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A remark on global solutions to nonlinear Klein-Gordon equation with a special quadratic nonlinearity in two space dimensions

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1 Introduction

In this note, we consider the Cauchy problem of the quadratic nonlinear Klein-Gordon equation in two space dimensions:

$$\partial_t^2 u - \Delta u + u = F(u, \partial u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \tag{1.1}$$

$$u(0,x) = \varepsilon u_0(x),\ \partial_t u(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^2, \tag{1.2}$$

where $\partial = (\partial_t, \partial_1, \partial_2)$, $\varepsilon > 0$ is a small parameter, and F is a smooth function of $(u, w) \in \mathbb{R} \times \mathbb{R}^3$ satisfying

$$F(u,w) = O(|u|^2 + |w|^2) \quad \text{near } (u,w) = (0,0).$$

There are many studies concerning the global existence and asymptotic behavior of solutions to the nonlinear Klein-Gordon equation. When spatial dimension $n \geq 5$, Klainerman-Ponce [6] and Shatah [10] showed that the Cauchy problem (1.1)-(1.2) has unique global solutions for small initial data and that the solutions asymptotically approach to the corresponding free solutions of the linear Klein-Gordon equation as $t \to \infty$. The proofs in [6] and [10] are based on the L^p-L^q decay estimate of solutions to the linear Klein-Gordon equation.

When $n \leq 4$, the L^p-L^q decay estimate does not provide us a sufficient time decay to construct global solutions. To overcome this difficulty, Klainerman [5] introduced the invariant Sobolev space with respect to the generators of the Lorentz group and showed the existence of global solution to (1.1)-(1.2) when $n = 3$, 4. Independently, Shatah [11] introduced the method of the normal forms, which is the extension of the Poincaré's theory of normal forms for the ordinary differential equations to the nonlinear Klein-Gordon equations, and showed the existence of global solution to (1.1)-(1.2) when $n = 3$, 4.

When $n = 2$, Georgiev-Popivanov [2] and Kosecki [4] showed the existence of global solution provided that the nonlinearity in (1.1) has the special form, which is called the null condition and the unit condition. Then, the general nonlinearities are treated by Simon-Taflin [12] and Ozawa-Tsutaya-Tsutsumi [8]. They showed that the existence of global solutions for small data without any special structure of the nonlinearity. The proof in [8] is based on the method of normal forms and the decay estimate due to Georgiev [1]. See also [9].

Above results require much regularity and decay on the initial data other than the smallness, which is due to the use of vector fields such as

$$\Omega_{ij} = x_i\partial_j - x_j\partial_i, \quad L_j = x_j\partial_t + t\partial_j.$$

The purpose of this note is to give another proof and to relax the condition on the initial data for the existence of global solutions to (1.1)-(1.2), when the nonlinearity takes a special form. Especially, no decay assumption on the data is required. Our method is based on the use of the normal forms and the endpoint Strichartz estimates in mixed norms on the polar coordinates, which is proved in [3].

2 Main Result

In what follows, we focus on the following problem:

$$\partial_t^2 u - \Delta u + u = Q(u, \partial u), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \tag{2.1}$$

$$u(0,x) = \varepsilon u_0(x), \ \partial_t u(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^2, \tag{2.2}$$

where Q is given by

$$Q(u, \partial u) = (\partial_t u)^2 - |\nabla u|^2 - u^2/2. \tag{2.3}$$

This Q is the simplest quadratic nonlinearity according to the method of normal form. In fact, if we set

$$v = u - u^2/2, \tag{2.4}$$

then v satisfies

$$\partial_t^2 v - \Delta v + v = -uQ(u, \partial u) \equiv C(u, \partial u). \tag{2.5}$$

