\}.$$

Then we have the following decomposition theorem. For a detail, we refer Kozono-Yanagisawa [2]

**Proposition 2.1 ([2])** Let $\Omega$ be as in the Assumption. Let $1 < r < \infty$.

(1) Both $X_{\text{har}}(\Omega)$ and $V_{\text{har}}(\Omega)$ are finite dimensional vector spaces.

(2) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_{\sigma}(\Omega)$ and $h \in X_{\text{har}}(\Omega)$ such that $u$ can be represented as

$$u = h + \text{rot} w + \nabla p. \tag{2.1}$$

Such a triplet $\{p,w,h\}$ is subordinate to the estimate

$$\|p\|_{W^{1,r}} + \|w\|_{W^{1,r}} + \|h\|_r \leq C\|u\|_r \tag{2.2}$$

with the constant $C = C(\Omega, r)$ independent of $u$. The above decomposition (2.1) is unique. In fact, if $u$ has another expression

$$u = \tilde{h} + \text{rot} \tilde{w} + \nabla \tilde{p}$$

for $\tilde{p} \in W^{1,r}(\Omega)$, $\tilde{w} \in V^r_{\sigma}(\Omega)$ and $\tilde{h} \in X_{\text{har}}(\Omega)$, then we have

$$h = \tilde{h}, \quad \text{rot} w = \text{rot} \tilde{w}, \quad \nabla p = \nabla \tilde{p}. \tag{2.3}$$

(3) For every $u \in L^r(\Omega)$, there are $p \in W^{1,r}_0(\Omega)$, $w \in X^r_{\sigma}(\Omega)$ and $h \in V_{\text{har}}(\Omega)$ such that $u$ can be represented as

$$u = h + \text{rot} w + \nabla p. \tag{2.4}$$

Such a triplet $\{p,w,h\}$ is subordinate to the estimate

$$\|p\|_{W^{1,r}} + \|w\|_{W^{1,r}} + \|h\|_r \leq C\|u\|_r \tag{2.5}.$$
with the constant $C = C(\Omega, r)$ independent of $u$. The above decomposition (2.4) is unique. In fact, if $u$ has another expression

$$u = \tilde{h} + \text{rot } \tilde{w} + \nabla \tilde{p}$$

for $\tilde{p} \in W^{1,r}_0(\Omega)$, $\tilde{w} \in X^r(\Omega)$ and $\tilde{h} \in V_{\text{har}}(\Omega)$, then we have

$$h = \tilde{h}, \quad \text{rot } w = \text{rot } \tilde{w}, \quad p = \tilde{p}.$$  

An immediate consequence of the above theorem is

**Corollary 2.1** Let $\Omega$ be as in the Assumption.

(1) By the unique decompositions (2.1) and (2.4) we have two kinds of direct sums in algebraic and topological sense

\begin{align*}
L^r(\Omega) &= X_{\text{har}}(\Omega) \oplus \text{rot } V^r_\sigma(\Omega) \oplus \nabla W^{1,r}(\Omega), \\
L^r(\Omega) &= V_{\text{har}}(\Omega) \oplus \text{rot } X^r_\sigma(\Omega) \oplus \nabla W^{1,r}_0(\Omega)
\end{align*}

for $1 < r < \infty$.

(2) Let $S_r$, $R_r$ and $Q_r$ be projection operators associated with both (2.1) and (2.4) from $L^r(\Omega)$ onto $X_{\text{har}}(\Omega)$, $\text{rot } V^r_\sigma(\Omega)$ and $\nabla W^{1,r}(\Omega)$, and from $L^r(\Omega)$ onto $V_{\text{har}}(\Omega)$, $\text{rot } X^r_\sigma(\Omega)$ and $\nabla W^{1,r}_0(\Omega)$, respectively, i.e.,

\begin{align*}
S_r u &\equiv h, \\
R_r u &\equiv \text{rot } w, \\
Q_r u &\equiv \nabla p.
\end{align*}