Note that $C(u, \partial u)$ is cubic in u and ∂u. The integral equation associated with (2.5) is

$$v(t) = \dot{U}(t)v_0 + U(t)v_1 + \int_0^t U(t-t')C(u(t'), \partial u(t'))\, dt', \tag{2.6}$$

where

$$\dot{U}(t) = \cos[t\langle\nabla\rangle], \quad U(t) = \langle\nabla\rangle^{-1}\sin[t\langle\nabla\rangle], \ \langle\nabla\rangle = (1-\Delta)^{1/2},$$

and $v_0 = u_0 - u_0^2/2$, $v_1 = u_1 - u_0u_1$. From (2.4) we derive the integral equation in u as

$$u(t) = \dot{U}(t)v_0 + U(t)v_1 + \int_0^t U(t-t')C(u(t'), \partial u(t'))\, dt' + u(t)^2/2. \tag{2.7}$$

By using the transformation (2.4), we reduce the problem (2.1)-(2.2) to (2.7) which has the cubic nonlinearity in the integrand. Such nonlinearity is easily handled by using endpoint Strichartz estimates which will be introduced in the next section.

Remark 2.1. General quadratic nonlinearities are also canceled out if we choose a special quadratic transformation

$$v = u - [u, K_1, u] - [\partial_t u, K_2, u] - [u, K_3, \partial_t u] - [\partial_t u, K_4, \partial_t u],$$

where $K_1, \ldots, K_4$ are kernel functions on $\mathbb{R}^2 \times \mathbb{R}^2$ and

$$[f, K, g](x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(y)K(x-y, x-z)g(z)\, dydz$$

is a quadratic integral transformation with kernel function K. We notice that such method also works well for the quasilinear case. For the details, see [8, 9].

Before stating our main result, we prepare some notation. Throughout this note we denote the polar coordinates $x = r\omega$, $r = |x|$ and $\omega \in S^1$. We denote by Δ_{S^1} the Laplace-Beltrami operator on S^1 and we set $\langle \nabla_\omega \rangle^s = (I - \Delta_{S^1})^{s/2}$. For $m \in \mathbb{N}$ and $s > 0$, we define the space

$$H^m(H^s_\omega) = \{f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_{H^m(H^s_\omega)} := \|\langle \nabla_\omega \rangle^s f\|_{H^m_x} < \infty\},$$

where H^m_x is the usual Sobolev space on $\mathbb{R}^2$. We also use the following type of norm

$$\|f\|_{L^p_r L^q_\omega} := \Big(\int_0^\infty \|f(r\omega)\|^p_{L^q_\omega} r\, dr \Big)^{1/p},$$

where $L^q_\omega = L^q(S^1)$.

Our main result is the following.

Theorem 2.2. *Let $u_0 \in H^2(H^\delta_\omega)$, $u_1 \in H^1(H^\delta_\omega)$ for $\delta > 0$. If $\varepsilon > 0$ is sufficiently small, then there exists a unique solution $u \in C(\mathbb{R}; H^2(H^\delta_\omega)) \cap C^1(\mathbb{R}; H^1(H^\delta_\omega))$ to* (2.7) *satisfying*

$$\|\partial u\|_{L^\infty_t H^1 \cap L^2_t L^\infty_r L^q_\omega} + \|u\|_{L^\infty_t H^1 \cap L^2_t L^\infty_r L^q_\omega} \leq C\varepsilon,$$

for some $C > 0$, where $\delta > 1/q > 0$.

3 Endpoint Strichartz estimates

In this section, we describe endpoint Strichartz estimates for the Klein-Gordon equation in two space dimensions without proof. For the proof, we refer to [3]. In the following, we denote

$$\dot{U}(t) = \cos[\,t\langle\nabla\rangle], \quad U(t) = \langle\nabla\rangle^{-1}\sin[\,t\langle\nabla\rangle],$$

which are the fundamental solution of the Klein-Gordon equation and its derivative in t.

The endpoint Strichartz estimates which we use for the proof of Theorem 2.2 is the following. Such type of estimates for the wave and the Klein-Gordon equation in three space dimensions have been introduced by Machihara-Nakamura-Nakanishi-Ozawa in [7].

Theorem 3.1. *Let $n = 2$ and let $1 \leq q < \infty$. Then, the following estimates hold.*

$$\|\dot{U}(t)f\|_{L_t^2 L_r^\infty L_\omega^q} \lesssim \sqrt{q}\,\|f\|_{H^1}, \tag{3.1}$$

$$\|U(t)g\|_{L_t^2 L_r^\infty L_\omega^q} \lesssim \sqrt{q}\,\|g\|_{L^2}. \tag{3.2}$$

Moreover, we have

$$\Big\|\int_0^t \dot{U}(t-t')F(t')dt'\Big\|_{L_t^2 L_r^\infty L_\omega^q} \lesssim \sqrt{q}\ \|F\|_{L_t^1 H_x^1}, \tag{3.3}$$

$$\Big\|\int_0^t U(t-t')G(t')dt'\Big\|_{L_t^2 L_r^\infty L_\omega^q} \lesssim \sqrt{q}\ \|G\|_{L_t^1 L_x^2}. \tag{3.4}$$

For the proof of Theorem 3.1 we refer to [3].

4 Proof of Theorem 2.2

In this section, we give a proof of Theorem 2.2. For simplicity we consider the case $t > 0$ since the other case is treated similarly. We construct the solution in the space

$$X = \{u \in C([0,\infty);\, H^2(H_\omega^\delta)) \cap C^1([0,\infty);\, H^1(H_\omega^\delta)) \mid \|u\|_X < \infty\},$$

where

$$\|u\|_X = \|\partial u\|_{L_t^\infty H^1(H_\omega^\delta)} + \|u\|_{L_t^\infty H^1(H_\omega^\delta)} + \|\partial\langle\nabla_\omega\rangle^\delta u\|_{L_t^2 L_r^\infty L_\omega^q} + \|\langle\nabla_\omega\rangle^\delta u\|_{L_t^2 L_r^\infty L_\omega^q},$$

with $\delta > 1/q > 0$. To prove Theorem 2.2 we set the right hand side of (2.7) as

$$\Phi[u](t) = \dot{U}(t)v_0 + U(t)v_1 + \int_0^t U(t-t')C(u(t'), \partial u(t'))\,dt' + u(t)^2/2,$$

and show that Φ is a contraction map on a small closed ball in X. Then, we are able to find a unique solution u to (2.7) as a fixed point of Φ.

Since

$$\|\partial f\|_{H^1(H^\delta_\omega)} + \|f\|_{H^1(H^\delta_\omega)} \lesssim \|f\|_{H^2(H^\delta_\omega)} + \|\partial_t f\|_{H^1(H^\delta_\omega)},$$

we first estimate

$$\begin{aligned}
&\|\Phi[u](t)\|_{H^2(H^\delta_\omega)} \\
&\le \|\dot U(t)v_0\|_{H^2(H^\delta_\omega)} + \|U(t)v_1\|_{H^2(H^\delta_\omega)} \\
&\quad + \int_0^\infty \|U(t-t')C(u(t'),\partial u(t'))\|_{H^2(H^\delta_\omega)}\,dt' + \frac{1}{2}\|u(t)^2\|_{H^2(H^\delta_\omega)} \\
&\lesssim \|\langle\nabla_\omega\rangle^\delta v_0\|_{H^2_x} + \|\langle\nabla_\omega\rangle^\delta v_1\|_{H^1_x} + \|\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t H^1_x} + \|\langle\nabla_\omega\rangle^\delta u^2\|_{L^\infty_t H^2_x}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\|\partial_t\Phi[u](t)\|_{H^1(H^\delta_\omega)} \\
&\le \|\ddot U(t)v_0\|_{H^1(H^\delta_\omega)} + \|\dot U(t)v_1\|_{H^1(H^\delta_\omega)} \\
&\quad + \int_0^\infty \|\dot U(t-t')C(u(t'),\partial u(t'))\|_{H^1(H^\delta_\omega)}\,dt' + \frac{1}{2}\|\partial_t(u(t)^2)\|_{H^1(H^\delta_\omega)} \\
&\lesssim \|\langle\nabla_\omega\rangle^\delta v_0\|_{H^2_x} + \|\langle\nabla_\omega\rangle^\delta v_1\|_{H^1_x} + \|\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t H^1_x} + \|\langle\nabla_\omega\rangle^\delta u\partial_t u\|_{L^\infty_t H^1_x}.
\end{aligned}$$