Then we have

\begin{align*}
\|S_r u\|_r &\leq C \|u\|_r, \\
\|R_r u\|_r &\leq C \|u\|_r, \\
\|Q_r u\|_r &\leq C \|u\|_r
\end{align*}

for all $u \in L^r(\Omega)$, where $C = C(r)$ is the constant depending only on $1 < r < \infty$. Moreover, there holds

\begin{align*}
S^2_r = S_r, &\quad S^* = S_r', \\
R^2_r = R_r, &\quad R^* = R_r', \\
Q^2_r = Q_r, &\quad Q^* = Q_r'.
\end{align*}

where $S^*_r$, $R^*_r$ and $Q^*_r$ denote the adjoint operators on $L^{r'}(\Omega)$ of $S_r$, $R_r$ and $Q_r$, respectively.

If $u$ has an additional regularity such as $\text{div } u \in L^q(\Omega)$ and $\text{rot } u \in L^q(\Omega)$ for some $1 < q \leq r$, then we may choose the scalar and the vector potentials $p$ and $w$ in (2.1) and (2.4) in the class $W^{2,q}(\Omega)$. More precisely, we have

**Proposition 2.2** Let $\Omega$ be as in the Assumption and let $1 < r < \infty$. Suppose that $u \in L^r(\Omega)$.

(1) Let us consider the decomposition (2.1).

(i) If, in addition, $\text{rot } u \in L^q(\Omega)$ for some $1 < q \leq r$, then the vector potential $w$ of $u$ in (2.1) can be chosen as $w \in W^{2,q}(\Omega) \cap V^r_\sigma(\Omega)$ with the estimate

$$\|w\|_{W^{2,q}} \leq C(\|\text{rot } u\|_q + \|u\|_r).$$

(ii) If, in addition, $\text{div } u \in L^q(\Omega)$ with $\gamma_{\nu}u \in W^{1-1/q,q}(\partial \Omega)$ for some $1 < q \leq r$, then the scalar potential $p$ of $u$ in (2.1) can be chosen as $p \in W^{2,q}(\Omega) \cap W^{1,r}(\Omega)$ with the estimate

$$\|p\|_{W^{2,q}} \leq C(\|\text{div } u\|_q + \|u\|_r + \|\gamma_{\nu} u\|_{W^{1-1/q,q}(\partial \Omega)}).$$
(2) Let us consider the decomposition (2.4).

(i) If, in addition, \( \text{div} \, u \in L^{q}(\Omega) \) for some \( 1 < q \leq r \), then the scalar potential \( p \) of \( u \) in (2.4) can be chosen as \( p \in W^{2,q}(\Omega) \cap W^{1,r}_{0}(\Omega) \) with the estimate

\[
||p||_{W^{2,q}} \leq C ||\text{div} \, u||_{q}.
\]

(ii) If, in addition, \( \text{rot} \, u \in L^{q}(\Omega) \) with \( \tau_{\nu}u \in W^{1-1/q,q}(\partial\Omega) \) for some \( 1 < q \leq r \), then the vector potential \( w \) of \( u \) in (2.4) can be chosen as \( w \in W^{2,q}(\Omega) \cap X^{r}_{\sigma}(\Omega) \) with the estimate

\[
||w||_{W^{2,q}} \leq C (||\text{rot} \, u||_{q} + ||u||_{r} + ||\tau_{\nu}u||_{W^{1-1/q,q}(\partial\Omega)}).
\]

Here \( C = C(\Omega, r, q) \) is the constant depending only on \( \Omega \), \( r \) and \( q \).