Moreover, applying Theorem 3.1 we obtain

$$\begin{aligned}
&\|\nabla\langle\nabla_\omega\rangle^\delta\Phi[u]\|_{L^2_t L^\infty_r L^q_\omega} \\
&\le \|\dot U(t)\nabla\langle\nabla_\omega\rangle^\delta v_0\|_{L^2_t L^\infty_r L^q_\omega} + \|U(t)\nabla\langle\nabla_\omega\rangle^\delta v_1\|_{L^2_t L^\infty_r L^q_\omega} \\
&\quad + \left\|\int_0^t U(t-t')\nabla\langle\nabla_\omega\rangle^\delta C(u(t'),\partial u(t'))\,dt'\right\|_{L^2_t L^\infty_r L^q_\omega} + \frac{1}{2}\|\nabla\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega} \\
&\lesssim \|\nabla\langle\nabla_\omega\rangle^\delta v_0\|_{H^1_x} + \|\nabla\langle\nabla_\omega\rangle^\delta v_1\|_{L^2_x} + \|\nabla\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t L^2_x} + \|\nabla\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega}
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_t\langle\nabla_\omega\rangle^\delta\Phi[u]\|_{L^2_t L^\infty_r L^q_\omega} \\
&\le \|\ddot U(t)\langle\nabla_\omega\rangle^\delta v_0\|_{L^2_t L^\infty_r L^q_\omega} + \|\dot U(t)\langle\nabla_\omega\rangle^\delta v_1\|_{L^2_t L^\infty_r L^q_\omega} \\
&\quad + \left\|\int_0^t \dot U(t-t')\langle\nabla_\omega\rangle^\delta C(u(t'),\partial u(t'))\,dt'\right\|_{L^2_t L^\infty_r L^q_\omega} + \frac{1}{2}\|\partial_t\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega} \\
&\lesssim \|\langle\nabla_\omega\rangle^\delta v_0\|_{H^2_x} + \|\langle\nabla_\omega\rangle^\delta v_1\|_{H^1_x} + \|\nabla\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t H^1_x} + \|\partial_t\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega}.
\end{aligned}$$

Since the estimate on $\|\langle\nabla_\omega\rangle^\delta\Phi[u]\|_{L^2_t L^\infty_r L^q_\omega}$ is similar as above, we conclude that

$$\begin{aligned}
\|\Phi[u]\|_X \lesssim{}& \|\langle\nabla_\omega\rangle^\delta v_0\|_{H^2_x} + \|\langle\nabla_\omega\rangle^\delta v_1\|_{H^1_x} + \|\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t H^1_x} \\
&+ \|\langle\nabla_\omega\rangle^\delta u^2\|_{L^\infty_t H^2_x} + \|\langle\nabla_\omega\rangle^\delta u\partial_t u\|_{L^\infty_t H^1_x} + \|\nabla\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega} \\
&+ \|\partial_t\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega} + \|\langle\nabla_\omega\rangle^\delta u^2\|_{L^2_t L^\infty_r L^q_\omega}.
\end{aligned} \tag{4.1}$$

In the following, we estimate the right hand side of (4.1). For that purpose, we prepare the following lemma.

Lemma 4.1. *Let $s \geq 0$ and $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$, $1 < p < \infty$, $q_1 \neq \infty$, $r_2 \neq \infty$. Then, we have*

$$\|fg\|_{H^{s,p}(S^1)} \lesssim \|f\|_{H^{s,q_1}(S^1)}\|g\|_{L^{r_1}(S^1)} + \|f\|_{L^{q_2}(S^1)}\|g\|_{H^{s,r_2}(S^1)}.$$

For the proof of this lemma, see e.g. [7, §2].