3 Proof of Theorem 1.

(i) Let us first consider the case when \( \{\gamma_{\nu}u_{j}\}_{j=1}^{\infty} \) is bounded in \( W^{1-1/q,q}(\partial\Omega) \). In such a case, we make use of the decomposition (2.1). Let \( S_{r} \), \( R_{r} \) and \( Q_{r} \) be the projection operators from \( L^{r}(\Omega) \) onto \( X_{\text{har}}(\Omega) \), \( \text{rot} \, V^{r}_{\sigma}(\Omega) \) and \( \nabla W^{1,r}(\Omega) \) defined by (2.9), respectively. Notice that the identity

\[
(u, v) = (S_{r}u, S_{r'}v) + (R_{r}u, R_{r'}v) + (Q_{r}u, Q_{r'}v)
\]

holds for all \( u \in L^{r}(\Omega) \) and all \( v \in L^{r'}(\Omega) \). Indeed, by the generalized Stokes formula (1.2) and (1.3), we have

\[
(\nabla p, h) = -(p, \text{div} \, h) + \langle \gamma_{\nu}h, \gamma_{0}p \rangle_{\partial\Omega} = 0,
\]

\[
(\text{rot} \, w, h) = (w, \text{rot} \, h) + \langle \tau_{\nu}w, \gamma_{0}h \rangle_{\partial\Omega} = 0
\]

for all \( p \in W^{1,r}(\Omega) \), \( w \in V^{r}_{\sigma}(\Omega) \) and \( h \in X_{\text{har}}(\Omega) \). Similarly, we have

\[
(\text{rot} \, w, \nabla p) = \langle \gamma_{\nu}(\text{rot} \, w), \gamma_{0}p \rangle_{\partial\Omega} = 0 \quad \text{for all} \ w \in V^{r}_{\sigma}(\Omega), \ p \in W^{1,r'}(\Omega).
\]

Thus we obtain (3.1).

Now, by (3.1), we see that the convergence (1.7) can be reduced to

\[
(S_{r}u_{j}, S_{r'}v_{j}) \rightarrow (S_{r}u, S_{r'}v),
\]

\[
(R_{r}u_{j}, R_{r'}v_{j}) \rightarrow (R_{r}u, R_{r'}v),
\]

\[
(Q_{r}u_{j}, Q_{r'}v_{j}) \rightarrow (Q_{r}u, Q_{r'}v).
\]

By Proposition 2.1 (1), the ranges of \( S_{r} \) and \( S_{r'} \) are of finite dimension, which means that both \( S_{r} \) and \( S_{r'} \) are finite rank operators, therefore compact. Hence, we have by (1.4) that

\[
S_{r}u_{j} \rightarrow S_{r}u \quad \text{strongly in} \ L^{r}(\Omega), \quad S_{r'}v_{j} \rightarrow S_{r'}v \quad \text{strongly in} \ L^{r'}(\Omega),
\]

from which it follows (3.2).

Next, we apply Proposition 2.2 (1) to (3.3) and (3.4). Since \( \Omega \) is bounded, we may assume that

\[
\max \left\{ 1, \frac{3r}{3+r} \right\} < q \leq r, \quad \max \left\{ 1, \frac{3r'}{3+r'} \right\} < s \leq r'.
\]
By (1.6) and (2.12) with $q$ and $r$ replaced by $s$ and $r'$, respectively, we see that $R_{r'}v_{j}\equiv \text{rot } \tilde{w}_{j}$ with $\tilde{w}_{j}\in V_{\sigma}^{r'}(\Omega)$ and $\overline{w}_{j}\in W^{2,s}(\Omega)\cap V_{\sigma}^{r'}(\Omega)$ with the estimate
\[
\|\tilde{w}_{j}\|_{W^{2,s}} \leq C(\|\text{rot } v_{j}\|_{s} + \|v_{j}\|_{r'}) \leq M, \quad \text{for all } j = 1, 2, \cdots
\]
with a constant $M$ independent of $j$. Since $1/r' > 1/s - 1/3$, the embedding $W^{2,s}(\Omega) \subset W^{1,r'}(\Omega)$ is compact, and hence we see that $\{\tilde{w}_{j}\}_{j=1}^{\infty}$ has a strongly convergent subsequence in $W^{1,r'}(\Omega)$, and hence $\{R_{r'}v_{j}\}_{j=1}^{\infty}$ has a strongly convergent subsequence in $L^{r'}(\Omega)$. Since (1.4) yields $\text{rot } \tilde{w}_{j} = R_{r'}v_{j} \rightharpoonup R_{r'}v$ weakly in $L^{r'}(\Omega)$, it holds, in fact, that
\[
(3.5) \quad R_{r'}v_{j} \rightharpoonup R_{r'}v \quad \text{strongly in } L^{r'}(\Omega).
\]
Obviously by (1.4), $R_{r'}u_{j} \rightharpoonup R_{r'}u$ weakly in $L^{r'}(\Omega)$, and hence (3.3) follows.