Estimate of the first, the second, the fourth, and the fifth term on RHS of (4.1).

Since $v_0 = u_0 - u_0^2/2$, we have

$$\|\langle\nabla_\omega\rangle^\delta v_0\|_{H_x^2} \lesssim \|u_0\|_{H^2(H_\omega^\delta)} + \|\langle\nabla_\omega\rangle^\delta u_0^2\|_{H_x^2}. \tag{4.2}$$

Applying Lemma 4.1 we estimate the second term on the ritht hand side of (4.2) as follows. We first consider

$$\begin{aligned}
&\|\partial_j\partial_k\langle\nabla_\omega\rangle^\delta u_0^2\|_{L_x^2} \\
&\lesssim \|\nabla\langle\nabla_\omega\rangle^\delta u_0\|_{L^4}\|\nabla u_0\|_{L^4} + \|\langle\nabla_\omega\rangle^\delta u_0\|_{L_r^\infty L_\omega^q}\|\nabla^2 u_0\|_{L_r^2 L_\omega^p} + \|u_0\|_{L^\infty}\|\nabla^2\langle\nabla_\omega\rangle^\delta u_0\|_{L^2} \\
&\lesssim \|\langle\nabla_\omega\rangle^\delta u_0\|_{H^{3/2}}\|u_0\|_{H^{1/2}} + \|\langle\nabla_\omega\rangle^\delta u_0\|_{L_x^\infty}\|\langle\nabla_\omega\rangle^{1/q}\nabla^2 u_0\|_{L_x^2} + \|u_0\|_{H^2}\|u_0\|_{H^2(H_\omega^\delta)} \\
&\lesssim \|u_0\|_{H^2(H_\omega^\delta)}^2,
\end{aligned}$$

where we take $1/p + 1/q = 1$ and we applied the Sobolev embedding $H^{1/q}(S^1) \hookrightarrow L^p(S^1)$. The other terms $\|\nabla\langle\nabla_\omega\rangle^\delta u_0^2\|_{L_x^2}$ and $\|\langle\nabla_\omega\rangle^\delta u_0^2\|_{L_x^2}$ are easily handled to yield

$$\|\langle\nabla_\omega\rangle^\delta v_0\|_{H_x^2} \lesssim \|u_0\|_{H^2(H_\omega^\delta)} + \|u_0\|_{H^2(H_\omega^\delta)}^2.$$

Meanwhile, since $v_1 = u_1 - u_0 u_1$, we have

$$\|\langle\nabla_\omega\rangle^\delta v_1\|_{H_x^1} \lesssim \|u_1\|_{H^1(H_\omega^\delta)} + \|\langle\nabla_\omega\rangle^\delta(u_0 u_1)\|_{H_x^1}. \tag{4.3}$$

For the estimate of the second term on the right hand side of (4.3), we first consider

$$\begin{aligned}
\|\nabla\langle\nabla_\omega\rangle^\delta(u_0u_1)\|_{L_x^2} &\lesssim \|\langle\nabla_\omega\rangle^\delta\nabla u_0\|_{L^4}\|u_1\|_{L^4} + \|\nabla u_0\|_{L^4}\|\langle\nabla_\omega\rangle^\delta u_1\|_{L^4} \\
&\quad + \|\langle\nabla_\omega\rangle^\delta u_0\|_{L_r^\infty L_\omega^q}\|\nabla u_1\|_{L_r^2 L_\omega^p} + \|u_0\|_{L^\infty}\|\langle\nabla_\omega\rangle^\delta\nabla u_1\|_{L^2} \\
&\lesssim \|u_0\|_{H^2(H_\omega^\delta)}\|u_1\|_{H^1(H_\omega^\delta)}.
\end{aligned}$$

The other term $\|\langle\nabla_\omega\rangle^\delta(u_0u_1)\|_{L_x^2}$ is easily handled to yield

$$\|\langle\nabla_\omega\rangle^\delta v_1\|_{H_x^1} \lesssim \|u_1\|_{H^1(H_\omega^\delta)} + \|u_0\|_{H^2(H_\omega^\delta)}\|u_1\|_{H^1(H_\omega^\delta)}.$$

The estimate of the fourth and the fifth term are essentially the same as above, and we obtain

$$\|\langle\nabla_\omega\rangle^\delta u^2\|_{L_t^\infty H_x^2} + \|\langle\nabla_\omega\rangle^\delta u\partial_t u\|_{L_t^\infty H_x^1} \lesssim \|u\|_X^2.$$

Estimate of the third term on RHS (4.1).

Now we estimate the cubic nonlinearity. Recall that

$$C(u,\partial u) = -u((\partial_t u)^2 - |\nabla u|^2 - u^2/2).$$

So, to estimate $\|\langle\nabla_\omega\rangle^\delta C(u,\partial u)\|_{L^1_t H^1_x}$ we begin with

$$\begin{aligned}
&\|\nabla\langle\nabla_\omega\rangle^\delta u(\partial_t u)^2\|_{L^1_t L^2_x}\\
&\lesssim \|\langle\nabla_\omega\rangle^\delta \nabla u\|_{L^\infty_t L^2_x}\|\partial_t u\|^2_{L^2_t L^\infty_x} + \|\nabla u\|_{L^\infty_t L^2_r L^p_\omega}\|\partial_t u\|_{L^2_t L^\infty_x}\|\langle\nabla_\omega\rangle^\delta \partial_t u\|_{L^2_t L^\infty_r L^q_\omega}\\
&\quad + \|\langle\nabla_\omega\rangle^\delta u\|_{L^2_t L^\infty_r L^q_\omega}\|\partial_t u\|_{L^2_t L^\infty_x}\|\nabla\partial_t u\|_{L^\infty_t L^2_r L^p_\omega}\\
&\quad + \|u\|_{L^2_t L^\infty_x}\|\langle\nabla_\omega\rangle^\delta \partial_t u\|_{L^2_t L^\infty_r L^q_\omega}\|\nabla\partial_t u\|_{L^\infty_t L^2_r L^p_\omega}\\
&\quad + \|u\|_{L^2_t L^\infty_x}\|\partial_t u\|_{L^2_t L^\infty_x}\|\langle\nabla_\omega\rangle^\delta \nabla\partial_t u\|_{L^\infty_t L^2_x}\\
&\lesssim \|u\|_{L^\infty_t H^1(H^\delta_\omega)}\|\langle\nabla_\omega\rangle^\delta \partial_t u\|^2_{L^2_t L^\infty_r L^q_\omega}\\
&\quad + \|\langle\nabla_\omega\rangle^\delta u\|_{L^2_t L^\infty_r L^q_\omega}\|\langle\nabla_\omega\rangle^\delta \partial_t u\|_{L^2_t L^\infty_r L^q_\omega}\|\partial_t u\|_{L^\infty_t H^1(H^\delta_\omega)}\\
&\lesssim \|u\|^3_X,
\end{aligned}$$

where we take $1/p+1/q=1$ and we applied the Sobolev embedding $H^{1/q}(S^1)\hookrightarrow L^p(S^1)$. Since it is easier to show $\|\langle\nabla_\omega\rangle^\delta u(\partial_t u)^2\|_{L^1_t L^2_x}\lesssim\|u\|^3_X$, we obtain

$$\|\langle\nabla_\omega\rangle^\delta u(\partial_t u)^2\|_{L^1_t H^1_x}\lesssim\|u\|^3_X.$$