Since $\{\gamma_{\nu}u_{j}\}_{j=1}^{\infty}$ is bounded in $W^{1-1/q,q}(\partial\Omega)$, we see from (1.5) and (2.13) that $Q_{r}u_{j} = \nabla p_{j}$ satisfies that $p_{j} \in W^{2,q}(\Omega)$ with the estimate
\[
\|p_{j}\|_{W^{2,q}} \leq C(\|\text{div } u_{j}\|_{q} + \|u_{j}\|_{r'}) \leq M \quad \text{for all } j = 1, 2, \cdots
\]
with a constant $M$ independent of $j$. Since $1/r > 1/q - 1/3$, again by the compact embedding $W^{2,q}(\Omega) \subset W^{1,r}(\Omega)$ and by the weak convergence $\nabla p_{j} = Q_{r}u_{j} \rightharpoonup Q_{r}u$ in $L^{r}(\Omega)$, implied by (1.4), it holds that
\[
(3.6) \quad Q_{r}u_{j} \rightharpoonup Q_{r}u \quad \text{strongly in } L^{r}(\Omega).
\]
Since (1.4) yields $Q_{r}v_{j} \rightharpoonup Q_{r}v$ weakly in $L^{r'}(\Omega)$, we see that (3.4) follows.

(ii) We next consider the case when $\{\tau_{\nu}v_{j}\}_{j=1}^{\infty}$ is bounded in $W^{1-1/s,s}(\partial\Omega)$. In this case, we make use of the decomposition (2.4). Then the argument is quite similar to the former case (i) above. So, we may omit the proof. This proves Theorem 1.

Proof of Corollary 1.1. We may prove that for every $\varphi \in C_{0}^{\infty}(D)$
\[
\int_{D} \varphi u_{j} \cdot v_{j} dx \rightharpoonup \int_{D} \varphi u \cdot v dx.
\]
Let us take a bounded domain $\Omega \subset \mathbb{R}^{3}$ with the smooth boundary $\partial\Omega$ so that $\text{supp } \varphi \subset \Omega \subset D$. Then it suffices to prove that
\[
(3.7) \quad \int_{\Omega} \varphi u_{j} \cdot v_{j} dx \rightharpoonup \int_{\Omega} \varphi u \cdot v dx.
\]
Obviously by (1.8), it holds that
\[
(3.8) \quad \varphi u_{j} \rightharpoonup \varphi u \quad \text{weakly } L^{r}(\Omega), \quad v_{j} \rightharpoonup v \quad \text{weakly } L^{r'}(\Omega).
\]
Since $\text{div } (\varphi u_{j}) = \varphi \text{div } u_{j} + u_{j} \cdot \nabla \varphi$, we see by (1.8) and (1.9) that $\{\text{div } (\varphi u_{j})\}_{j=1}^{\infty}$ is bounded in $L^{r}(\Omega)$ with
\[
(3.9) \quad \gamma_{\nu}(\varphi u_{j}) = 0, \quad j = 1, 2, \cdots.
\]
Since (1.9) states that $\{\text{rot } v_{j}\}_{j=1}^{\infty}$ is also bounded in $L^{r'}(\Omega)$, by taking $q = r$ and $s = r'$ in (1.5) and (1.6), respectively, we see that the convergence (3.7) follows from (3.8), (3.9) and Theorem 1 (i). This proves Corollary 1.1.
References