We next consider

$$\begin{aligned}
\|\nabla\langle\nabla_\omega\rangle^\delta u|\nabla u|^2\|_{L^1_t L^2_x} &\lesssim \|\langle\nabla_\omega\rangle^\delta \nabla u\|_{L^\infty_t L^2_x}\|\nabla u\|^2_{L^2_t L^\infty_x}\\
&\quad + \|\langle\nabla_\omega\rangle^\delta u\|_{L^2_t L^\infty_r L^q_\omega}\|\nabla u\|_{L^2_t L^\infty_x}\|\nabla^2 u\|_{L^\infty_t L^2_r L^p_\omega}\\
&\quad + \|u\|_{L^2_t L^\infty_x}\|\langle\nabla_\omega\rangle^\delta \nabla u\|_{L^2_t L^\infty_r L^q_\omega}\|\nabla^2 u\|_{L^\infty_t L^2_r L^p_\omega}\\
&\quad + \|u\|_{L^2_t L^\infty_x}\|\nabla u\|_{L^2_t L^\infty_x}\|\langle\nabla_\omega\rangle^\delta \nabla^2 u\|_{L^\infty_t L^2_x}\\
&\lesssim \|u\|_{L^\infty_t H^1(H^\delta_\omega)}\|\langle\nabla_\omega\rangle^\delta \nabla u\|^2_{L^2_t L^\infty_r L^q_\omega}\\
&\quad + \|\langle\nabla_\omega\rangle^\delta u\|_{L^2_t L^\infty_r L^q_\omega}\|\langle\nabla_\omega\rangle^\delta \nabla u\|_{L^2_t L^\infty_r L^q_\omega}\|u\|_{L^\infty_t H^2(H^\delta_\omega)}\\
&\lesssim \|u\|^3_X.
\end{aligned}$$

Since it is easier to show $\|\langle\nabla_\omega\rangle^\delta u|\nabla u|^2\|_{L^1_t L^2_x}\lesssim\|u\|^3_X$, we obtain

$$\|\langle\nabla_\omega\rangle^\delta u|\nabla u|^2\|_{L^1_t H^1_x}\lesssim\|u\|^3_X.$$

Moreover, it is easier to show $\|u^3\|_{L^1_t H^1_x}\lesssim\|u\|^3_X$. Therefore, combining the above estimates we finally obtain

$$\|C(u,\partial u)\|_{L^1_t H^1_x}\lesssim\|u\|^3_X.$$

Estimate of the sixth, the seventh, and the eighth term on RHS (4.1).

Since those terms are similarly treated, we only estimate the sixth term here.

$$\begin{aligned}\|\nabla\langle\nabla_\omega\rangle^\delta u^2\|_{L_t^2L_r^\infty L_\omega^q} &\lesssim \|\langle\nabla_\omega\rangle^\delta u\|_{L_t^\infty L_r^\infty L_\omega^q}\|\nabla u\|_{L_t^2L_x^\infty} + \|u\|_{L_t^\infty L_x^\infty}\|\langle\nabla_\omega\rangle^\delta\nabla u\|_{L_t^2L_r^\infty L_\omega^q}\\ &\lesssim \|u\|_{L_t^\infty H^2(H_\omega^\delta)}\|\langle\nabla_\omega\rangle^\delta\nabla u\|_{L_t^2L_r^\infty L_\omega^q}\\ &\lesssim \|u\|_X^2,\end{aligned}$$

where we have used the Sobolev embedding $H^2(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$.

Combining the estimates above, we conclude that

$$\|\Phi[u]\|_X \le C_0\varepsilon + C_1\|u\|_X^3 + C_2\|u\|_X^2,$$

where $C_0 = C_0(\|u_0\|_{H^2(H_\omega^\delta)}, \|u_1\|_{H^1(H_\omega^\delta)})$. Furthermore, from a similar argument we also obtain

$$\|\Phi[u]-\Phi[w]\|_X \le C_1'(\|u\|_X + \|u\|_X^2 + \|w\|_X + \|w\|_X^2)\|u-w\|_X.$$

Form the above estimates, we are able to observe that Φ is a contraction map on a ball in X with radius $2C_0\varepsilon$ if $\varepsilon > 0$ is sufficiently small. This completes the proof of Theorem 2.2.

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